Area under a Parametric Curve

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Consider the non-self-intersecting plane curve defined by the parametric equations

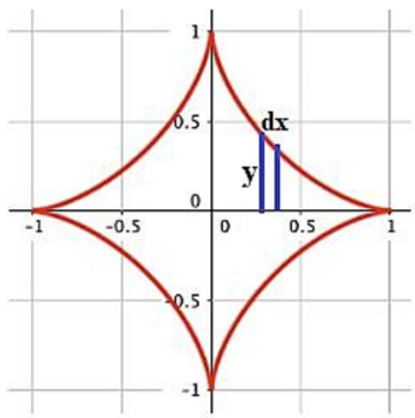
$$x=x\,(t),\quad y=y\,(t),\quad a\leq t\leq b$$

And assume that x (t) differentiable. The area under this curve is given by

$$\boldsymbol{A=\int_a^b y(t)\,\dot{x}\,(t)dt}$$

Example7: Find the area enclosed by the asteroid in figure.

$x=\ cos^3t,\quad y=sin^3t \qquad 0\leq t\leq 2\pi$



<u>Solution</u>

dA= ydx

$\boldsymbol{A=4\int_0^1 ydx}$

$\boldsymbol{y(t)}=sin^3t,\ \ \boldsymbol{dx}=\ \boldsymbol{-3}cos^2t\,sint\,dt$

At x=0 → 0= cos^3t → cos t= 0 → t= $\cos^{-1}0$ → t=$\frac{\pi}{2}$,

at x=1 → 1= cos^3t → cos t= 1 → t= $\cos^{-1}1$ → t=0

$\boldsymbol{A=4}\int_{\frac{\pi}{2}}^0 sin^3t\,(\boldsymbol{-3}cos^2t\,sint)\boldsymbol{dt}$

$\boldsymbol{A=-12}\int_{\frac{\pi}{2}}^0 sin^4t.cos^2t\,\boldsymbol{dt}\ \ \boldsymbol{=12}\int_0^{\frac{\pi}{2}}sin^4t.cos^2t\,\boldsymbol{dt}$

$sin^2t=\frac{1-cos2t}{2}\rightarrow sin^4t=\left(\frac{1-cos2t}{2}\right)^2,\ \ cos^2t=\frac{1+cos2t}{2}$

$\boldsymbol{A=12}\int_0^{\frac{\pi}{2}}\left(\frac{1-cos2t}{2}\right)^2.\frac{1+cos2t}{2}\ \boldsymbol{dt}\ =\ \frac{\boldsymbol{12}}{\boldsymbol{8}}\int_0^{\frac{\pi}{2}}(1-cos2t)^2.(1+cos2t)\mathbf{dt}$

$\boldsymbol{A}=\frac{\boldsymbol{3}}{\boldsymbol{2}}\int_0^{\frac{\pi}{2}}(1-2cos2t+cos^22t).(1+cos2t)\,\mathbf{dt}$

$$A = \frac{3}{2}\int_0^{\frac{\pi}{2}}(1 - 2cos2t + cos^2 2t + cos2t - 2cos^2 2t + cos^3 2t)dt$$

$$A = \frac{3}{2}\int_0^{\frac{\pi}{2}}(1 - cos2t - cos^2 2t + cos^3 2t)\mathbf{dt}$$

$$A = \frac{3}{2}\left[\int_0^{\frac{\pi}{2}}(1 - cos2t - \int_0^{\frac{\pi}{2}}(cos^2 2t + \int_0^{\frac{\pi}{2}}(cos^3 2t)\ dt\right] \quad \textbf{-----1}$$

$$\int_0^{\frac{\pi}{2}}(1 - cos2t)dt = t - \frac{1}{2}sin\ 2t \quad \textbf{------2}$$

$$\int_0^{\frac{\pi}{2}}(cos^2 2t\ dt = \int_0^{\frac{\pi}{2}}(\frac{1+cos4t}{2})\ dt = \frac{1}{2}(t + \frac{1}{4}sin4t) \quad \text{------3}$$

$$\int cos^3 2t\ dt = \int cos^2 2t . cos2tdt = \int (1 - sin^2 2t). cos2t\ dt$$

u=sin2t → du= 2cos2t dt → cos2t dt = $\frac{1}{2}du$

$$\frac{1}{2}\int (1 - u^2).du = \frac{1}{2}\left(u - \frac{u^3}{3}\right)$$

$$= \frac{1}{2}(\sin 2t - \frac{1}{3}\sin^3 2t) \quad \text{-------4}$$

Sub equations 2, 3 &4 in equation 1

$$A = \frac{3}{2}\left[(t - \frac{1}{2}sin\ 2t) - \frac{1}{2}(t + \frac{1}{4}sin4t) + \frac{1}{2}(\sin 2t - \frac{1}{3}\sin^3 2t)\right]_0^{\frac{\pi}{2}}$$

$$A = \frac{3}{2}\left[\left(\frac{\pi}{2} - 0 - 0 - 0\right) - \frac{1}{2}\left(\frac{\pi}{2} + 0 - 0 - 0\right) + \frac{1}{2}(0 - 0 - 0 + 0)\right]$$

$$A = \frac{3\pi}{8}$$

Arc length of parametric curves

Let x = f (t) and y = g (t) be parametric equations with $\frac{dx}{dt}$ and $\frac{dy}{dt}$ continuous on some open interval II containing t1t1 and t2t2 on which the graph traces itself only once. The arc length of the graph, from t=t1t=t1 to t=t2t=t2, is

$$L = \int_{t1}^{t2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

Example 8: Find the arc length of the circle parametrized by:

x=3cost, y=3sint for $0 < t < \frac{3\pi}{2}$

$\frac{dx}{dt} = -3sint \quad \frac{dy}{dt} = 3\ cost$

By direct application of Theorem of arc length

$$L = \int_0^{\frac{3\pi}{2}} \sqrt{(-3sint)^2 + (3cost)^2}\ dt$$

$$L = \int_0^{\frac{3\pi}{2}} \sqrt{9sin^2 t + 9cos^2 t}\ dt = \int_0^{\frac{3\pi}{2}} 3\sqrt{sin^2 t + cos^2 t}\ dt = \int_0^{\frac{3\pi}{2}} 3\sqrt{1}\ dt$$

$$L = \int_0^{\frac{3\pi}{2}} 3\ dt = 3t\Big]_0^{\frac{3\pi}{2}} = 9\frac{\pi}{2}$$