SHEAR AND MOMENT DIAGRAMS

Members that are slender and support loadings that are applied perpendicular to their longitudinal axis are called ***beams.*** In general, beams are long, straight bars having a constant cross-sectional area. Often they are classified as to how they are supported. For example, a *simply supported beam* is pinned at one end and roller supported at the other, Fig. 6–1, a *cantilevered beam* is fixed at one end and free at the other, and an *overhanging beam* has one or both of its ends freely extended over

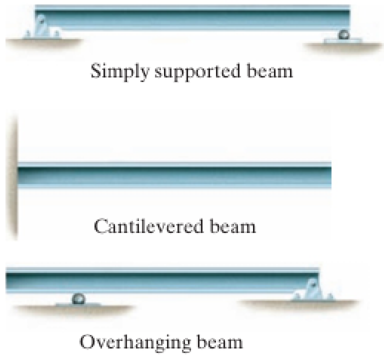


Fig. 6–1

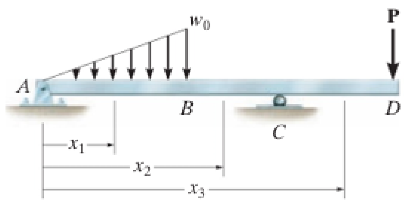


Fig. 6–2

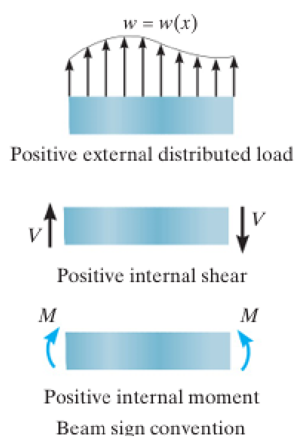


Fig. 6–3

the supports. Beams are considered among the most important of all structural elements. They are used to support the floor of a building, the deck of a bridge, or the wing of an aircraft. Also, the axle of an automobile, the boom of a crane, even many of the bones of the body act as beams.

Because of the applied loadings, beams develop an internal shear force and bending moment that, in general, vary from point to point along the axis of the beam. In order to properly design a beam it therefore becomes important to determine the *maximum* shear and moment in the beam. One way to do this is to express V and M as functions of their arbitrary position x along the beam's axis, and then plot these functions. They represent the ***shear and moment diagrams***, respectively. The maximum values of V and M can then be obtained directly from these graphs. Also, since the shear and moment diagrams provide detailed information about the *variation* of the shear and moment along the beam's axis, they are often used by engineers to decide where to place reinforcement materials within the beam or how to proportion the size of the beam at various points along its length.

In order to formulate V and M in terms of x we must choose the origin and the positive direction for x. Although the choice is arbitrary, most often the origin is located at the left end of the beam and the positive x direction is to the right.

Since beams can support portions of a distributed load and concentrated forces and couple moments, the internal shear and moment functions of x will be *discontinuous,* or their slopes will be discontinuous, at points where the loads are applied. Because of this, these functions must be determined for each region of the beam *between* any two discontinuities of loading. For example, coordinates x_1, x_2, and x_3 will have to be used to describe the variation of V and M throughout the length of the beam in Fig. 6–2. Here the coordinates are valid *only* within the regions from A to B for x_1, from B to C for x_2, and from C to D for x_3.

Beam Sign Convention. Before presenting a method for determining the shear and moment as functions of x, and later plotting these functions (shear and moment diagrams), it is first necessary to establish a *sign convention* in order to define "positive" and "negative" values for V and M. Although the choice of a sign convention is arbitrary, here we will use the one often used in engineering practice. It is shown in Fig. 6–3. The *positive directions* are as follows: the *distributed load* acts *upward* on the beam, the internal *shear force* causes a *clockwise* rotation of the beam segment on which it acts, and the internal *moment* causes *compression* in the *top fibers* of the segment such that it bends the segment so that it "holds water". Loadings that are opposite to these are considered negative.

IMPORTANT POINTS

- *Beams* are long straight members that are subjected to loads perpendicular to their longitudinal axis. They are classified according to the way they are supported, e.g., simply supported, cantilevered, or overhanging.
- In order to properly design a beam, it is important to know the *variation* of the internal shear and moment along its axis in order to find the points where these values are a maximum.
- Using an established sign convention for positive shear and moment, the shear and moment in the beam can be determined as a function of their position x on the beam, and then these functions can be plotted to form the shear and moment diagrams.

