Bending deformation of straight member: -

In this section, we will discuss the deformations that occur when a straight prismatic beam, made of homogeneous material, is subjected to bending. The discussion will be limited to beams having a cross-sectional area that is symmetrical with respect to an axis, and the bending moment is applied about an axis perpendicular to this axis of symmetry, as shown in Fig. 6–18. The behavior of members that have unsymmetrical cross sections, or are made of several different materials, is based on similar observations and will be discussed separately in later sections of this chapter.

Consider the undeformed bar in Fig. 6–19*a*, which has a square cross section and is marked with horizontal and vertical grid lines. When a bending moment is applied, it tends to distort these lines into the pattern shown in Fig. 6–19*b*. Here the horizontal lines become *curved*, while the vertical lines *remain straight* but undergo a *rotation*. The bending moment causes the material within the *bottom* portion of the bar to *stretch* and the material within the *top* portion to *compress*. Consequently, between these two regions there must be a surface, called the ***neutral surface,*** in which horizontal fibers of the material will not undergo a change in length, Fig. 6–18. As noted, we will refer to the *z* axis that lies along the neutral surface as the ***neutral axis.***

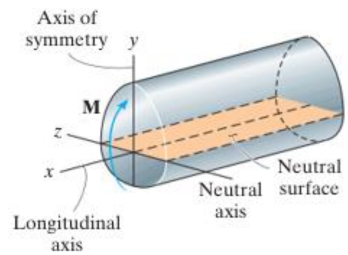


Fig. 6–18

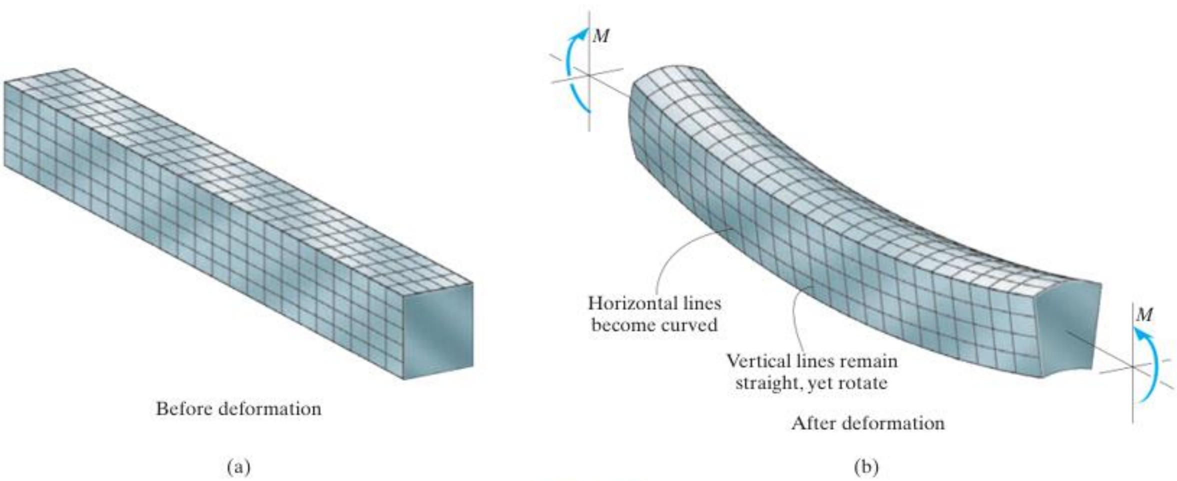
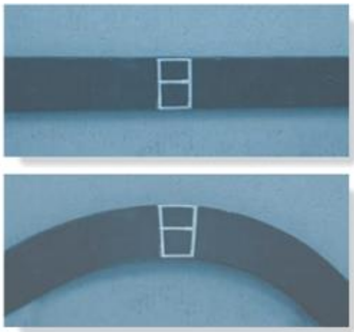


Fig. 6–19

From these observations we will make the following three assumptions regarding the way the moment deforms the material. First, the longitudinal axis, which lies within the neutral surface, Fig. 6–20*a*, does not experience any change in length. Rather the moment will tend to deform the beam so that this line becomes a curve that lies in the vertical plane of symmetry, Fig. 6–20*b*. Second, all cross sections of the beam remain plane and perpendicular to the longitudinal axis during the deformation. And third, the small lateral strains due to the Poisson effect discussed in Sec. 3.6 will be neglected. In other words, the cross section in Fig. 6–19 retains its shape.

