[2024]

[POWER SERIES]

[Fourth lecture]

1.1 Power series method

We consider the second-order linear homogeneous differential equation for $y = y(x)$

$$P(x)y'' + Q(x)y' + R(x)y = 0$$

where $P(x)$, $Q(x)$ and $R(x)$ are polynomials

The idea of power series method is to assume that the unknown function y can be expanded into a power series:

$$y = \sum_{n=0}^{\infty} a_n\, x^n = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \cdots$$

$$y' = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 + 6a_6x^5 + \cdots$$

$$y'' = 2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + 30a_6x^4 + 42a_7x^5 + \cdots$$

Substituting in the differential equation and requiring that the coefficients of each power of x sum to zero, we can find all the constants a_n $\forall\, n = 2,3,4,\cdots$ in terms of a_0 or a_1.

Power Series

Define a function f on the interval $(-1, 1)$

$$f(x) = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x} \quad \text{for } |x| < 1$$

As the Limit

f can be viewed as the limit of a sequence of polynomials:

$$f(x) = \lim_{n\to\infty} p_n(x),$$

where $p_n(x) = 1 + x + x^2 + x^3 + \cdots + x^n$.

Variations on the Geometric Series (I)

Closed forms for many power series can be found by relating the series to the geometric series

Examples 1.

$$f(x) = \sum_{k=0}^{\infty} (-1)^k x^k = 1 - x + x^2 - x^3 + \cdots$$

$$= \sum_{k=0}^{\infty} (-x)^k = \frac{1}{1-(-x)} = \frac{1}{1+x}, \quad \text{for } |x| < 1.$$

$$f(x) = \sum_{k=0}^{\infty} 2^k x^{k+2} = x^2 + 2x^3 + 4x^4 + 8x^5 + \cdots$$

$$= x^2 \sum_{k=0}^{\infty} (2x)^k = \frac{x^2}{1-2x} \quad \text{for } |2x| < 1.$$

Variations on the Geometric Series (II)
Closed forms for many power series can be found by relating the series to the geometric series
Examples 2.

$$f(x) = \sum_{k=0}^{\infty}(-1)^k x^{2k} = 1 - x^2 + x^4 - x^6 + \cdots$$
$$= \sum_{k=0}^{\infty}(-x^2)^k = \frac{1}{1-(-x^2)} = \frac{1}{1+x^2}, \quad \text{for } |x| < 1.$$

$$f(x) = \sum_{k=0}^{\infty}\frac{x^{2k+1}}{3^k} = x + \frac{1}{3}x^3 + \frac{1}{9}x^5 + \frac{1}{27}x^7 + \cdots$$
$$= x\sum_{k=0}^{\infty}\left(\frac{x^2}{3}\right)^k = \frac{x}{1-(x^2/3)} = \frac{3x}{3-x^2} \quad \text{for } |x^2/3| < 1.$$

Example 1: Solve $y'' + 2xy' + 2y = 0$ by power series method

Solution :

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \cdots$$
$$y' = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 + 6a_6x^5 + \cdots$$
$$y'' = 2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + 30a_6x^4 + 42a_7x^5 + \cdots$$

Now

$$y'' = 2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + 30a_6x^4 + 42a_7x^5 + \cdots$$
$$2xy' = 0 \quad + 2a_1x + 4a_2x^2 + 6a_3x^3 \quad + 8a_4x^4 \quad + 12a_5x^5 + \cdots$$
$$2y = 2a_0 \quad + 2a_1x + 2a_2x^2 + 2a_3x^3 \quad + 2a_4x^4 \quad + 2a_5x^5 + \cdots$$

$$2a_2 + 2a_0 = 0 \quad \Rightarrow \boxed{a_2 = -a_0} \quad \text{and} \quad 6a_3 + 2a_1 + 2a_1 = 0 \quad \Rightarrow \boxed{a_3 = -(2/3)a_1}$$

$$12a_4 + 4a_2 + 2a_2 = 0 \quad \Rightarrow a_4 = -(1/2)a_2 \quad \Rightarrow \boxed{a_4 = (1/2)a_0}$$

