



## محاضرات قسم الهندسة الكهربائية

المرحلة: الرابعة

عنوان المادة: السيطرة

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اسم التدريسي:

المحاضرة الاولى

تسلسل المحاضرة:

المحاور الرئيسية:

- 1- مقدمة
- 2- منظومات السيطرة
- 3- المخطط الكتلي لمنظومات السيطرة واختصاراتها
- 4- خطط سريان الاشارة
- 5- التحليل الزمني لمنظومات السيطرة
- 6- الخطء الثابت للمنظومات المغلقة
- 7- التحليل الزمن لمنظومة الدرجة الثانية
- 8- نمذجة وتمثيل منظومات السيطرة
- 9- قاعدة راوث هرتز للاستقرار
- 10- التصميم باستخدام المحل الهندسي
- 11- المحل الهندسي للجذور
- 12- المسيطر التناسبي التكاملي التفاضلي
- 13- معادلات فضاء الحالة
- 14- ازاحة الاقطاب
- 15- قاعدة نايكويست للاستقرارية Nyquist
- 16- الاستجابة الترددية لمنظومة السيطرة

محتويات  
المحاضرة

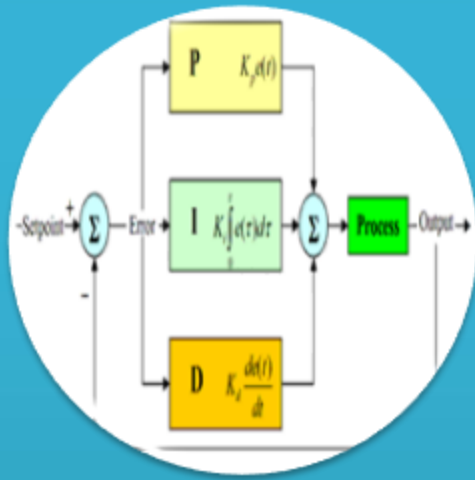
# Control Systems (CS)

المحاضرة الاولى  
مقدمة عامة للموضوع

أ م د عبدالاله خضر محمود  
كلية الهندسة  
قسم الكهرباء

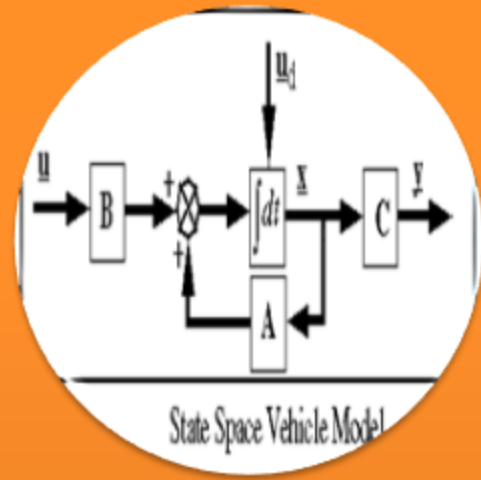
# المضمون

- Course Outline محتوى الموضوع
- Recommended Books الكتب الرئيسية
- Prerequisites of the subject اوليات واساسيات الموضوع
- Basic Definitions التعاريف الاساسية
- Types of Control Systems انواع انظمة السيطرة
- Examples امثلة



## Classical Control

- System Modelling
  - Transfer Function
  - Block Diagrams
  - Signal Flow Graphs
- System Analysis
  - Time Domain Analysis
  - Frequency Domain Analysis
    - Bode Plots, Nyquist Plots, Nichol's Chart
- Root Locus
- System Design
  - Compensation Techniques
  - PID Control



## Modern Control

- State Space Modelling
- Eigenvalue Analysis
- Observability and Controllability
- Solution of State Equations (state Transition Matrix)
- State Space to Transfer Function
- Transfer Function to State Space
  - Direct Decomposition of Transfer Function
  - Cascade Decomposition of Transfer Function
  - Parallel Decomposition of Transfer Function
- State Space Design Techniques



# الكتب المستخدمة Text Books

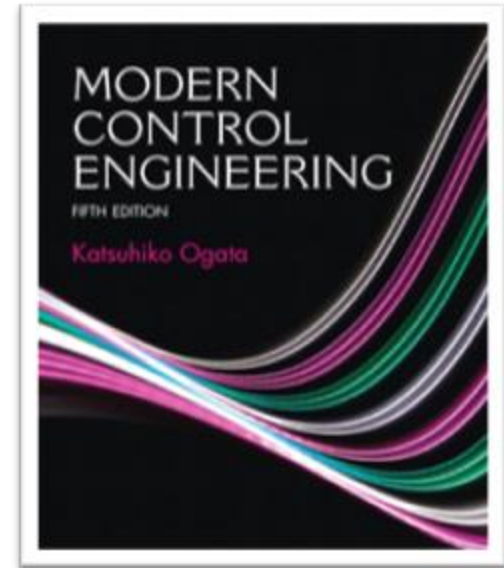
## 1. Modern Control Engineering, (5<sup>th</sup> Edition)

By: Katsuhiko Ogata.

(Prof Emeritus)

Mechanical Engineering

University of Minnesota



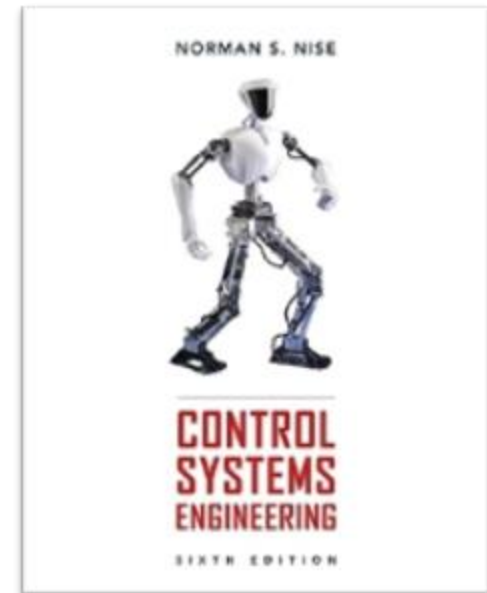
## 2. Control Systems Engineering, (6<sup>th</sup> Edition)

By: Norman S. Nise. (Professor Emeritus)

Electrical and Computer

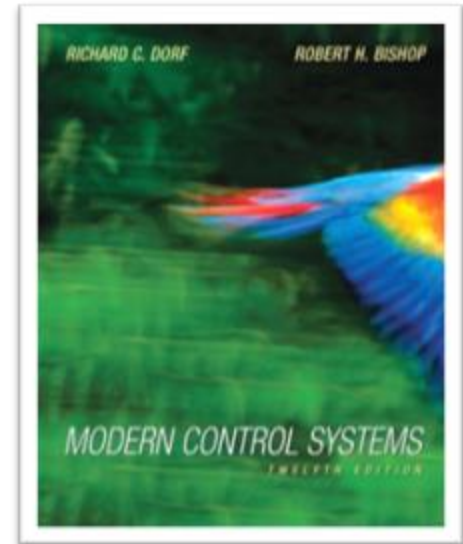
Engineering Department

at California State Polytechnic University

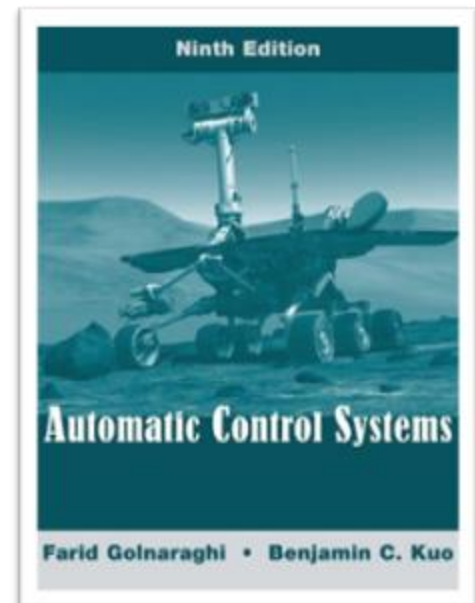


# Reference Books

1. Modern Control Systems, (12<sup>th</sup> Edition)  
By: Richard C. Dorf and Robert H. Bishop.



2. Automatic Control Systems, (9<sup>th</sup> Edition)  
By: Golnaraghi and B. C. Kuo.



# Prerequisites

- For Classical Control Theory
  - Differential Equations المعادلات التفاضلية
  - Laplace Transform تحويلات لابلاس
  - Basic Physics
  - Ordinary and Semi-logarithmic graph papers
- For Modern Control theory above &
  - Linear Algebra
  - Matrices

# What is Control System?

ماذا تعني منظومة السيطرة

- A system Controlling the operation of another system.
- A system that can regulate itself and another system.
- A control System is a device, or set of devices to manage, command, direct or regulate the behaviour of other device(s) or system(s).

# Definitions

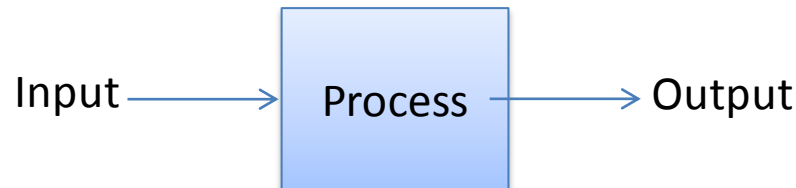
## التعاريف الرئيسية

**System** – An interconnection of elements and devices for a **desired purpose**.

المنظومة وهي التي تتكون من عدة عناصر واجهزة لتحقيق هدف او عمل ما

**Control System** – An interconnection of components forming a system configuration that will provide a desired response. منظومة السيطرة.

**Process** – العمليات The device, **plant**, or system under control. The input and output relationship represents the cause-and-effect relationship of the process.



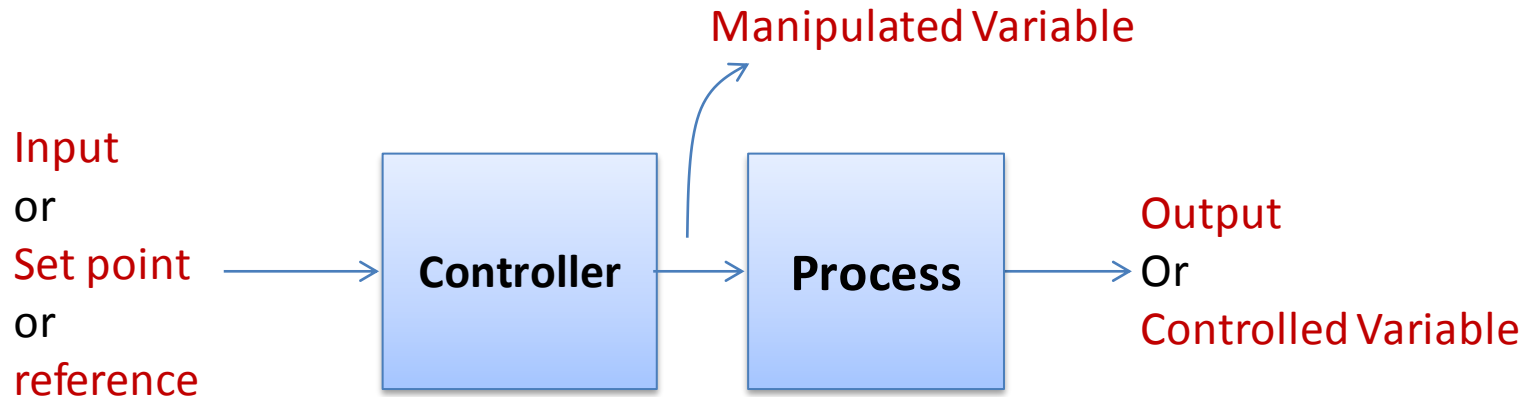
# Definitions

**Controlled Variable**— المتغيرات المسيطر عليها It is the quantity or condition that is measured and Controlled. Normally *controlled variable* is the output of the control system.

**Manipulated Variable**— It is the quantity of the condition that is varied by the controller so as to affect the value of *controlled variable*.

**Control** – Control means measuring the value of *controlled variable* of the system and applying the *manipulated variable* to the system to correct or limit the deviation of the measured value from a desired value.

# Definitions

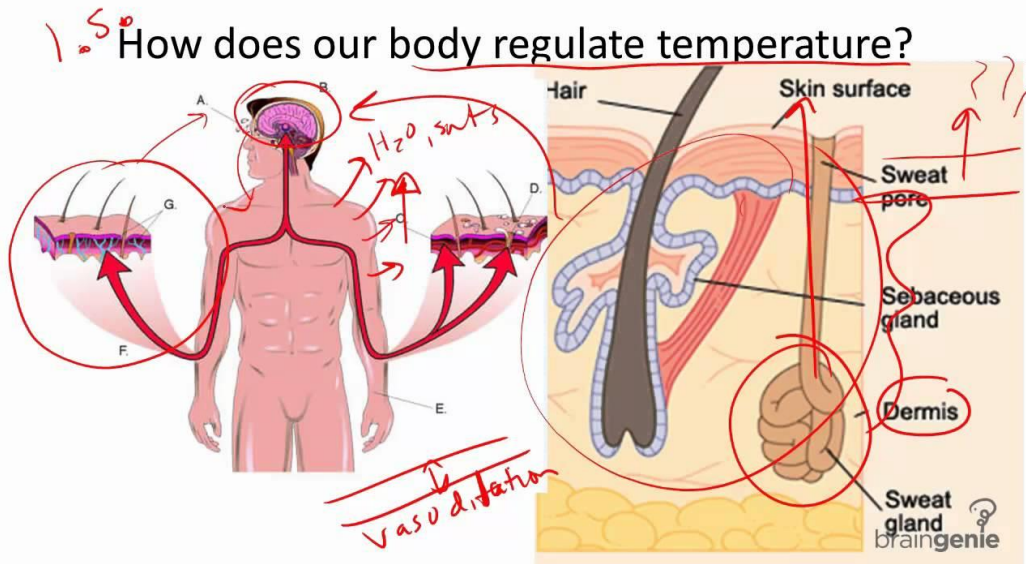


**Disturbances**— A disturbance is a signal that tends to adversely affect the value of the system. It is an unwanted input of the system.

- If a disturbance is generated within the system, it is called *internal disturbance*. While an *external disturbance* is generated outside the system.

# انواع نظم السيطرة Types of Control System

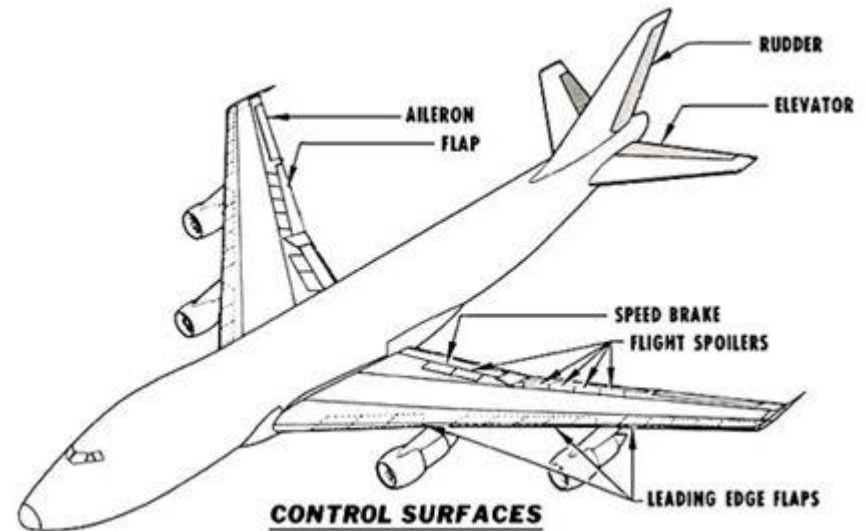
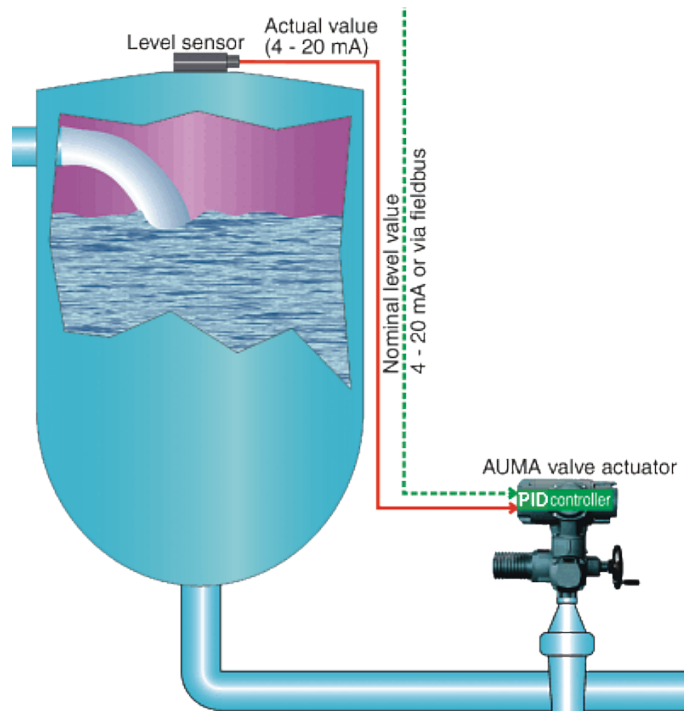
- Natural Control System
  - Universe
  - Human Body





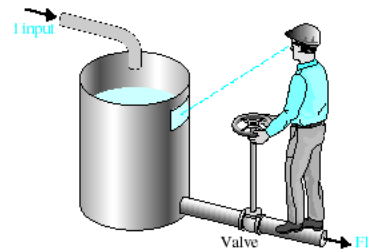
# Types of Control System

- Manmade Control System
  - Aeroplanes
  - Chemical Process



# Types of Control System

- Manual Control Systems
  - Room Temperature regulation Via Electric Fan
  - Water Level Control



- Automatic Control System
  - Home Water Heating Systems (Geysers)
  - Room Temperature regulation Via A.C
  - Human Body Temperature Control



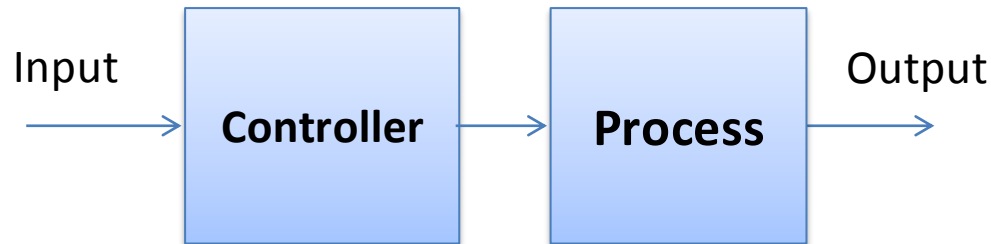
# Types of Control System

## Open-Loop Control Systems

منظومة الدارة المفتوحة

**Open-Loop Control Systems** utilize a controller or control actuator to obtain the desired response.

- Output has no effect on the control action.
- In other words output is neither measured nor fed back.



Open-loop control system (without feedback).

**Examples:-** Washing Machine, Toaster, Electric Fan, microwave oven, e.t.c

# Types of Control System

## Open-Loop Control Systems

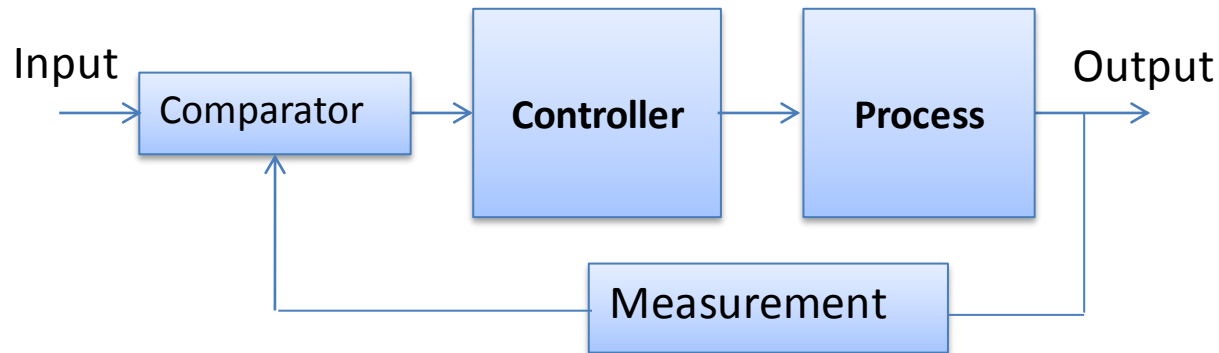
- Since in open loop control systems reference input is not compared with measured output, for each reference input there is fixed operating condition. Therefore, the accuracy of the system depends on calibration.
- The performance of open loop system is severely affected by the presence of disturbances, or variation in operating/ environmental conditions.

# Types of Control System

## Closed-Loop Control Systems

منظومة الدارة المغلقة

**Closed-Loop Control Systems** utilizes feedback تغذية عكسية to compare the actual output to the desired output response.



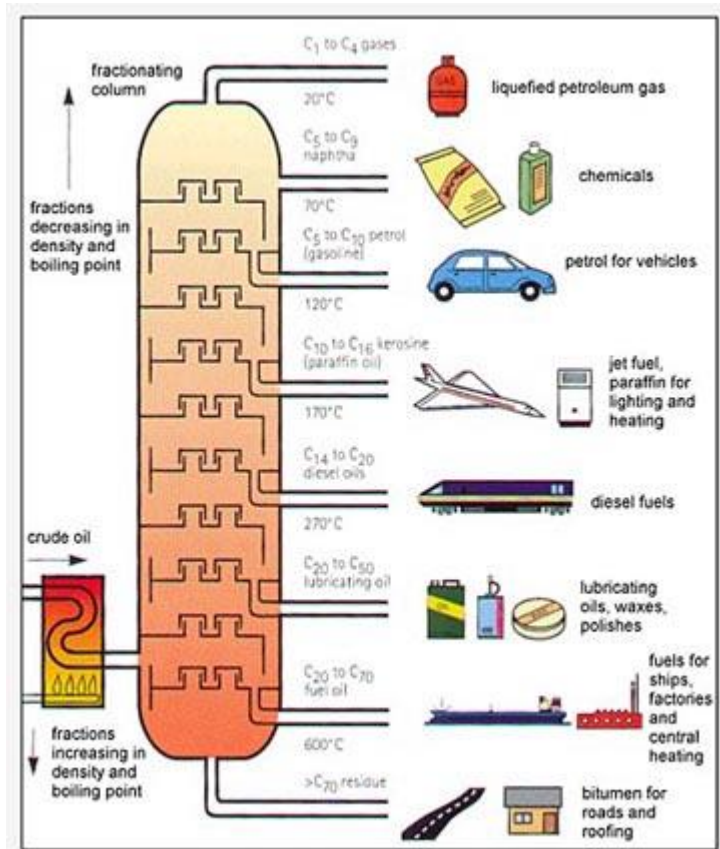
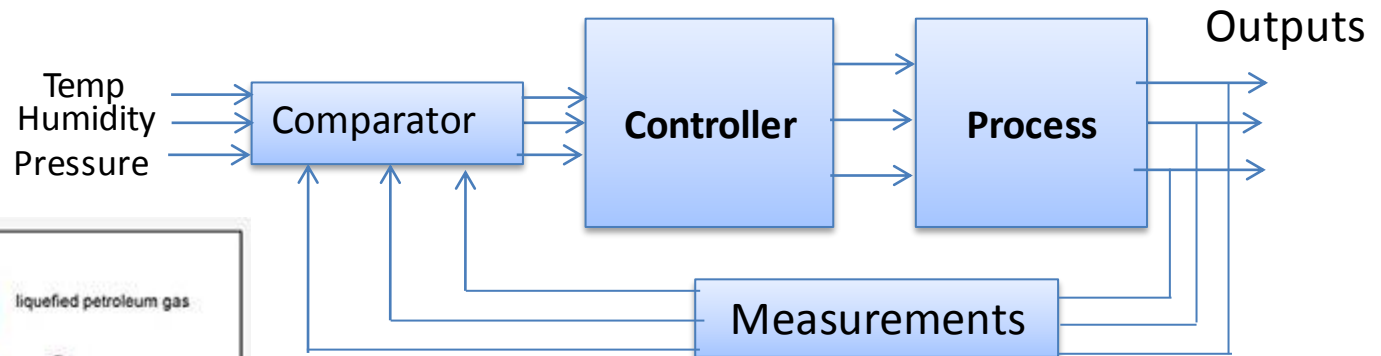
Closed-loop feedback control system (with feedback).

**Examples:-** Refrigerator, Electric Iron, Air conditioner

# Types of Control System

## Multivariable Control System

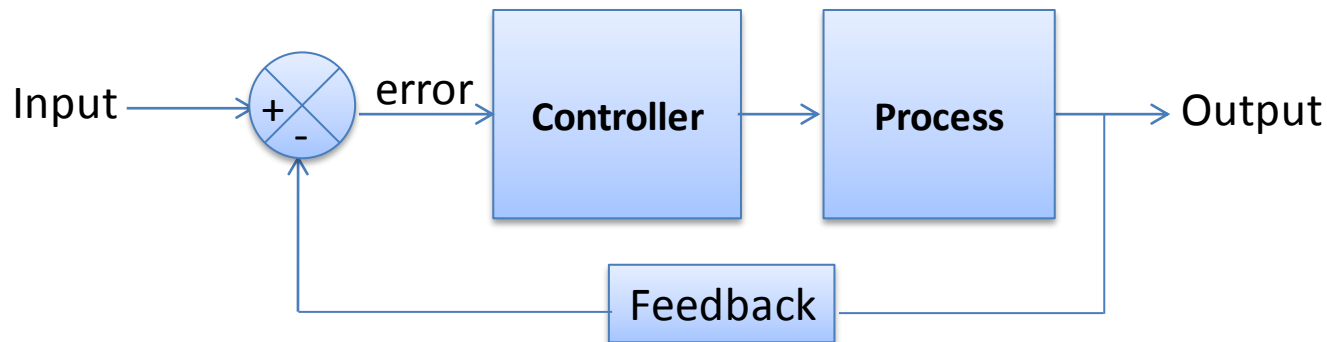
منظومات متعددة الادخال والاخراج



# Types of Control System

## Feedback Control System

- A system that maintains a prescribed relationship between the output and some reference input by comparing them and using the difference (i.e. error) as a means of control is called a feedback control system.

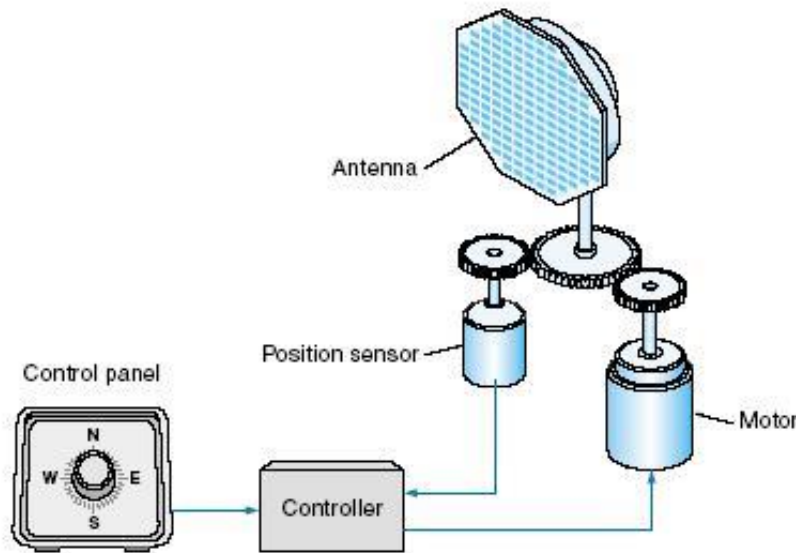


- Feedback can be positive or negative.

# Types of Control System

## Servo System

- A Servo System (or servomechanism) is a feedback control system in which the output is some mechanical position, velocity or acceleration.



Antenna Positioning System



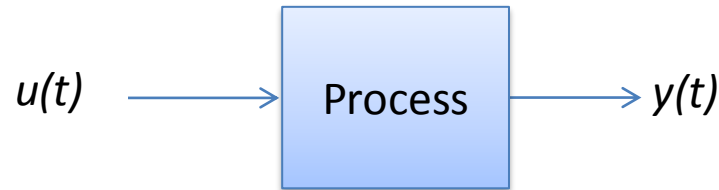
Modular Servo System (MS150)



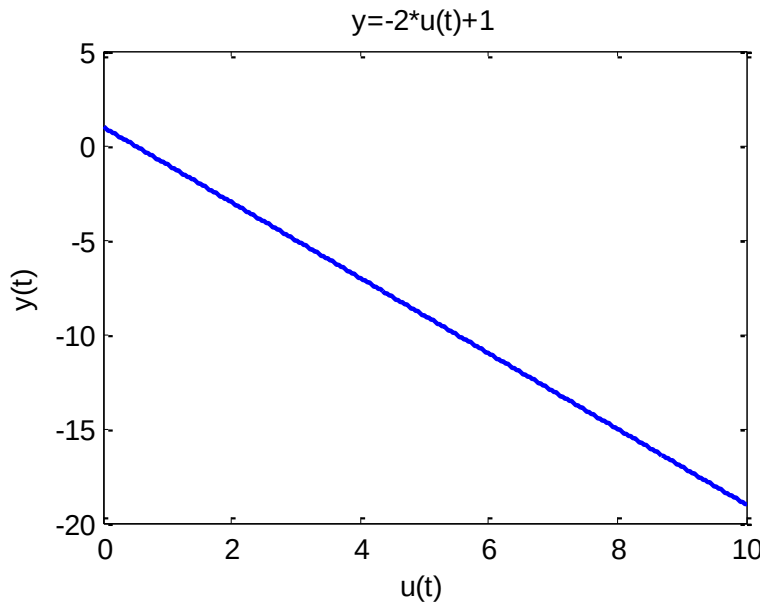
# Types of Control System

## Linear Vs Nonlinear Control System

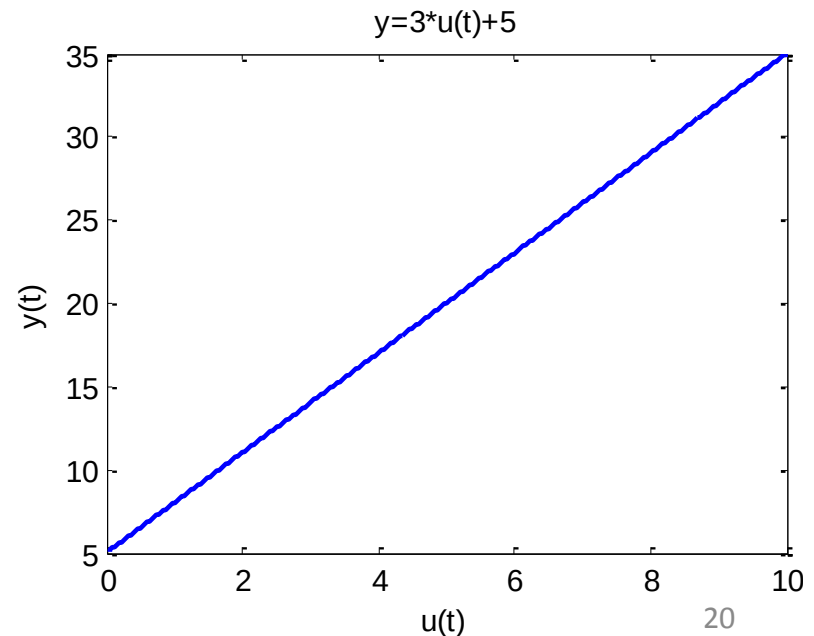
- A Control System in which output varies linearly with the input is called a linear control system . منظومات السيطرة الخطية



$$y(t) = -2u(t) + 1$$



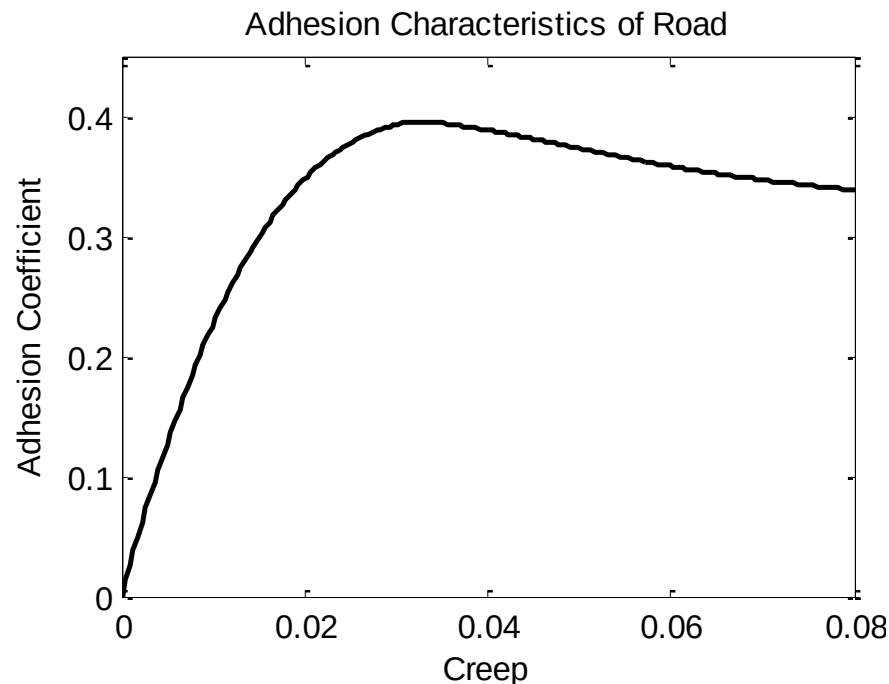
$$y(t) = 3u(t) + 5$$



# Types of Control System

## Linear Vs Nonlinear Control System

- When the input and output has nonlinear relationship the system is said to be nonlinear . منظومات غير خطية .



# Types of Control System

## Time invariant vs Time variant

- When the characteristics of the system do not depend upon time itself then the system is said to time invariant control system.

$$y(t) = -2u(t) + 1$$

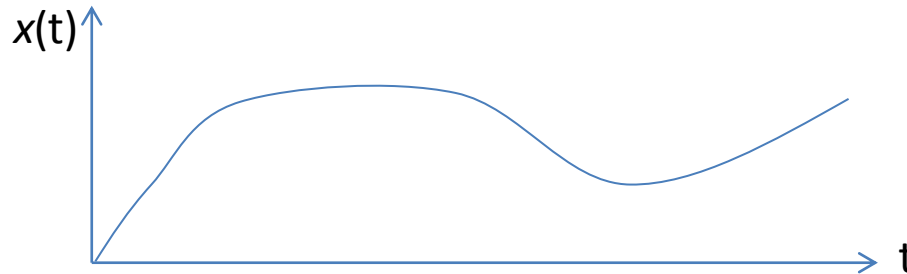
- Time varying control system is a system in which one or more parameters vary with time.

$$y(t) = 2u(t) - 3t$$

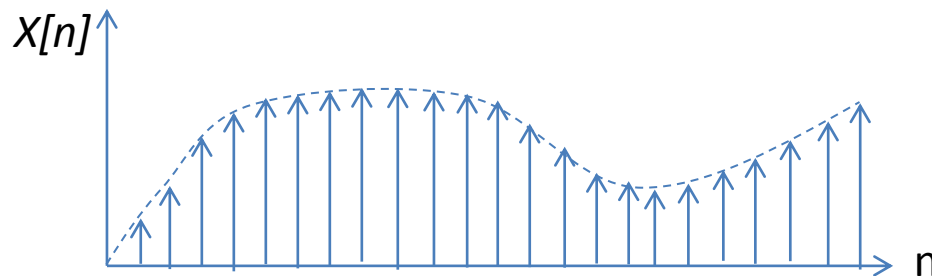
# Types of Control System

## Continuous Data Vs Discrete Data System

- In continuous data control system all system variables are function of a continuous time  $t$ . منظومات الزمن المستمر



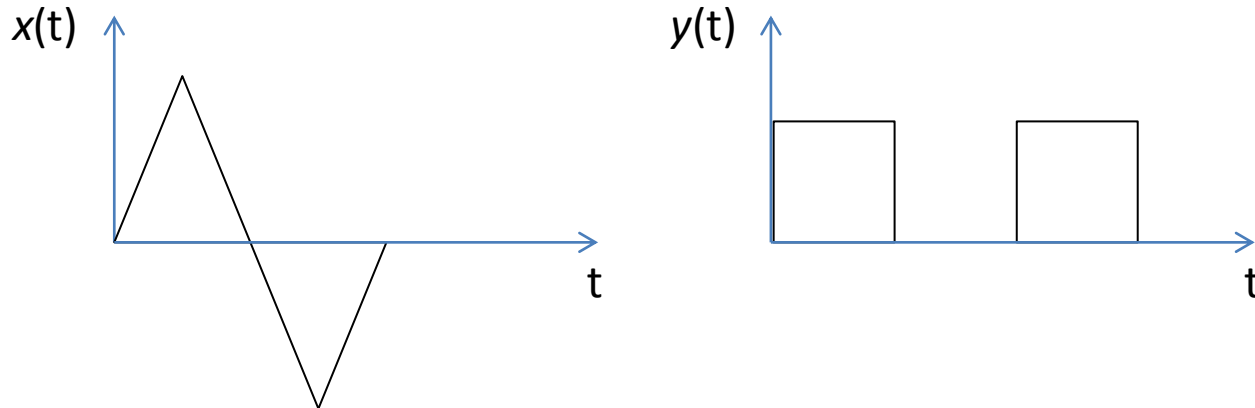
- A discrete time control system منظومات الزمن المتقطع involves one or more variables that are known only at discrete time intervals.