With the above assumptions, we will now consider how the bending moment distorts a small element of the beam located a distance x along the beam's length, Fig. 6–20. This element is shown in profile view in the undeformed and deformed positions in Fig. 6–21. Here the line segment



Note the distortion of the lines due to bending of this rubber bar. The top line stretches, the bottom line compresses, and the center line remains the same length. Furthermore the vertical lines rotate and yet remain straight.

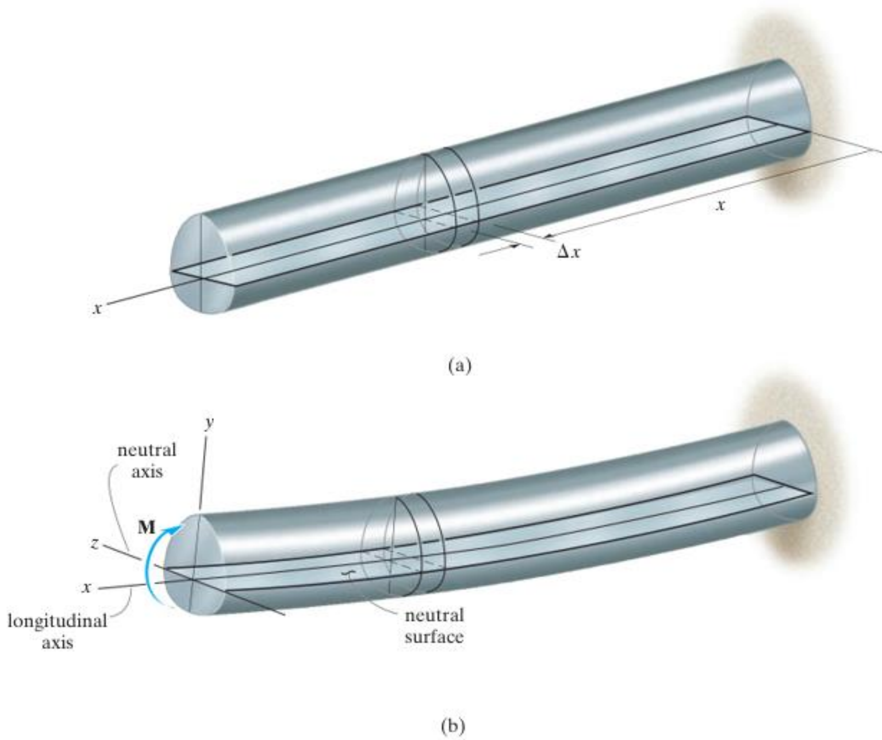


Fig. 6–20

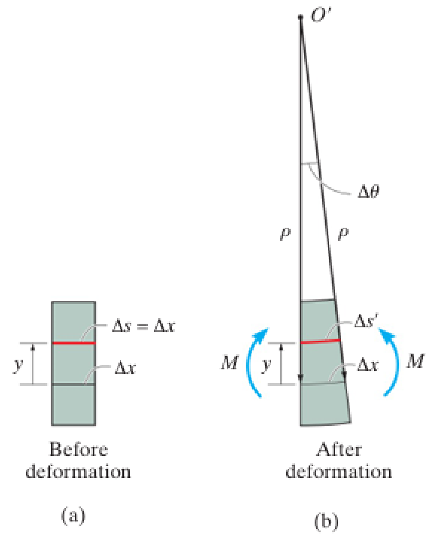


Fig. 6–21

Δx, located on the neutral surface, does not change its length, whereas any line segment Δs, located at the arbitrary distance y above the neutral surface, will contract and become $\Delta s'$ after deformation. By definition, the normal strain along Δs is determined from Eq. 2–2, namely,