$$20a_5 + 6a_3 + 2a_3 = 0 \quad \Rightarrow a_5 = -(2/5)a_3 \quad \Rightarrow a_5 = (4/15)a_1$$

$$y = a_0 + a_1x - a_0x^2 - (2/3)a_1x^3 + (1/2)a_0x^4 + (4/15)a_1x^5 + \cdots$$
$$y = a_0\left(1 - x^2 + \frac{1}{2}x^4 + \cdots\right) + a_1\left(x - \frac{2}{3}x^3 + \frac{4}{15}x^5 + \cdots\right)$$

Example 2: Solve $y'' - xy' + 4y = 0$ by using power series method

Solution :

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \cdots$$

$$y' = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 + 6a_6x^5 + \cdots$$

$$y'' = 2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + 30a_6x^4 + 42a_7x^5 + \cdots$$

Now

$$\begin{aligned} y'' &= \boxed{2a_2} \boxed{+6a_3} x \boxed{+12a_4} x^2 \boxed{+20a_5} x^3 \boxed{+30a_6} x^4 \boxed{+42a_7} x^5 + \cdots \\ -xy' &= \boxed{0} \boxed{-a_1} x \boxed{-2a_2} x^2 \boxed{-3a_3} x^3 \boxed{-4a_4} x^4 \boxed{-5a_5} x^5 - \cdots \\ 4y &= \boxed{4a_0} \boxed{+4a_1} x \boxed{+4a_2} x^2 \boxed{+4a_3} x^3 \boxed{+4a_4} x^4 \boxed{+4a_5} x^5 + \cdots \end{aligned}$$

$2a_2 + 4a_0 = 0 \Rightarrow \boxed{a_2 = -2a_0}$, $6a_3 - a_1 + 4a_1 = 0 \Rightarrow \boxed{a_3 = -(1/2)a_1}$

$12a_4 - 2a_2 + 4a_2 = 0 \Rightarrow a_4 = -(1/6)a_2 \Rightarrow \boxed{a_4 = (1/3)a_0}$

$20a_5 - 3a_3 + 4a_3 = 0 \Rightarrow a_5 = -(1/20)a_3 \Rightarrow \boxed{a_5 = (1/40)a_1}$

$30a_6 - 4a_4 + 4a_4 = 0 \Rightarrow \boxed{a_6 = 0}$ and $a_8 = a_{10} = \cdots = 0$

$$y = a_0 + a_1x - 2a_0x^2 - (1/2)a_1x^3 + (1/3)a_0x^4 + (1/40)a_1x^5 + \cdots$$

$$y = a_0\left(1 - 2x^2 + \frac{x^4}{3}\right) + a_1\left(x - \frac{x^3}{2} + \frac{x^5}{40} + \cdots\right)$$