PROCEDURE FOR ANALYSIS

The shear and moment diagrams for a beam can be constructed using the following procedure.

Support Reactions.

- Determine all the reactive forces and couple moments acting on the beam, and resolve all the forces into components acting perpendicular and parallel to the beam's axis.

Shear and Moment Functions.

- Specify separate coordinates x having an origin at the beam's *left end* and extending to regions of the beam between concentrated forces and/or couple moments, or where there is no discontinuity of distributed loading.
- Section the beam at each distance x, and draw the free-body diagram of one of the segments. Be sure **V** and **M** are shown acting in their positive sense, in accordance with the sign convention given in Fig. 6–3.
- The shear is obtained by summing forces perpendicular to the beam's axis.
- To eliminate V, the moment is obtained directly by summing moments about the sectioned end of the segment.

Shear and Moment Diagrams.

- Plot the shear diagram (V versus x) and the moment diagram (M versus x). If numerical values of the functions describing V and M are *positive,* the values are plotted above the x axis, whereas negative values are plotted below the axis.
- Generally it is convenient to show the shear and moment diagrams below the free-body diagram of the beam.

Draw the shear and moment diagrams for the beam shown in Fig. 6–4*a*.

SOLUTION

Support Reactions. The support reactions are shown in Fig. 6–4*c*.

Shear and Moment Functions. A free-body diagram of the left segment of the beam is shown in Fig. 6–4*b*. The distributed loading on this segment is represented by its resultant force $(3x)$ kN, which is found only *after* the segment is isolated as a free-body diagram. This force acts through the centroid of the area under the distributed loading, a distance of $x/2$ from the right end. Applying the two equations of equilibrium yields

$$+\uparrow \Sigma F_y = 0; \qquad 6\text{ kN} - (3x)\text{ kN} - V = 0$$

$$V = (6 - 3x)\text{ kN} \qquad (1)$$

$$\zeta + \Sigma M = 0; \qquad -6\text{ kN}(x) + (3x)\text{ kN}\left(\tfrac{1}{2}x\right) + M = 0$$

$$M = (6x - 1.5x^2)\text{ kN}\cdot\text{m} \qquad (2)$$

Shear and Moment Diagrams. The shear and moment diagrams shown in Fig. 6–4*c* are obtained by plotting Eqs. 1 and 2. The point of *zero shear* can be found from Eq. 1:

$$V = (6 - 3x)\text{ kN} = 0$$

$$x = 2\text{ m}$$

NOTE: From the moment diagram, this value of x represents the point on the beam where the *maximum moment* occurs, since by Eq. 6–2 (see Sec. 6.2) the *slope* $V = dM/dx = 0$. From Eq. 2, we have

$$M_{\max} = [6(2) - 1.5(2)^2]\text{ kN}\cdot\text{m}$$

$$= 6\text{ kN}\cdot\text{m}$$

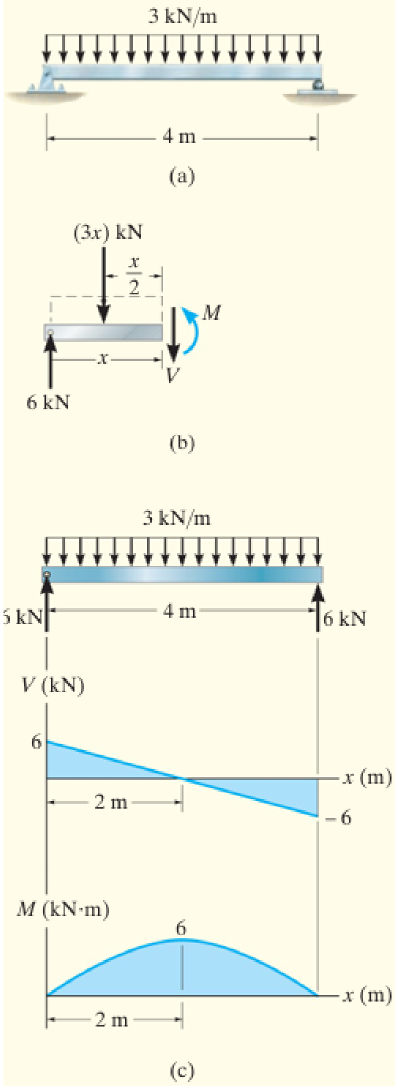
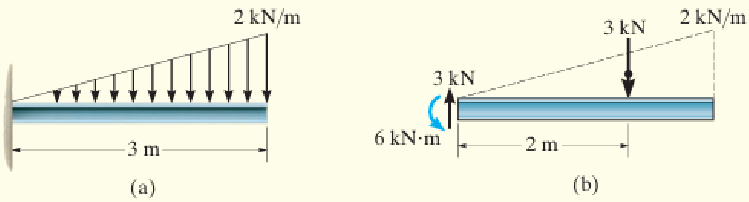
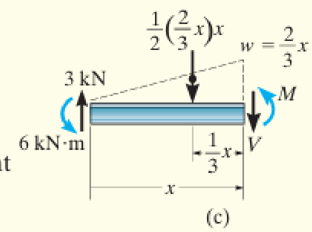