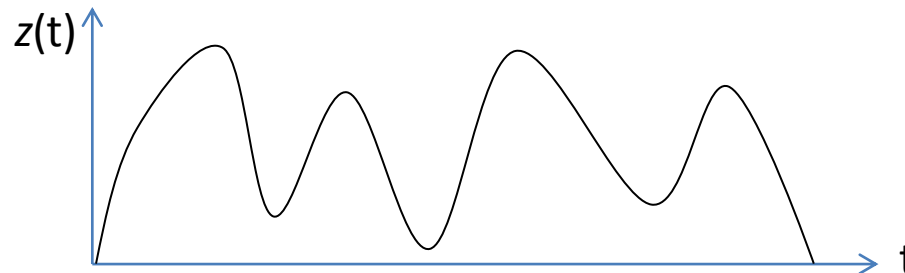
# Types of Control System

## Deterministic vs Stochastic Control System

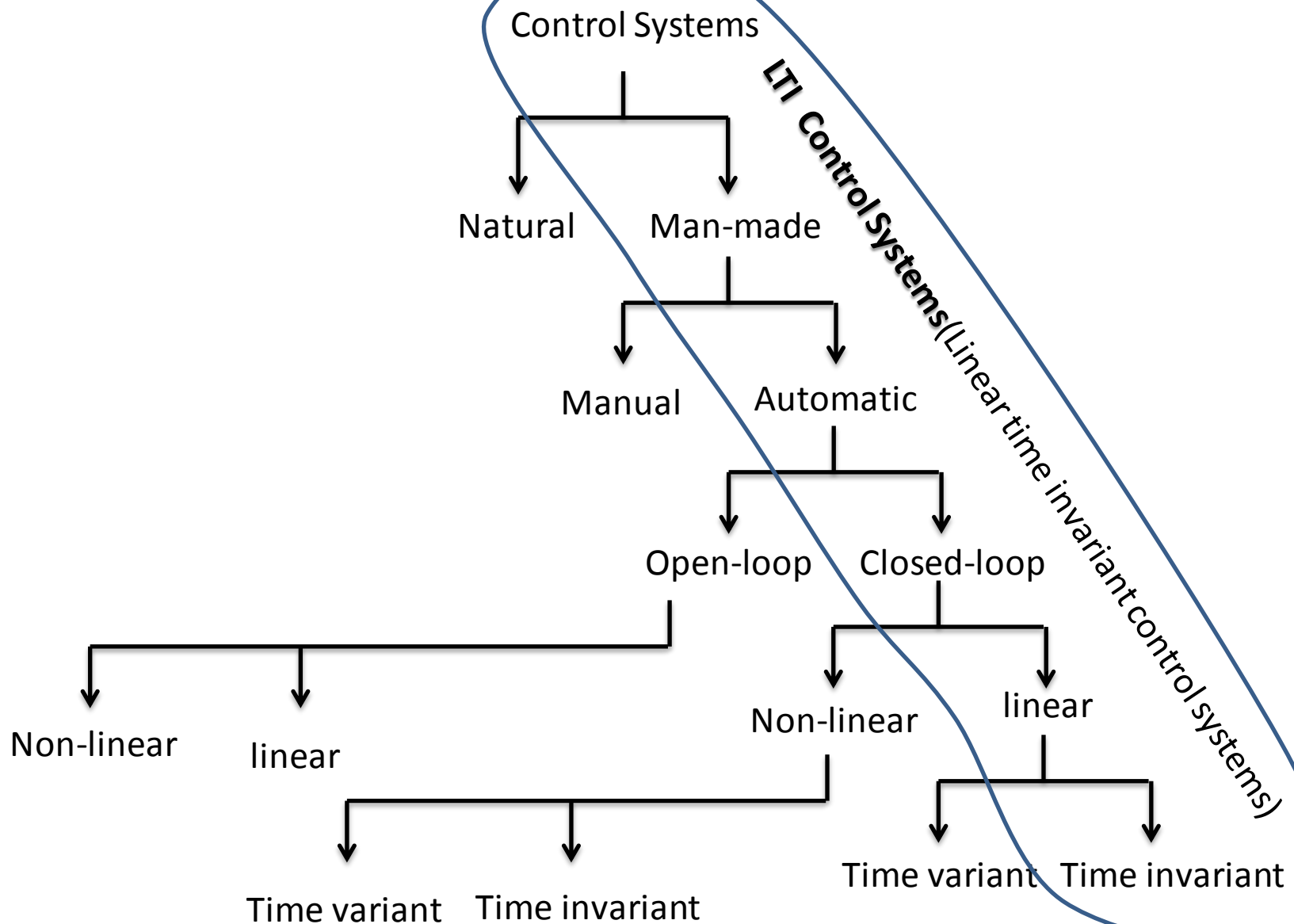
- A control System is deterministic if the response to input is predictable and repeatable.



- If not, the control system is a stochastic control system

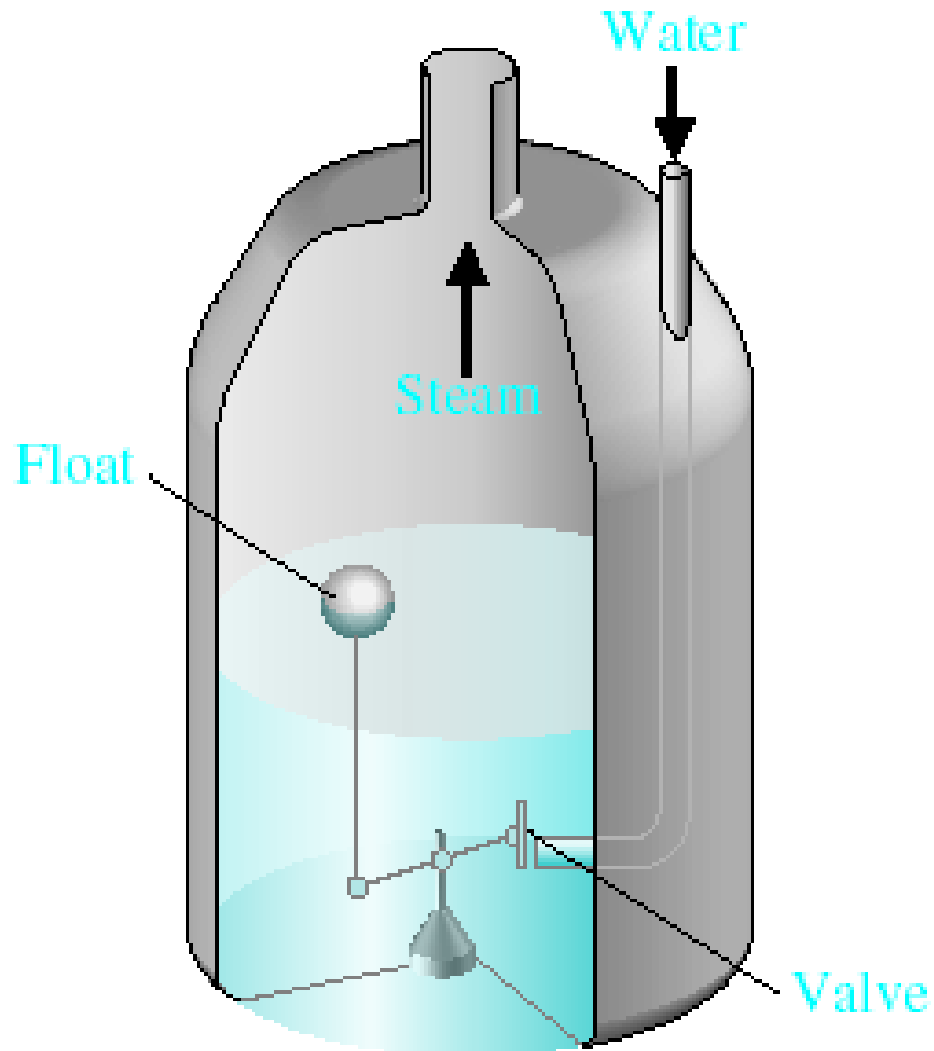


# Classification of Control Systems

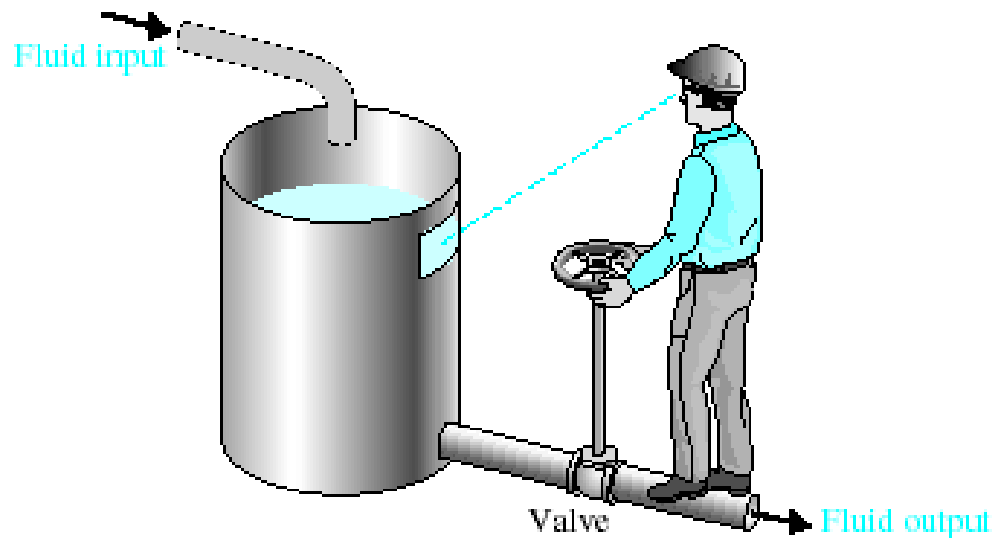


# Examples of Control Systems

Water-level float regulator



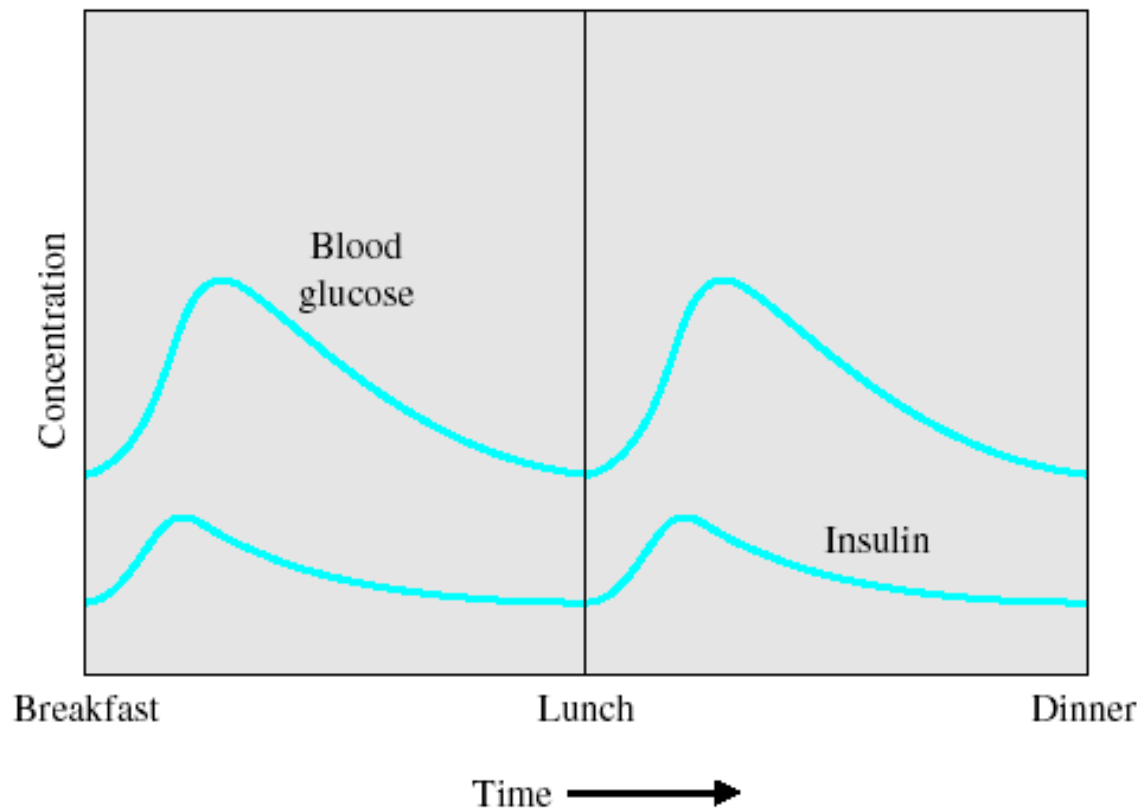
# Examples of Control Systems



A manual control system for regulating the level of fluid in a tank by adjusting the output valve. The operator views the level of fluid through a port in the side of the tank.



# Examples of Modern Control Systems



The blood glucose and insulin levels for a healthy person.

# LAPLASE TRANSFORM

$$L\{f(t)\} = F(s) = \int_0^{\infty} e^{-st} f(t) dt$$

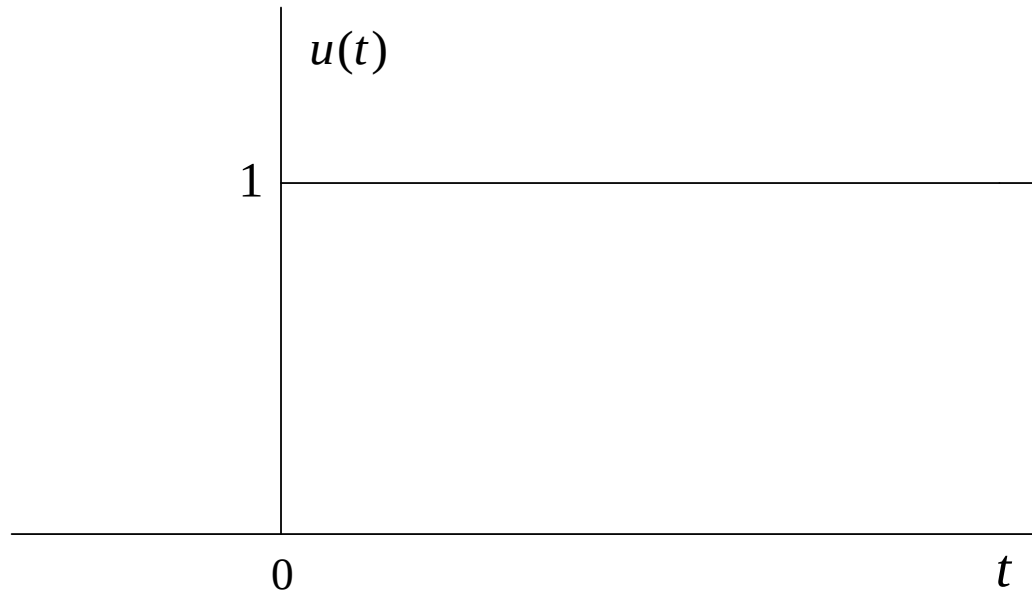
# Laplace Transformation

$$L[f(t)] = F(s)$$

$$L^{-1}[F(s)] = f(t)$$

$$F(s) = \int_0^{\infty} f(t)e^{-st} dt$$

Figure 10-2. Illustration of the unit step function.



Example 10-1. Derive the Laplace transform of the unit step function.

$$F(s) = \int_0^{\infty} (1)e^{-st} dt$$

$$F(s) = \left. \frac{e^{-st}}{-s} \right]_0^{\infty} = 0 - \left( \frac{e^{-0}}{-s} \right) = \frac{1}{s}$$

Example 10-2. Derive the Laplace transform of the exponential function

$$\begin{aligned} F(s) &= \int_0^{\infty} e^{-\alpha t} e^{-st} dt = \int_0^{\infty} e^{-(\alpha+s)t} dt \\ &= \left. \frac{e^{-(s+\alpha)t}}{-(s+\alpha)} \right]_0^{\infty} = 0 - \frac{e^{-0}}{-(s+\alpha)} \\ &= \frac{1}{s+\alpha} \end{aligned}$$

Table 10-1. Common transform pairs.

| $f(t)$                        | $F(s) = L[f(t)]$                               |      |
|-------------------------------|------------------------------------------------|------|
| 1 or $u(t)$                   | $\frac{1}{s}$                                  | T-1  |
| $e^{-\alpha t}$               | $\frac{1}{s + \alpha}$                         | T-2  |
| $\sin \omega t$               | $\frac{\omega}{s^2 + \omega^2}$                | T-3  |
| $\cos \omega t$               | $\frac{s}{s^2 + \omega^2}$                     | T-4  |
| $e^{-\alpha t} \sin \omega t$ | $\frac{\omega}{(s + \alpha)^2 + \omega^2}$     | T-5* |
| $e^{-\alpha t} \cos \omega t$ | $\frac{s + \alpha}{(s + \alpha)^2 + \omega^2}$ | T-6* |
| $t$                           | $\frac{1}{s^2}$                                | T-7  |
| $t^n$                         | $\frac{n!}{s^{n+1}}$                           | T-8  |
| $e^{-\alpha t} t^n$           | $\frac{n!}{(s + \alpha)^{n+1}}$                | T-9  |
| $\delta(t)$                   | 1                                              | T-10 |

\*Use when roots are complex.

Example 10-3. A force in newtons (N) is given below. Determine the Laplace transform.

$$f(t) = 50u(t)$$

$$F(s) = \frac{50}{s}$$



Example 10-9. Determine  
inverse transform of  
function below.

$$F(s) = \frac{s+6}{s^2+3s+2} = \frac{s+6}{(s+1)(s+2)}$$

$$F(s) = \frac{s+6}{(s+1)(s+2)} = \frac{A_1}{s+1} + \frac{A_2}{s+2}$$

$$A_1 = (s+1)F(s) \Big|_{s=-1} = \frac{s+6}{s+2} \Big|_{s=-1} = \frac{-1+6}{-1+2} = 5$$

## Example 10-9.

### Continuation.

$$A_2 = (s + 2)F(s) \Big|_{s=-2} = \frac{s + 6}{s + 1} \Big|_{s=-2} = \frac{-2 + 6}{-2 + 1} = -4$$

$$F(s) = \frac{5}{s + 1} - \frac{4}{s + 2}$$

$$f(t) = 5e^{-t} - 4e^{-2t}$$

Example 10-16. Solve DE shown below.

$$\frac{d^2 y}{dt^2} + 2 \frac{dy}{dt} + 5y = 20$$

$$y(0) = 0 \text{ and } y'(0) = 10$$

$$s^2 Y(s) - 0 - 10 + 2[sY(s) - 0] + 5Y(s) = \frac{20}{s}$$

$$Y(s) = \frac{20}{s(s^2 + 2s + 5)} + \frac{10}{s^2 + 2s + 5}$$

## Example 10-16. Continuation.

$$\frac{20}{s(s^2 + 2s + 5)} = \frac{4}{s} + \frac{As + B}{(s^2 + 2s + 5)}$$

$$\frac{20}{(1)(1 + 2 + 5)} = \frac{4}{1} + \frac{A + B}{(1 + 2 + 5)}$$

$$\frac{20}{(-1)(1 - 2 + 5)} = \frac{4}{-1} + \frac{-A + B}{(1 - 2 + 5)}$$

$$A = -4 \quad B = -8$$

## Example 10-16. Continuation.

$$Y(s) = \frac{4}{s} + \frac{-4s-8}{s^2+2s+5} + \frac{10}{s^2+2s+5} = \frac{4}{s} + \frac{-4s+2}{s^2+2s+5}$$

$$s^2 + 2s + 5 = s^2 + 2s + 1 + 5 - 1 = (s+1)^2 + (2)^2$$

$$Y(s) = \frac{4}{s} + \frac{-4(s+1)}{(s+1)^2 + (2)^2} + \frac{3(2)}{(s+1)^2 + (2)^2}$$

$$y(t) = 4 - 4e^{-t} \cos 2t + 3e^{-t} \sin 2t$$

# Examples of Control Systems

Control systems find numerous and widespread applications from everyday to extraordinary in science, industry, and home. Here are a few examples:

(a) Residential heating and air-conditioning systems controlled by a thermostat

(b) The cruise (speed) control of an automobile

(c) Manual control:

(i) Opening or closing of a window for regulating air temperature or air quality

(ii) Activation of a light switch to regulate the illumination in a room

(iii) Human controlling the speed of an automobile by regulating the gas supply to the engine

(d) Automatic traffic control (signal) system at roadway intersections

(e) Control system which automatically turns on a room lamp at dusk, and turns it off in daylight

(f) Automatic hot water heater

- (*g*) Environmental test-chamber temperature control system
- (*h*) An automatic positioning system for a missile launcher
- (*i*) An automatic speed control for a field-controlled dc motor
- (*j*) The attitude control system of a typical space vehicle
- (*k*) Automatic position-control system of a high speed automated train system
- (*l*) Human heart using a pacemaker
- (*m*) An elevator-position control system used in high-rise multilevel buildings.



# Types of Systems

- **Static System:** If a system does not change with time, it is called a static system.
- **Dynamic System:** If a system changes with time, it is called a dynamic system.

# Dynamic Systems

A system is said to be dynamic if its current output may depend on the past history as well as the present values of the input variables. Mathematically,

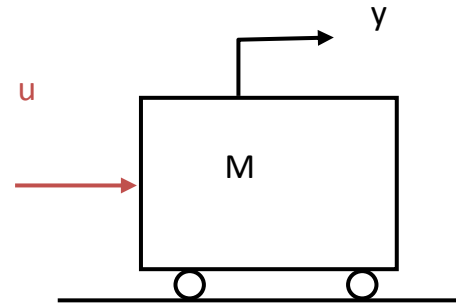
$$\mathbf{y}(t) = \boldsymbol{\varphi}[\mathbf{u}(\tau), 0 \leq \tau \leq t]$$

**$\mathbf{u}$  : Input,  $t$  : Time**

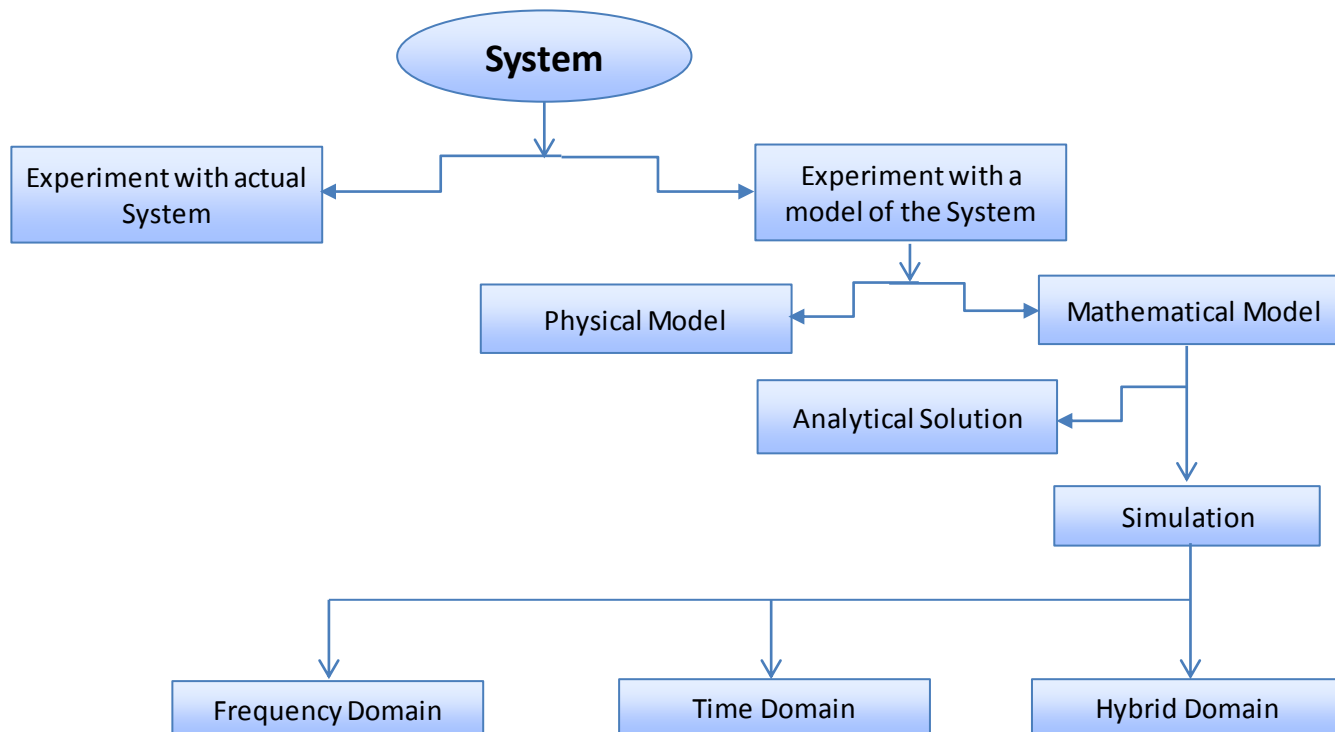
Example: A moving mass

Model: Force=Mass x Acceleration

$$M\ddot{y} = u$$



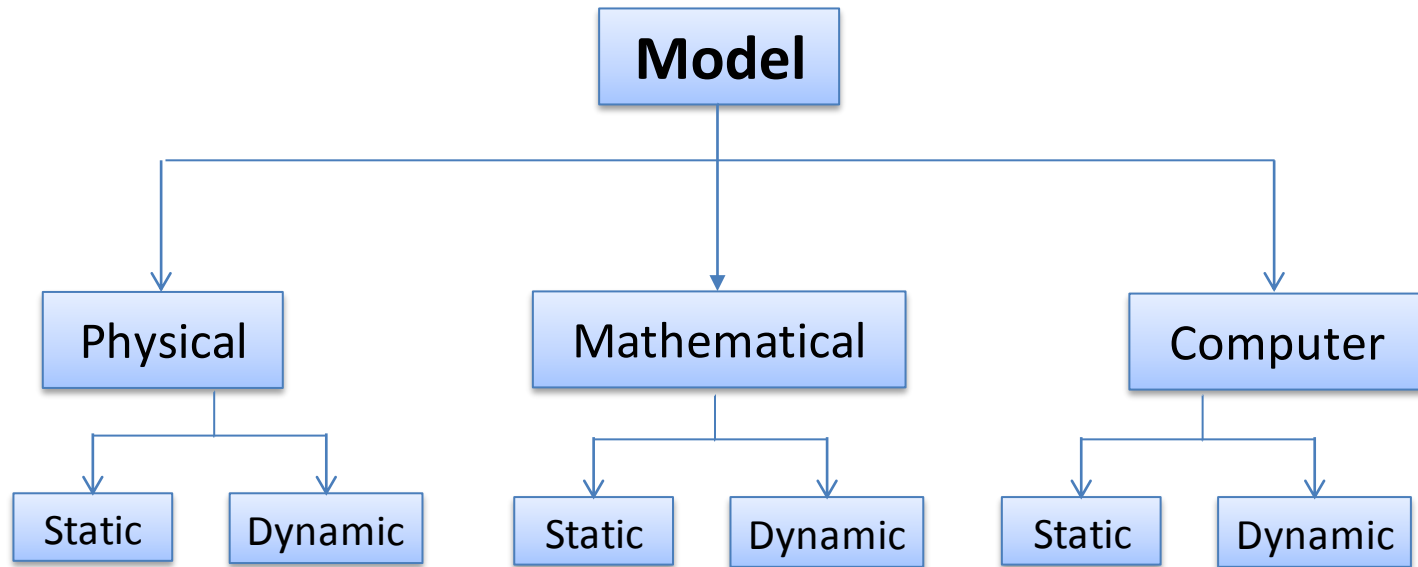
# Ways to Study a System



## Model

- A *model* is a simplified representation or abstraction of reality.
- Reality is generally too complex to copy exactly.
- Much of the complexity is actually *irrelevant* in problem solving.

# Types of Models



# What is Mathematical Model?

A set of mathematical equations (e.g., differential eqs.) that describes the input-output behavior of a system.

What is a model used for?

- Simulation
- Prediction/Forecasting
- Prognostics/Diagnostics
- Design/Performance Evaluation
- Control System Design

## **Classification of Mathematical Models**

- Linear vs. Non-linear
- Deterministic vs. Probabilistic (Stochastic)
- Static vs. Dynamic
- Discrete vs. Continuous
- White box, black box and gray box

# Black Box Model

- When only input and output are known.
- Internal dynamics are either too complex or unknown.

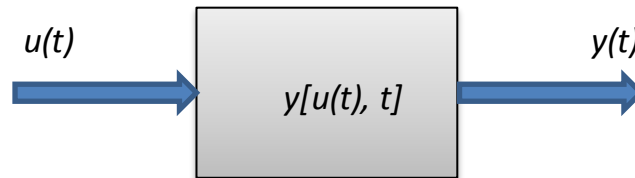


- Easy to Model



# Grey Box Model

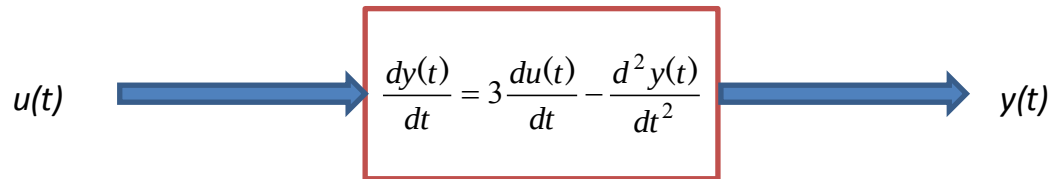
- When input and output and some information about the internal dynamics of the system is known.



- Easier than white box Modelling.

# White Box Model

- When input and output and internal dynamics of the system is known.



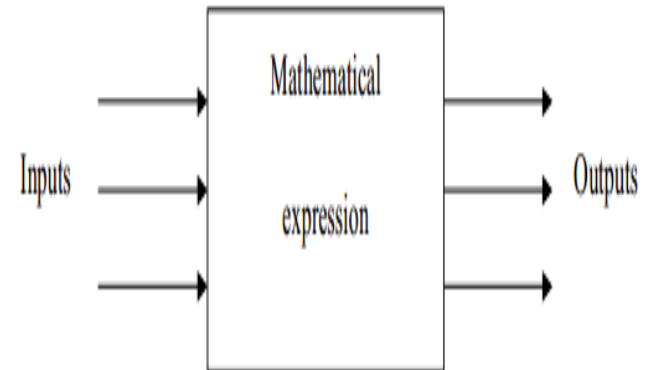
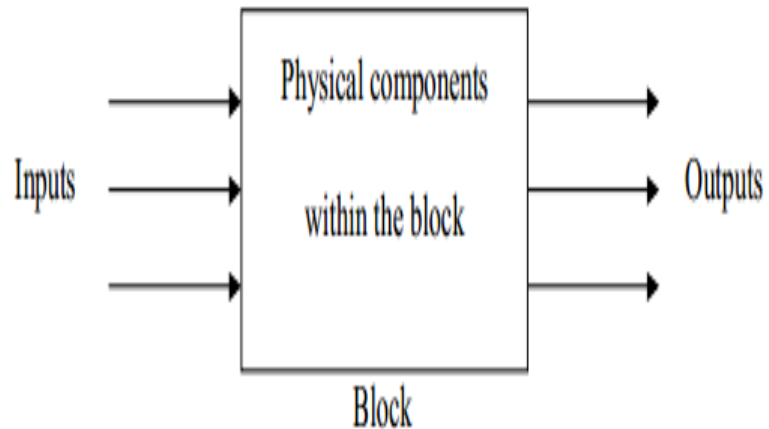
- One should know have complete knowledge of the system to derive a white box model.

# Mathematical Modelling Basics

Mathematical model of a real world system is derived using a combination of *physical laws* and/or *experimental* means

- Physical laws are used to determine the model structure (linear or nonlinear) and order.
- The parameters of the model are often estimated and/or validated experimentally.
- Mathematical model of a dynamic system can often be expressed as a system of differential (difference in the case of discrete-time systems) equations

# CONTROL SYSTEM CONFIGURATION



***Transfer Function.*** The transfer function is a property of the system elements only, and is not dependent on the excitation and initial conditions. The transfer function of a system (or a block) is defined as the ratio of output to input as shown in Fig.1.5.

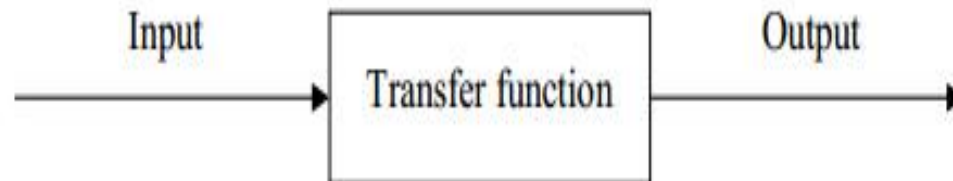


**Fig. 1.5 Transfer function**

**Open-loop Control System.** Open-loop control systems represent the simplest form of controlling devices. A general block diagram of open-loop system is shown in Fig. 1.6.