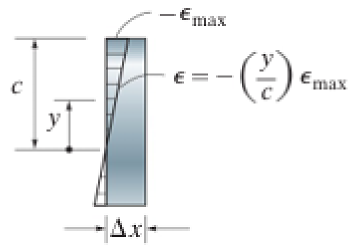
$$\epsilon = \lim_{\Delta s \to 0} \frac{\Delta s' - \Delta s}{\Delta s}$$

Now let's represent this strain in terms of the location y of the segment and the radius of curvature ρ of the longitudinal axis of the element. Before deformation, $\Delta s = \Delta x$, Fig. 6–21*a*. After deformation, Δx has a radius of curvature ρ, with center of curvature at point O', Fig. 6–21*b*, so that $\Delta x = \Delta s = \rho \Delta \theta$. Also, since $\Delta s'$ has a radius of curvature of $\rho - y$, then $\Delta s' = (\rho - y)\Delta \theta$. Substituting these results into the above equation, we get

$$\epsilon = \lim_{\Delta \theta \to 0} \frac{(\rho - y)\Delta \theta - \rho \Delta \theta}{\rho \Delta \theta}$$

or

$$\epsilon = -\frac{y}{\rho} \qquad (6\text{–}7)$$



Normal strain distribution

Fig. 6–22

Since $1/\rho$ is constant at x, this important result, $\epsilon = -y/\rho$, indicates that the *longitudinal normal strain will vary linearly* with y measured from the neutral axis. A contraction $(-\epsilon)$ will occur in fibers located above the neutral axis $(+y)$, whereas elongation $(+\epsilon)$ will occur in fibers located below the axis $(-y)$. This variation in strain over the cross section is shown in Fig. 6–22. Here the maximum strain occurs at the outermost fiber, located a distance of $y = c$ from the neutral axis. Using Eq. 6–7, since $\epsilon_{max} = c/\rho$, then by division,

$$\frac{\epsilon}{\epsilon_{max}} = -\left(\frac{y/\rho}{c/\rho}\right)$$

So that

$$\epsilon = -\left(\frac{y}{c}\right)\epsilon_{max} \tag{6–8}$$

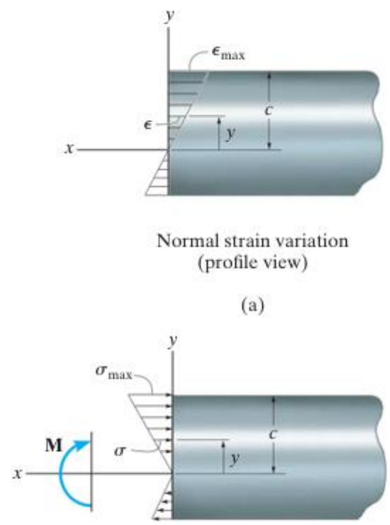
This normal strain depends only on the assumptions made with regard to the deformation.

THE FLEXURE FORMULA

In this section, we will develop an equation that relates the stress distribution within a straight beam to the bending moment acting on its cross section. To do this we will assume that the material behaves in a linear elastic manner, so that by Hooke's law, a linear variation of normal strain, Fig. 6–23*a*, must result in a linear variation in normal stress, Fig. 6–23*b*. Hence, like the normal strain variation, σ will vary from zero at the member's neutral axis to a maximum value, σ_{max}, a distance c farthest from the neutral axis. Because of the proportionality of triangles, Fig. 6–23*b*, or by using Hooke's law, $\sigma = E\epsilon$, and Eq. 6–8, we can write

$$\sigma = -\left(\frac{y}{c}\right)\sigma_{max} \tag{6–9}$$

This equation describes the stress distribution over the cross-sectional area. The sign convention established here is significant. For positive **M,** which acts in the $+z$ direction, positive values of y give negative values for σ, that is, a compressive stress, since it acts in the negative x direction. Similarly, negative y values will give positive or tensile values for σ.