Fig. 6–4

Draw the shear and moment diagrams for the beam shown in Fig. 6–5a.



SOLUTION



Support Reactions. The distributed load is replaced by its resultant force, and the reactions have been determined, as shown in Fig. 6–5b.

Shear and Moment Functions. A free-body diagram of a beam segment of length x is shown in Fig. 6–5c. The intensity of the triangular load at the section is found by proportion, that is, $w/x = (2\text{ kN/m})/3\text{ m}$ or $w = \left(\frac{2}{3}x\right)\text{ kN/m}$. The resultant of the distributed loading is found from the area under the diagram. Thus,

$$+\uparrow \Sigma F_y = 0; \qquad 3\text{ kN} - \frac{1}{2}\left(\frac{2}{3}x\right)x - V = 0$$

$$V = \left(3 - \frac{1}{3}x^2\right)\text{kN} \qquad (1)$$

$$\circlearrowleft\!+\Sigma M = 0; \qquad 6\text{ kN}\cdot\text{m} - (3\text{ kN})(x) + \frac{1}{2}\left(\frac{2}{3}x\right)x\left(\frac{1}{3}x\right) + M = 0$$

$$M = \left(-6 + 3x - \frac{1}{9}x^3\right)\text{kN}\cdot\text{m} \qquad (2)$$

Shear and Moment Diagrams. The graphs of Eqs. 1 and 2 are shown in Fig. 6–5d.

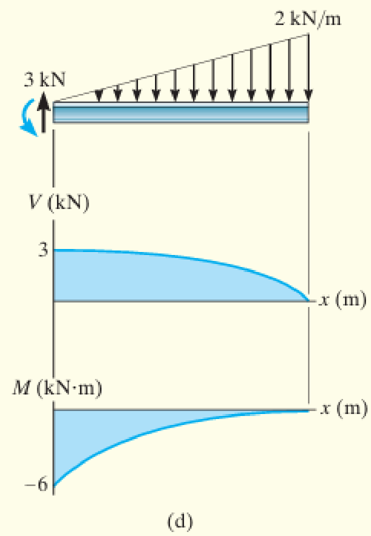


Fig. 6–5

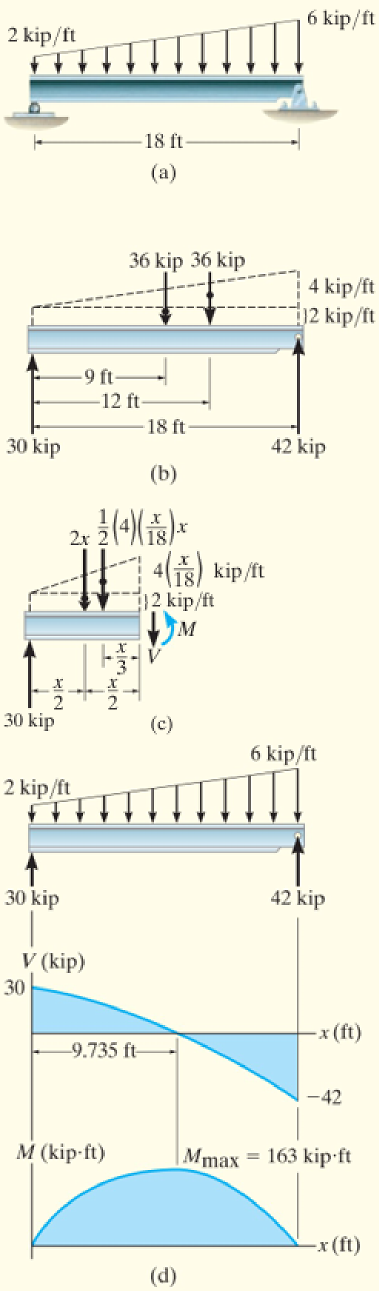


Fig. 6–6

Draw the shear and moment diagrams for the beam shown in Fig. 6–6*a*.