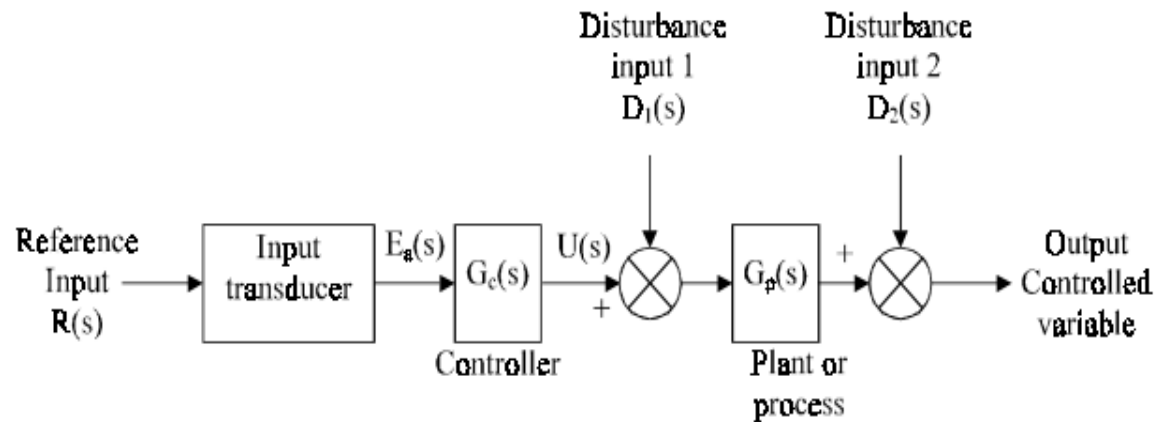
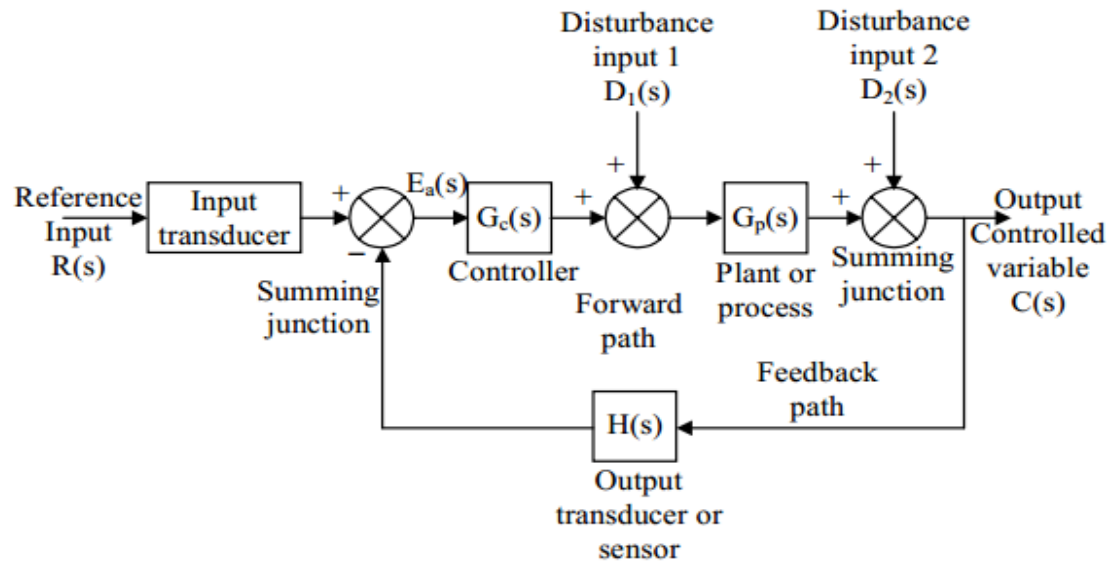


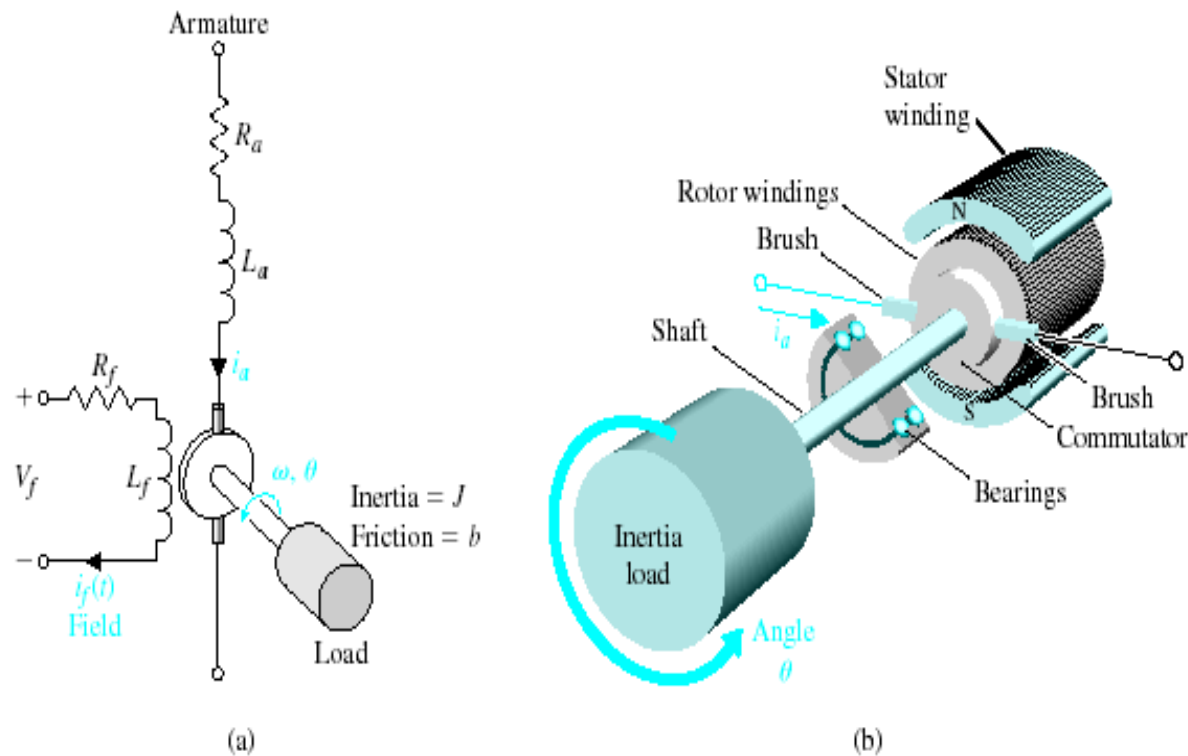
Fig. 1.6 Block diagram of

**Closed-loop (Feedback Control) System.** Closed-loop control systems derive their valuable accurate reproduction of the input from feedback comparison. The general architecture of a closed-loop control system is shown in Fig. 1.7. A system with one or more feedback paths is called a **closed-loop system**.



**General block diagram of closed-loop control system**

## The Transfer Function of Linear Systems



A dc motor (a) wiring diagram and (b) sketch.



# The Transfer Function of Linear Systems

$$\phi = K_f \cdot i_f$$

$$T_m = K_1 \cdot K_f \cdot i_f(t) \cdot i_a(t)$$

field controlled motor - Laplace Transfc

$$T_m(s) = (K_1 \cdot K_f \cdot I_a) \cdot I_f(s)$$

$$V_f(s) = (R_f + L_f \cdot s) \cdot I_f(s)$$

$$T_m(s) = T_L(s) + T_d(s)$$

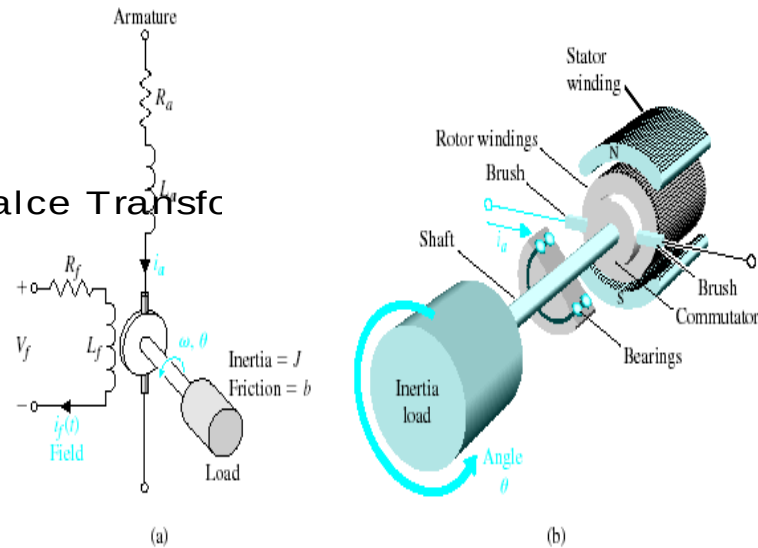
$$T_L(s) = J \cdot s^2 \cdot \theta(s) + b \cdot s \cdot \theta(s)$$

rearranging equations

$$T_L(s) = T_m(s) - T_d(s)$$

$$T_m(s) = K_m \cdot I_f(s)$$

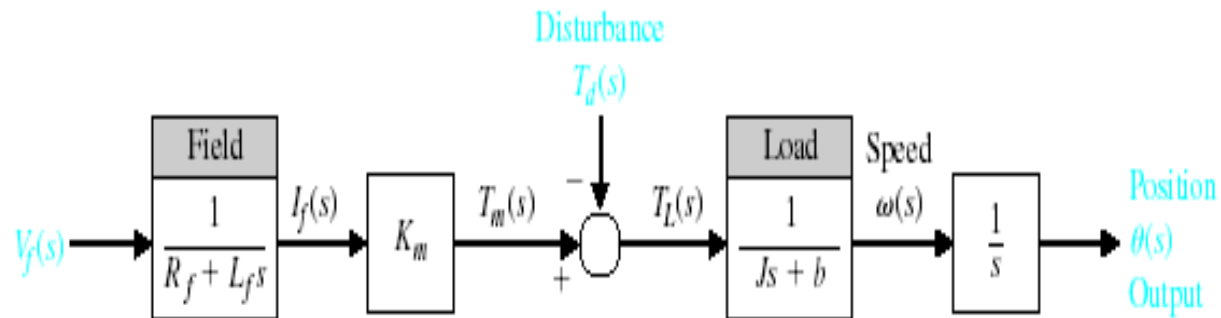
$$I_f(s) = \frac{V_f(s)}{R_f + L_f \cdot s}$$



$$T_d(s) = 0$$

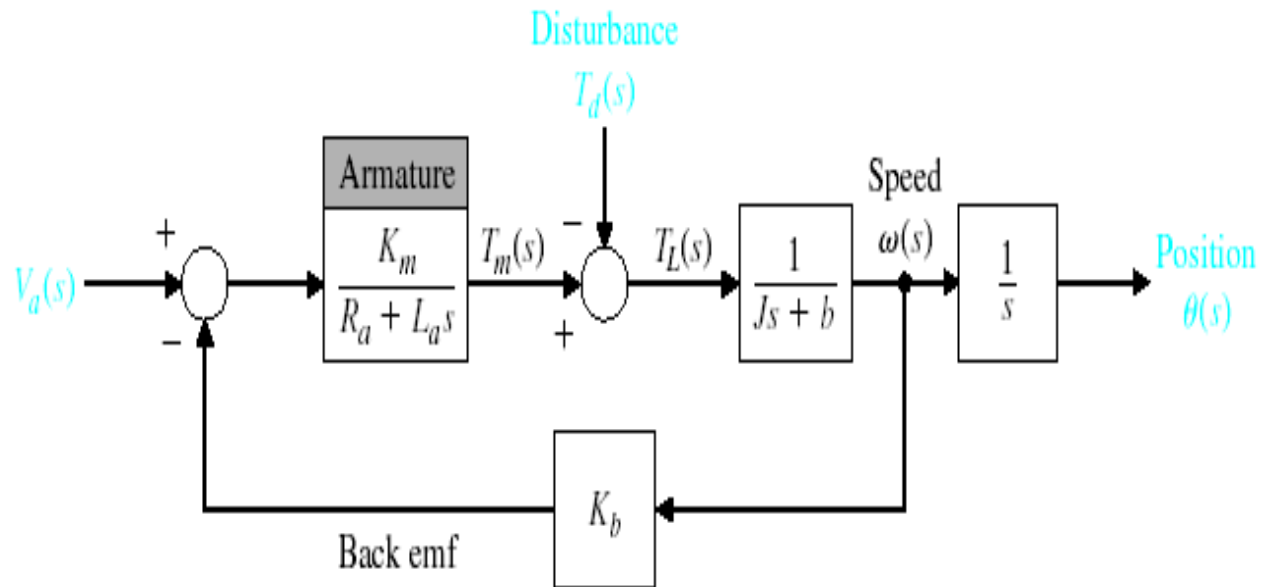
$$\frac{\theta(s)}{V_f(s)} = \frac{K_m}{s \cdot (J \cdot s + b) \cdot (L_f s + R_f)}$$

## The Transfer Function of Linear Systems



Block diagram model of field-controlled dc motor.

## The Transfer Function of Linear Systems



Armature-controlled dc motor.

# Chapter 1: Introduction to Control Systems مقدمة

## الى منظومات السيطرة

### Objectives

In this chapter we describe a general process for designing a control system.

A control system consisting of interconnected components is designed to achieve a desired purpose. To understand the purpose of a control system, it is useful to examine examples of control systems through the course of history. These early systems incorporated many of the same ideas of feedback that are in use today.

Modern control engineering practice includes the use of control design strategies for improving manufacturing processes, the efficiency of energy use, advanced automobile control, including rapid transit, among others.

We also discuss the notion of a design gap. The gap exists between the complex physical system under investigation and the model used in the control system synthesis.

The iterative nature of design allows us to handle the design gap effectively while accomplishing necessary tradeoffs in complexity, performance, and cost in order to meet the design specifications.

# Introduction

**System** منظومة – An interconnection of elements and devices for a desired purpose.

**Control System** منظومة السيطرة – An interconnection of components forming a system configuration that will provide a desired response.

**Process** العمليات – The device, plant, or system under control. The input and output relationship represents the cause-and-effect relationship of the process.



Process to be controlled.

# Introduction

## الحلقة المفتوحة Open-Loop

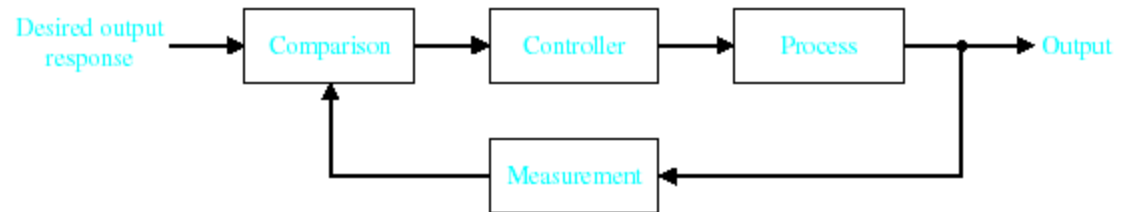
**Control Systems** utilize a controller or control actuator to obtain the desired response.



Open-loop control system (without feedback).

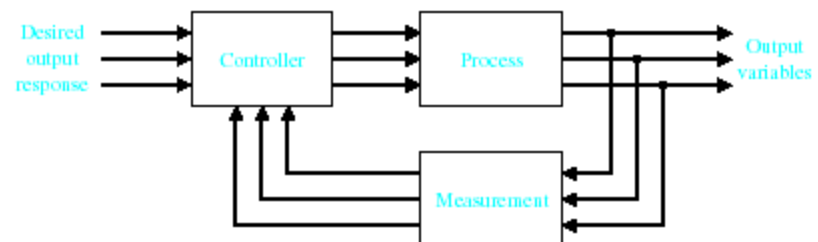
## الحلقة المغلقة Closed-Loop

**Control Systems** utilizes feedback to compare the actual output to the desired output response.

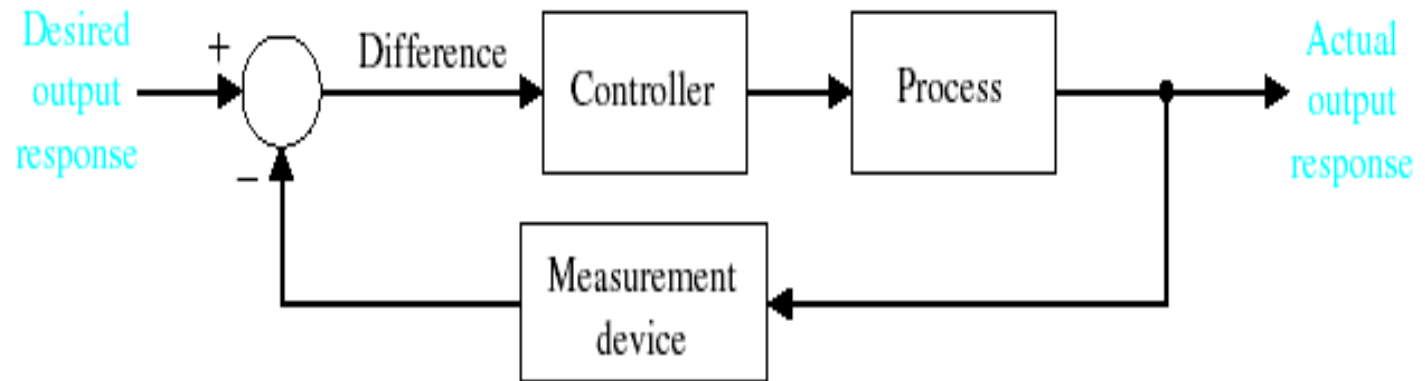


Closed-loop feedback control system (with feedback).

## Multivariable Control System متعدد الادخال والايخراج

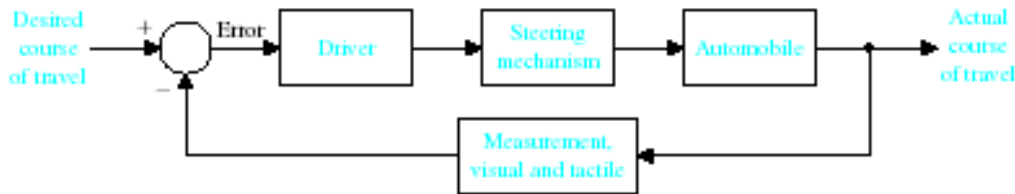


# History

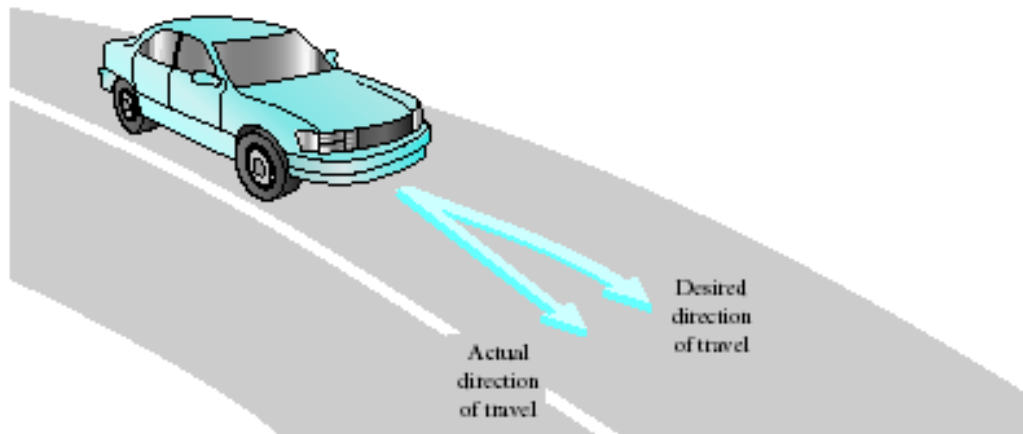


Closed-loop feedback system.

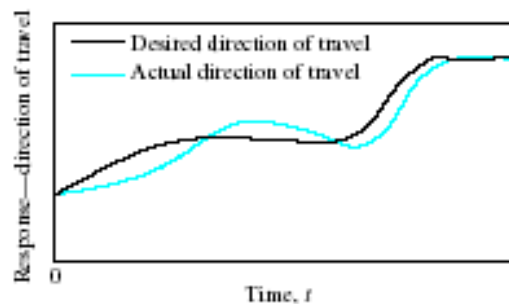
# Examples of Modern Control Systems



(a)



(b)



(c)

(a) Automobile steering control system.

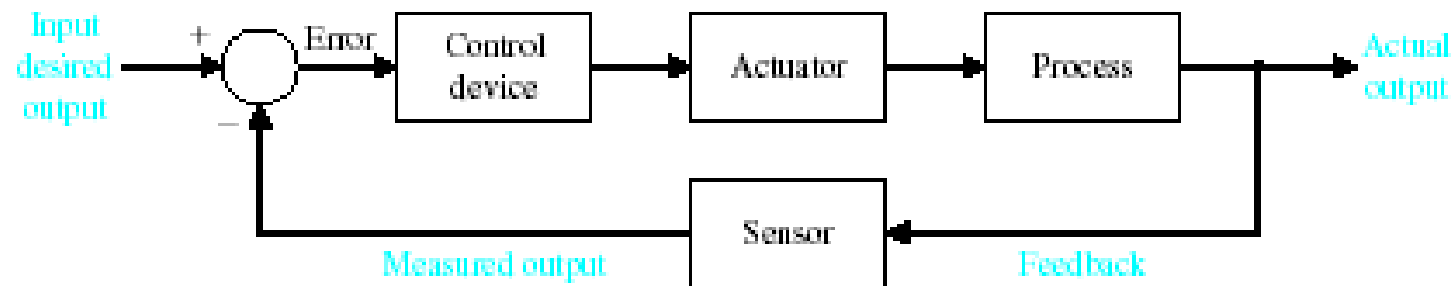
(b) The driver uses the difference between the actual and the desired direction of travel

to generate a controlled adjustment of the steering wheel.

(c) Typical direction-of-travel response.

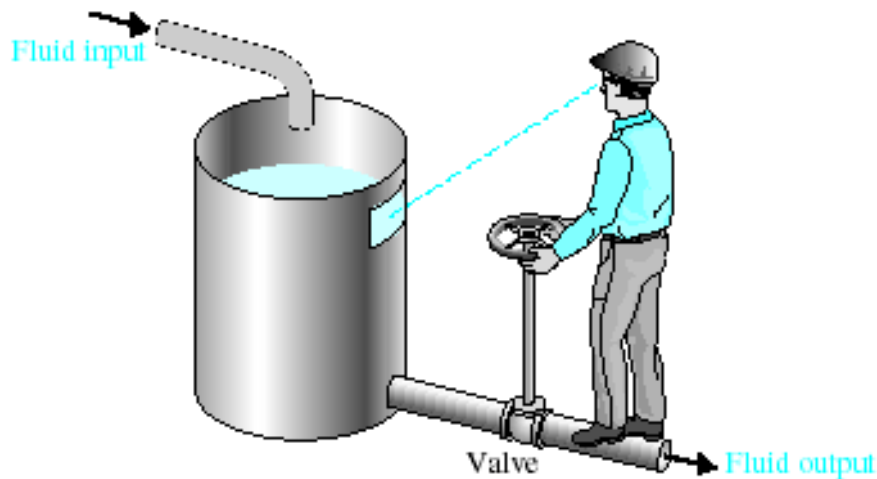


# Examples of Modern Control Systems



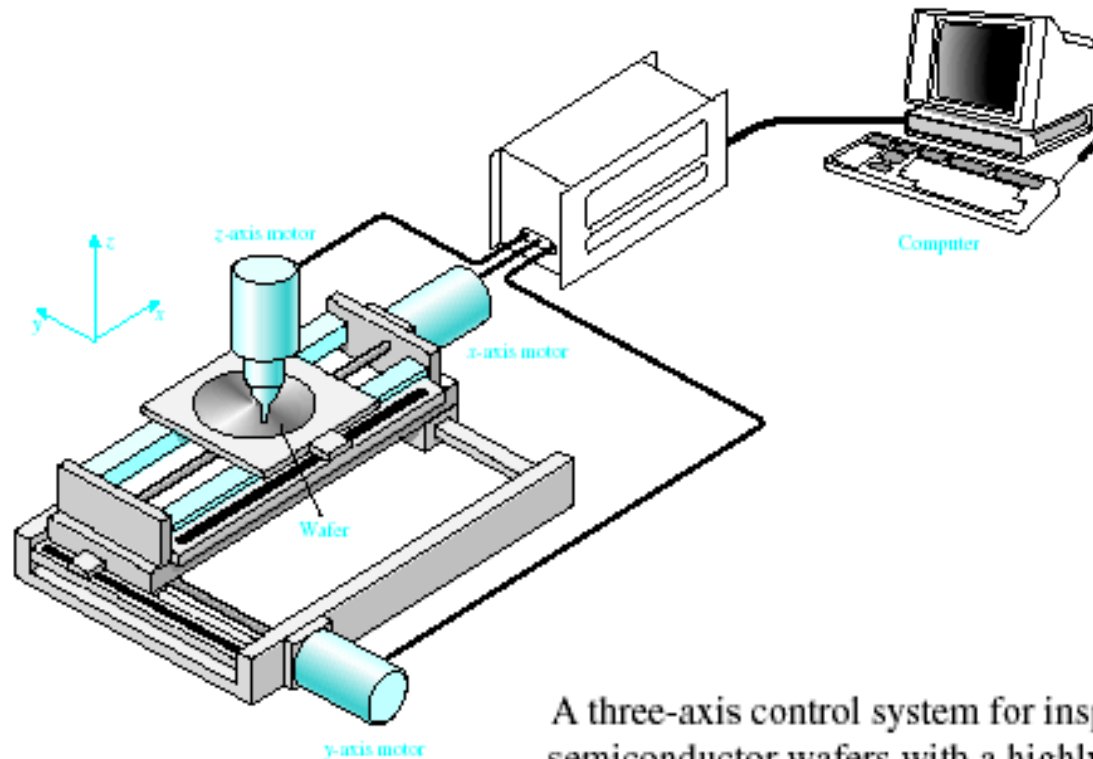
A negative feedback system block diagram depicting a basic closed-loop control system.  
The control device is often called a “controller.”

# Examples of Modern Control Systems



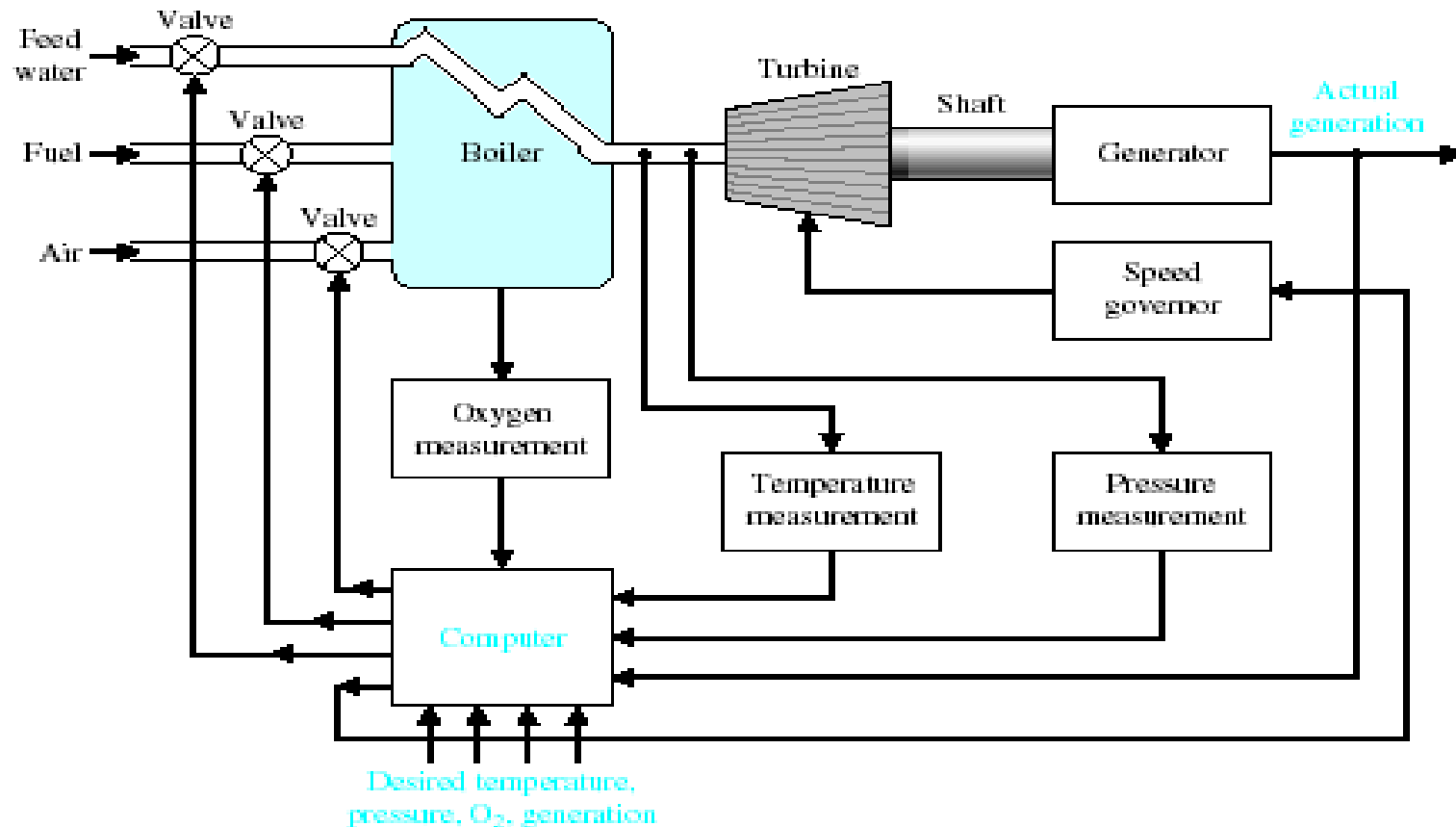
A manual control system for regulating the level of fluid in a tank by adjusting the output valve. The operator views the level of fluid through a port in the side of the tank.

# Examples of Modern Control Systems



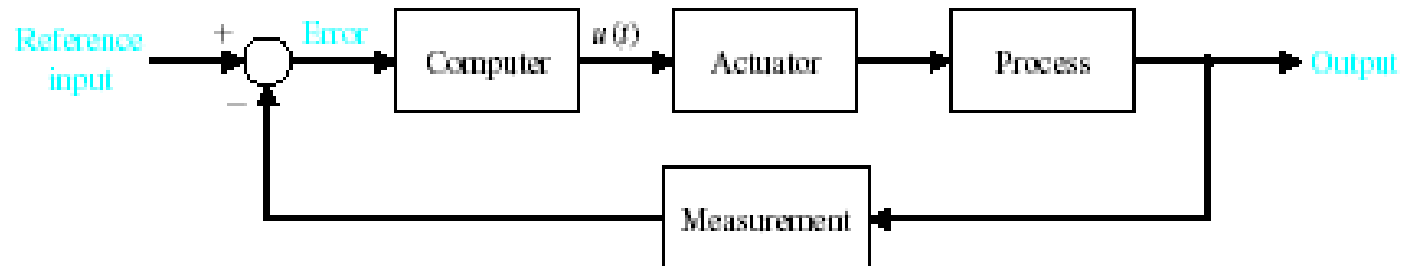
A three-axis control system for inspecting individual semiconductor wafers with a highly sensitive camera.

# Examples of Modern Control Systems



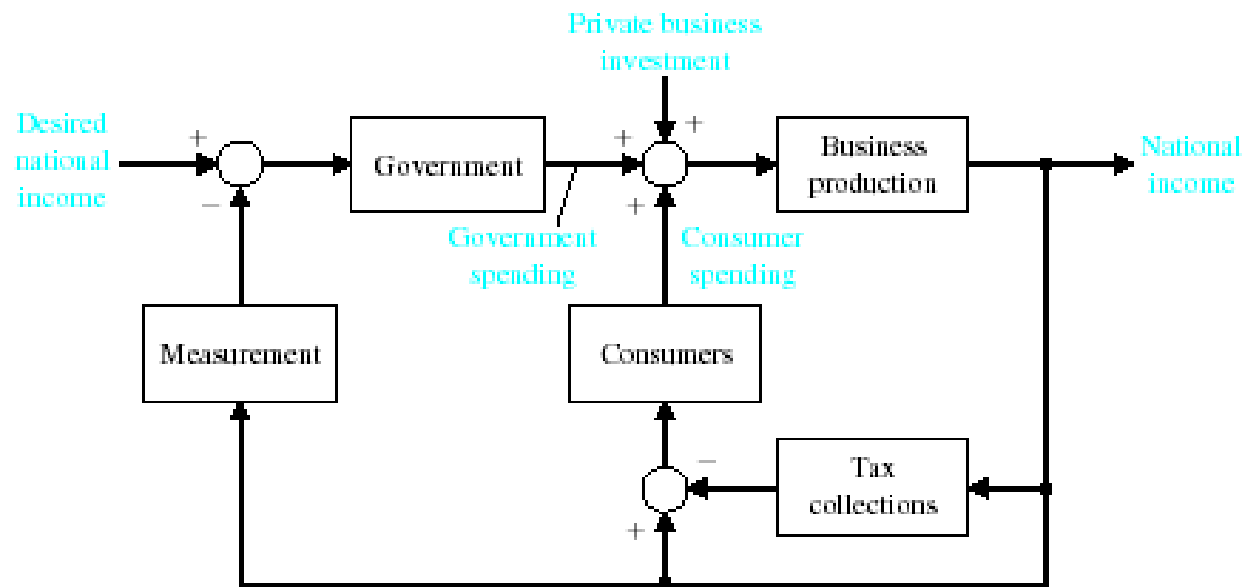
Coordinated control system for a boiler-generator.

# Examples of Modern Control Systems



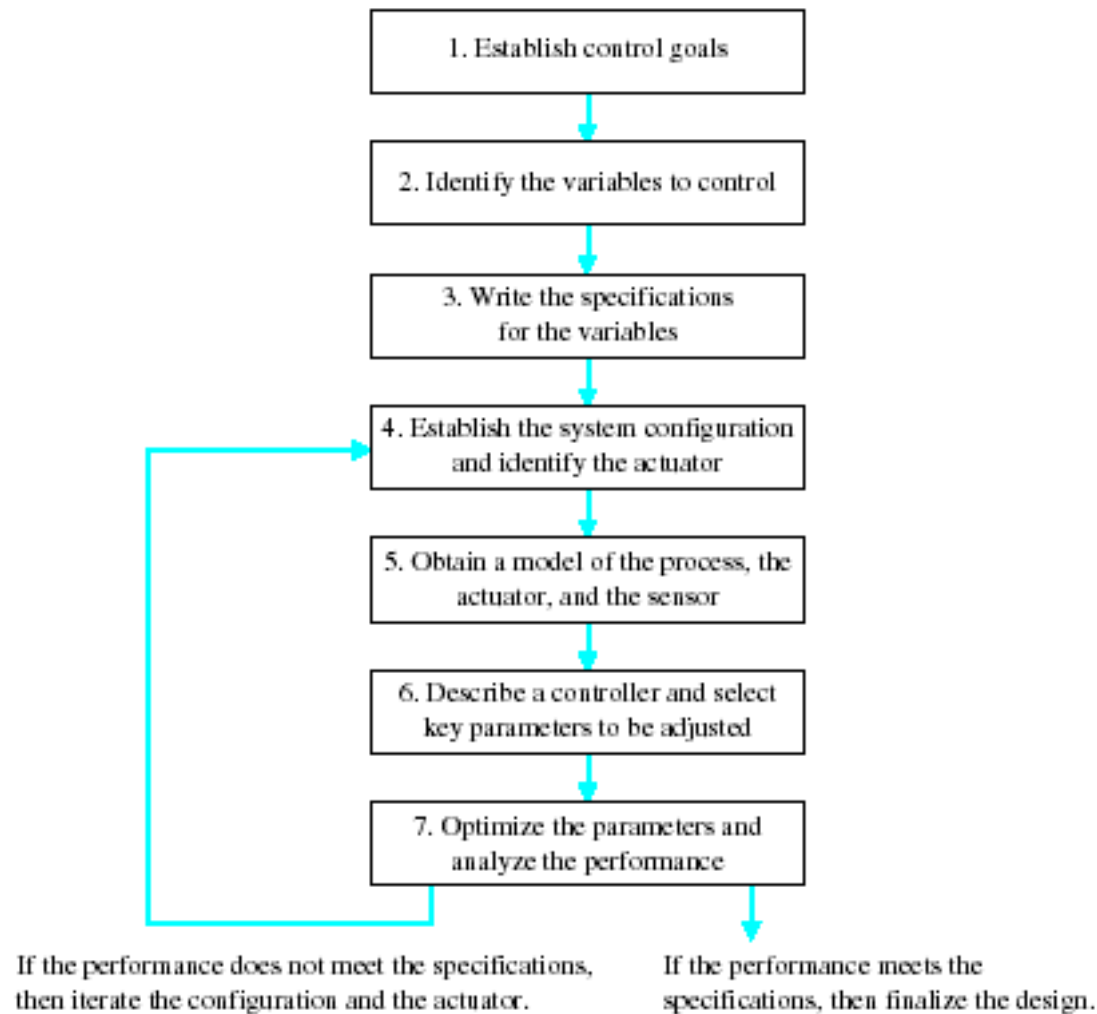
A computer control system.