Exercises

Solve the differential equations by using power series method

(1) $x y'' + y' + xy = 0$

(2) $x^2y'' + y' + x^2y = 0$

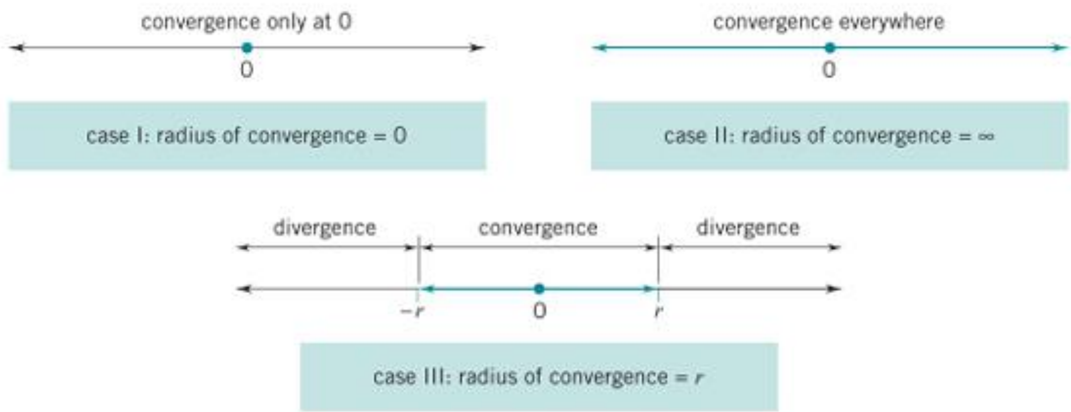
(3) $y'' + 3xy' - y = 0$

(4) $(x-1)y'' + 2y' = 0$

1.2 Radius of Convergence

Radius of Convergence

There are exactly three possibilities for a power series: $\sum a_k x^k$.



Radius of Convergence: Ratio Test (I)

The radius of convergence of a power series can usually be found by applying the ratio test. In some cases the root test is easier.

Example 3.

$$f(x) = \sum_{k=1}^{\infty} k^2 x^k = x + 4x^2 + 9x^3 + \cdots$$

$$\text{Ratio Test}: \left|\frac{a_{k+1}}{a_k}\right| = \left|\frac{(k+1)^2 x^{k+1}}{k^2 x^k}\right|$$

$$= \frac{(k+1)^2}{k^2}|x| \to |x| \quad \text{as } k \to \infty$$

Thus the series converges absolutely when $|x| < 1$ and diverges when $|x| > 1$.

Radius of Convergence: Ratio Test (II)

The radius of convergence of a power series can usually be found by applying the ratio test. In some cases the root test is easier.
Example 4.

$$f(x)=\sum_{k=1}^{\infty}\frac{(-1)^k}{k!}x^k = 1-x+\frac{1}{2}x^2-\frac{1}{6}x^3+\cdots = e^{-x}$$

$$\text{Ratio Test}: \left|\frac{a_{k+1}}{a_k}\right| = \left|\frac{x^{k+1}/(k+1)!}{x^k/k!}\right|$$
$$= \frac{k!}{(k+1)!}\left|\frac{x^{k+1}}{x^k}\right| = \frac{1}{k+1}|x| \to 0<1 \quad \text{for all } x$$

Thus the series converges absolutely for all x.

Radius of Convergence: Ratio Test (III)

The radius of convergence of a power series can usually be found by applying the ratio test. In some cases the root test is easier.
Example 5.

$$f(x)=\sum_{k=1}^{\infty}\left(\frac{k+1}{k}\right)^{k^2}x^k = 2x+(3/2)^4x^2+(4/3)^9x^3+\cdots$$

$$\text{Ratio Test}:(|a_k|)^{\frac{1}{k}} = \left(\left(\frac{k+1}{k}\right)^{k^2}|x|^k\right)^{\frac{1}{k}} = \left(\frac{k+1}{k}\right)^{k}|x|$$
$$= \left(1+\frac{1}{k}\right)^k|x| \to e|x|<1 \quad \text{if } |x|<1/e$$

Thus the series converges absolutely when $|x|<1/e$ and diverges when $|x|>1/e$.

Interval of Convergence

For a series with radius of convergence r, the interval of convergence can be $[-r,r]$, $(-r,r]$, $[-r,r)$, or $(-r,r)$.

Example 6. In general, *the behavior of a power series at $-r$ and at r is not predictable.* For example, the series

$$\sum x^k,\quad \sum\frac{(-1)^k}{k}x^k,\quad \sum\frac{1}{k}x^k,\quad \sum\frac{1}{k^2}x^k$$

all have radius of convergence 1, but the first series converges only on $(-1,1)$, the second converges on $(-1,1]$, but the third converges on $[-1,1)$, the fourth on $[-1,1]$.

Interval of Convergence

$$f(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} x^k$$

$$\text{Ratio Test : } \left|\frac{a_{k+1}}{a_k}\right| = \left|\frac{x^{k+1}/(k+1)}{x^k/k}\right| = \frac{k}{k+1}|x| \to |x|$$

Thus the series converges absolutely when $|x| < 1$ and diverges when $|x| > 1$.
So the radius of convergence is 1

$$x = -1 : \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} (-1)^k = \sum_{k=1}^{\infty} \frac{-1}{k} \text{ diverges}$$

$$x = 1 : \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} (1)^k = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \text{ converges conditionally}$$

The interval of convergence is $(-1, 1]$.

