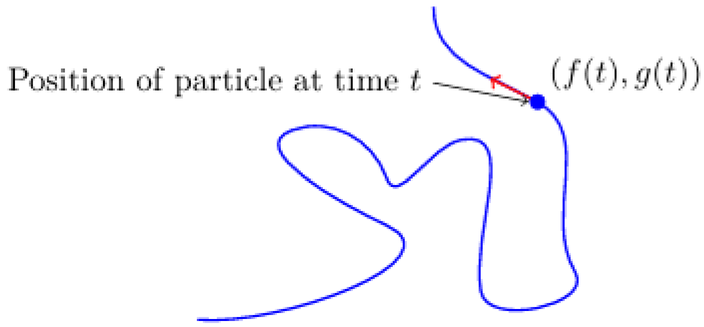
Parametrisations of Plane Curves

Sixth lecture

A.L. Raghad .S A.L. Zaidoun Muhannad

Parametrisations of Plane Curves

Parametric Equations: Below we have the path of a moving particle on the xy-plane. We can sometimes describe such a path by a pair of equations, x = f(t) and y = g(t), where f(t) and g(t) are continuous functions. Equations like these describe more general curves than those described by a single function, and they provide not only the graph of the path traced out but also the location of the particle (x,y) = (f(t),g(t)) at any time t.



Definitions: If x and y are given as functions

$$x = f(t) \qquad y = g(t),$$

over an interval I of t-values, then the set of points $(x, y) = (f(t), g(t))$ defined by these equations is a parametric curve.

The equations are parametric equations for the curve.

The variable t is the parameter for the curve and its domain I is the parameter interval.

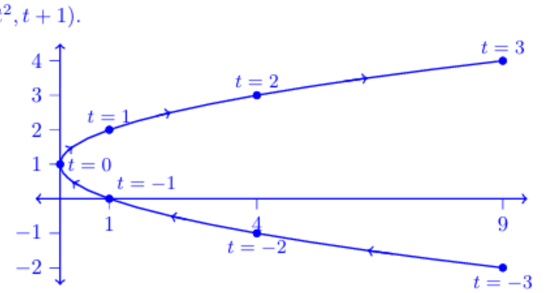
If I is a closed interval, $a \le t \le b$, the initial point of the curve is the point $(f(a), g(a))$ and the terminal point of the curve is $(f(b), g(b))$.

Example 1: Sketch the curve defined by the parametric equations

$$x = t^2, \quad y = t + 1, \quad -\infty < t < \infty.$$

The (x, y) coordinates are determined by values for t, $(t^2, t + 1)$.

t	x	y
-3	9	-2
-2	4	-1
-1	1	0
0	0	1
1	1	2
2	4	3
3	9	4



Here the arrows indicate the direction of travel.

Example 2: Identify geometrically the curve in Example 1 by eliminating the parameter t and obtaining an algebraic equation in x and y.

Since both x and y are defined in terms of t, we can use substitution to eliminate the parameter:

Option 1:

$$\begin{array}{lll} & y = t+1 & x = t^2 \\ \implies & y - 1 = t & x = (y-1)^2 \\ & \implies & \boxed{x = y^2 - 2y + 1} \end{array}$$

Option 2:

$$\begin{array}{lll} & x = t^2 & y = t+1 \\ \implies & \pm\sqrt{x} = t & y = \pm\sqrt{x} + 1 \\ & \implies & \boxed{y = \sqrt{x} + 1, \quad y = -\sqrt{x} + 1} \end{array}$$

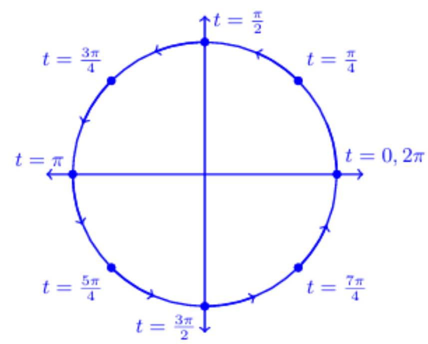
Example 3: Graph the parametric curves

$$\text{(a)}\ \ x = \cos(t), \qquad y = \sin(t), \qquad 0 \le t \le 2\pi,$$

$$\text{(b)}\ \ x = a\cos(t), \qquad y = a\sin(t), \qquad 0 \le t \le 2\pi, \qquad a \in \mathbb{R}.$$

(a)

t	x	y
0	1	0
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
$\frac{\pi}{2}$	0	1
$\frac{3\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
π	-1	0
$\frac{5\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$
$\frac{3\pi}{2}$	0	-1
$\frac{7\pi}{4}$	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$
2π	1	0



Here the arrows indicate the direction of travel.