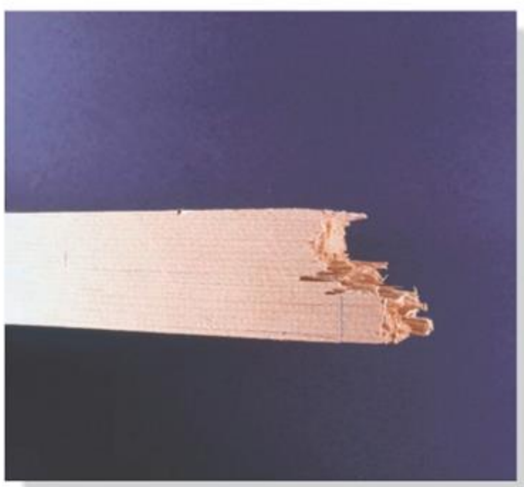
Normal strain variation
(profile view)

(a)

Bending stress variation
(profile view)

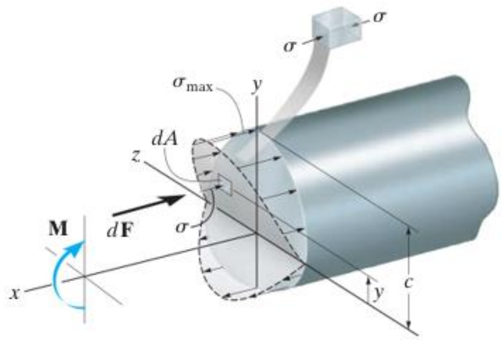
(b)

Fig. 6–23



This wood specimen failed in bending due to its fibers being crushed at its top and torn apart at its bottom.

Location of Neutral Axis. To locate the position of the neutral axis, we require the *resultant force* produced by the stress distribution acting over the cross-sectional area to be equal to *zero*. Noting that the force $dF = \sigma\, dA$ acts on the arbitrary element dA in Fig. 6–24, we have



Bending stress variation

Fig. 6–24

$$F_R = \Sigma F_x; \qquad 0 = \int_A dF = \int_A \sigma\, dA$$

$$= \int_A -\left(\frac{y}{c}\right)\sigma_{\max}\, dA$$

$$= \frac{-\sigma_{\max}}{c}\int_A y\, dA$$

Since $\sigma_{\max}/c$ is not equal to zero, then

$$\int_A y\, dA = 0 \qquad (6\text{–}10)$$

In other words, the first moment of the member's cross-sectional area about the neutral axis must be zero. This condition can only be satisfied if the neutral axis is also the horizontal centroidal axis for the cross section.* Therefore, once the centroid for the member's cross-sectional area is determined, the location of the neutral axis is known.

Bending Moment. We can determine the stress in the beam if we require the moment M to be equal to the moment produced by the stress distribution about the neutral axis. The moment of $d\mathbf{F}$ in Fig. 6–24 is $dM = y\, dF$. Since $dF = \sigma\, dA$, using Eq. 6–9, we have for the entire cross section,

$$(M_R)_z = \Sigma M_z; \qquad M = \int_A y\, dF = \int_A y\,(\sigma\, dA) = \int_A y\left(\frac{y}{c}\sigma_{\max}\right) dA$$

or

$$M = \frac{\sigma_{\max}}{c}\int_A y^2\, dA \qquad (6\text{–}11)$$

*Recall that the location $\bar{y}$ for the centroid of an area is defined from the equation $\bar{y} = \int y\, dA / \int dA$. If $\int y\, dA = 0$, then $\bar{y} = 0$, and so the centroid lies on the reference (neutral) axis. See Appendix A.