2 Differentiation and Integration

2.1 Differentiation and Integration

Differentiation and Integration Theorem

Let $f(x) = \sum a_k x^k$ be a power series with a nonzero radius of convergence r. Then

$$f'(x) = \sum a_k \, k \, x^{k-1} \quad \text{for } |x| < r$$

$$\int f(x)\, dx = \sum \frac{a_k}{k+1} x^{k+1} + C \quad \text{for } |x| < r$$

$$\text{Geometric series:} \quad \frac{1}{1-x} = \sum_{k=0}^{\infty} x^k \quad \text{for } |x| < 1$$

$$\text{Differentiation:} \quad \frac{1}{(1-x)^2} = \sum_{k=0}^{\infty} k \, x^{k-1} \sum_{k=0}^{\infty} (k+1)\, x^k \ \text{ for } |x| < 1$$

$$\text{Integration:} \quad -\ln(1-x) = \sum_{k=0}^{\infty} \frac{1}{k+1}\, x^{k+1} = \sum_{k=1}^{\infty} \frac{1}{k}\, x^k \ \text{ for } |x| < 1$$

2.2 Examples

Power Series Expansion of $\ln(1+x)$

$$\text{Note: } \frac{d}{dx}\ln(1+x) = \frac{1}{1+x} = \sum_{k=0}^{\infty}(-1)^k x^k \text{ for } |x|<1$$

$$\text{Integration: } \ln(1+x) = \sum_{k=0}^{\infty}\frac{(-1)^k}{k+1}x^{k+1}(+C=0)$$

$$= \sum_{k=1}^{\infty}\frac{(-1)^k}{k}x^k = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots$$

The interval of convergence is $(-1,1]$. At $x=1$,

$$\ln 2 = \sum_{k=1}^{\infty}\frac{(-1)^k}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

Power Series Expansion of $\tan^{-1} x$

$$\text{Note: } \frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2} = \sum_{k=0}^{\infty}(-1)^k x^{2k} \text{ for } |x|<1$$

$$\text{Integration: } \tan^{-1} x = \sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}x^{2k+1}(+C=0)$$

$$= x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots$$

The interval of convergence is $(-1,1]$. At $x=1$,

$$\tan^{-1} 1 = \sum_{k=1}^{\infty}\frac{(-1)^k}{2k+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}$$

Practice Problems

Determine the interval of convergence of the following power series.

1. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n+1}$
2. $\displaystyle\sum_{n=1}^{\infty} \sqrt{n} x^n$
3. $\displaystyle\sum_{n=1}^{\infty} n^n x^n$
4. $\displaystyle\sum_{n=1}^{\infty} \frac{10^n x^n}{n^3}$
5. $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{5^n n^5}$
6. $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$
7. $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{(x-3)^n}{2n+1}$
8. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{4^n} (x+1)^n$
9. $\displaystyle\sum_{n=1}^{\infty} \frac{(3x-2)^n}{n3^n}$
10. $\displaystyle\sum_{n=1}^{\infty} \frac{n(x-4)^n}{n^3+1}$
11. $\displaystyle\sum_{n=1}^{\infty} n!(2x-1)^n$
12. $\displaystyle\sum_{n=1}^{\infty} \frac{(4x+1)^n}{n^2}$

Answers to Practice Problems

1. $-1 < x \leq 1$
2. $-1 < x < 1$
3. $x = 0$
4. $-\frac{1}{10} \leq x \leq \frac{1}{10}$
5. $-5 \leq x \leq 5$
6. $-\infty < x < \infty$
7. $2 < x \leq 4$
8. $-5 < x < 3$
9. $-\frac{1}{3} \leq x < \frac{5}{3}$
10. $3 \leq x \leq 5$
11. $x = \frac{1}{2}$
12. $-\infty < x < \infty$