# Examples of Modern Control Systems

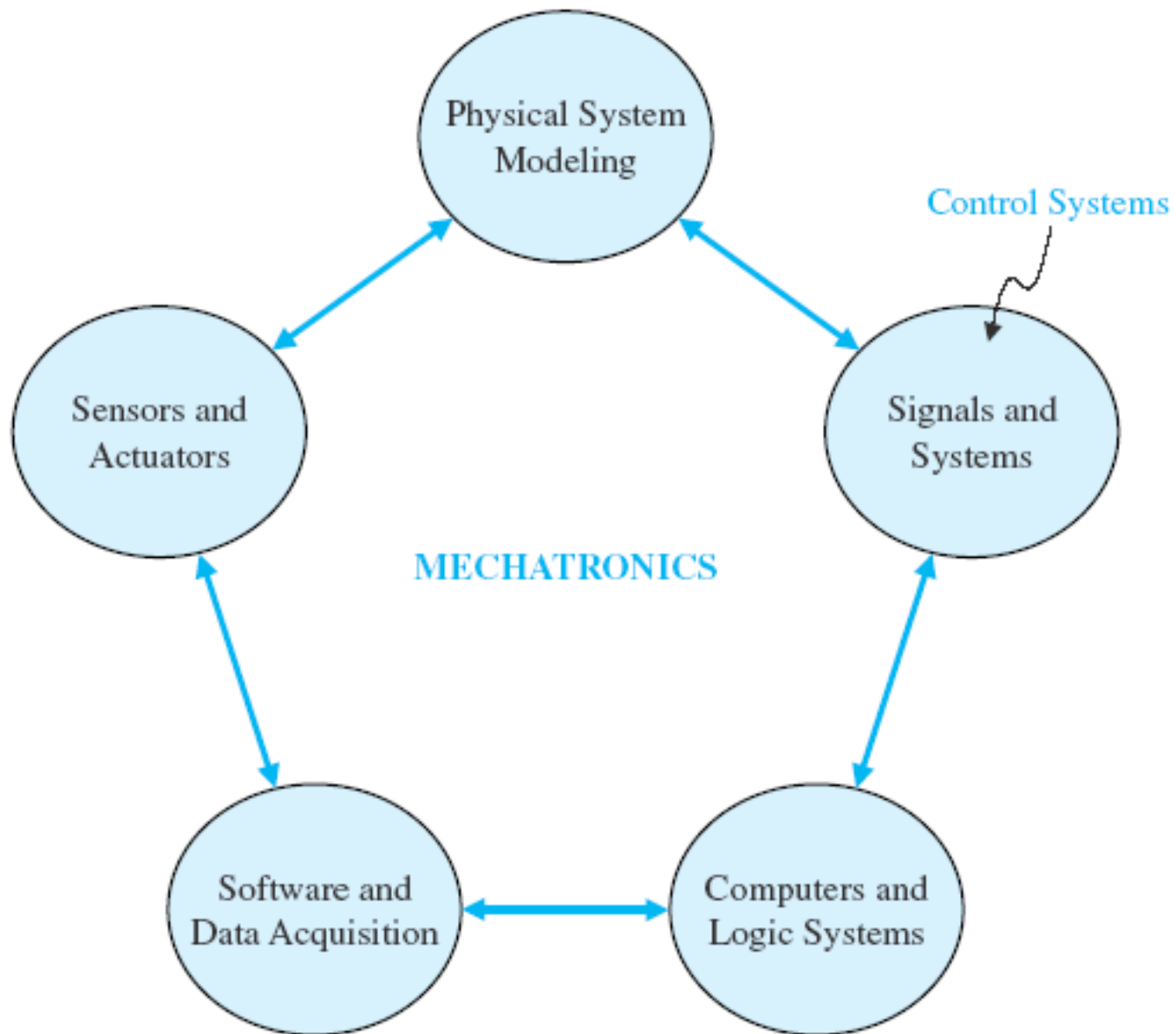


A feedback control system model of the national income.

# Control System Design

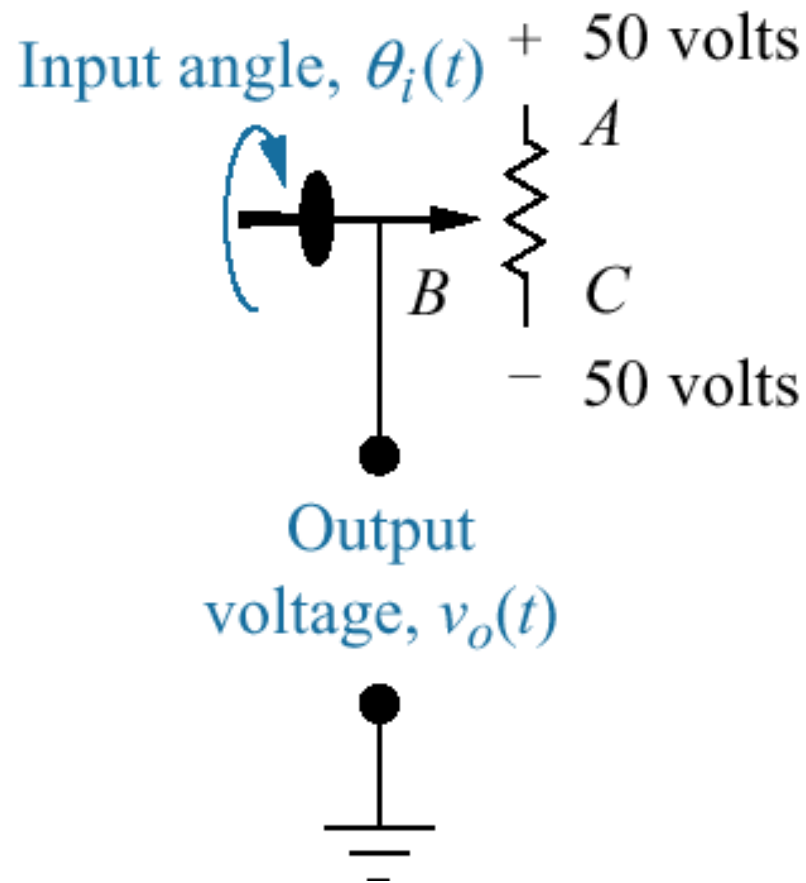


The control system design process.





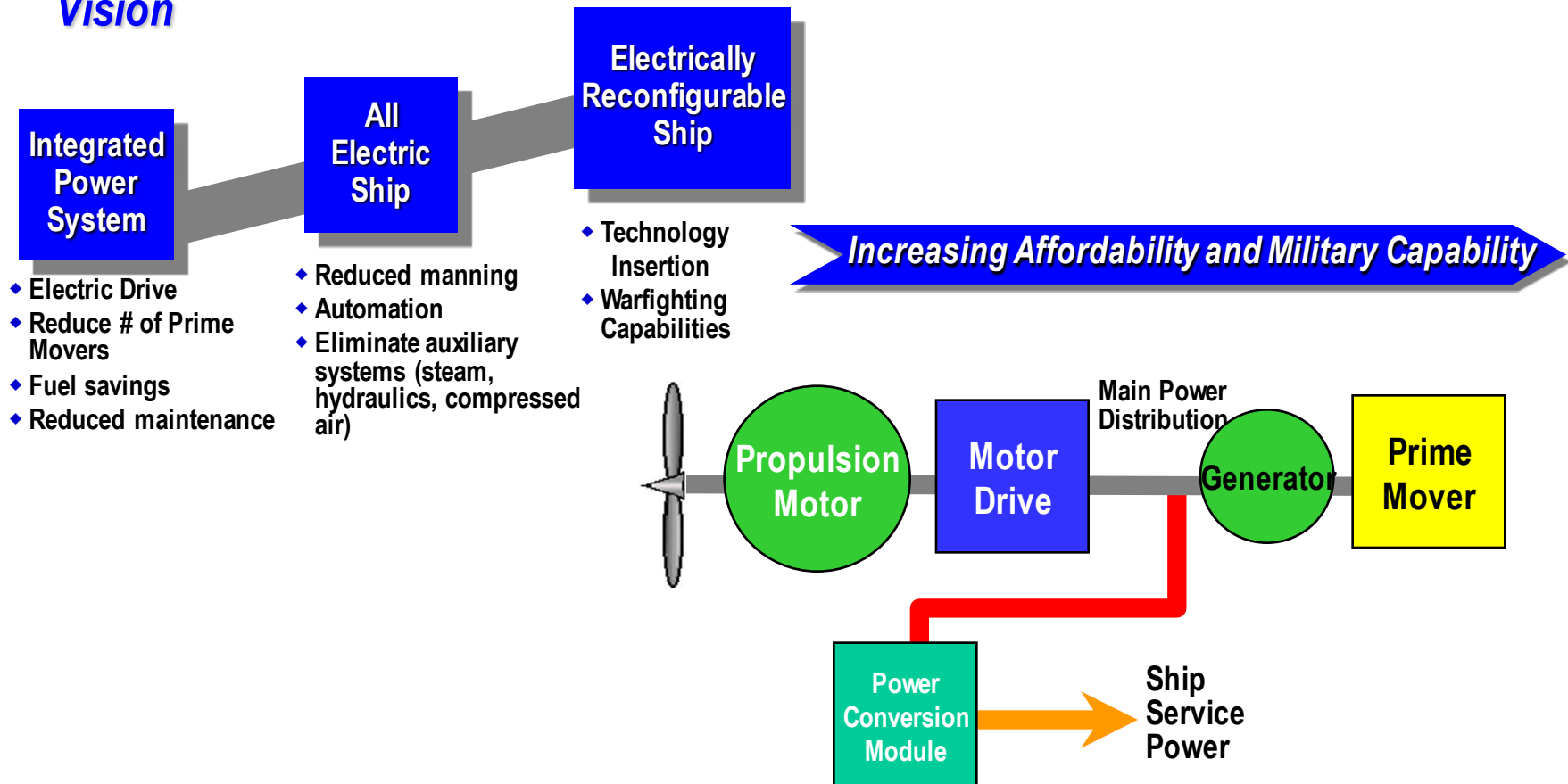
# Design Example



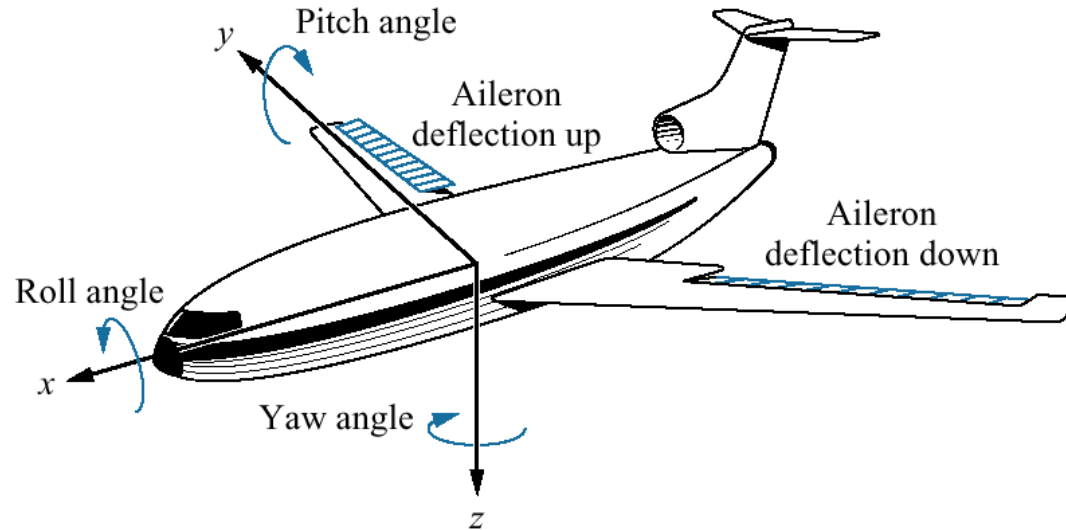
# Design Example

## ELECTRIC SHIP CONCEPT

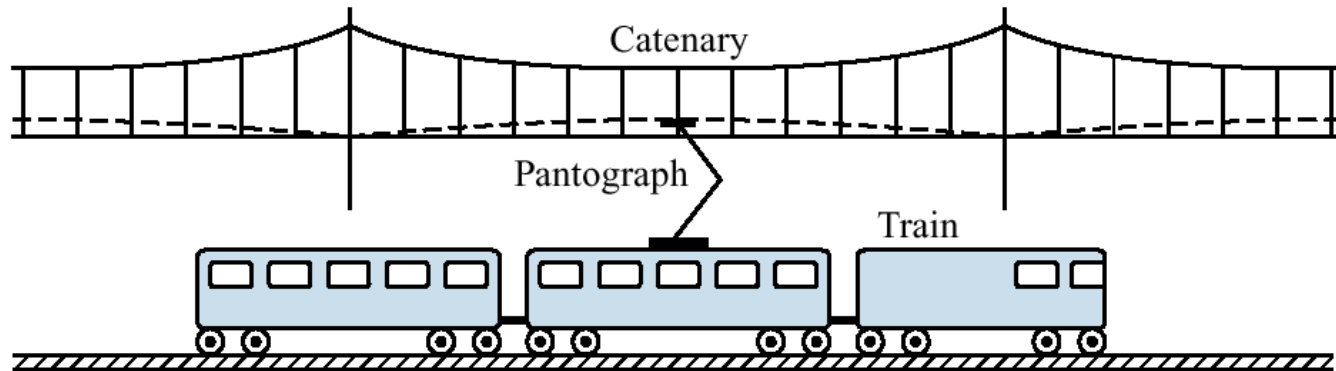
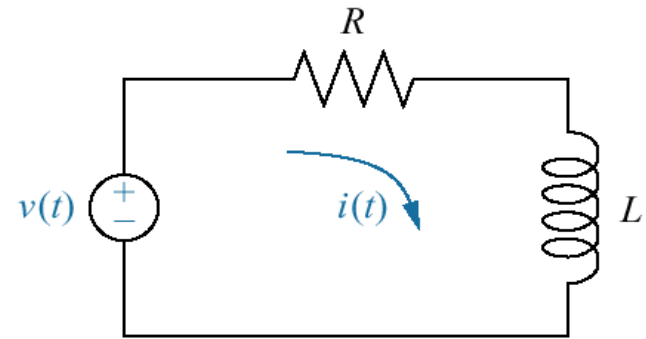
### *Vision*



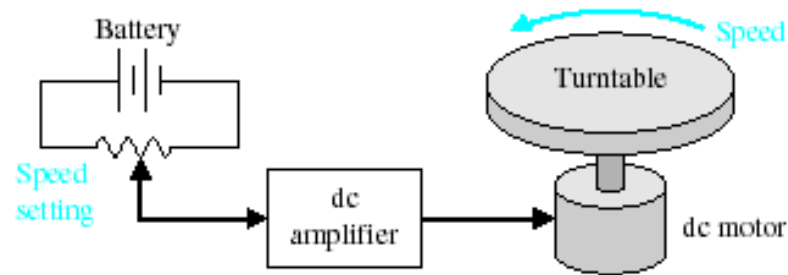
# Design Example



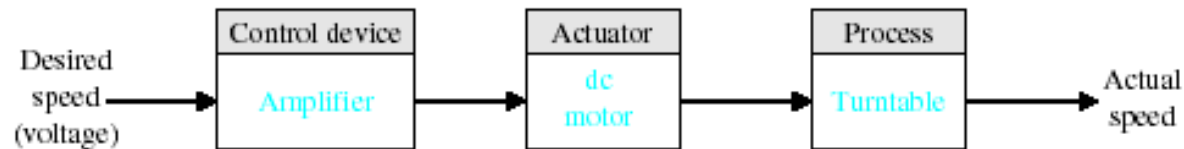
# Design Example



# Design Example



(a)

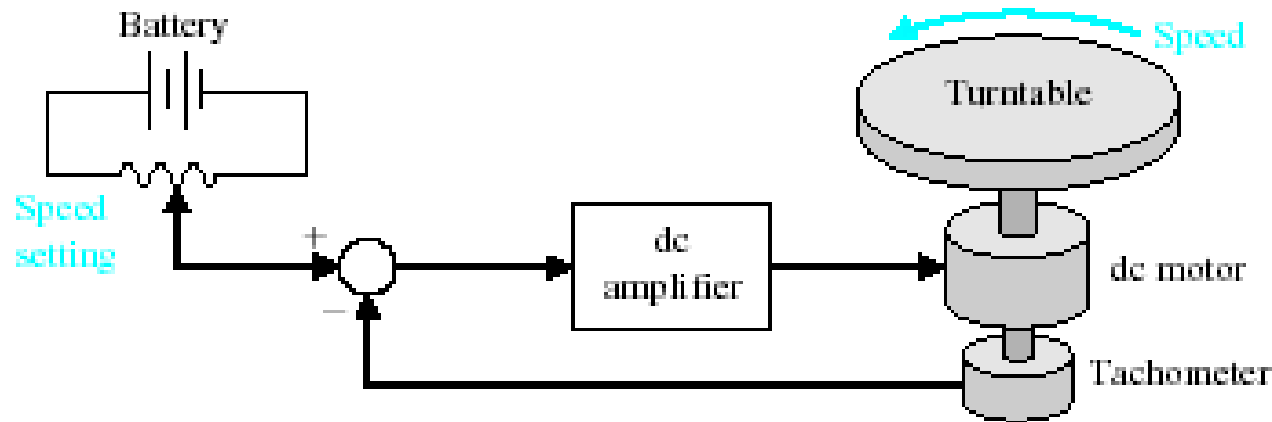


(b)

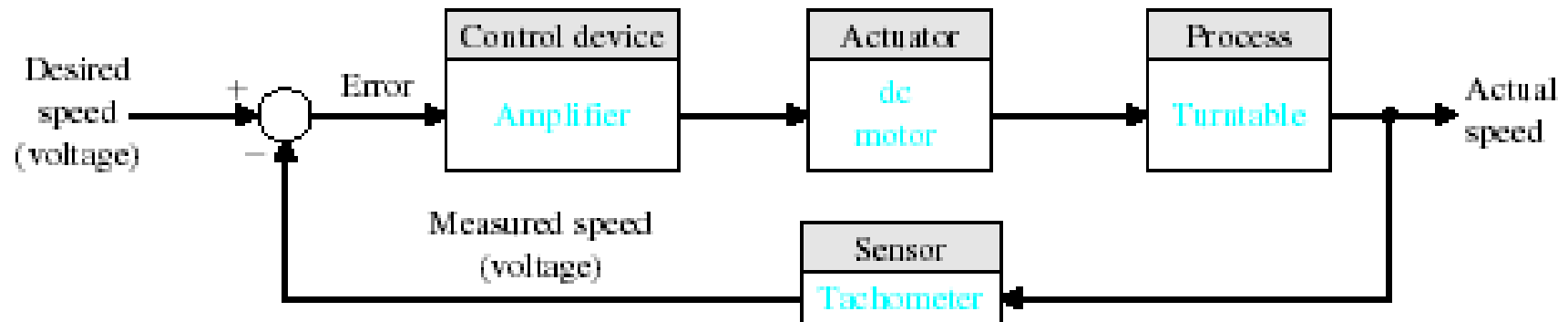
(a) Open-loop (without feedback) control of the speed of a turntable.

(b) Block diagram model.

# Design Example



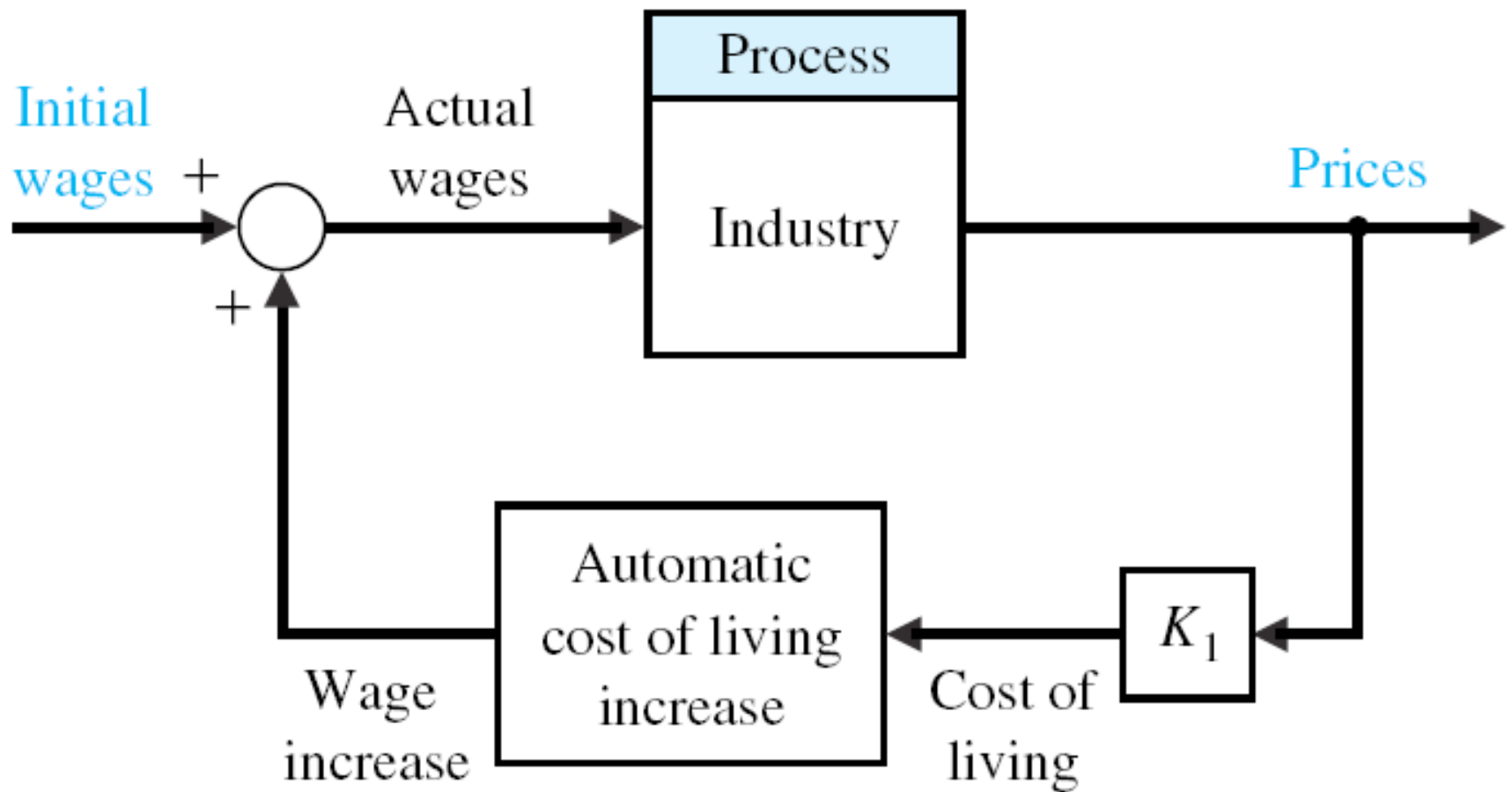
(a)



(b)

(a) Closed-loop control of the speed of a turntable.

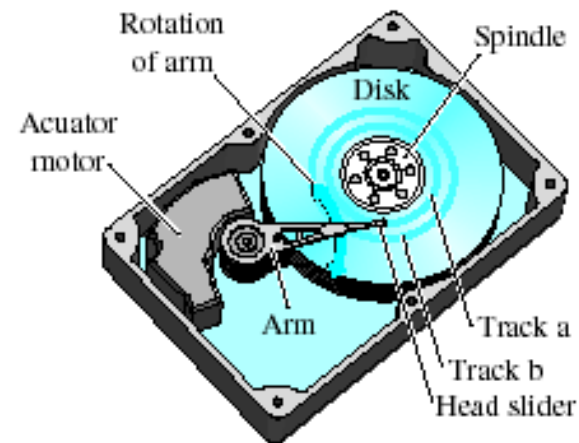
(b) Block diagram model.



# Sequential Design Example



(a)



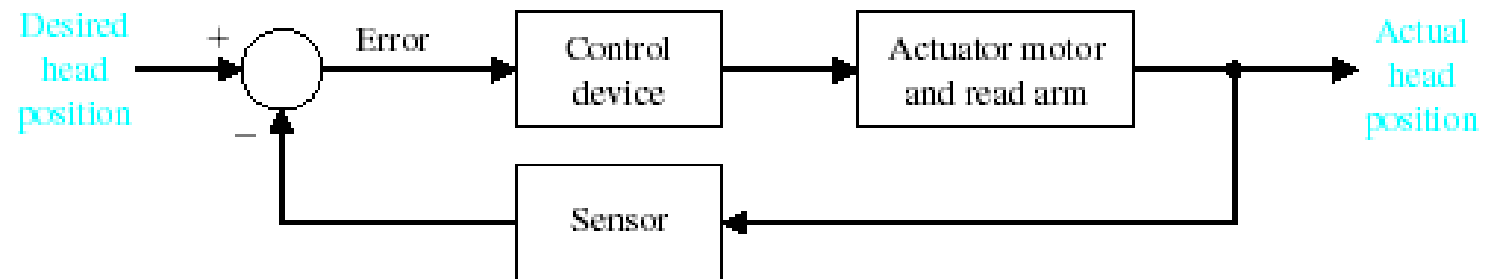
(b)

(a) A disk drive ©1999 Quantum Corporation. All rights reserved.

(b) Diagram of a disk drive.



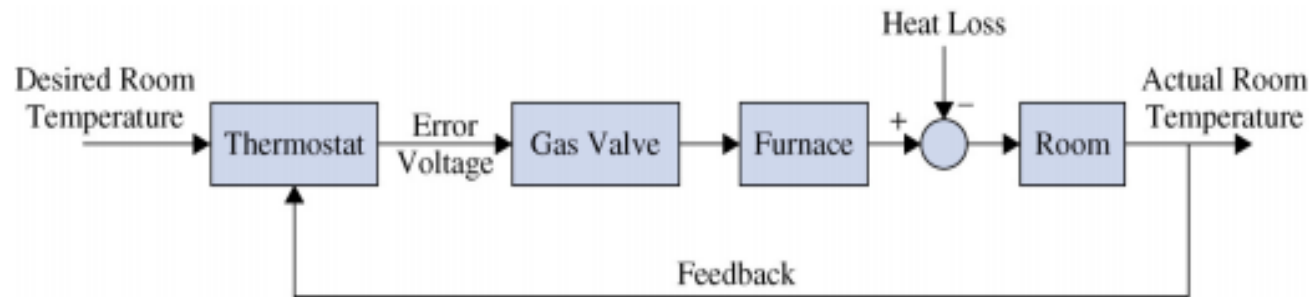
# Sequential Design Example



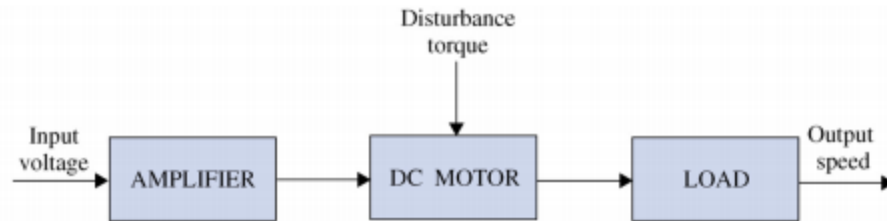
Closed-loop control system for disk drive.



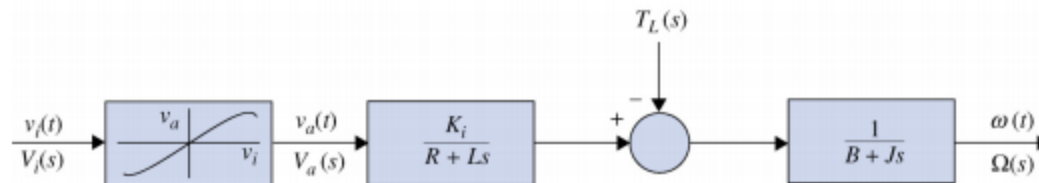




Heating system block diagram (simplified). Actual temp. (output) measured by sensor in the thermostat. Simple electronic circuit (comparator) compares temps. Generates error voltage that acts as switch to open the gas valve to turn out the furnace. Opening windows and doors etc. in room causes heat loss (disturbance). Process of sensing output and comparing with input to generate error signal called Feedback.

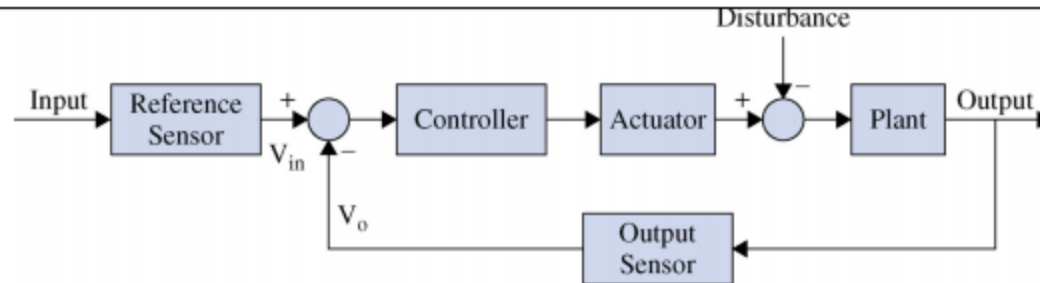


(a)



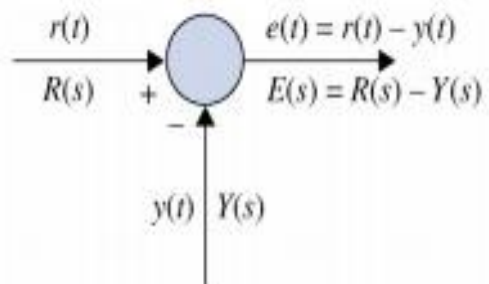
(b)

(a) Open loop, dc-motor, speed control system (b) Block diagram; input voltage to the motor, output of (non linear) power amp. Representation issue for NL blocks.

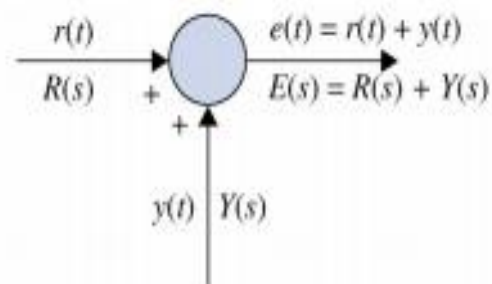


Common elements in block diagram of most control systems  
(Compare to Figs.3-1, 3-2)

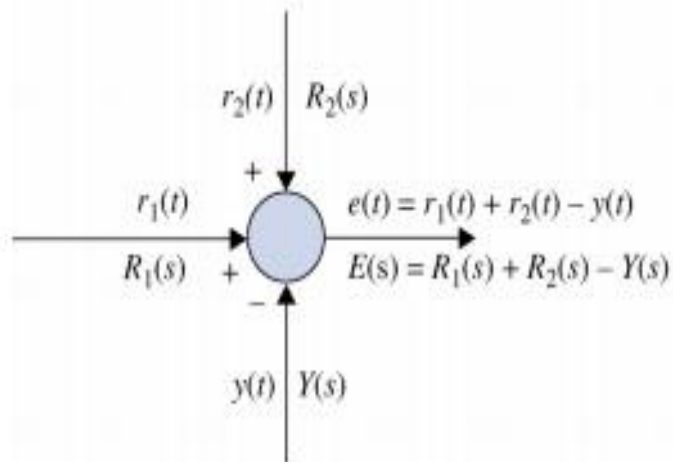
- . Comparators (electronic circuit measures error)
- . BLOCKS representing transfer functions
  - . Reference sensor
  - . Output sensor
  - . Actuator
  - . Controller
  - . Plant
- . INPUT or reference signals
- . OUTPUT signals
- . Disturbance signal
- . Feedback loops



(a)

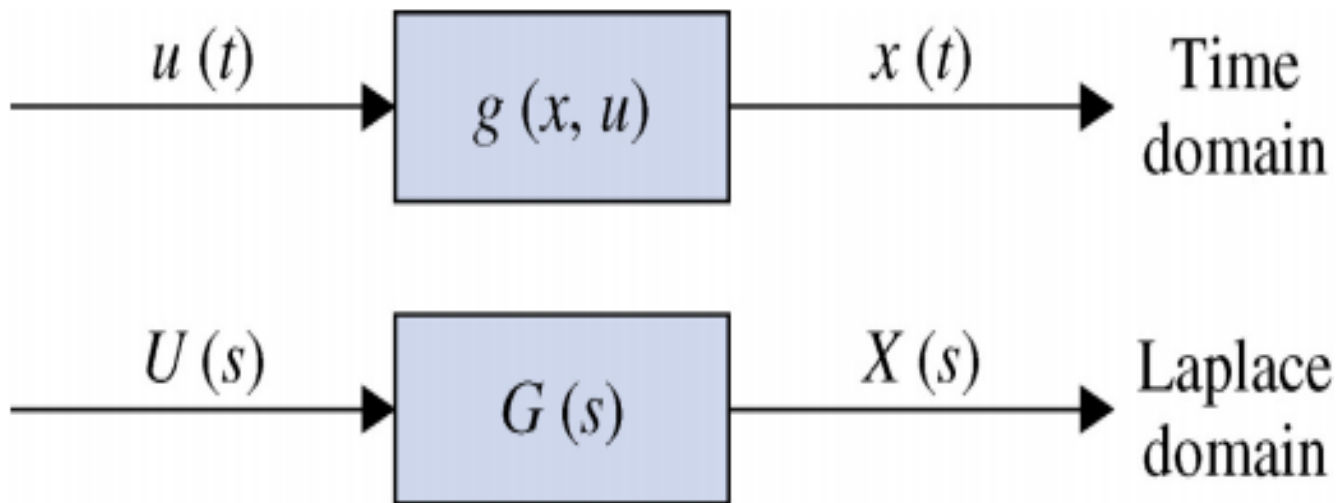


(b)



(c)