SOLUTION

Support Reactions. The distributed load is divided into triangular and rectangular component loadings, and these loadings are then replaced by their resultant forces. The reactions have been determined as shown on the beam's free-body diagram, Fig. 6–6*b*.

Shear and Moment Functions. A free-body diagram of the left segment is shown in Fig. 6–6*c*. As above, the trapezoidal loading is replaced by rectangular and triangular distributions. Here the intensity of the triangular load at the section is found by proportion. The resultant force and the location of each distributed loading are also shown. Applying the equilibrium equations, we have

$$+\uparrow \Sigma F_y = 0; \qquad 30 \text{ kip} - (2 \text{ kip/ft})x - \frac{1}{2}(4 \text{ kip/ft})\left(\frac{x}{18 \text{ ft}}\right)x - V = 0$$

$$V = \left(30 - 2x - \frac{x^2}{9}\right) \text{kip} \qquad (1)$$

$$\circlearrowleft + \Sigma M = 0;$$

$$-30 \text{ kip}(x) + (2 \text{ kip/ft})x\left(\frac{x}{2}\right) + \frac{1}{2}(4 \text{ kip/ft})\left(\frac{x}{18 \text{ ft}}\right)x\left(\frac{x}{3}\right) + M = 0$$

$$M = \left(30x - x^2 - \frac{x^3}{27}\right) \text{kip} \cdot \text{ft} \qquad (2)$$

Shear and Moment Diagrams. Equations 1 and 2 are plotted in Fig. 6–6*d*. Since the point of maximum moment occurs when $dM/dx = V = 0$ (Eq. 6–2), then, from Eq. 1,

$$V = 0 = 30 - 2x - \frac{x^2}{9}$$

Choosing the positive root,

$$x = 9.735 \text{ ft}$$

Thus, from Eq. 2,

$$M_{\max} = 30(9.735) - (9.735)^2 - \frac{(9.735)^3}{27} = 163 \text{ kip} \cdot \text{ft}$$

Draw the shear and moment diagrams for the beam shown in Fig. 6–7*a*.

SOLUTION

Support Reactions. The reactions at the supports are shown on the free-body diagram of the beam, Fig. 6–7*d*.

Shear and Moment Functions. Since there is a discontinuity of distributed load and also a concentrated load at the beam's center, two regions of x must be considered in order to describe the shear and moment functions for the entire beam.

$0 \le x_1 < 5$ m, Fig. 6–7*b*:

$$+\uparrow \Sigma F_y = 0; \qquad 5.75 \text{ kN} - V = 0$$

$$V = 5.75 \text{ kN} \qquad (1)$$

$$\circlearrowleft + \Sigma M = 0; \qquad -80 \text{ kN}\cdot\text{m} - 5.75 \text{ kN}\, x_1 + M = 0$$

$$M = (5.75x_1 + 80) \text{ kN}\cdot\text{m} \qquad (2)$$

$5 \text{ m} < x_2 \le 10$ m, Fig. 6–7*c*:

$$+\uparrow \Sigma F_y = 0; \quad 5.75 \text{ kN} - 15 \text{ kN} - 5 \text{ kN/m}(x_2 - 5 \text{ m}) - V = 0$$

$$V = (15.75 - 5x_2) \text{ kN} \qquad (3)$$

$$\circlearrowleft + \Sigma M = 0; \quad -80 \text{ kN}\cdot\text{m} - 5.75 \text{ kN}\, x_2 + 15 \text{ kN}(x_2 - 5 \text{ m})$$

$$+ 5 \text{ kN/m}(x_2 - 5 \text{ m})\left(\frac{x_2 - 5 \text{ m}}{2}\right) + M = 0$$

$$M = (-2.5x_2^2 + 15.75x_2 + 92.5) \text{ kN}\cdot\text{m} \qquad (4)$$

Shear and Moment Diagrams. Equations 1 through 4 are plotted in Fig. 6–7*d*.

