# Parametric Equations and Polar Coordinates

Parametric Equations and Polar Coordinates

In the previous lesson we presented the concept of parametric equations and the concept of parameterized curve. And we graphed several sets of parametric equations and discussed how to eliminate the parameter to get an algebraic equation which will often help with the graphing process. Today we will apply calculus to parametric curves. Specifically, we find tangents, slopes, lengths, and areas associated with parametrized curves.

# # Calculus with Parametric Curves

Tangent is a line which locally touches a curve at one and only one point. A parametrized curve $x = f(t)$ and $y = g(t)$ is differentiable at t if $f$ and g are differentiable at t. At a point on a differentiable parametrized curve where y is also a differentiable function of x, the derivatives $dy\ dt/$, $dx\ dt/$, and $dy\ dx/$ are related by the Chain Rule.

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

If $\frac{dx}{dt} \neq 0$, we may divide both sides of this equation by $\frac{dx}{dt}$ to solve for $\frac{dy}{dx}$ .

## Parametric Equations and Polar Coordinates

**Parametric Formula for $dy/dx$**

If all three derivatives exist and $dx/dt \neq 0$,

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}. \qquad (1)$$

If parametric equations define $y$ as a twice-differentiable function of $x$, we can apply Equation (1) to the function $dy/dx = y'$ to calculate $d^2y/dx^2$ as a function of $t$:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(y') = \frac{dy'/dt}{dx/dt} \qquad \text{Eq. (1) with } y' \text{ in place of } y$$

**Parametric Formula for $d^2y/dx^2$**

If the equations $x = f(t)$, $y = g(t)$ define $y$ as a twice-differentiable function of $x$, then at any point where $dx/dt \neq 0$ and $y' = dy/dx$,

$$\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt}. \qquad (2)$$

**Ex1**: Find the tangent to the curve اوجد المماس او الظل للمنحني

$$x = sec\, t, \qquad y = tan\, t, \qquad -\frac{\pi}{2} \le t \le \frac{\pi}{2}$$

at the point $(\sqrt{2}, 1)$, where $t = \frac{\pi}{4}$.

**Sol.** The slope of the curve at $t$ is ميل المنحني عند $t$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{sec^2 t}{sec\, t\, tant} = \frac{sect}{tant}$$

Setting $t = \frac{\pi}{4}$ gives

**Parametric Equations and Polar Coordinates**

$$\left.\frac{dy}{dx}\right|_{t=\pi/4} = \frac{\sec(\pi/4)}{\tan(\pi/4)}$$

$$= \frac{\sqrt{2}}{1} = \sqrt{2}.$$

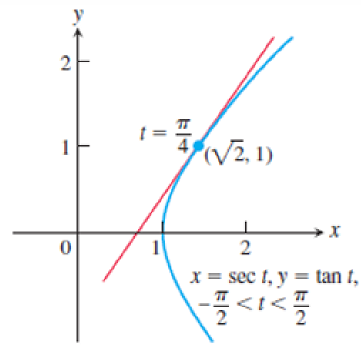
$\therefore$ The tangent line when $a = \sqrt{2}$, $b = 1$ and $m = \sqrt{2}$

$$y - b = m(x - a)$$

$$y - 1 = \sqrt{2}(x - \sqrt{2})$$

$$y = \sqrt{2}x - 2 + 1$$

$$y = \sqrt{2}x - 1.$$




**Ex.2**: Find the $\frac{dy}{dx}$ to the parametric curve

$$x = t + \sin t, \qquad y = t - \cos t$$

**Sol.**

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1-(-\sin t)}{1+\cos t} = \frac{1+\sin t}{1+\cos t}$$

**Ex3:** Fine the slope of the tangent line to

$x = \ln t, \qquad y = 1 + t^2$ at $t = 1$

**Sol.**

$$m = \left.\frac{dy}{dx}\right|_{t=1} = \left.\frac{dy/dt}{dx/dt}\right|_{t=1} = \frac{2t}{\frac{1}{t}} = \frac{2(1)}{\frac{1}{1}} = 2$$

**<u>Ex.4</u>**: Find the equation of the tangent to the parametric curve

$$x = 1 - t^3, \qquad y = t^2 - 3t + 1 \quad \text{at } t = 1.$$

**<u>Sol.</u>** To find the equation of the tangent we need to find the point and the slope

so to find the point when $t = 1$ we have,

$$\begin{aligned} x &= 1 - t^3 && \Rightarrow x = 1 - 1^3 = 0 \\ y &= t^2 - 3t + 1 && \Rightarrow y = 1^2 - 3(1) + 1 = -1 \\ & && \therefore (0, -1) \end{aligned}$$

now find the slope

$$\left.\frac{dy}{dx}\right|_{t=1} = \left.\frac{dy/dt}{dx/dt}\right|_{t=1} = \frac{2t-3}{-3t^2} = \frac{2(1)-3}{-3(1^2)} = \frac{1}{3} = m$$

$\therefore$ the equation of the tangent is

$$y - b = m(x - a)$$

$$y + 1 = \frac{1}{3}(x - 0)$$

$$y = \frac{1}{3}x - 1$$

**<u>Ex 5:</u>** Find $\frac{d^2y}{dx^2}$ as a function of $t$ if $\quad x = t - t^2, \quad y = t - t^3$ .

**<u>Sol.</u>**

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1. Express $y' = \frac{dy}{dx}$ in terms of $t$.

$$y' = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1-3t^2}{1-2t}$$

2. Differentiate $y'$ with respect to $t$.

$$\frac{dy'}{dt} = \frac{d}{dt}\left(\frac{1-3t^2}{1-2t}\right) = \frac{(1-2t).(-6t)-(1-3t^2).(-2)}{(1-2t)^2} = \frac{-6t+12t^2+2-6t^2}{(1-2t)^2} = \frac{2-6t+6t^2}{(1-2t)^2}$$

3. Divide $\frac{dy'}{dt}$ by $\frac{dy}{dx}$.

$$\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt} = \frac{(2-6t+6t^2)/(1-2t)^2}{1-2t} = \frac{2-6t+6t^2}{(1-2t)^3}$$

**Ex 6:** Find $\frac{d^2y}{dx^2}$ as a function of $t$ if $x = t^2, \quad y = t^3$ .

**Sol.** using the chain rule,

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2}{2t} = \frac{3}{2}t$$

Now find the second derivative,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{dy'/dt}{dx/dt} = \frac{\frac{3}{2}}{2t} = \frac{3}{4t}$$

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**H.W**

1. Find an equation for the line tangent to the curve at the point defined by the given value of $t$. Also, find the value of $\frac{d^2y}{dx^2}$ at this point.

$x = 2\cos t, \quad y = 2\sin t, \quad t = \frac{\pi}{4}$

2. Find the slope of the curve $x^3 + 2t^2 = 9, \quad 2y^3 - 3t^2 = 4$ at the given value of $t = 2$.