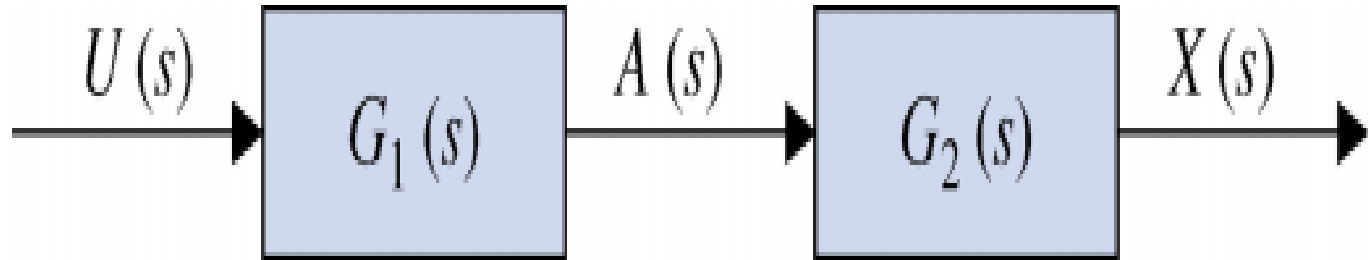
**A comparator**  
performs addition  
and subtraction



fig\_03\_05

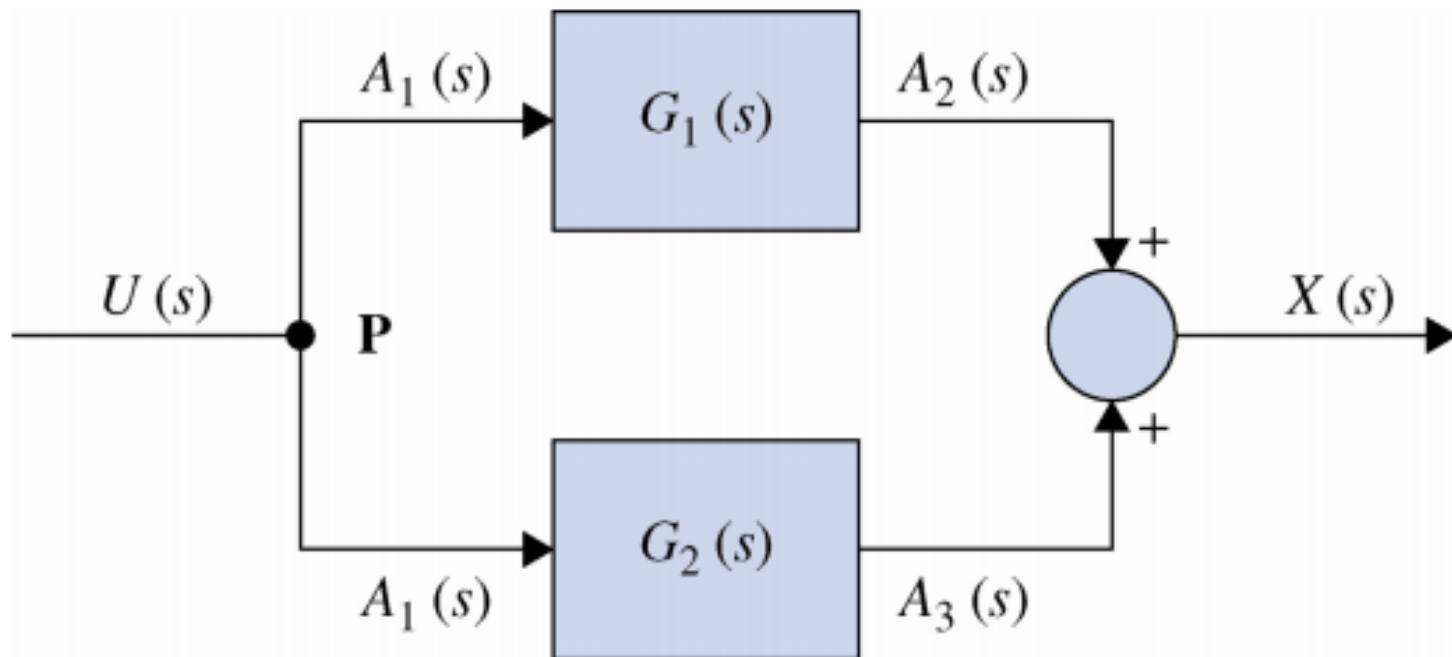
$$X(s) = G(s) U(s)$$





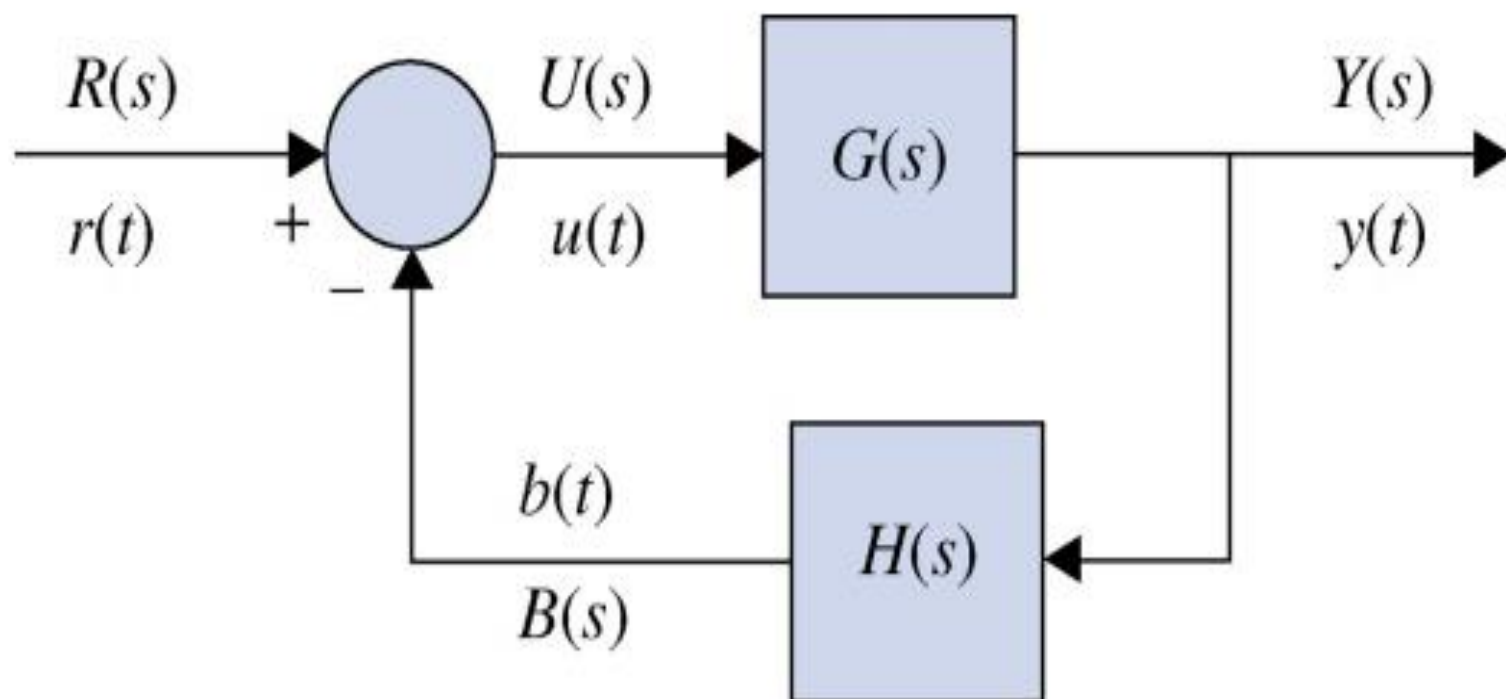
fig\_03\_06

$$X(s) = G_1(s) G_2(s) U(s)$$



fig\_03\_07

$$G(s) = G_1(s) + G_2(s)$$

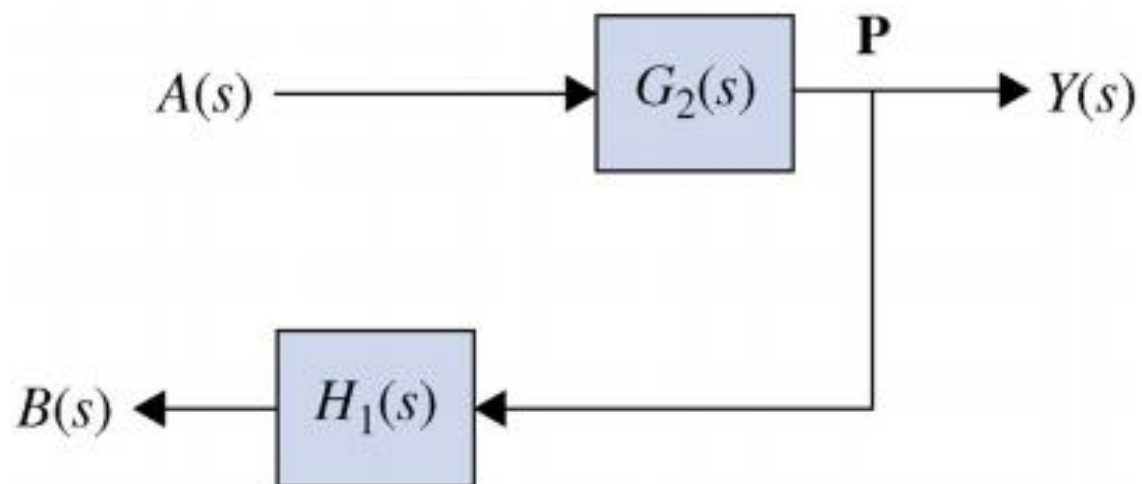


$$M(s) = \frac{Y(s)}{R(s)} = \frac{G(s)}{1 + G(s) H(s)} \quad \text{negative feedback}$$

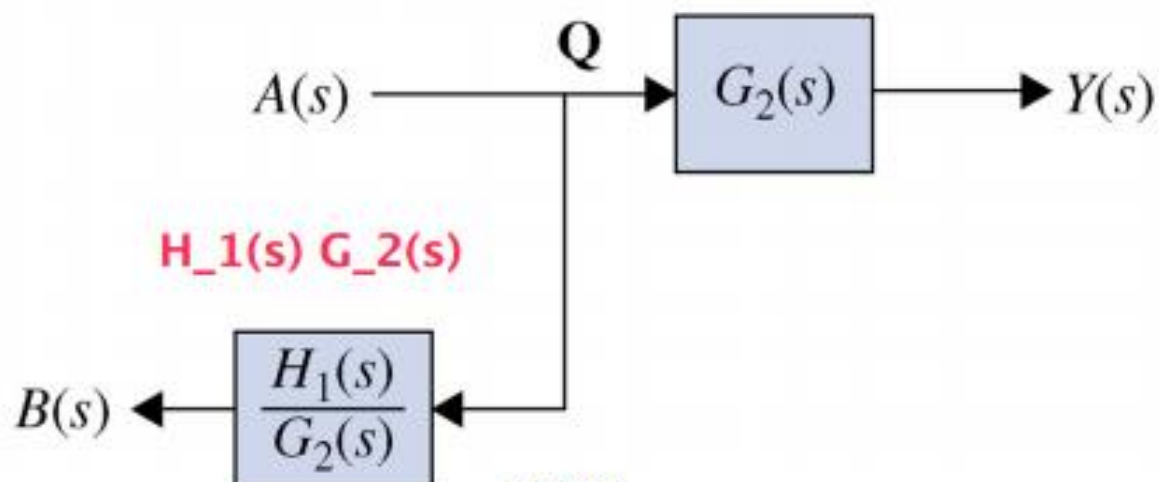
$$M(s) = \frac{Y(s)}{R(s)} = \frac{G(s)}{1 - G(s) H(s)} \quad \text{positive feedback}$$

## Moving a branch point

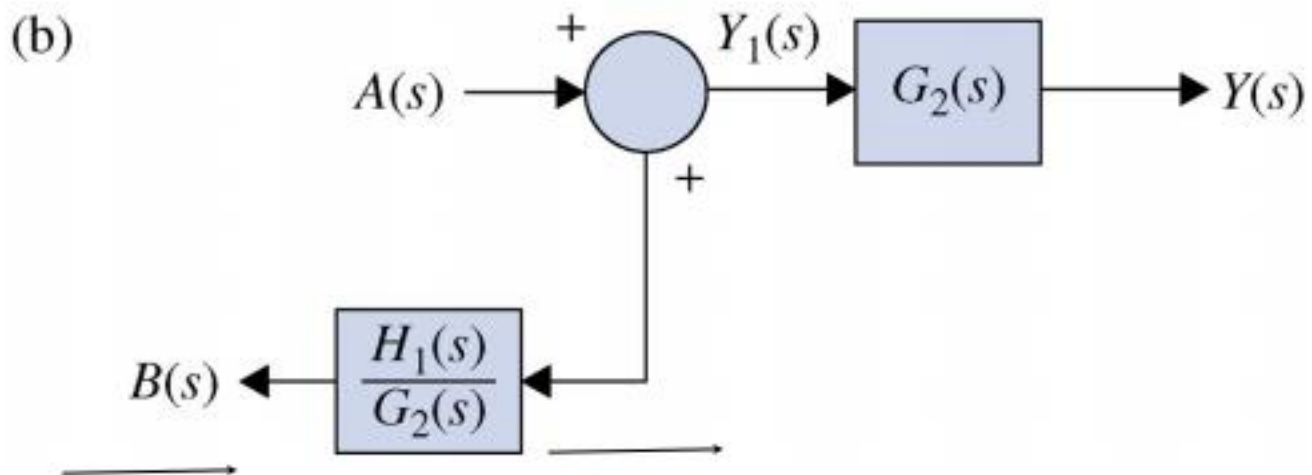
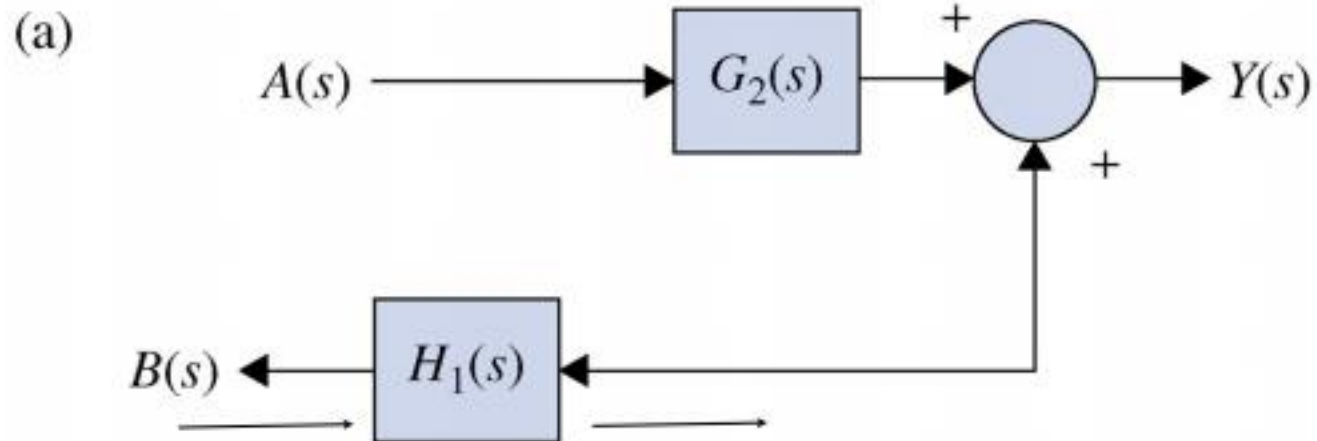
(a)



(b)



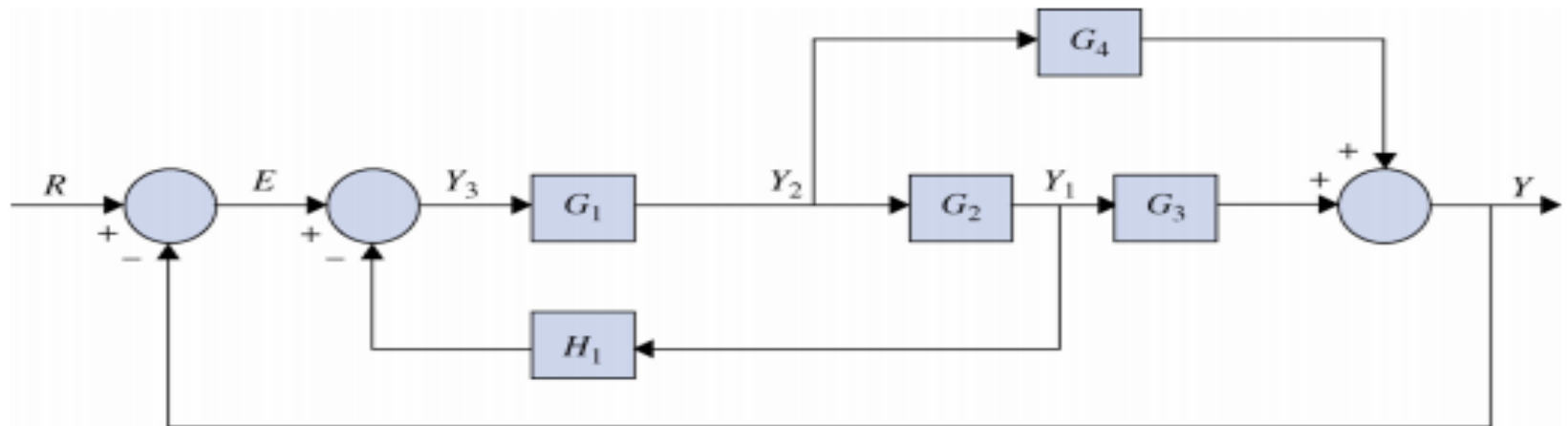
## Moving a comparator from RHS of $G_2(s)$ to its LHS



fig\_03\_17

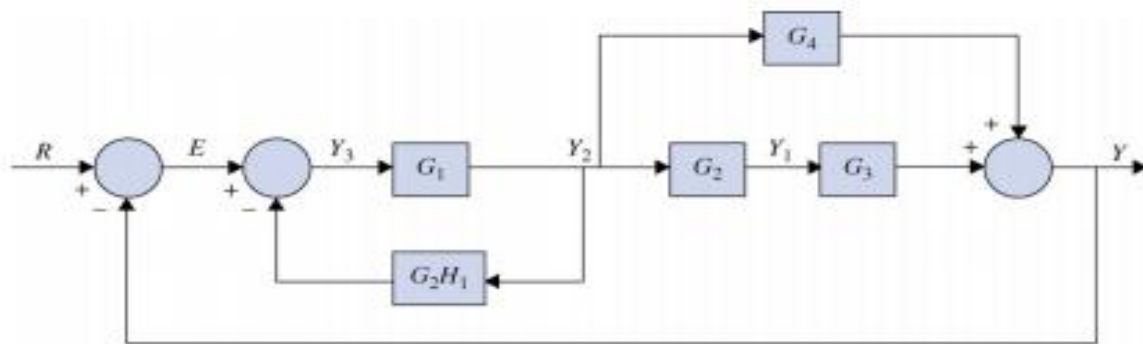
## Block diagram reduction

first move branch pt. at  $Y_1$  to left of  $G_2$

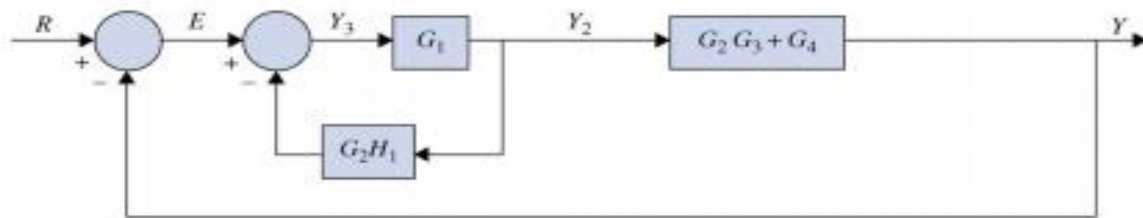


(a)

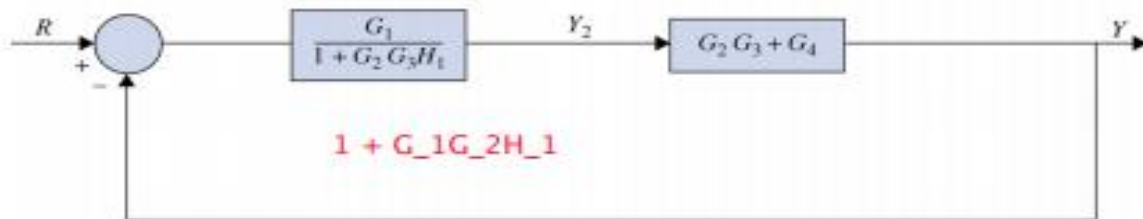
fig\_03\_18a



(b)



(c)

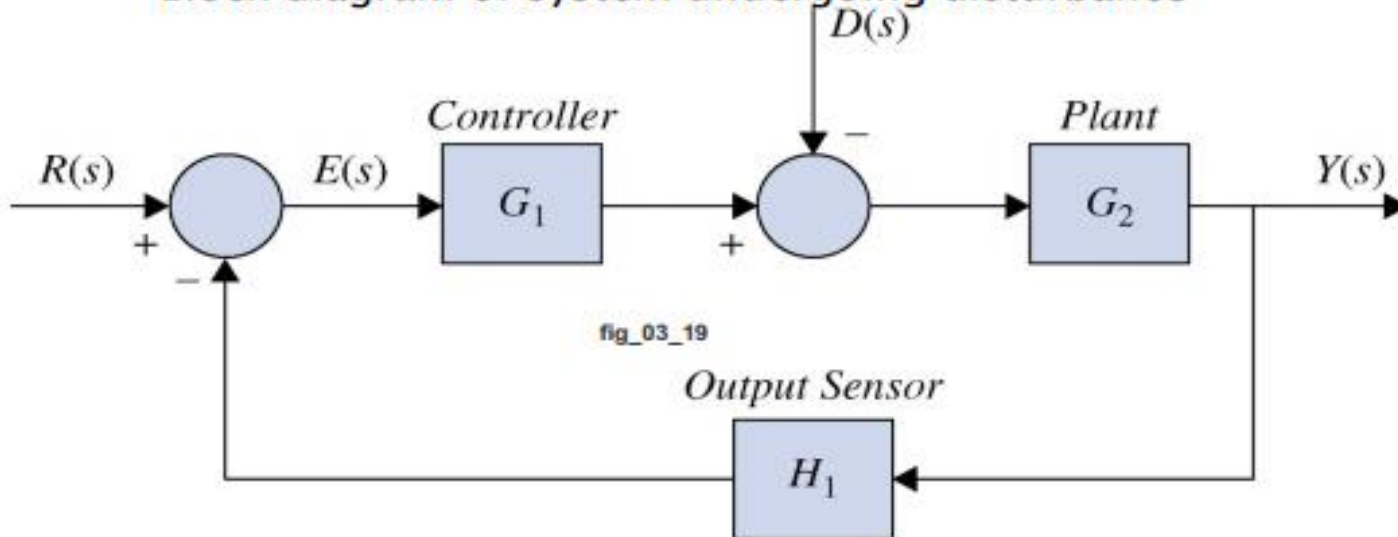


(d)

fig\_03\_18b

$$1 + G_1G_2H_1$$

### Block diagram of system undergoing disturbance



We need to determine effects of  $D(s)$  on the system

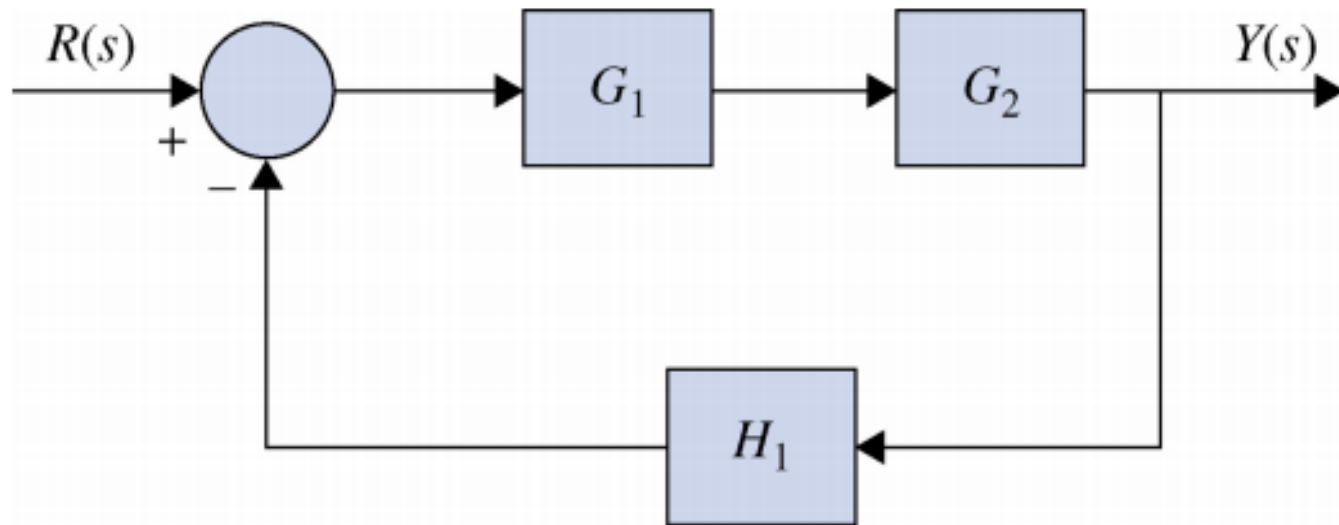
$$Y_{total} = Y_R|_{D=0} + Y_D|_{R=0}$$

$$\frac{Y(s)}{R(s)} = \frac{G_1(s)G_2(s)}{1 + G_1(s)G_2(s)H_1(s)}$$

$$\frac{Y(s)}{R(s)} = \frac{-G_2(s)}{1 + G_1(s)G_2(s)H_1(s)}$$

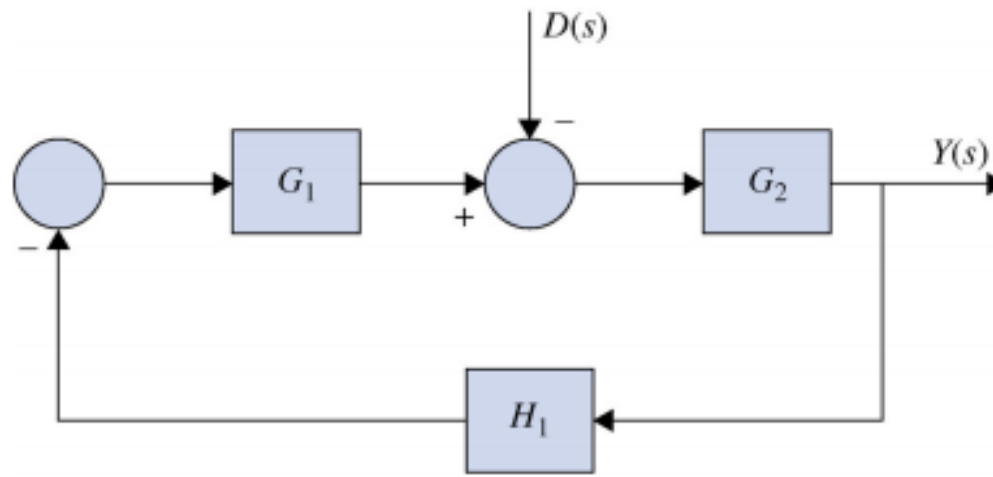
So disturbance interferes with controller signal and adversely affects system performance.



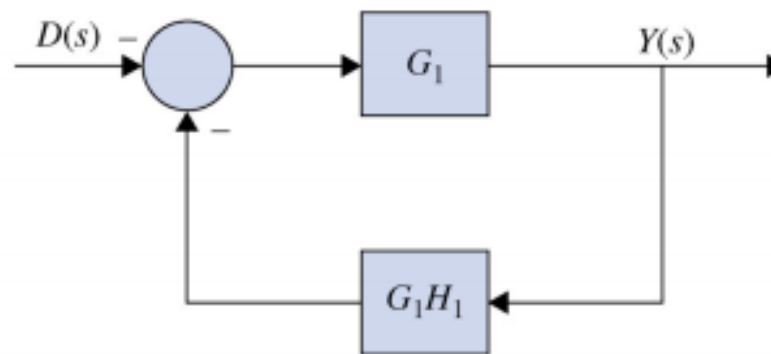


fig\_03\_20

Block diagram for  $D(s) = 0$



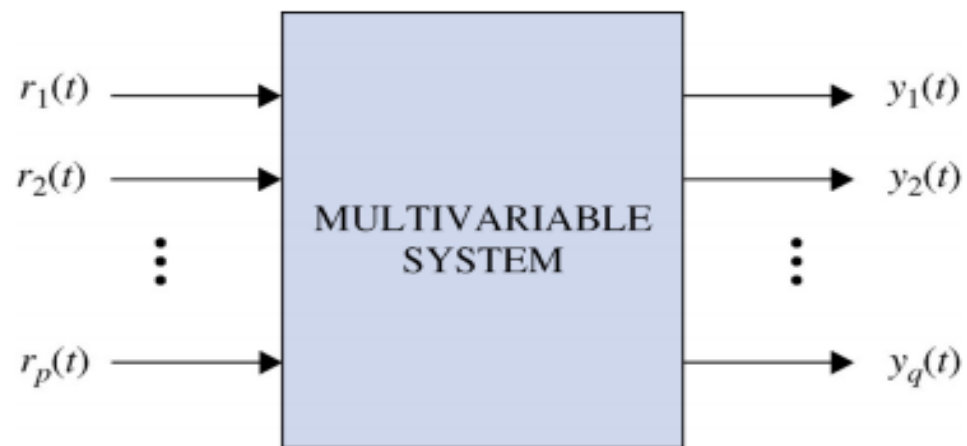
(a)



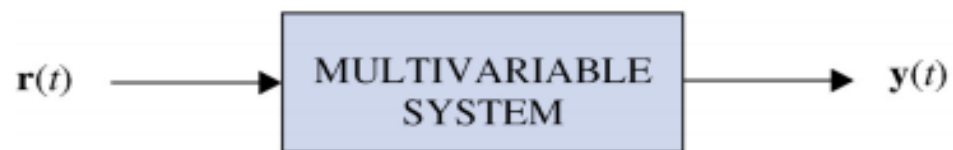
(b)

Block diagram for  $R(s)=0$

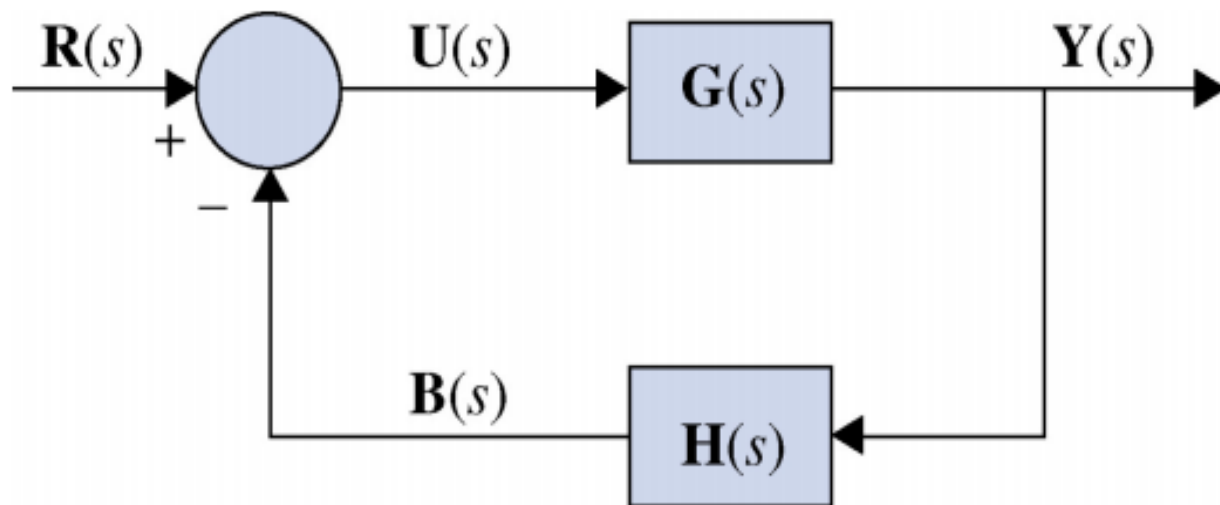
fig\_03\_21



(a)



(b)



fig\_03\_23

$$Y(s) = [I + G(s)H(s)]^{-1}G(s)R(s)$$

# References, and Resources

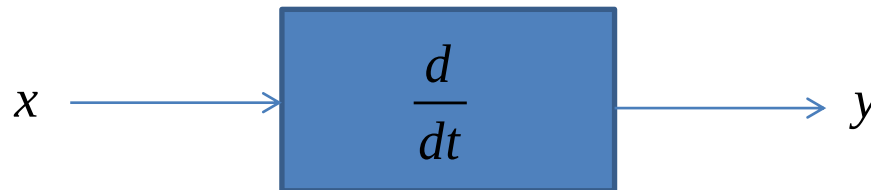
<http://www.ieeecss.org/siteindex/SITEindex.html>

[http://www-control.eng.cam.ac.uk/extras/Virtual\\_Library/Control\\_VL.html](http://www-control.eng.cam.ac.uk/extras/Virtual_Library/Control_VL.html)

# المخطط الكتلي لمنظومات السيطرة واختصاراتها

# Introduction

- A Block Diagram المخطط الكتلي is a shorthand pictorial representation of the cause-and-effect relationship of a system.
- The interior of the rectangle representing the block usually contains a description of or the name of the element, or the symbol for the mathematical operation to be performed on the input to yield the output.
- The arrows represent the direction of information or signal flow.



# Introduction

## Open-Loop Control Systems

utilize a controller or control actuator to obtain the desired response. منظومة الحلقة المفتوحة.

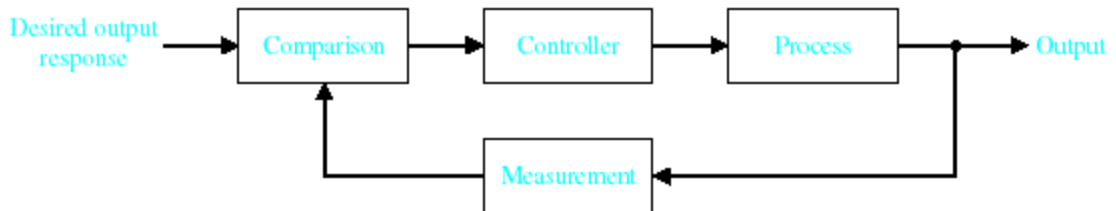


Open-loop control system (without feedback).

## Closed-Loop Control

منظومة الحلقة المغلقة ويحتوي  
Systems على التغذية العكسية

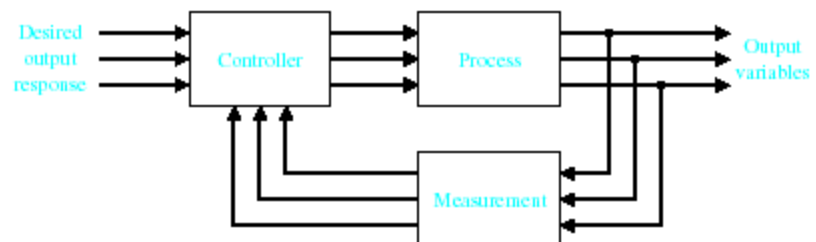
utilizes feedback to compare the actual output to the desired output response.



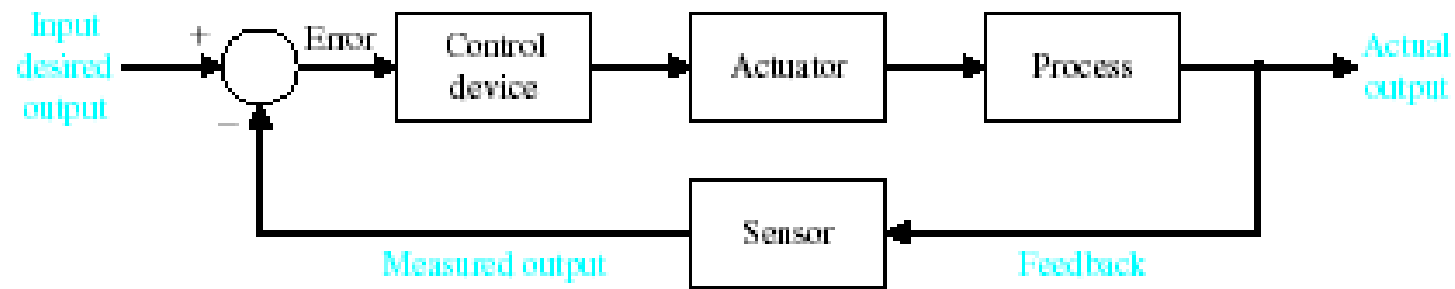
Closed-loop feedback control system (with feedback).

## Multivariable Control System

منظومات متعددة الادخال والايخراج

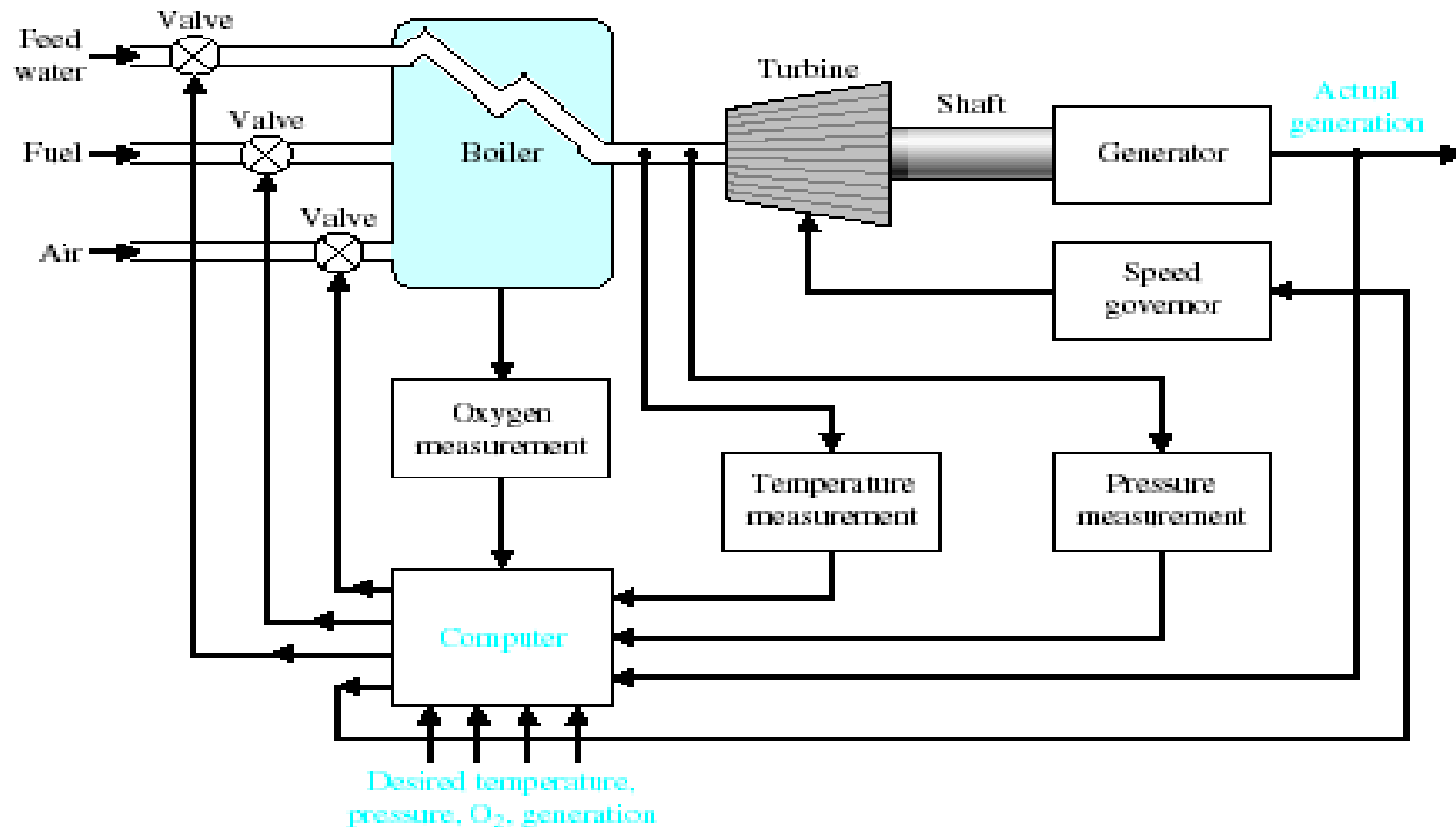






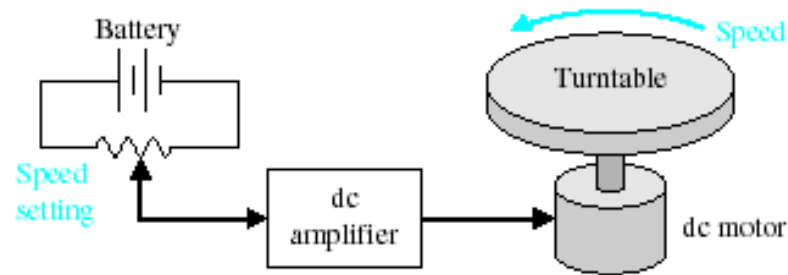
A negative feedback system block diagram depicting a basic closed-loop control system.  
The control device is often called a “controller.”