The integral represents the ***moment of inertia*** of the cross-sectional area about the neutral axis.* We will symbolize its value as I. Hence, Eq. 6–11 can be solved for σ_{max} and written as

$$\sigma_{max} = \frac{Mc}{I} \tag{6–12}$$

Here

σ_{max} = the maximum normal stress in the member, which occurs at a point on the cross-sectional area *farthest away* from the neutral axis

M = the resultant internal moment, determined from the method of sections and the equations of equilibrium, and calculated about the neutral axis of the cross section

c = perpendicular distance from the neutral axis to a point farthest away from the neutral axis. This is where σ_{max} acts.

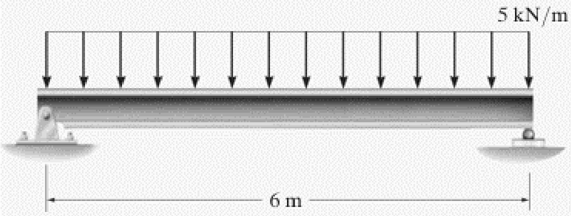
I = moment of inertia of the cross-sectional area about the neutral axis

Since $\sigma_{max}/c = -\sigma/y$, Eq. 6–9, the normal stress at any distance y can be determined from an equation similar to Eq. 6–12. We have

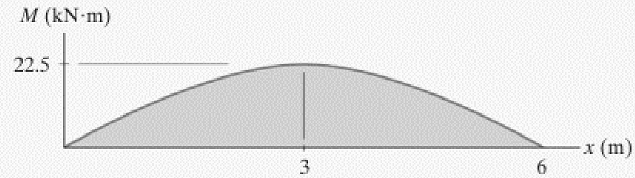
$$\sigma = -\frac{My}{I} \tag{6–13}$$

Either of the above two equations is often referred to as the ***flexure formula.*** Although we have assumed that the member is prismatic, we can conservatively also use the flexure formula to determine the normal stress in members that have a *slight taper.* For example, using a mathematical analysis based on the theory of elasticity, a member having a rectangular cross section and a length that is tapered 15° will have an actual maximum normal stress that is about 5.4% *less* than that calculated using the flexure formula.

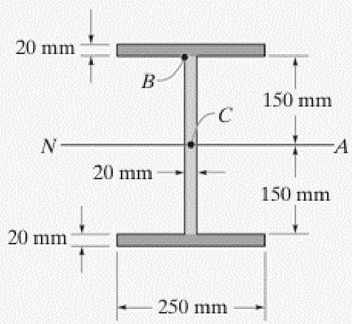
The simply supported beam in Fig. 6–26*a* has the cross-sectional area shown in Fig. 6–26*b*. Determine the absolute maximum bending stress in the beam and draw the stress distribution over the cross section at this location. Also, what is the stress at point B?



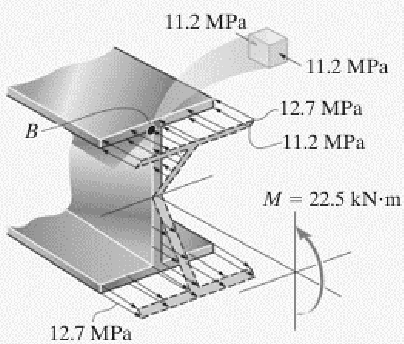
(a)



(c)



(b)



(d)

Fig. 6–26

SOLUTION

Maximum Internal Moment. The maximum internal moment in the beam, $M = 22.5\ \text{kN}\cdot\text{m}$, occurs at the center, as indicated on the moment diagram, Fig. 6–26*c*.

Section Property. By reasons of symmetry, the neutral axis passes through the centroid C at the midheight of the beam, Fig. 6–26*b*. The area is subdivided into the three parts shown, and the moment of inertia of each part is calculated about the neutral axis using the parallel-axis theorem. (See Eq. A–5 of Appendix A.) Choosing to work in meters, we have

$$I = \Sigma(\bar{I} + Ad^2)$$

$$= 2\left[\frac{1}{12}(0.25\text{ m})(0.020\text{ m})^3 + (0.25\text{ m})(0.020\text{ m})(0.160\text{ m})^2\right] + \left[\frac{1}{12}(0.020\text{ m})(0.300\text{ m})^3\right]$$

$$= 301.3(10^{-6})\text{ m}^4$$

$$\sigma_{\max} = \frac{Mc}{I};\quad \sigma_{\max} = \frac{22.5(10^3)\text{ N}\cdot\text{m}(0.170\text{ m})}{301.3(10^{-6})\text{ m}^4} = 12.7\text{ MPa} \quad \textit{Ans.}$$