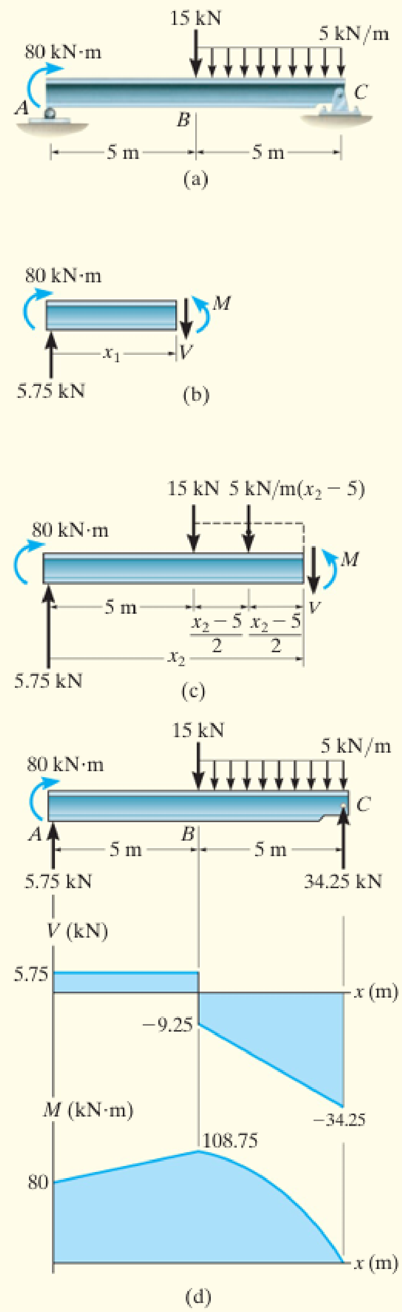


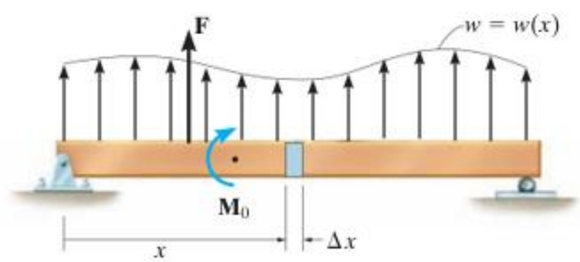
Fig. 6–7

Relationship between shear force diagram and bending moment diagram

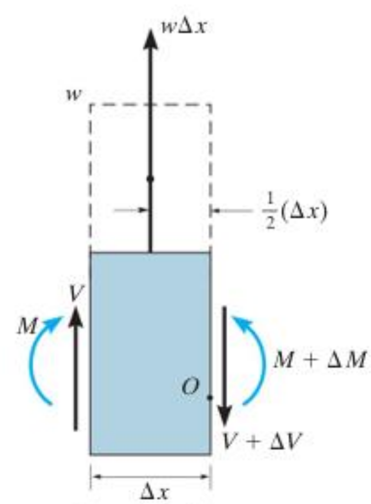
In cases where a beam is subjected to *several* different loadings, determining V and M as functions of x and then plotting these equations can become quite tedious. In this section a simpler method for constructing the shear and moment diagrams is discussed—a method based on two differential relations, one that exists between the distributed load and shear, and the other between the shear and moment.

Regions of Distributed Load. For purposes of generality, consider the beam shown in Fig. 6–8*a*, which is subjected to an arbitrary loading. A free-body diagram for a very small segment Δx of the beam is shown in Fig. 6–8*b*. Since this segment has been chosen at a position x where there is no concentrated force or couple moment, the results to be obtained will *not* apply at these points.

Notice that all the loadings shown on the segment act in their positive directions according to the established sign convention, Fig. 6–3. Also, both the internal resultant shear and moment, acting on the right face of the segment, must be changed by a small amount in order to keep the segment in equilibrium. The distributed load, which is approximately constant over Δx, has been replaced by a resultant force $w\Delta x$ that acts at $\frac{1}{2}(\Delta x)$ from the right side. Applying the equations of equilibrium to the segment, we have



(a)



Free-body diagram of segment Δx

(b)