# Examples of Modern Control Systems

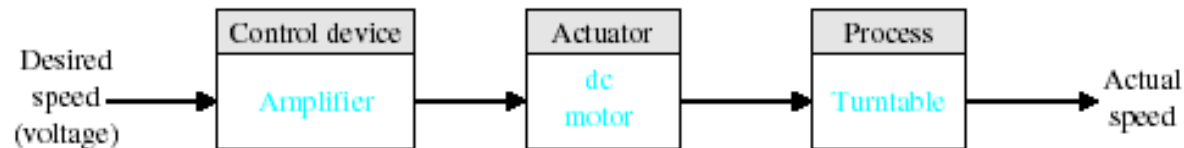


Coordinated control system for a boiler-generator.

# Design Example



(a)

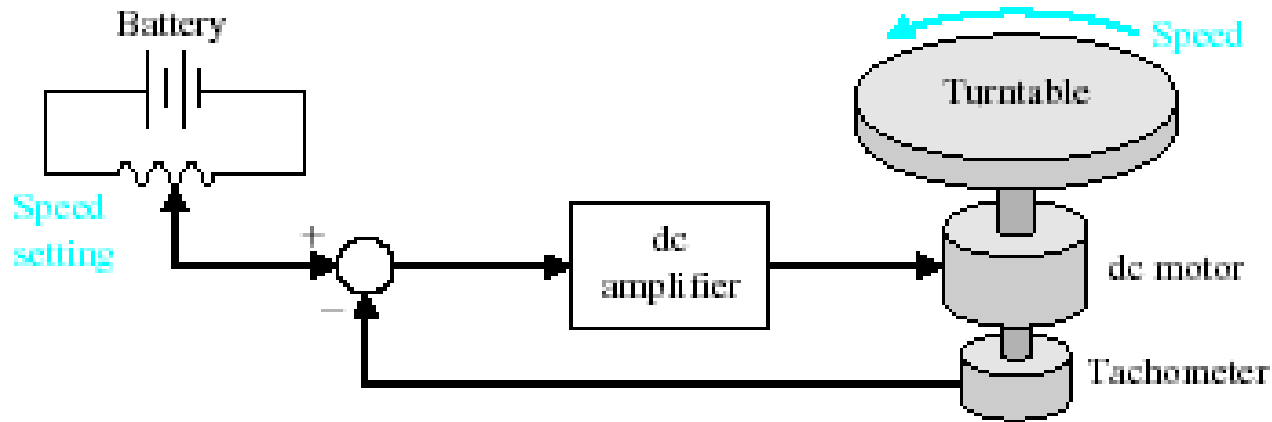


(b)

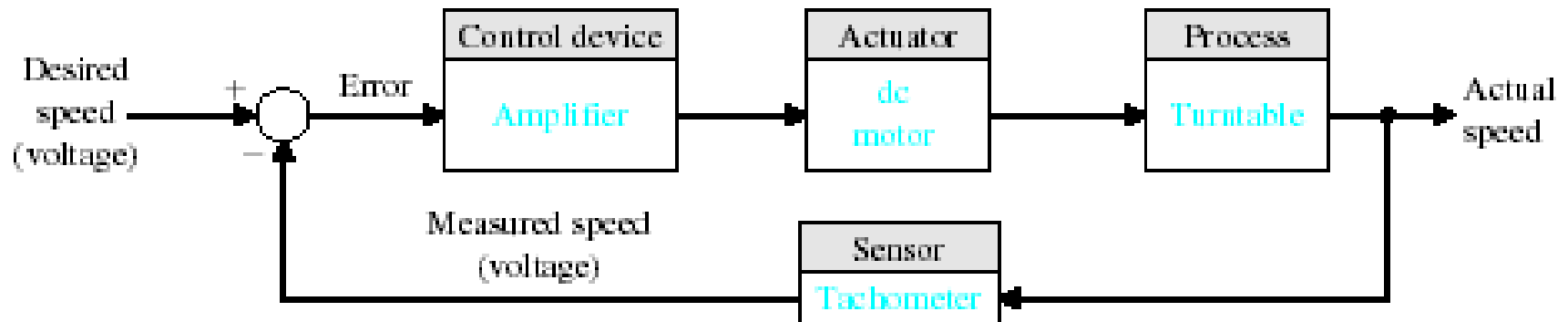
(a) Open-loop (without feedback) control of the speed of a turntable.

(b) Block diagram model.

# Design Example



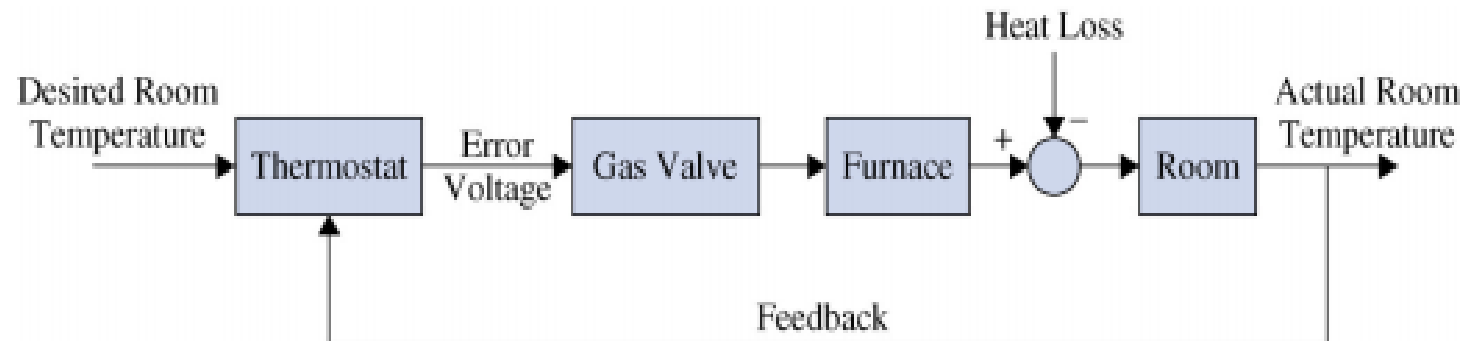
(a)



(b)

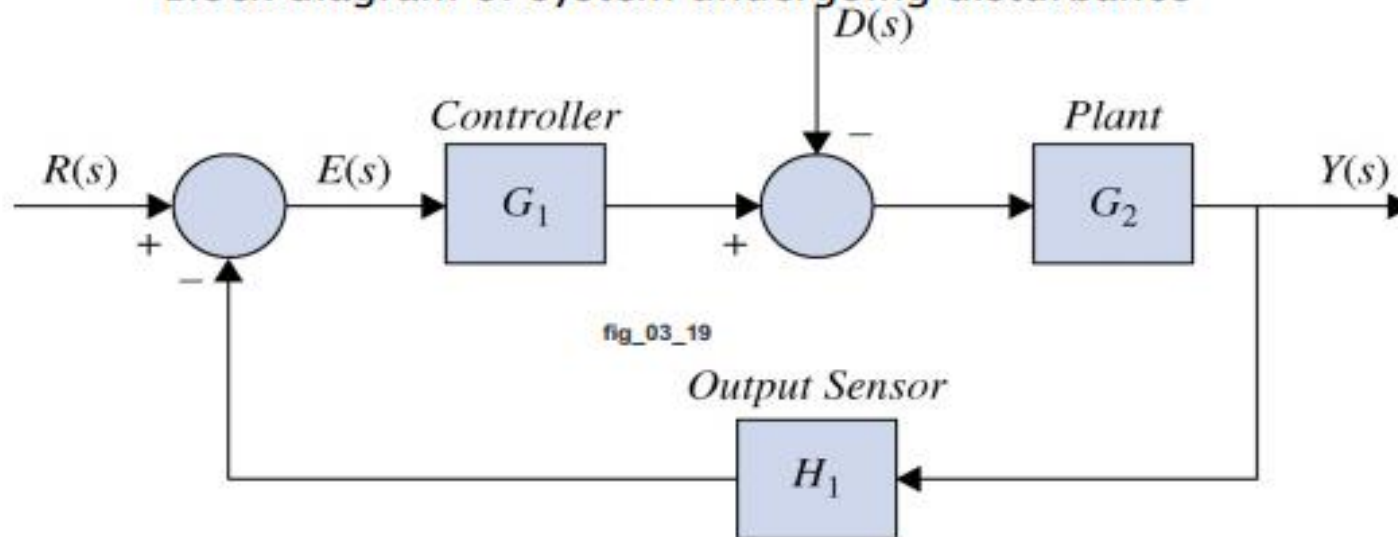
(a) Closed-loop control of the speed of a turntable.

(b) Block diagram model.



Heating system block diagram (simplified). Actual temp. (output) measured by sensor in the thermostat. Simple electronic circuit (comparator) compares temps. Generates error voltage that acts as a switch to open the gas valve to turn out the furnace. Opening windows and doors etc. in room causes heat loss (disturbance). Process of sensing output and comparing with input to generate error signal called Feedback.

### Block diagram of system undergoing disturbance



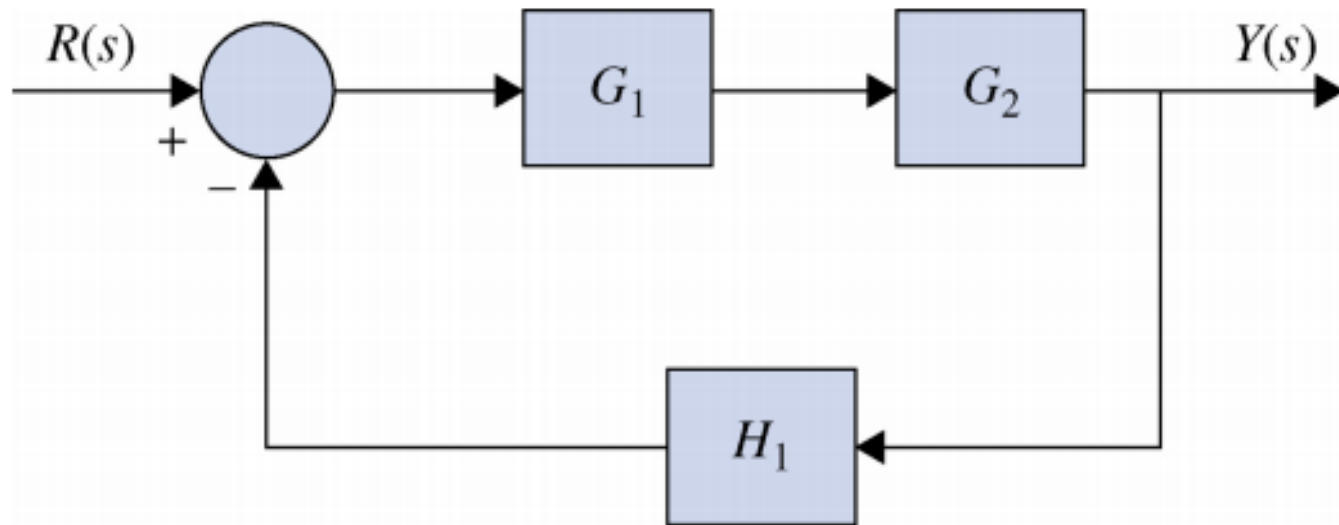
We need to determine effects of  $D(s)$  on the system

$$Y_{total} = Y_R|_{D=0} + Y_D|_{R=0}$$

$$\frac{Y(s)}{R(s)} = \frac{G_1(s)G_2(s)}{1 + G_1(s)G_2(s)H_1(s)}$$

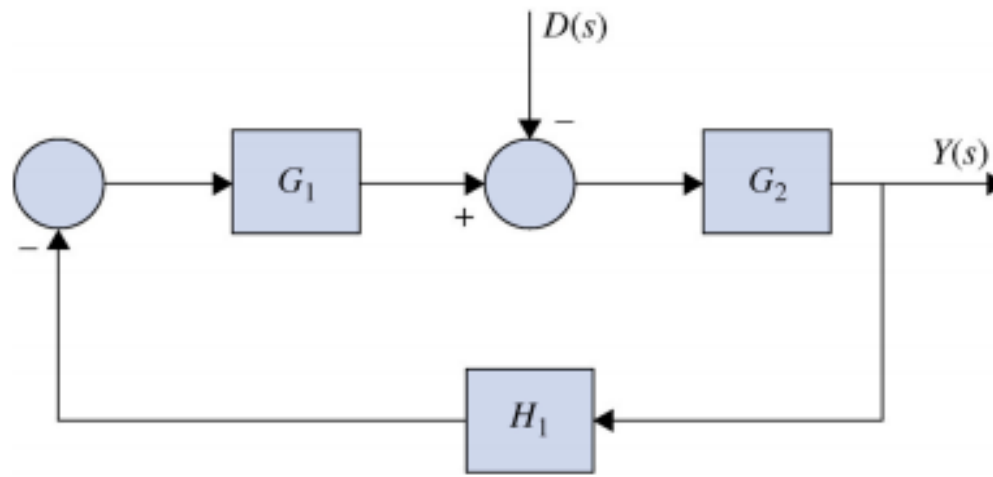
$$\frac{Y(s)}{R(s)} = \frac{-G_2(s)}{1 + G_1(s)G_2(s)H_1(s)}$$

So disturbance interferes with controller signal and adversely affects system performance.

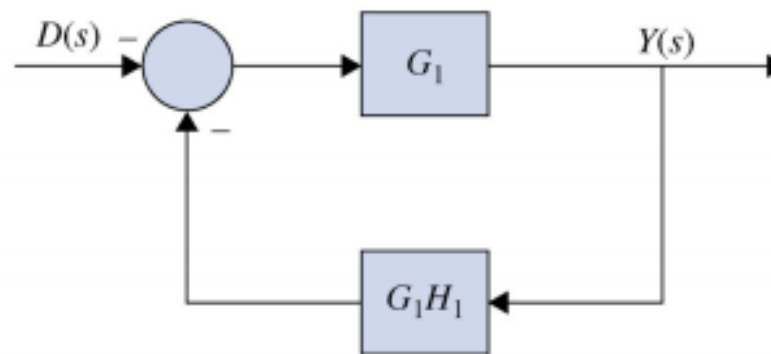


fig\_03\_20

Block diagram for  $D(s) = 0$



(a)

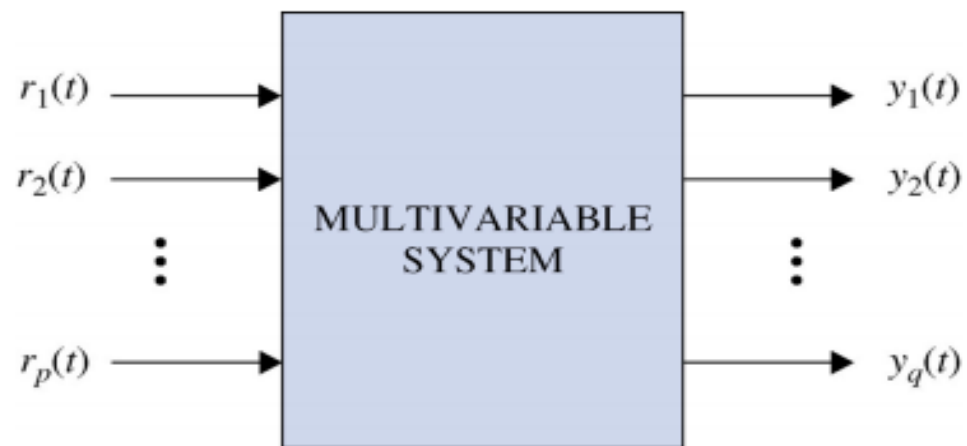


(b)

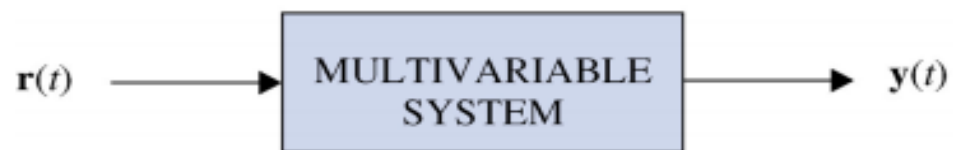
Block diagram for  $R(s)=0$

fig\_03\_21

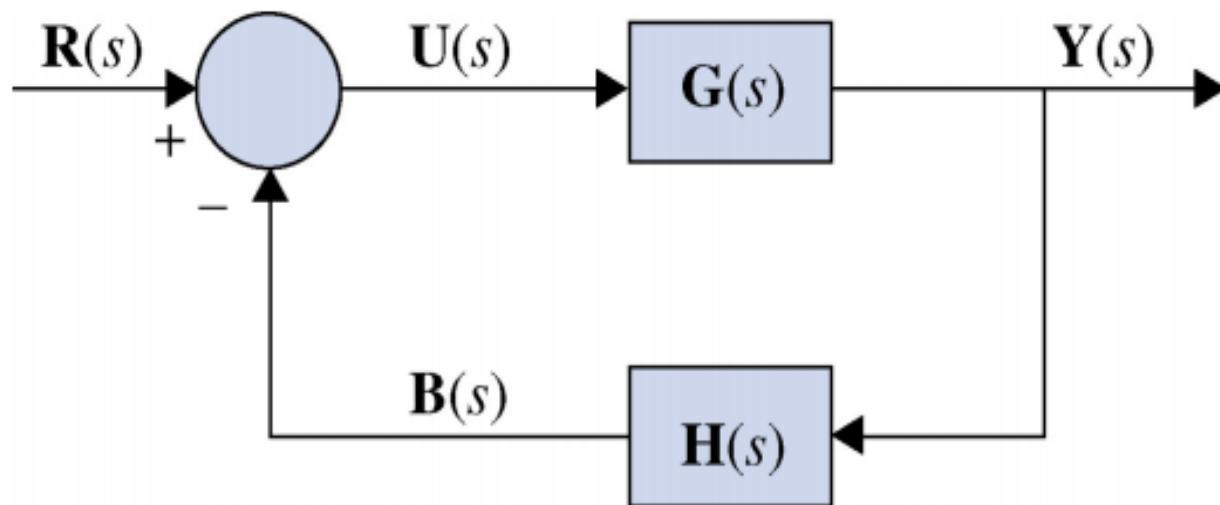




(a)



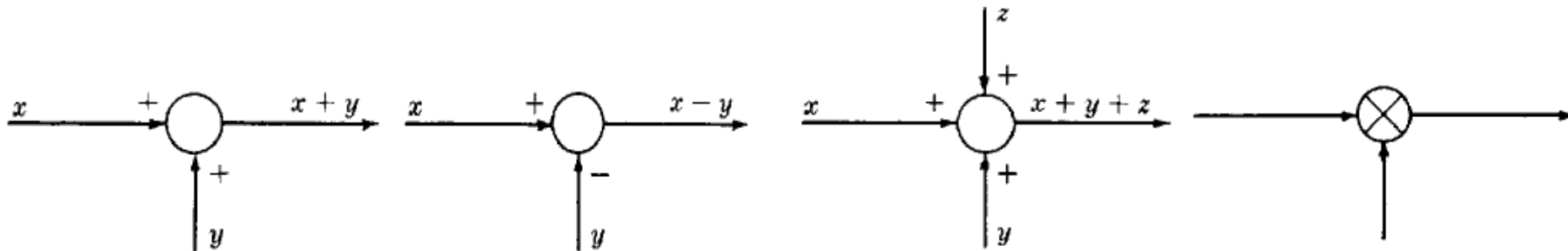
(b)



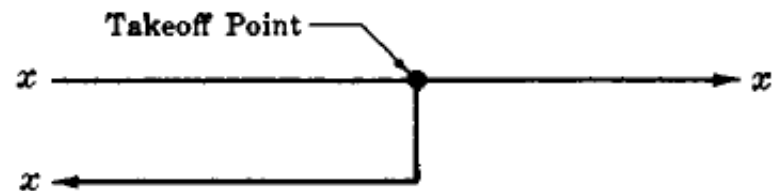
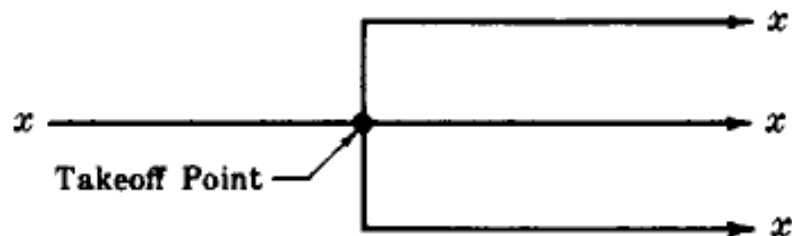
fig\_03\_23

$$Y(s) = [I + G(s)H(s)]^{-1}G(s)R(s)$$

- The operations of addition and subtraction have a special representation.
- The block becomes a small circle, called a summing point, with the appropriate plus or minus sign associated with the arrows entering the circle.
- The output is the algebraic sum of the inputs.
- Any number of inputs may enter a summing point.
- Some books put a cross in the circle.



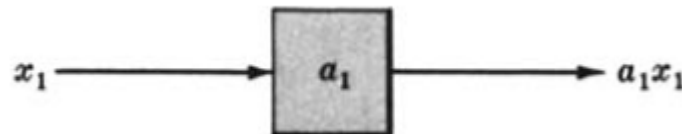
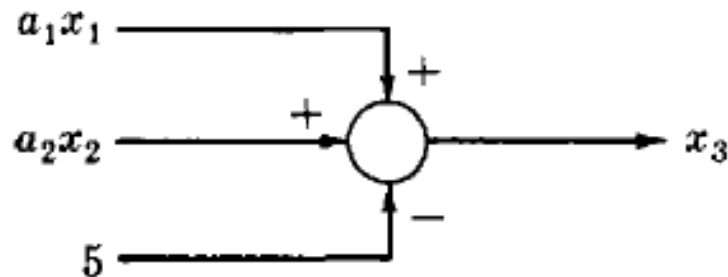
- In order to have the same signal or variable be an input to more than one block or summing point, a takeoff point is used.
- This permits the signal to proceed unaltered along several different paths to several destinations.



# Example-1

- Consider the following equations in which  $x_1, x_2, x_3$ , are variables, and  $a_1, a_2$  are general coefficients or mathematical operators.

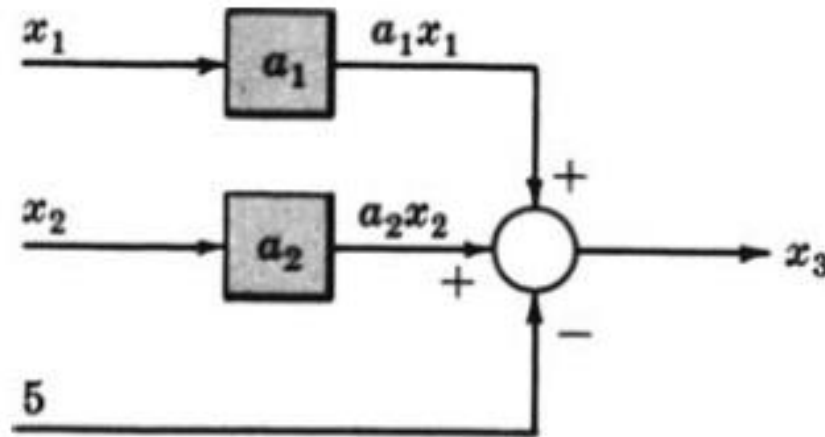
$$x_3 = a_1x_1 + a_2x_2 - 5$$



# Example-1

- Consider the following equations in which  $x_1, x_2, x_3$ , are variables, and  $a_1, a_2$  are general coefficients or mathematical operators.

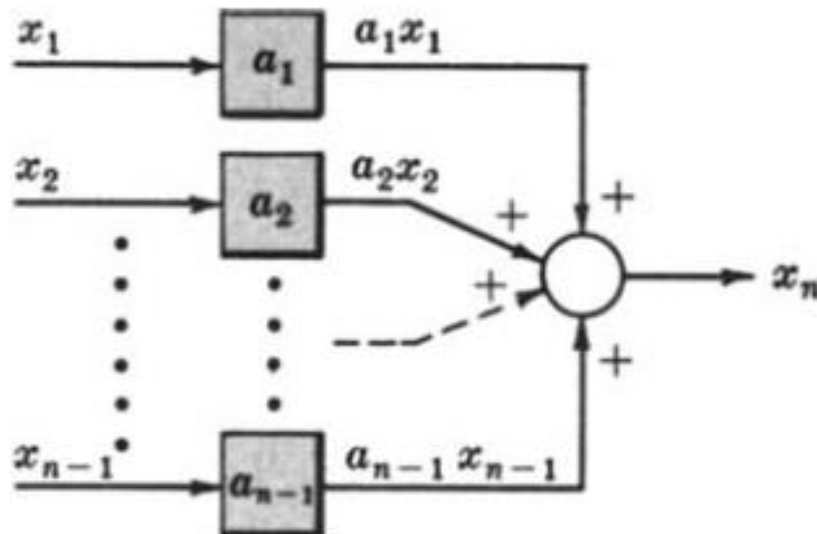
$$x_3 = a_1 x_1 + a_2 x_2 - 5$$



# Example-2

- Consider the following equations in which  $x_1, x_2, \dots, x_n$ , are variables, and  $a_1, a_2, \dots, a_n$ , are general coefficients or mathematical operators.

$$x_n = a_1 x_1 + a_2 x_2 + a_{n-1} x_{n-1}$$



# Example-3

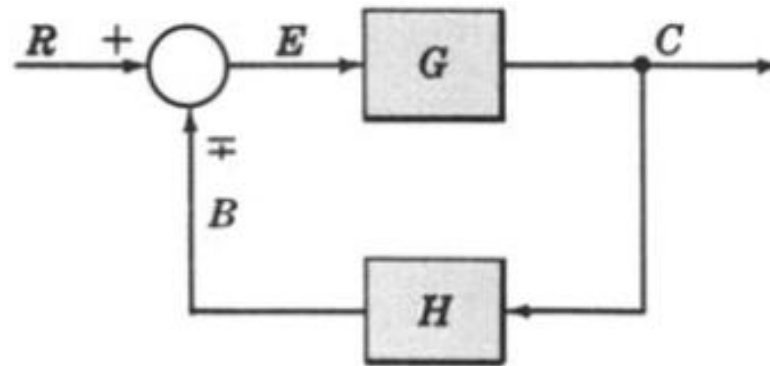
- Draw the Block Diagrams of the following equations.

$$(1) \quad x_2 = a_1 \frac{dx_1}{dt} + \frac{1}{b} \int x_1 dt$$

$$(2) \quad x_3 = a_1 \frac{d^2 x_2}{dt^2} + 3 \frac{dx_1}{dt} - bx_1$$



# Canonical Form of A Feedback Control System



$G \equiv$  direct transfer function  $\equiv$  forward transfer function

$H \equiv$  feedback transfer function

$GH \equiv$  loop transfer function  $\equiv$  open-loop transfer function

$C/R \equiv$  closed-loop transfer function  $\equiv$  control ratio  $\frac{C}{R} = \frac{G}{1 \pm GH}$

$E/R \equiv$  actuating signal ratio  $\equiv$  error ratio  $\frac{E}{R} = \frac{1}{1 \pm GH}$

$B/R \equiv$  primary feedback ratio  $\frac{B}{R} = \frac{GH}{1 \pm GH}$

# Characteristic Equation

## المعادلة المميزة

- The control ratio is the closed loop transfer function of the system.

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 \pm G(s)H(s)}$$

- The denominator of closed loop transfer function determines the characteristic equation of the system. عند مساوات المقام لصفر تدعى المعادلة المميزة

- Which is usually determined as:

$$1 \pm G(s)H(s) = 0$$

# Example-4

1. Open loop transfer function  $\frac{B(s)}{E(s)} = G(s)H(s)$

2. Feed Forward Transfer function  $\frac{C(s)}{E(s)} = G(s)$

3. control ratio  $\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$

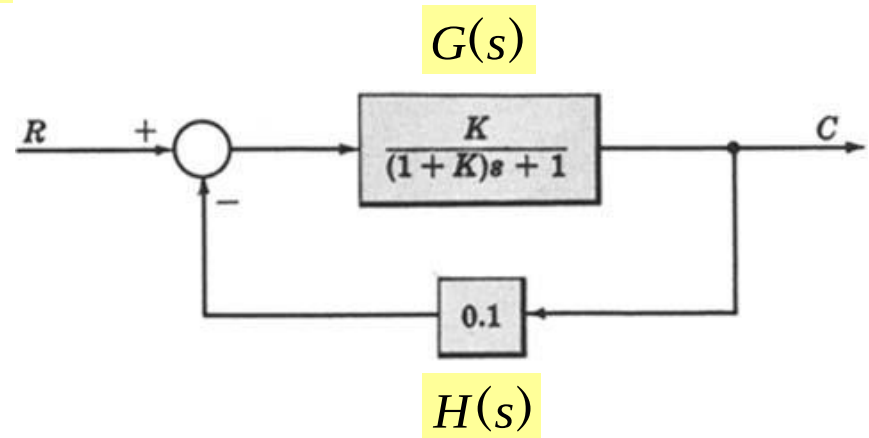
4. feedback ratio  $\frac{B(s)}{R(s)} = \frac{G(s)H(s)}{1 + G(s)H(s)}$

5. error ratio  $\frac{E(s)}{R(s)} = \frac{1}{1 + G(s)H(s)}$

6. closed loop transfer function  $\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$

7. characteristic equation  $1 + G(s)H(s) = 0$

8. closed loop poles and zeros if  $K=10$ .

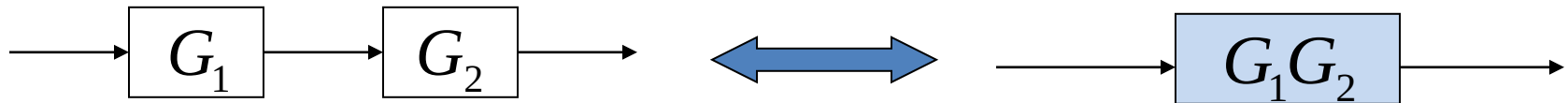


# Reduction of Complicated Block Diagrams تحويلات واختصارات المخطط الكتلي

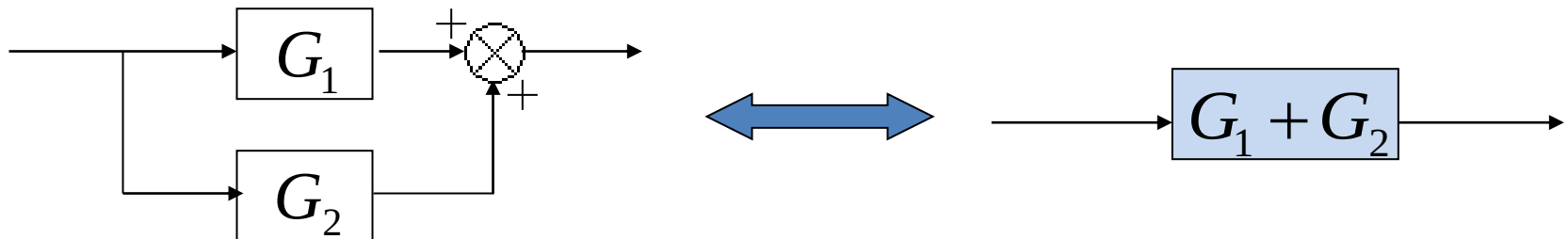
- The block diagram of a practical control system is often quite complicated.
- It may include several feedback or feedforward loops, and multiple inputs.
- By means of systematic block diagram reduction, every multiple loop linear feedback system may be reduced to canonical form.