A three-dimensional view of the stress distribution is shown in Fig. 6–26*d*. Specifically, at point B, $y_B = 150$ mm, and so as shown in Fig. 6–26*d*,

$$\sigma_B = -\frac{My_B}{I};\quad \sigma_B = -\frac{22.5(10^3)\text{ N}\cdot\text{m}(0.150\text{ m})}{301.3(10^{-6})\text{ m}^4} = -11.2\text{ MPa} \quad \textit{Ans.}$$

The beam shown in Fig. 6–27*a* has a cross-sectional area in the shape of a channel, Fig. 6–27*b*. Determine the maximum bending stress that occurs in the beam at section *a–a*.

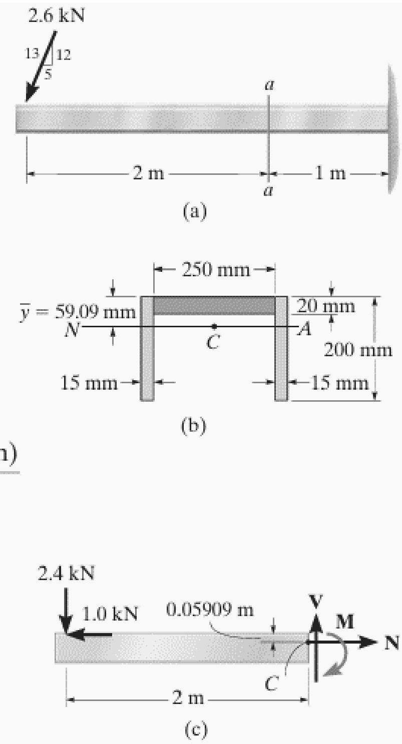


Fig. 6–27

SOLUTION

Internal Moment. Here the beam's support reactions do not have to be determined. Instead, by the method of sections, the segment to the left of section *a–a* can be used, Fig. 6–27*c*. It is important that the resultant internal axial force **N** passes through the centroid of the cross section. Also, realize that the resultant internal moment must be calculated about the beam's neutral axis at section *a–a*.

To find the location of the neutral axis, the cross-sectional area is subdivided into three composite parts as shown in Fig. 6–27*b*. Using Eq. A–2 of Appendix A, we have

$$\bar{y} = \frac{\Sigma \tilde{y} A}{\Sigma A} = \frac{2[0.100 \text{ m}](0.200 \text{ m})(0.015 \text{ m}) + [0.010 \text{ m}](0.02 \text{ m})(0.250 \text{ m})}{2(0.200 \text{ m})(0.015 \text{ m}) + 0.020 \text{ m}(0.250 \text{ m})}$$

$$= 0.05909 \text{ m} = 59.09 \text{ mm}$$

This dimension is shown in Fig. 6–27*c*.

Applying the moment equation of equilibrium about the neutral axis, we have

$$\zeta+\Sigma M_{NA} = 0; \quad 2.4 \text{ kN}(2 \text{ m}) + 1.0 \text{ kN}(0.05909 \text{ m}) - M = 0$$

$$M = 4.859 \text{ kN} \cdot \text{m}$$

Section Property. The moment of inertia of the cross-sectional area about the neutral axis is determined using $I = \Sigma(\bar{I} + Ad^2)$ applied to each of the three composite parts of the area. Working in meters, we have

$$I = \left[\frac{1}{12}(0.250 \text{ m})(0.020 \text{ m})^3 + (0.250 \text{ m})(0.020 \text{ m})(0.05909 \text{ m} - 0.010 \text{ m})^2\right]$$

$$+ 2\left[\frac{1}{12}(0.015 \text{ m})(0.200 \text{ m})^3 + (0.015 \text{ m})(0.200 \text{ m})(0.100 \text{ m} - 0.05909 \text{ m})^2\right]$$