Fig. 6–8

$$+\uparrow \Sigma F_y = 0; \qquad V + w\,\Delta x - (V + \Delta V) = 0$$

$$\Delta V = w\,\Delta x$$

$$\zeta + \Sigma M_O = 0; \qquad -V\,\Delta x - M - w\,\Delta x\left[\tfrac{1}{2}(\Delta x)\right] + (M + \Delta M) = 0$$

$$\Delta M = V\,\Delta x + w\,\tfrac{1}{2}(\Delta x)^2$$

Dividing by Δx and taking the limit as $\Delta x \rightarrow 0$, the above two equations become

$$\frac{dV}{dx} = w \qquad (6\text{–}1)$$

slope of shear diagram at each point = distributed load intensity at each point

$$\frac{dM}{dx} = V \qquad (6\text{–}2)$$

slope of moment diagram at each point = shear at each point

Equation 6–1 states that at any point the *slope* of the shear diagram equals the intensity of the distributed loading. For example, consider the beam in Fig. 6–9*a*. The distributed loading is negative and increases from zero to w_B. Knowing this provides a quick means for drawing the shape of the shear diagram. It must be a curve that has a *negative slope,* increasing from zero to $-w_B$. Specific slopes $w_A = 0$, $-w_C$, $-w_D$, and $-w_B$ are shown in Fig. 6–9*b*.

In a similar manner, Eq. 6–2 states that at any point the *slope* of the moment diagram is equal to the shear. Since the shear diagram in Fig. 6–9*b* starts at $+V_A$, decreases to zero, and then becomes negative and decreases to $-V_B$, the moment diagram (or curve) will then have an initial slope of $+V_A$ which decreases to zero, then the slope becomes negative and decreases to $-V_B$. Specific slopes V_A, V_C, V_D, 0, and $-V_B$ are shown in Fig. 6–9*c*.

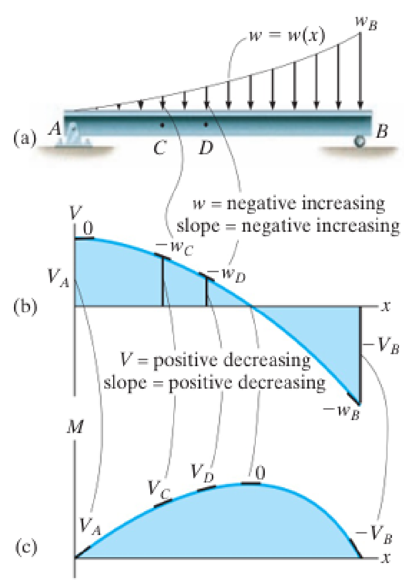


Fig. 6–9

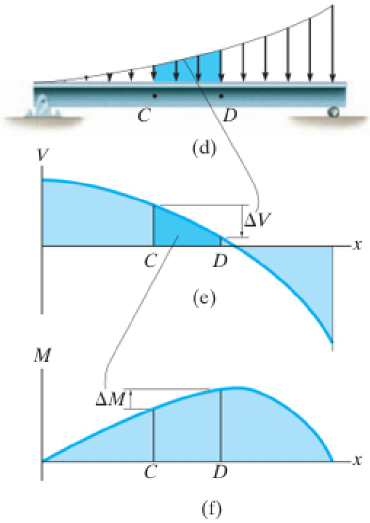


Fig. 6–9 (cont.)

Equations 6–1 and 6–2 may also be rewritten in the form $dV = w\,dx$ and $dM = V\,dx$. Since $w\,dx$ and $V\,dx$ represent differential areas under the distributed loading and the shear diagram, we can then integrate these areas between any two points C and D on the beam, Fig. 6–9d, and write

$$\Delta V = \int w\,dx \tag{6–3}$$

change in shear — area under distributed loading

$$\Delta M = \int V\,dx \tag{6–4}$$

change in moment — area under shear diagram

Equation 6–3 states that the *change in shear* between C and D is equal to the *area* under the distributed-loading curve between these two points, Fig. 6–9d. In this case the change is negative since the distributed load acts downward. Similarly, from Eq. 6–4, the change in moment between C and D, Fig. 6–9f, is equal to the area under the shear diagram within the region from C to D. Here the change is positive.

Regions of Concentrated Force and Moment.

A free-body diagram of a small segment of the beam in Fig. 6–8a taken from under the force is shown in Fig. 6–10a. Here force equilibrium requires

$$+\uparrow \Sigma F_y = 0; \qquad V + F - (V + \Delta V) = 0$$

$$\Delta V = F \tag{6–5}$$

Thus, when **F** acts *upward* on the beam, then the change in shear, ΔV, is *positive* so the values of the shear on the shear diagram will "jump" *upward*. Likewise, if **F** acts *downward*, the jump (ΔV) will be *downward*.