We see then that these parametric equations correspond to travelling around the unit circle anticlockwise. Algebraically we can verify this to see that

$$\cos^2(t) + \sin^2(t) = x^2 + y^2 = 1$$

which is precisely the equation for a circle of radius 1, centred at the origin.

(b) It should come at no surprise that these parametric equations correspond to travelling around the circle of radius a, centred at the origin, anticlockwise.

Example 4: The position $P(x, y)$ of a particle moving in the xy-plane is given by the equations and parameter interval

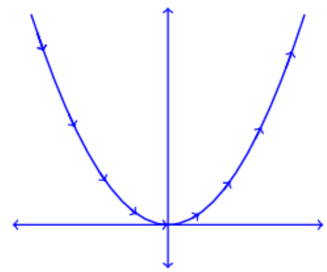
$$x = \sqrt{t}, \quad y = t, \quad t \geq 0.$$

Identify the path traced by the particle and describe the motion.

We can either find a table of values and plot or we can find a Cartesian equation. The latter is more straight forward and we see that $x = \sqrt{y}$ for $y \geq 0$ (or $y = x^2$ for $x \geq 0$). So the curve is the part of $y = x^2$ lying in the first quadrant of the xy-plane.

Example 5 - Natural Parametrisation: A parametrisation of the function $f(x) = x^2$ is given by

Let $x = t$. Then $y = x^2 = t^2$ and so the *natural parametrisation* of the curve $y = x^2$ is (t, t^2) where $-\infty < t < \infty$.



Example 6: Find a parametrisation for the line through the point (a, b) having slope m.

A Cartesian equation of the line through (a, b) with slope m is

$$y - b = m(x - a).$$

Let $t = x - a$. Then $y - b = mt$ so $y = mt + b$. Therefore a parametrisation is

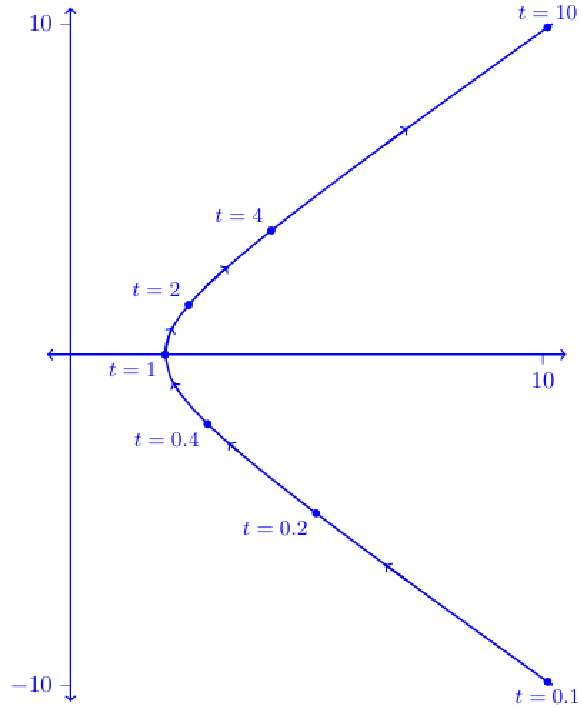
$$(x, y) = (t + a, mt + b), \quad -\infty < t < \infty.$$

It is important that the usage of the phrase "<u>a</u> parametrisation" is precise here since parametrisations are <u>not</u> unique. Here we could also use the *natural parametrisation* to obtain $(x, y) = (t, mt - (ma - b))$, $-\infty < t < \infty$.

Example 7: Sketch and identify the path traced by the point $P(x, y)$ if

$$x = t + \frac{1}{t}, \quad y = t - \frac{1}{t}, \quad t > 0.$$

t	x	y
0.1	10.1	−9.9
0.2	5.2	−4.8
0.4	2.9	2.1
1	2	0
2	2.5	1.5
4	4.25	3.75
10	10.1	9.9



(1) $x + y = \left(t + \frac{1}{t}\right) + \left(t - \frac{1}{t}\right) = 2t$

(2) $x - y = \left(t + \frac{1}{t}\right) - \left(t - \frac{1}{t}\right) = \frac{2}{t}$

(3) $x^2 - y^2 = (x+y)(x-y) = (2t)\left(\frac{2}{t}\right) = 4$

The Cartesian equation $x^2 - y^2 = 4$ is the standard form for the equation of a hyperbola.