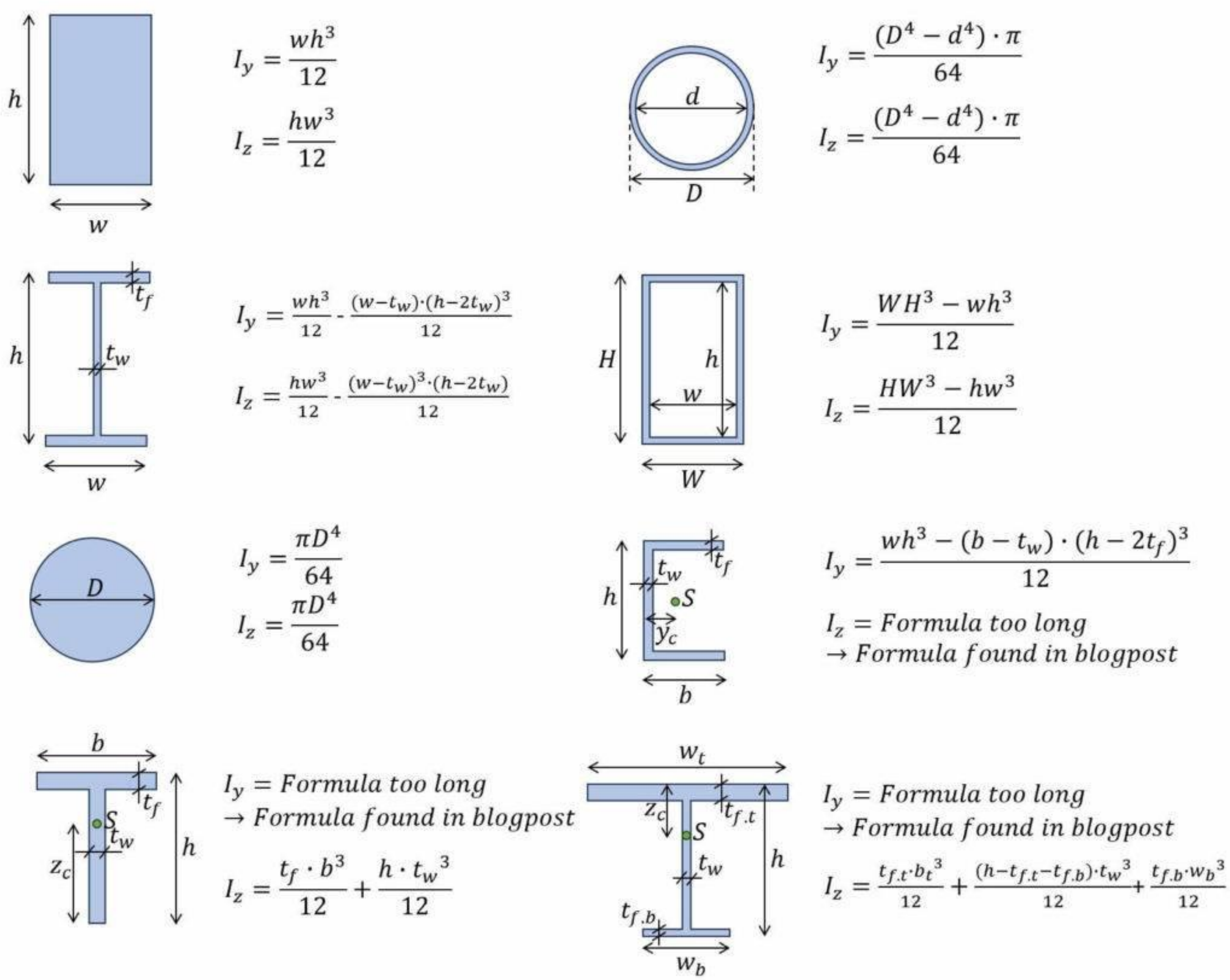
$$= 42.26(10^{-6}) \text{ m}^4$$

Maximum Bending Stress. The maximum bending stress occurs at points farthest away from the neutral axis. This is at the bottom of the beam, $c = 0.200 \text{ m} - 0.05909 \text{ m} = 0.1409 \text{ m}$. Here the stress is compressive. Thus,