When the beam segment includes the couple moment M_0, Fig. 6–10b, then moment equilibrium requires the change in moment to be

$$\circlearrowleft + \Sigma M_O = 0; \quad M + \Delta M - M_0 - V\,\Delta x - M = 0$$

Letting $\Delta x \approx 0$, we get

$$\Delta M = M_0 \tag{6–6}$$

In this case, if $\mathbf{M}_0$ is applied *clockwise*, the change in moment, ΔM, is *positive* so the moment diagram will "jump" *upward*. Likewise, when $\mathbf{M}_0$ acts *counterclockwise*, the jump (ΔM) will be *downward*.

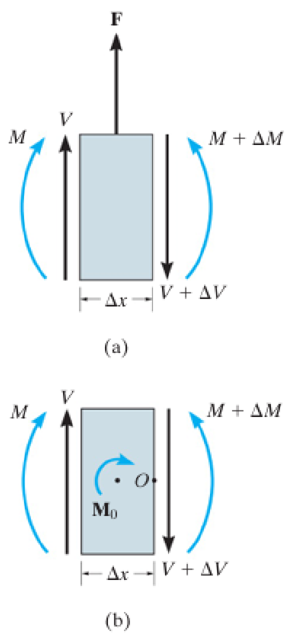
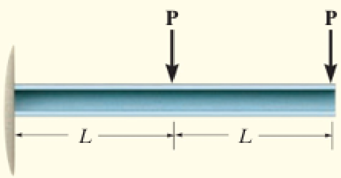


Fig. 6–10



(a)

Draw the shear and moment diagrams for the beam shown in Fig. 6–11*a*.

SOLUTION

Support Reactions. The reaction at the fixed support is shown on the free-body diagram, Fig. 6–11*b*.

Shear Diagram. The shear at each end of the beam is plotted first, Fig. 6–11*c*. Since there is no distributed loading on the beam, the slope of the shear diagram is zero as indicated. Note how the force P at the center of the beam causes the shear diagram to jump downward an amount P, since this force acts downward.

Moment Diagram. The moments at the ends of the beam are plotted, Fig. 6–11*d*. Here the moment diagram consists of two sloping lines, one with a slope of $+2P$ and the other with a slope of $+P$.

The value of the moment in the center of the beam can be determined by the method of sections, or from the area under the shear diagram. If we choose the left half of the shear diagram,

$$M|_{x=L} = M|_{x=0} + \Delta M$$
$$M|_{x=L} = -3PL + (2P)(L) = -PL$$

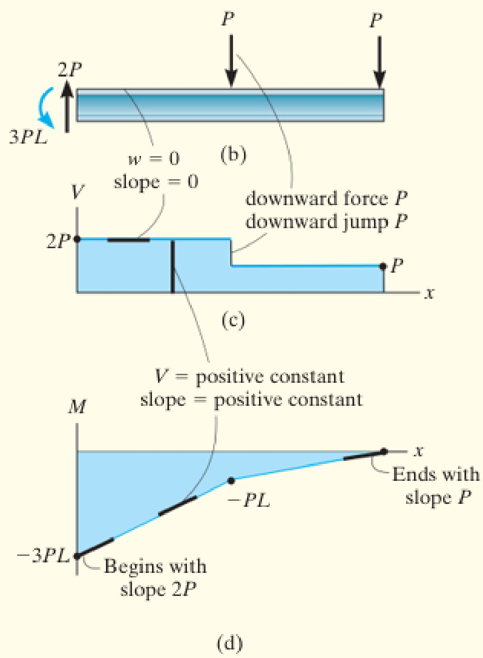
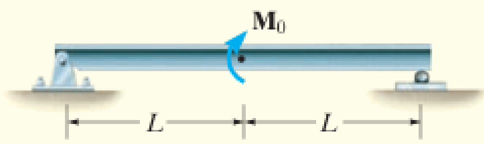


Fig. 6–11

Draw the shear and moment diagrams for the beam shown in Fig. 6–12*a*.



(a)

SOLUTION

Support Reactions. The reactions are shown on the free-body diagram in Fig. 6–12*b*.

Shear Diagram. The shear at each end is plotted first, Fig. 6–12*c*. Since there is no distributed load on the beam, the shear diagram has zero slope and is therefore a horizontal line.