# Reduction techniques

## 1. Combining blocks in cascade

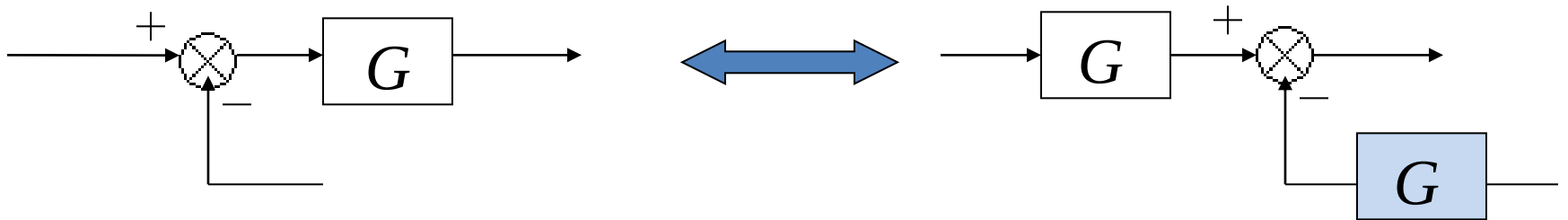


## 2. Combining blocks in parallel

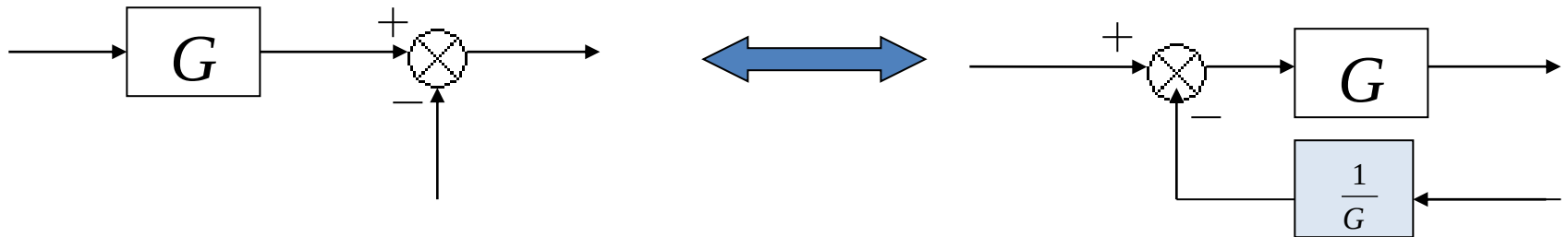


# Reduction techniques

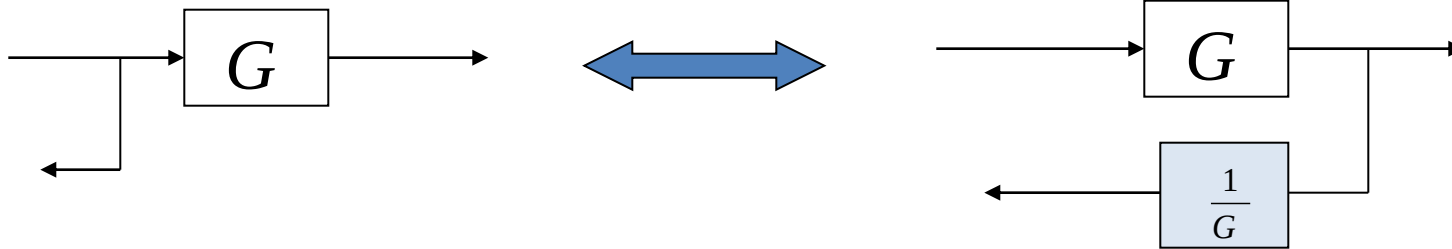
## 3. Moving a summing point behind a block



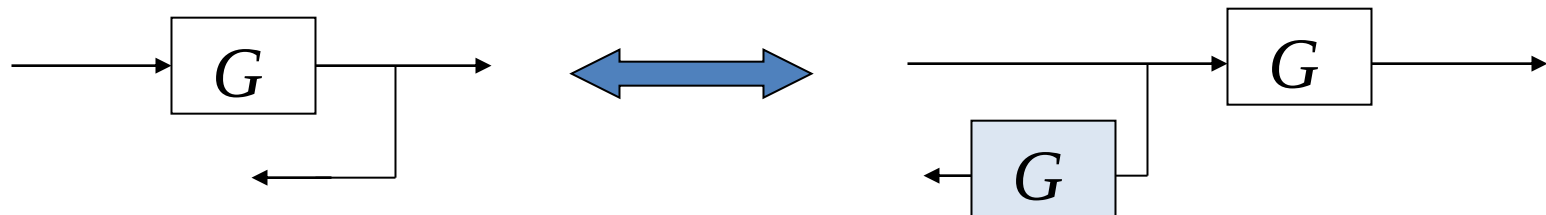
## 4. Moving a summing point ahead of a block



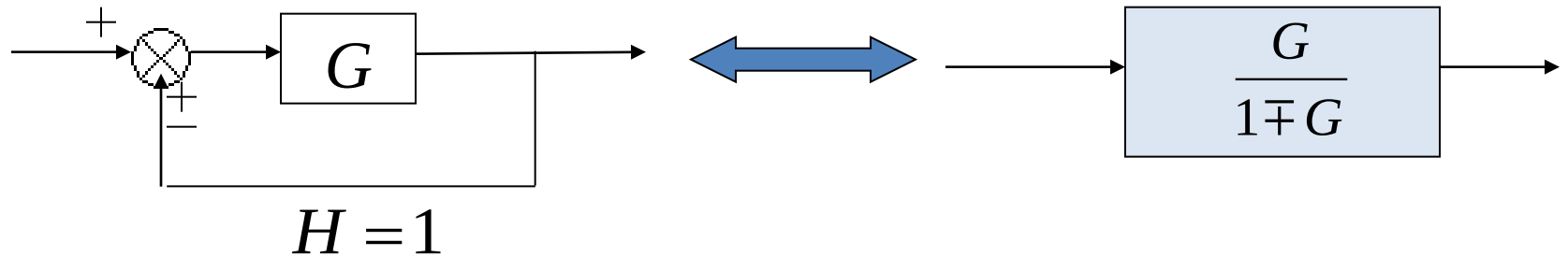
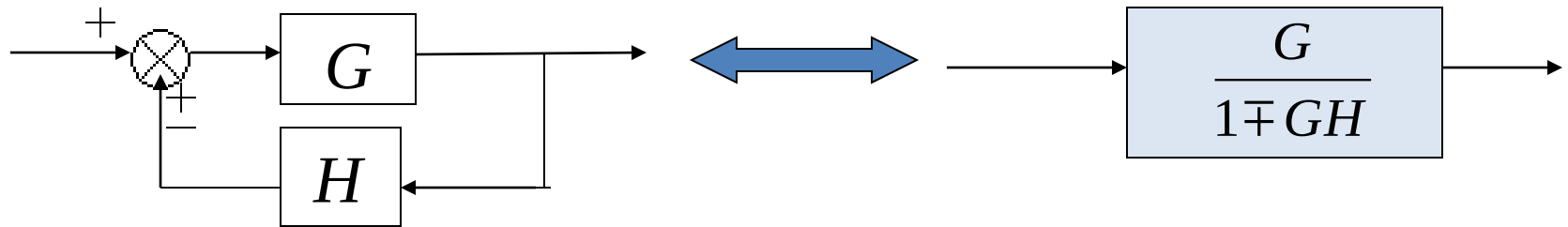
## 5. Moving a pickoff point behind a block



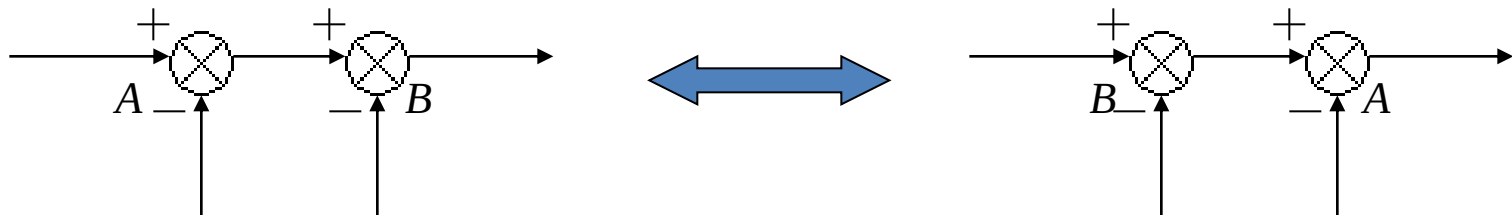
## 6. Moving a pickoff point ahead of a block



## 7. Eliminating a feedback loop

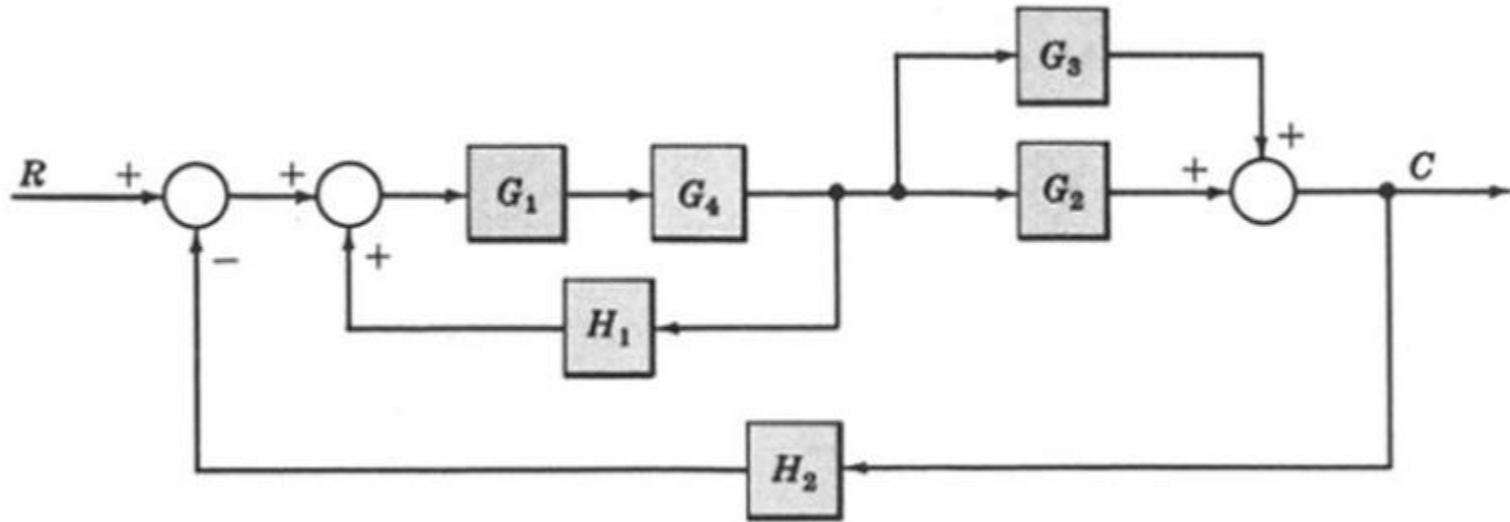


## 8. Swap with two neighboring summing points





## Example-5: Reduce the Block Diagram to Canonical Form.



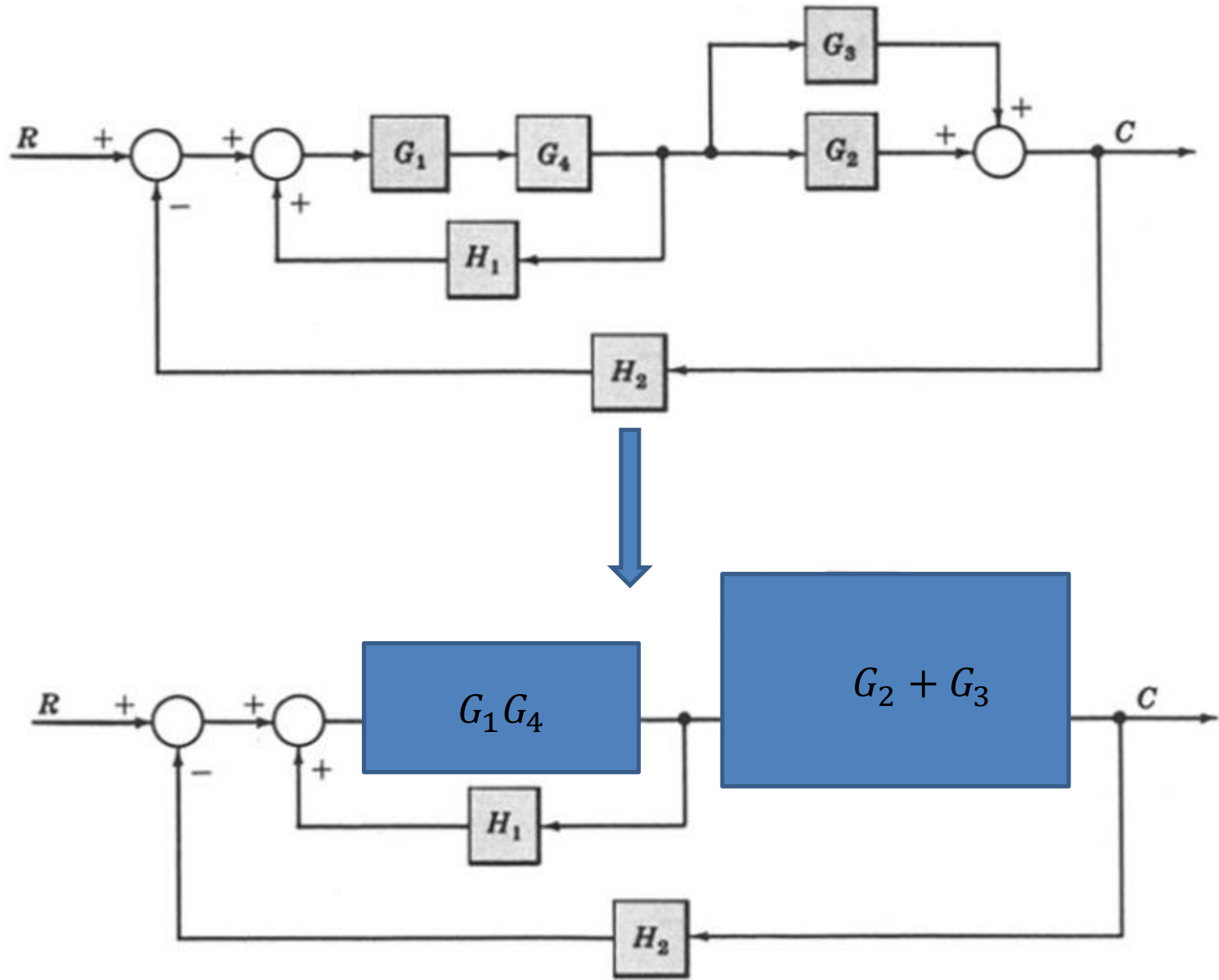
- Combine all cascade block using rule-1



- Combine all parallel block using rule-2



Example-5: Reduce the Block Diagram to **Canonical Form**.

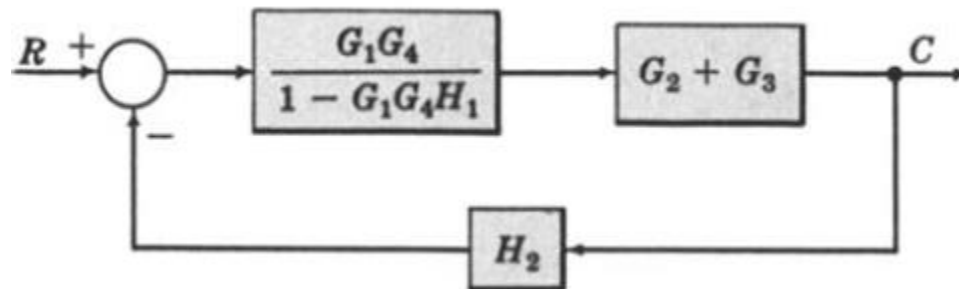


## Example-5: Continue.

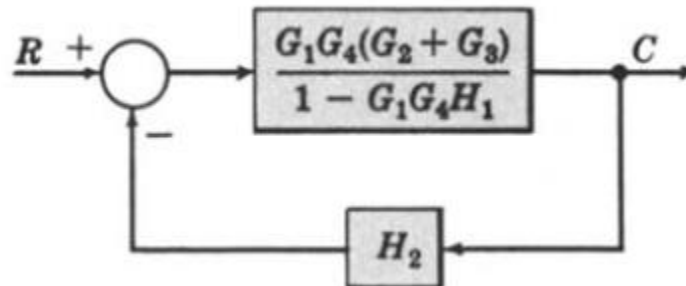
- Eliminate all minor feedback loops using rule-7



- After the elimination of minor feedback loop the block diagram is reduced to as shown below

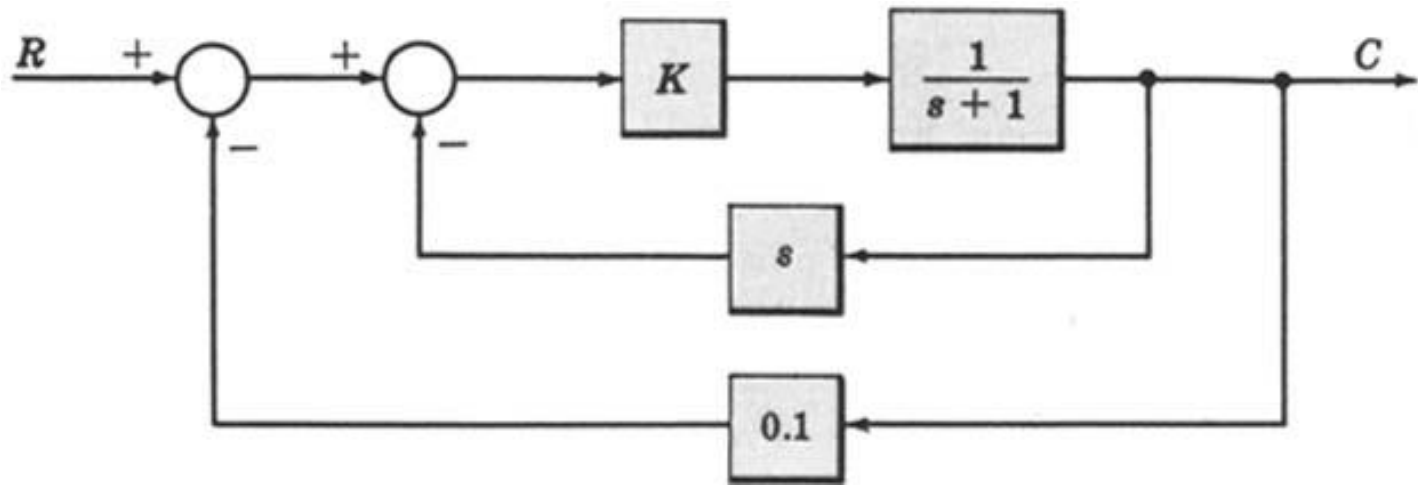


- Again blocks are in cascade are removed using rule-1



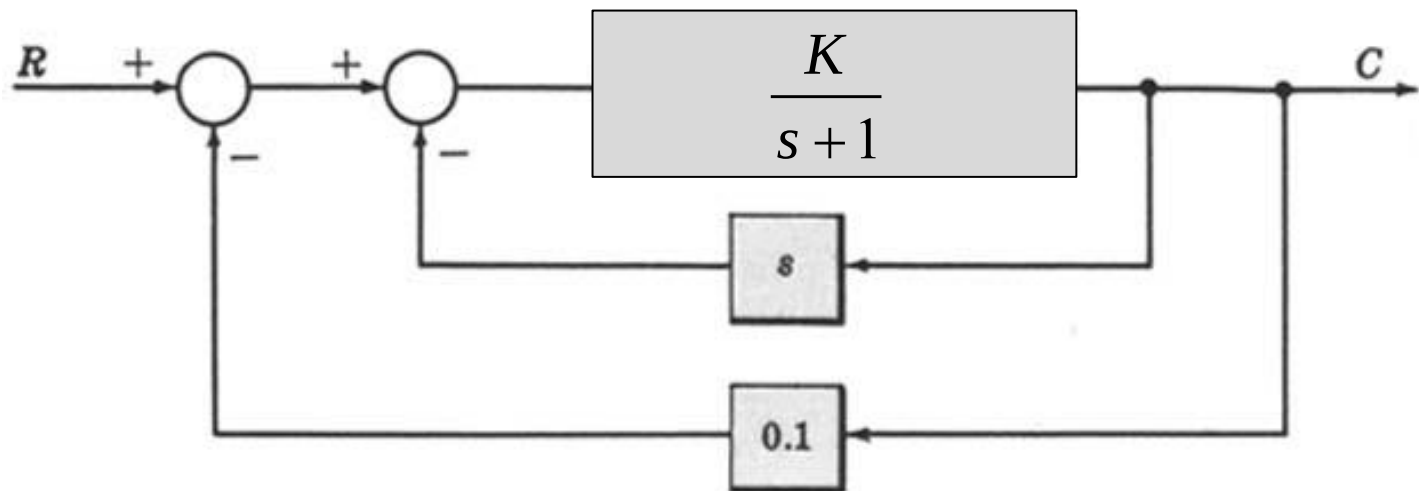
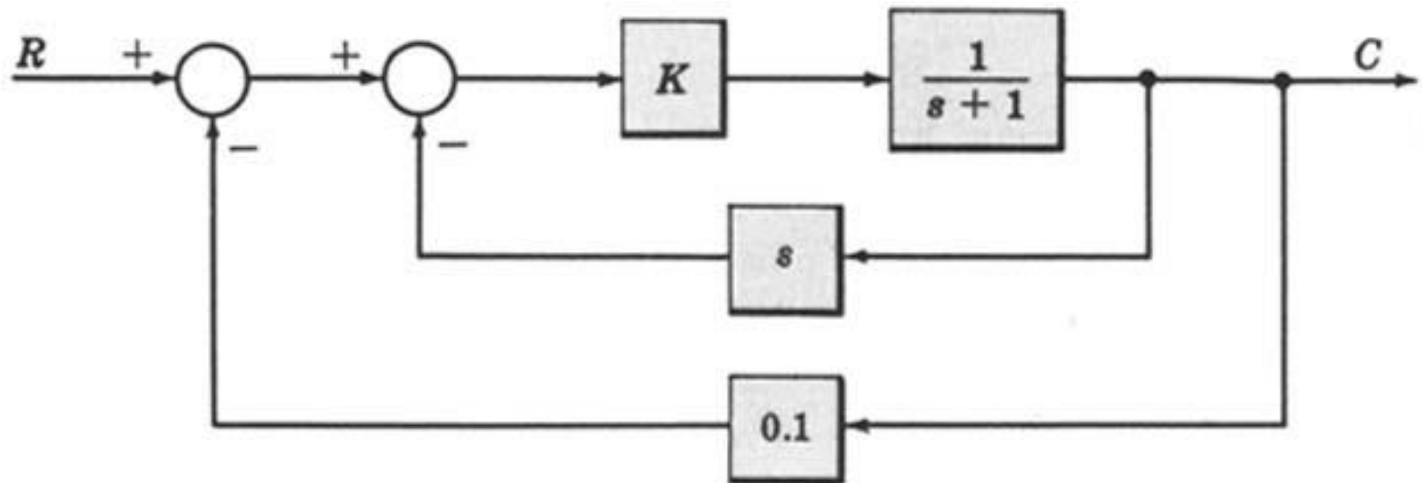
# Example-6

- For the system represented by the following block diagram determine:
  - Open loop transfer function
  - Feed Forward Transfer function
  - control ratio
  - feedback ratio
  - error ratio
  - closed loop transfer function
  - characteristic equation
  - closed loop poles and zeros if  $K=10$ .

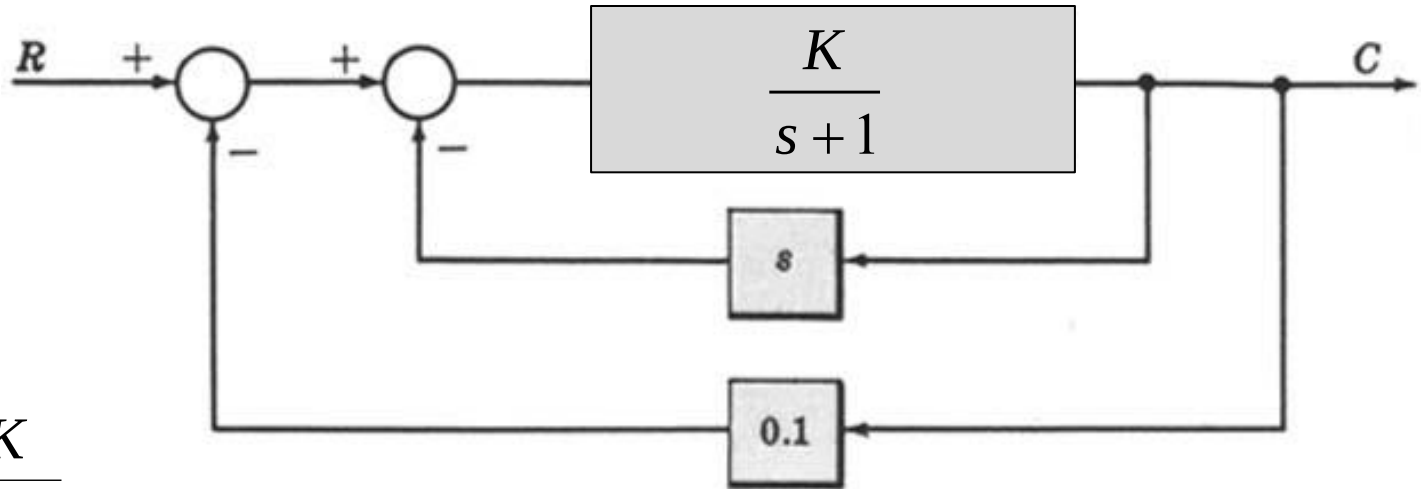


# Example-6

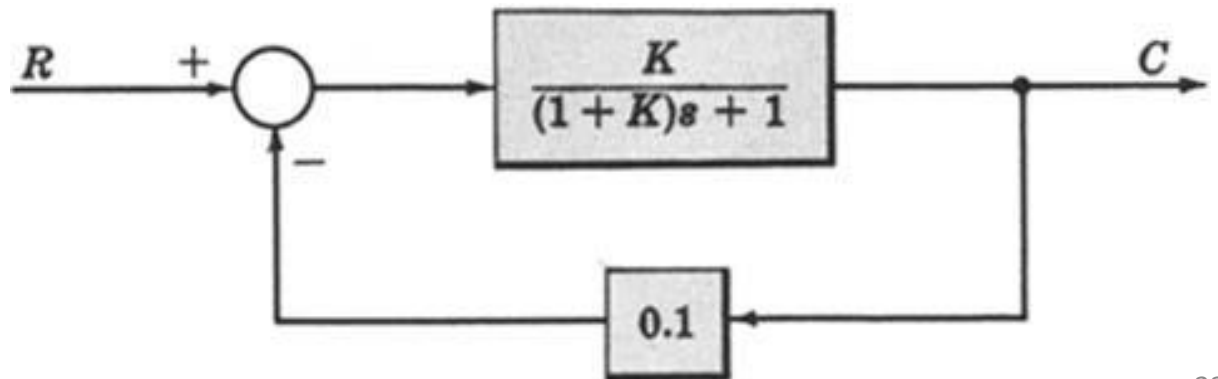
- First we will reduce the given block diagram to canonical form



# Example-6



$$\frac{G}{1+GH} = \frac{\frac{K}{s+1}}{1 + \frac{K}{s+1}s}$$



# Example-6

1. Open loop transfer function  $\frac{B(s)}{E(s)} = G(s)H(s)$

2. Feed Forward Transfer function  $\frac{C(s)}{E(s)} = G(s)$

3. control ratio  $\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$

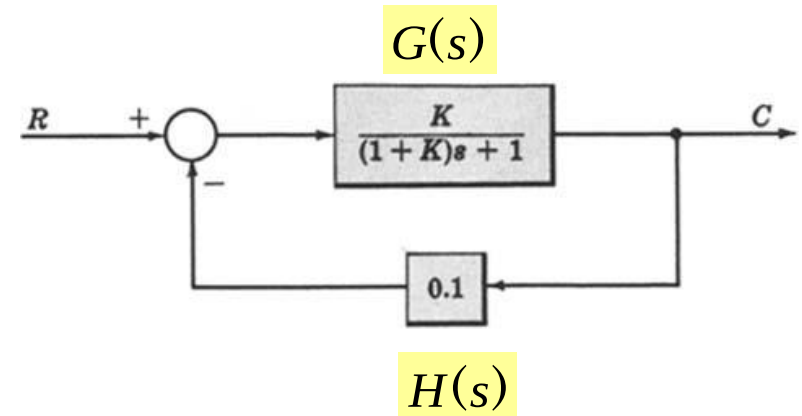
4. feedback ratio  $\frac{B(s)}{R(s)} = \frac{G(s)H(s)}{1 + G(s)H(s)}$

5. error ratio  $\frac{E(s)}{R(s)} = \frac{1}{1 + G(s)H(s)}$

6. closed loop transfer function  $\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$

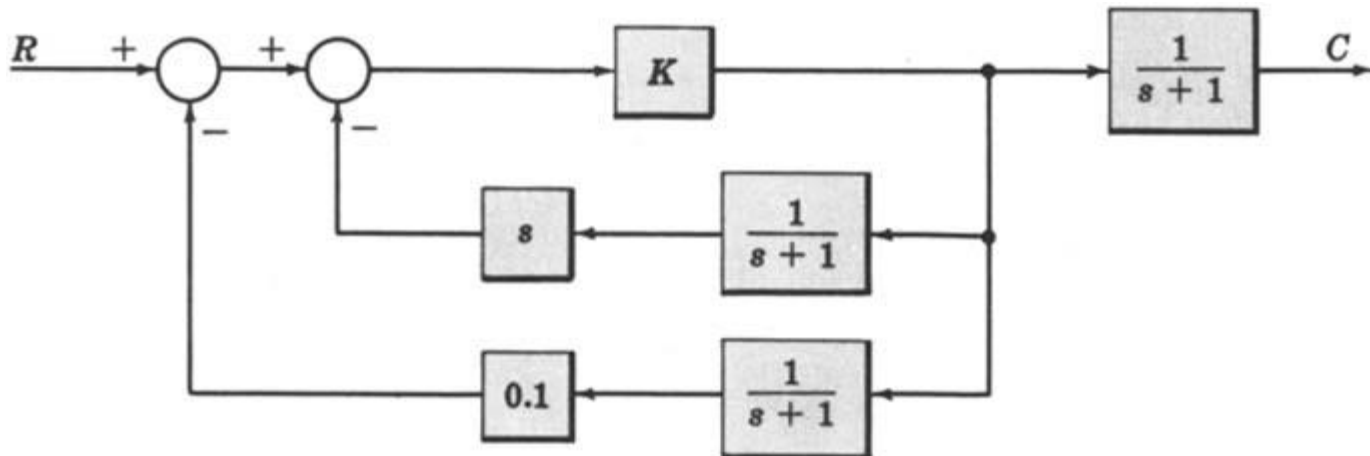
7. characteristic equation  $1 + G(s)H(s) = 0$

8. closed loop poles and zeros if K=10.



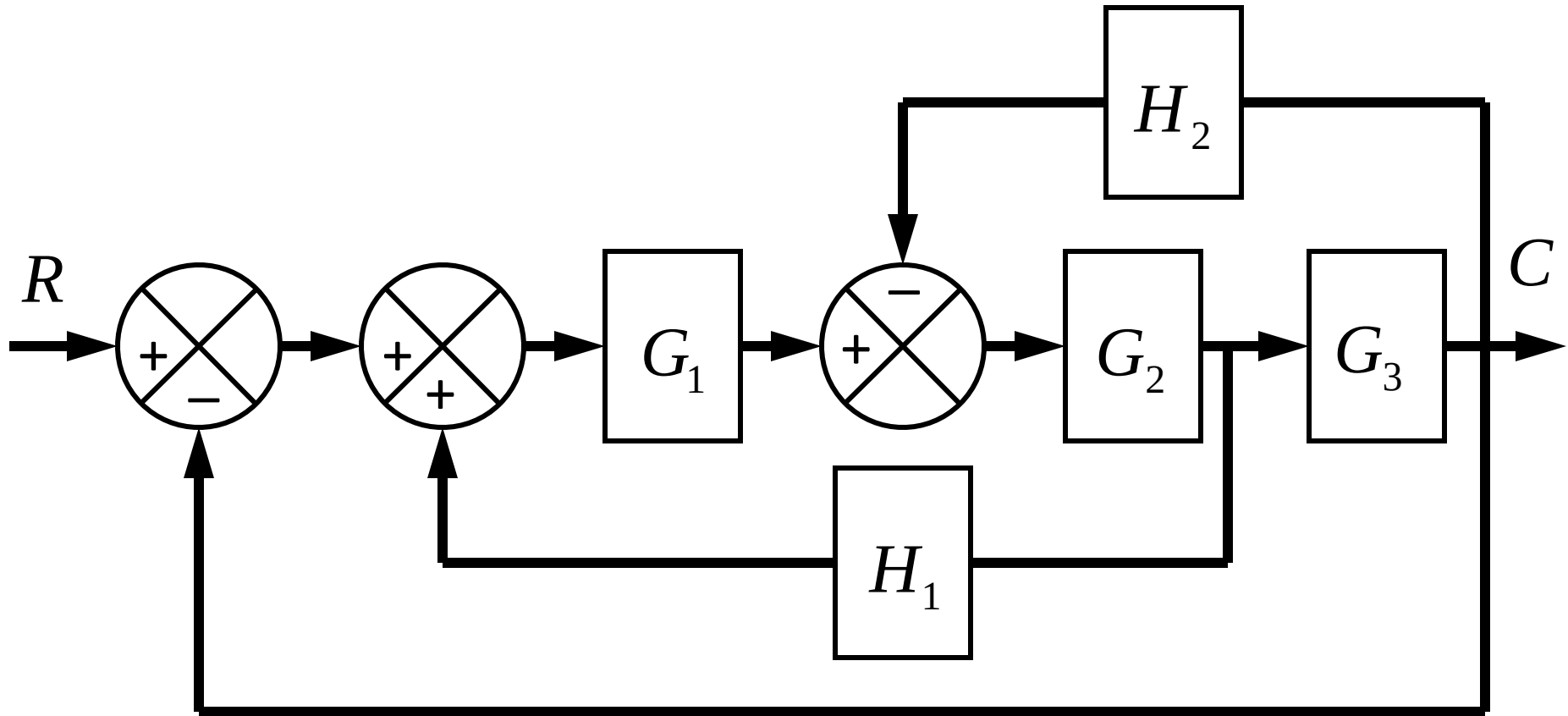
# Example-7

- For the system represented by the following block diagram determine:
  - Open loop transfer function
  - Feed Forward Transfer function
  - control ratio
  - feedback ratio
  - error ratio
  - closed loop transfer function
  - characteristic equation
  - closed loop poles and zeros if  $K=100$ .