$$\sigma_{\max} = \frac{Mc}{I} = \frac{4.859(10^3) \text{ N} \cdot \text{m}(0.1409 \text{ m})}{42.26(10^{-6}) \text{ m}^4} = 16.2 \text{ MPa (C)} \qquad \textit{Ans.}$$

Show that at the top of the beam the bending stress is $\sigma' = 6.79$ MPa.

NOTE: The normal force of $N = 1$ kN and shear force $V = 2.4$ kN will also contribute additional stress on the cross section. The superposition of all these effects will be discussed in Chapter 8.



h
w
$I_y = \frac{wh^3}{12}$
$I_z = \frac{hw^3}{12}$
d
D
$I_y = \frac{(D^4 - d^4) \cdot \pi}{64}$
$I_z = \frac{(D^4 - d^4) \cdot \pi}{64}$
t_f
h
t_w
w
$I_y = \frac{wh^3}{12} - \frac{(w-t_w)\cdot(h-2t_w)^3}{12}$
$I_z = \frac{hw^3}{12} - \frac{(w-t_w)^3\cdot(h-2t_w)}{12}$
H
h
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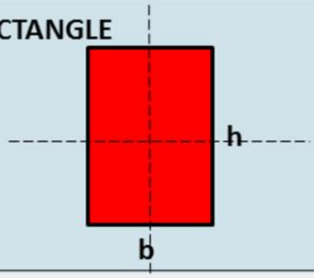
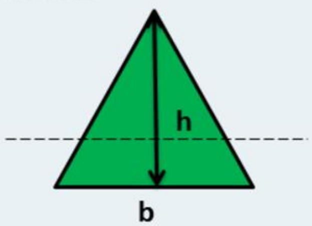
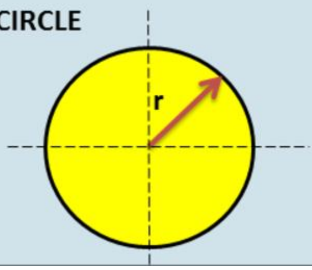
W
$I_y = \frac{WH^3 - wh^3}{12}$
$I_z = \frac{HW^3 - hw^3}{12}$
D
$I_y = \frac{\pi D^4}{64}$
$I_z = \frac{\pi D^4}{64}$
t_f
t_w
h
S
y_c
b
$I_y = \frac{wh^3 - (b - t_w) \cdot (h - 2t_f)^3}{12}$
$I_z =$ Formula too long
→ Formula found in blogpost
b
t_f
S
t_w
z_c
h
$I_y =$ Formula too long
→ Formula found in blogpost
$I_z = \frac{t_f \cdot b^3}{12} + \frac{h \cdot t_w{}^3}{12}$
w_t
z_c
$t_{f.t}$
S
t_w
h
$t_{f.b}$
w_b
$I_y =$ Formula too long
→ Formula found in blogpost
$I_z = \frac{t_{f.t}\cdot b_t{}^3}{12} + \frac{(h-t_{f.t}-t_{f.b})\cdot t_w{}^3}{12} + \frac{t_{f.b}\cdot w_b{}^3}{12}$

SHAPE	MOMENT OF INERTIA	RADIUS OF GYRATION
RECTANGLE 	$I_x = \frac{bh^3}{12}$	$\frac{h}{\sqrt{12}}$
TRIANGLE 	$I_x = \frac{bh^3}{36}$	$\frac{h}{\sqrt{18}}$
CIRCLE 	$\frac{\pi r^4}{4}$ OR $\frac{\pi D^4}{64}$	$\frac{r}{2}$