Moment Diagram. The moment is zero at each end, Fig. 6–12*d*. The moment diagram has a constant negative slope of $-M_0/2L$ since this is the shear in the beam at each point. However, here the couple moment M_0 causes a jump in the moment diagram at the beam's center.

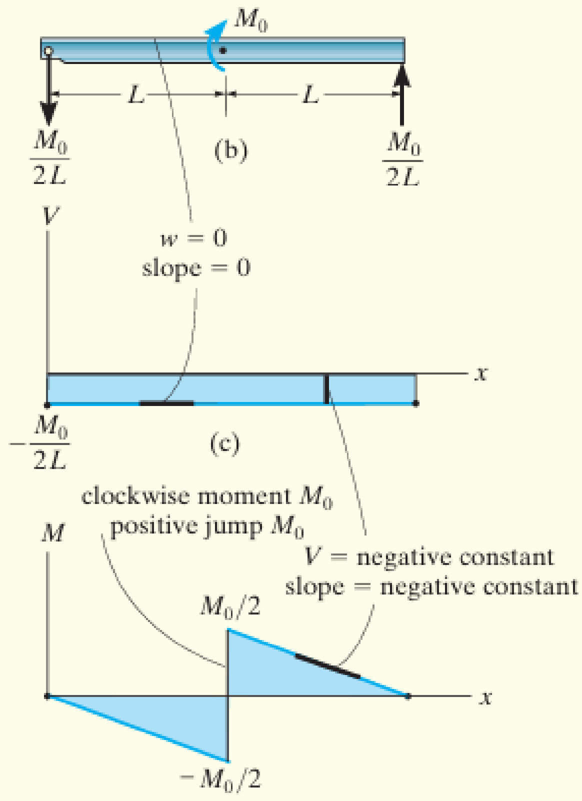
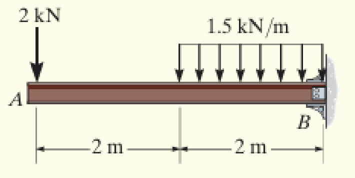


Fig. 6–12

Draw the shear and moment diagrams for the cantilever beam in Fig. 6–15*a*.



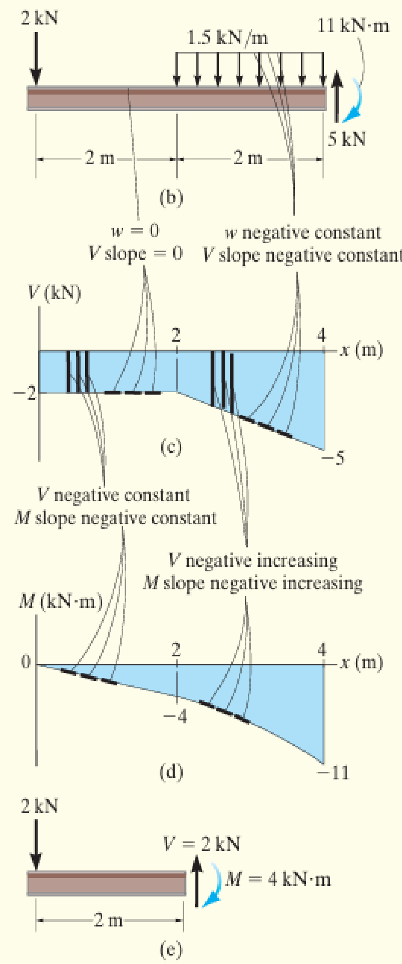


Fig. 6–15

SOLUTION

Support Reactions. The support reactions at the fixed support *B* are shown in Fig. 6–15*b*.

Shear Diagram. The shear at the ends is plotted first, Fig. 6–15*c*. Notice how the shear diagram is constructed by following the slopes defined by the loading *w*.

Moment Diagram. The moments at the ends of the beam are plotted first, Fig. 6–15*d*. Notice how the moment diagram is constructed based on knowing its slope, which is equal to the shear at each point. The moment at $x = 2$ m can be found from the area under the shear diagram. We have

$$M|_{x=2\text{ m}} = M|_{x=0} + \Delta M = 0 + [-2\text{ kN}(2\text{ m})] = -4\text{ kN}\cdot\text{m}$$

Of course, this same value can be determined from the method of sections, Fig. 6–15*e*.