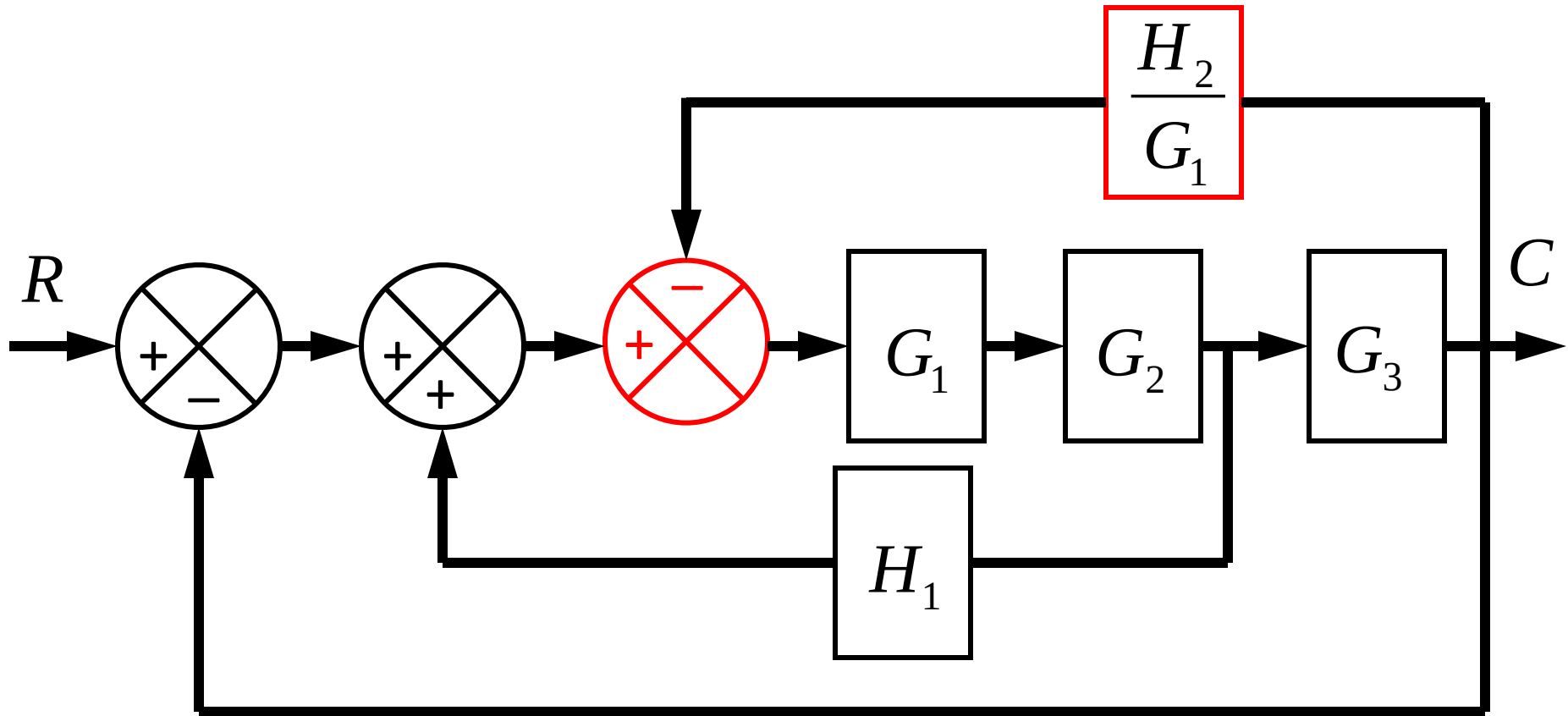




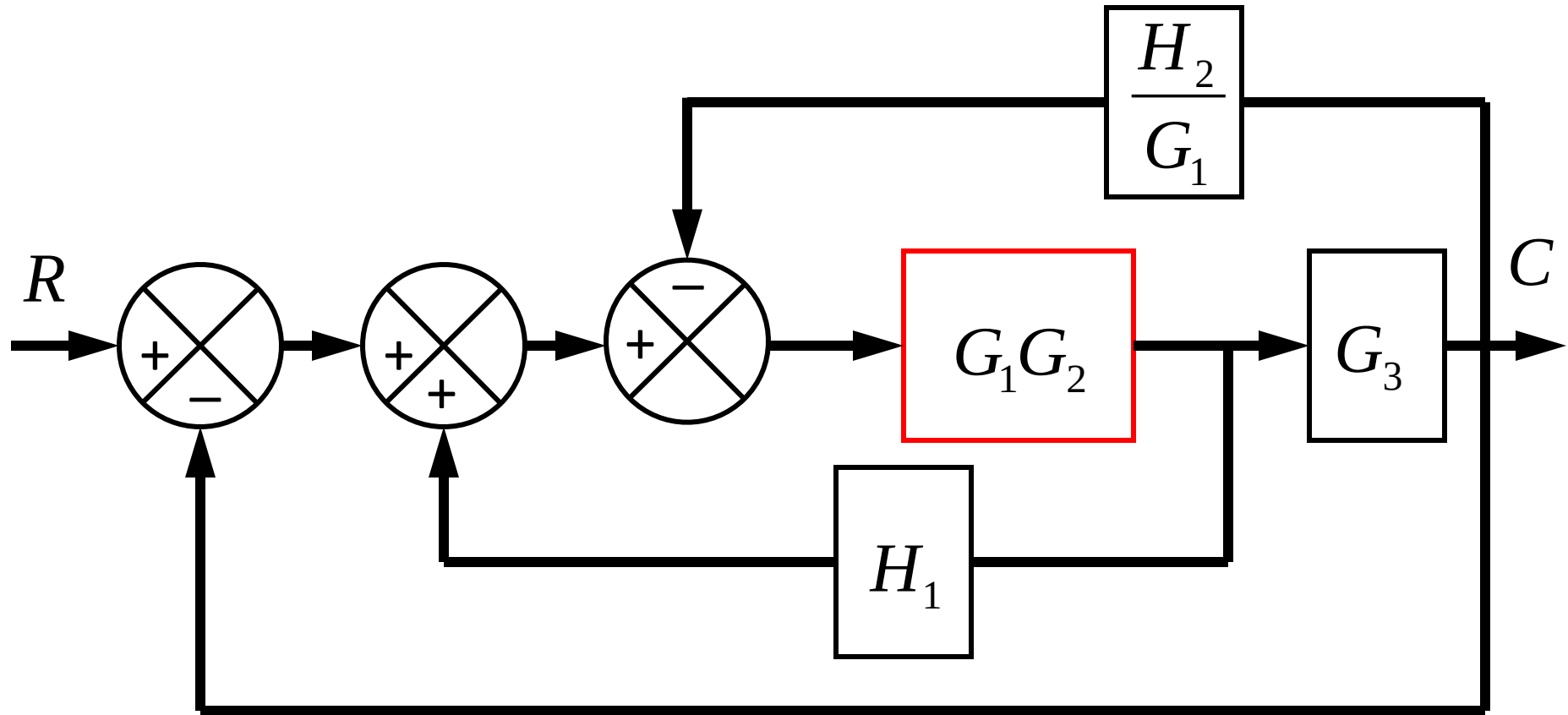
# Example-8



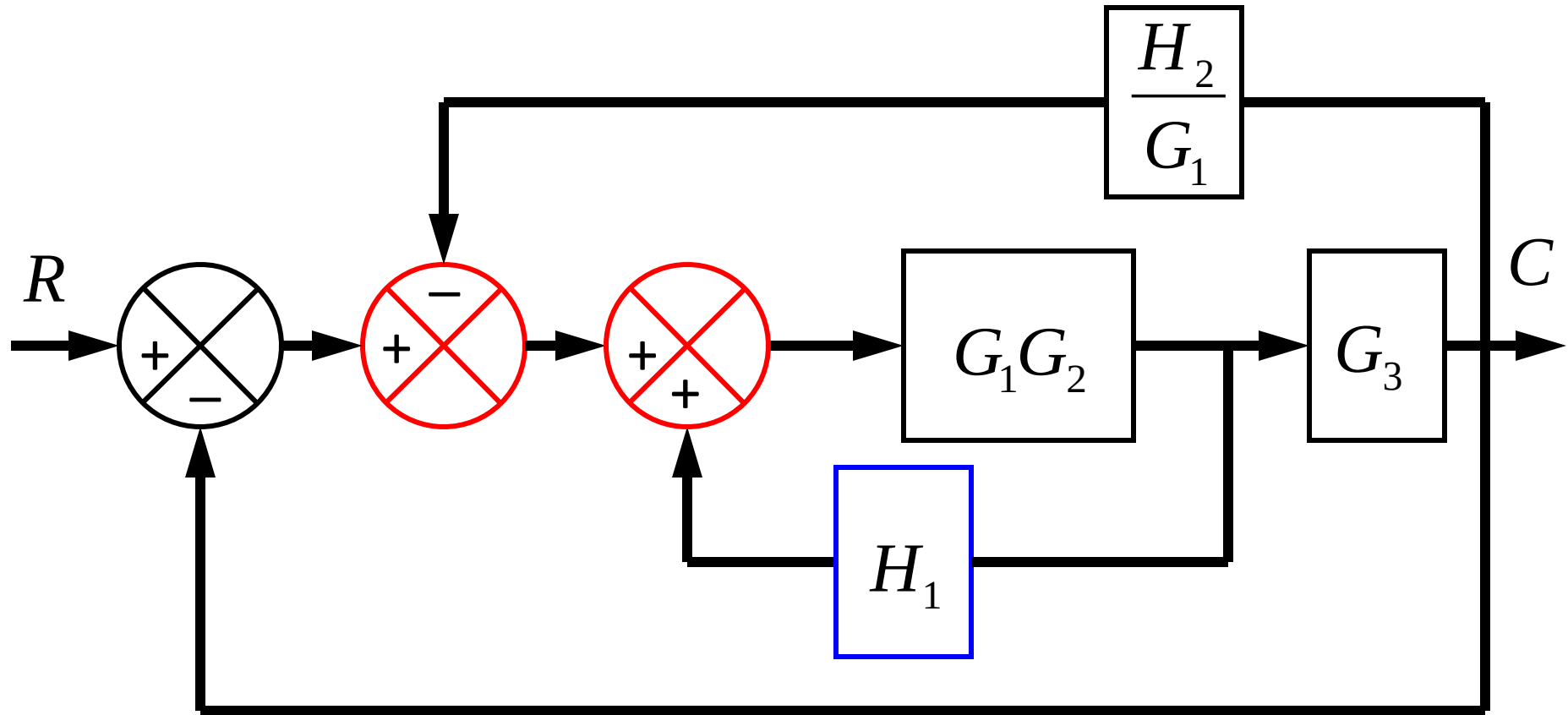
# Example-8



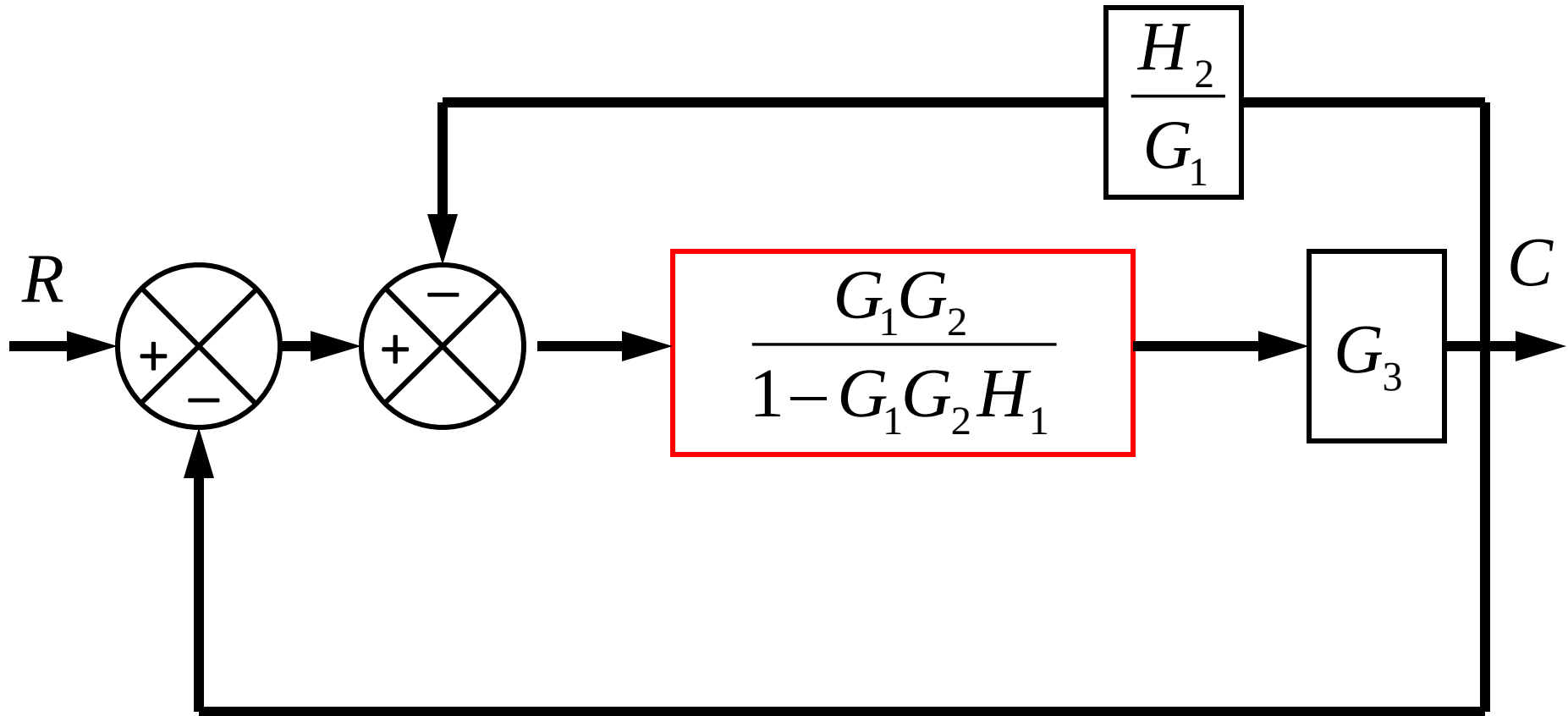
# Example-8



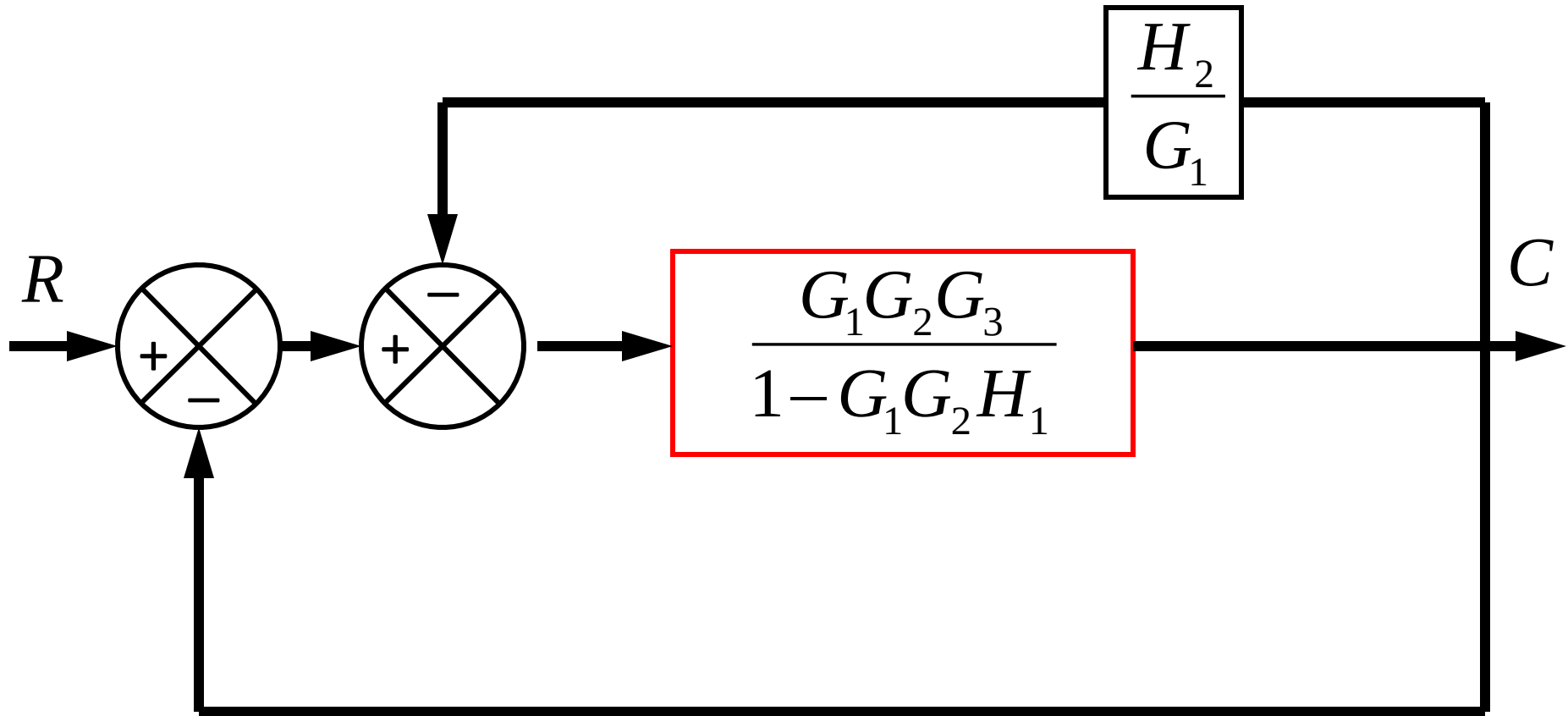
# Example-8



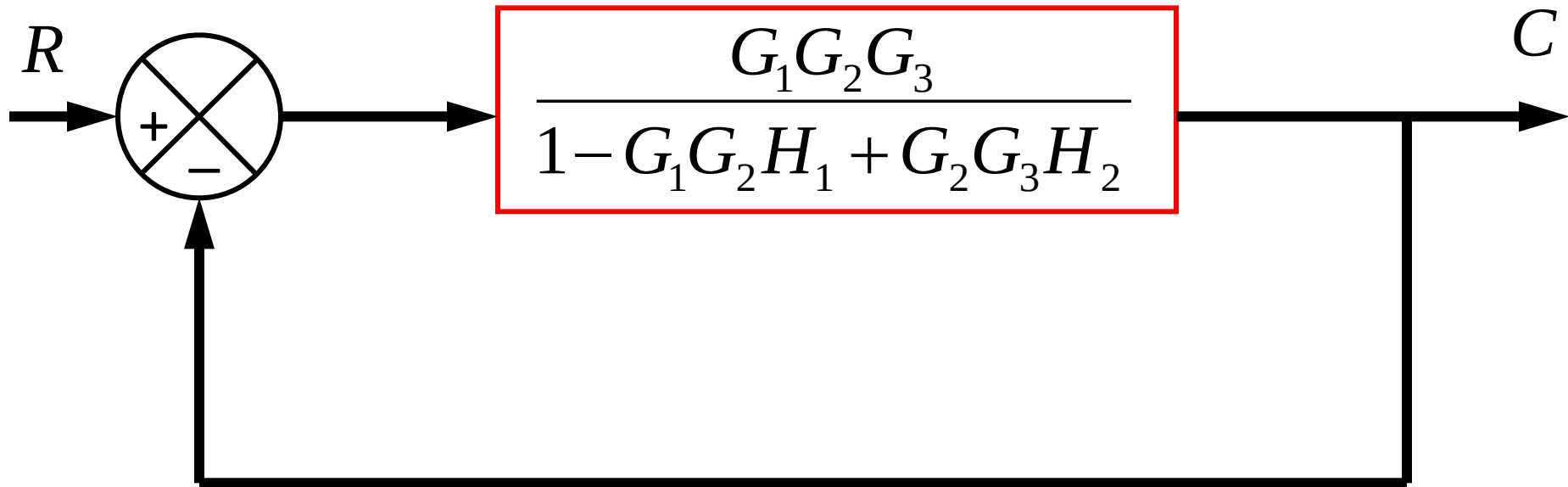
# block diagram: reduction example



# block diagram: reduction example

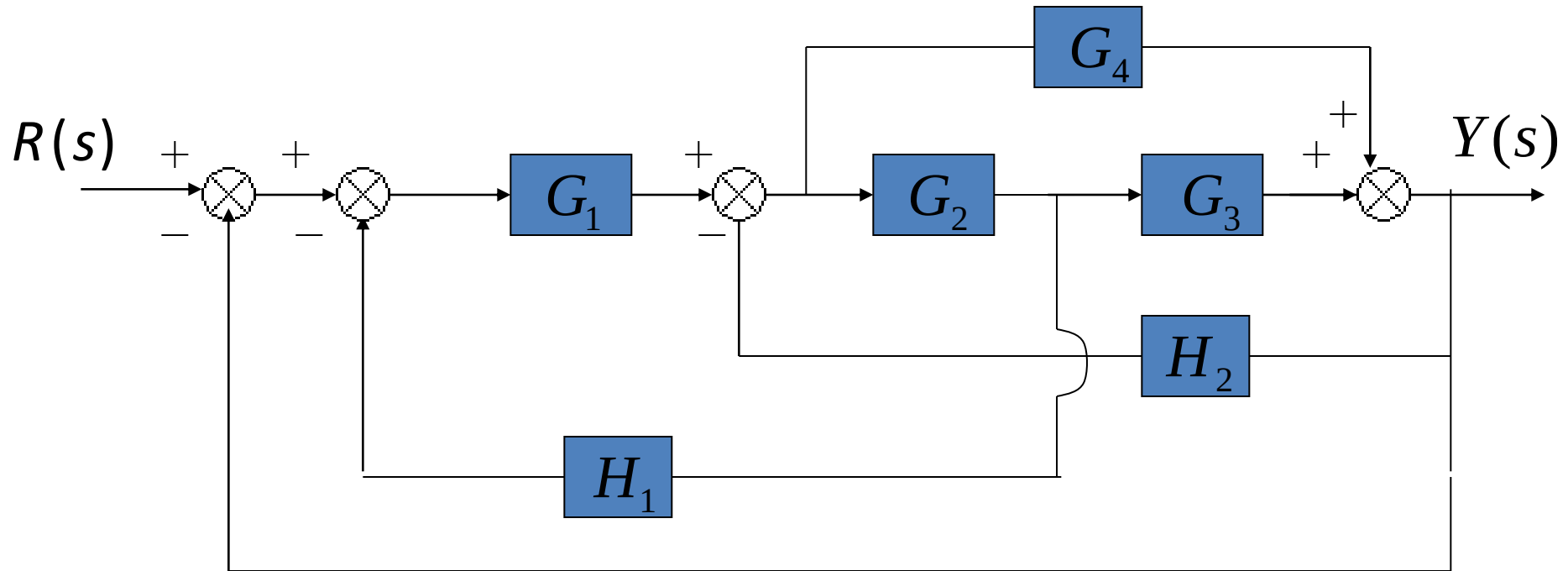


# block diagram: reduction example



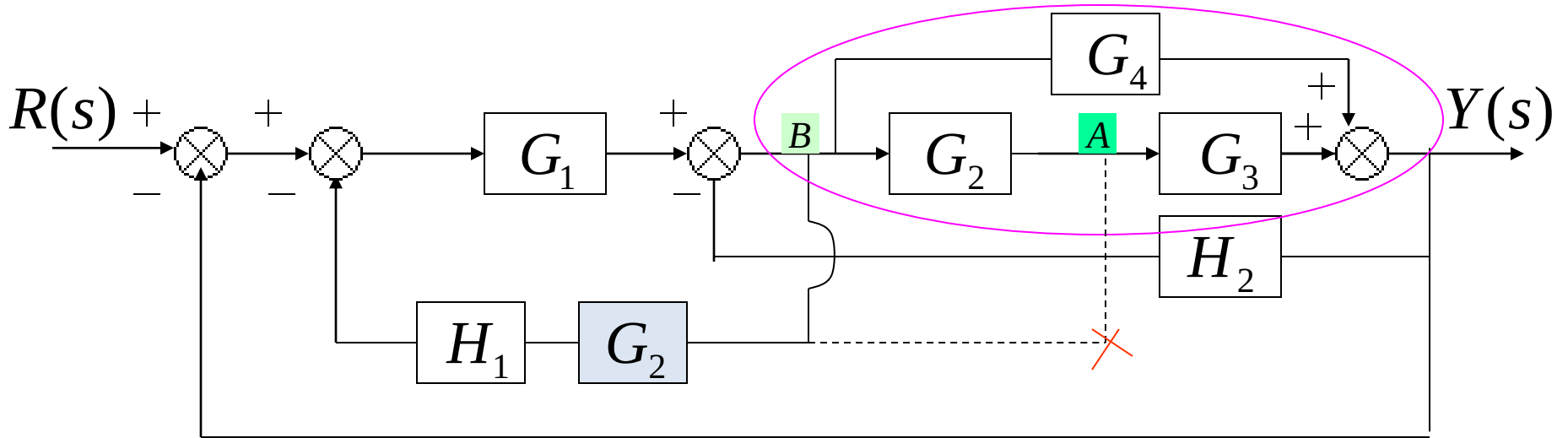
## Example 9

Find the transfer function of the following block diagram





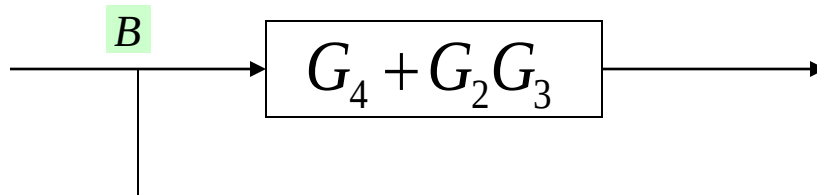
$I$

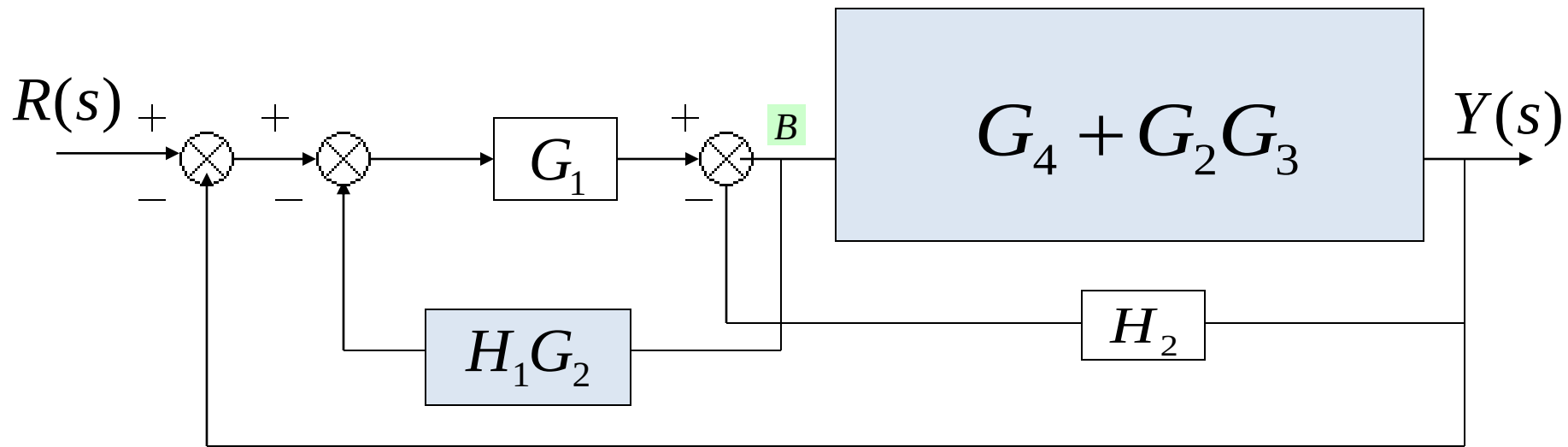


Solution:

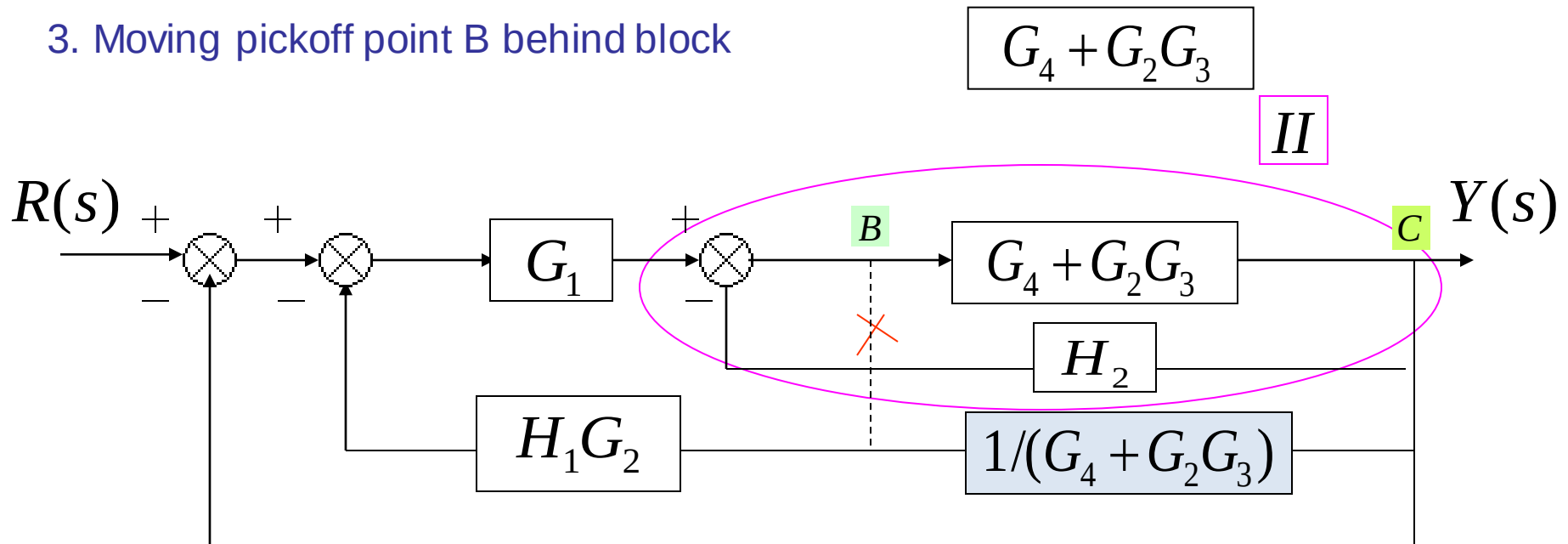
1. Moving pickoff point A ahead of block  $G_2$

2. Eliminate loop I & simplify

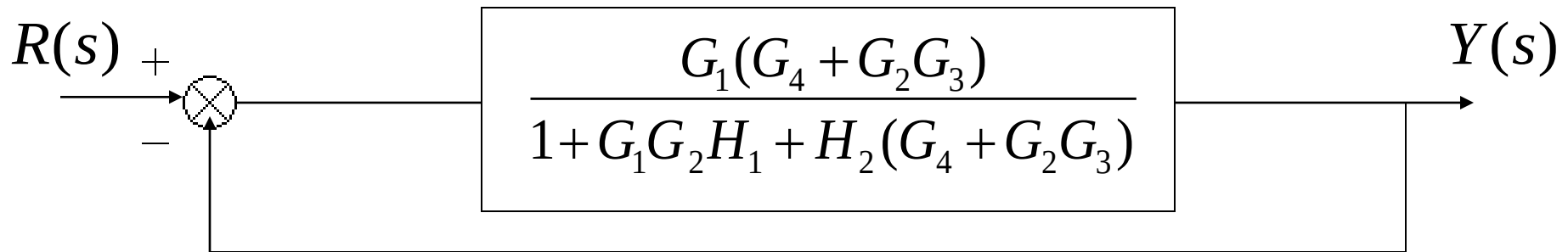
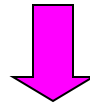
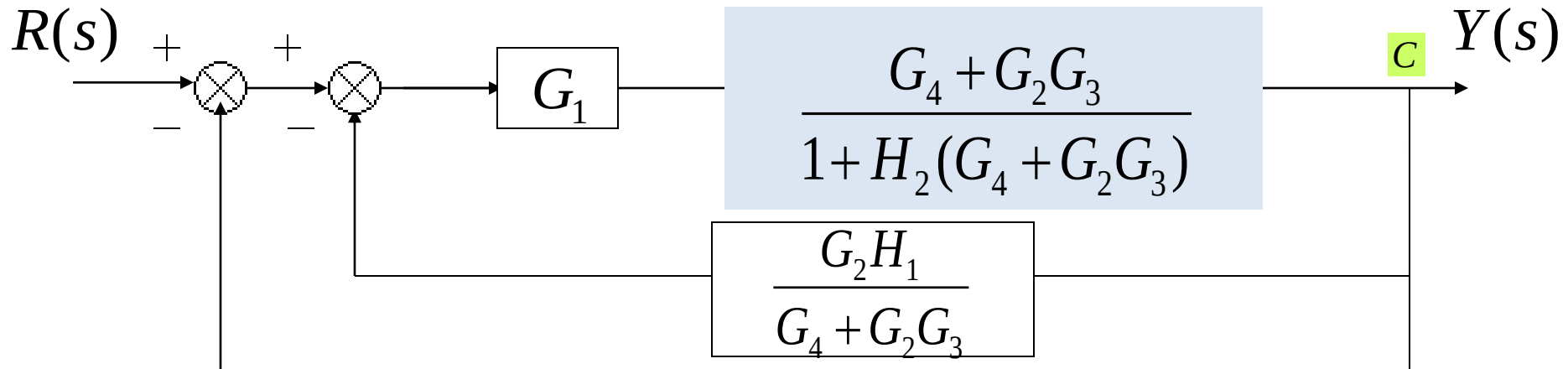




### 3. Moving pickoff point B behind block



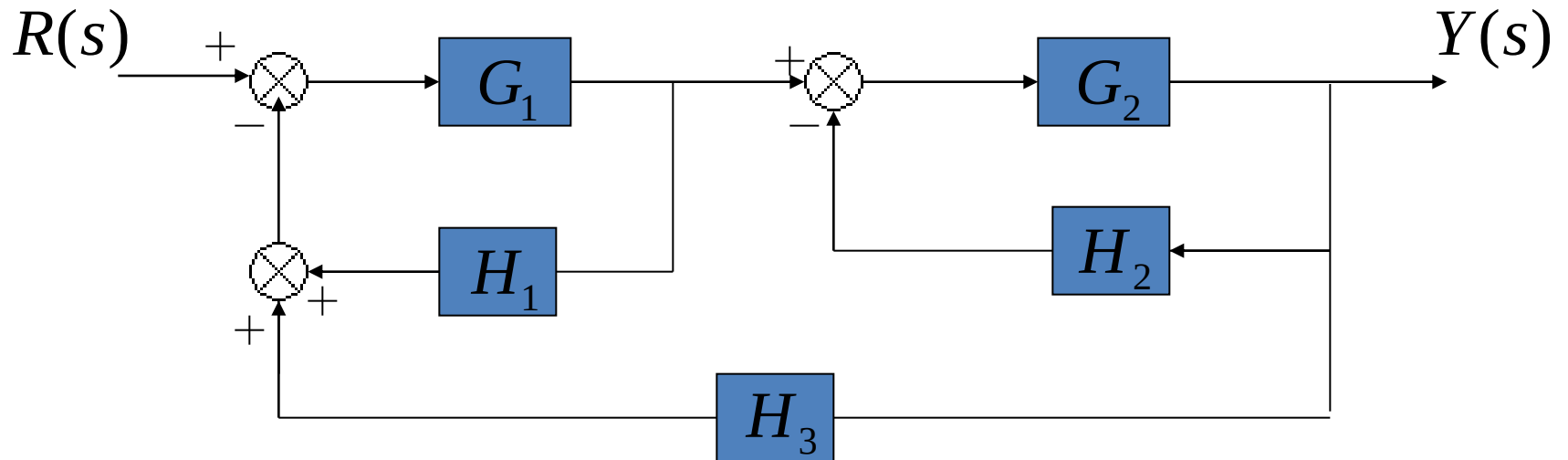
#### 4. Eliminate loop III



$$\frac{Y(s)}{R(s)} = \frac{G_1(G_4 + G_2G_3)}{1 + G_1G_2H_1 + H_2(G_4 + G_2G_3) + G_1(G_4 + G_2G_3)}$$

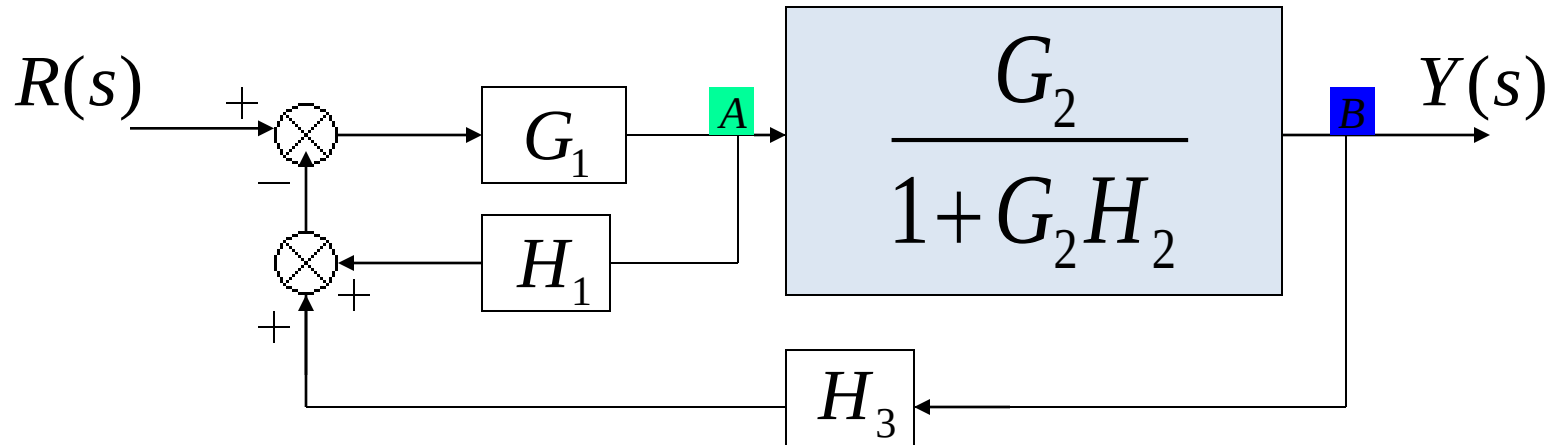
# Example 10

Find the transfer function of the following block diagrams

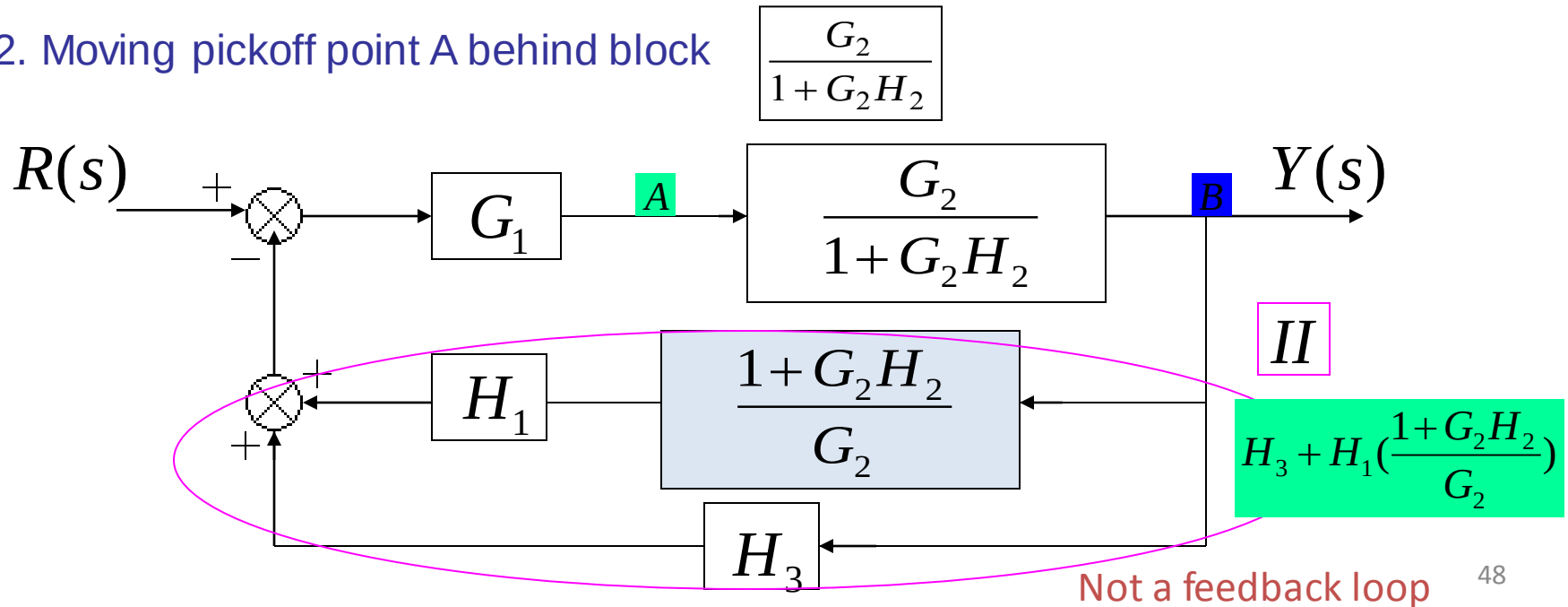


Solution:

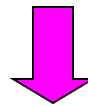
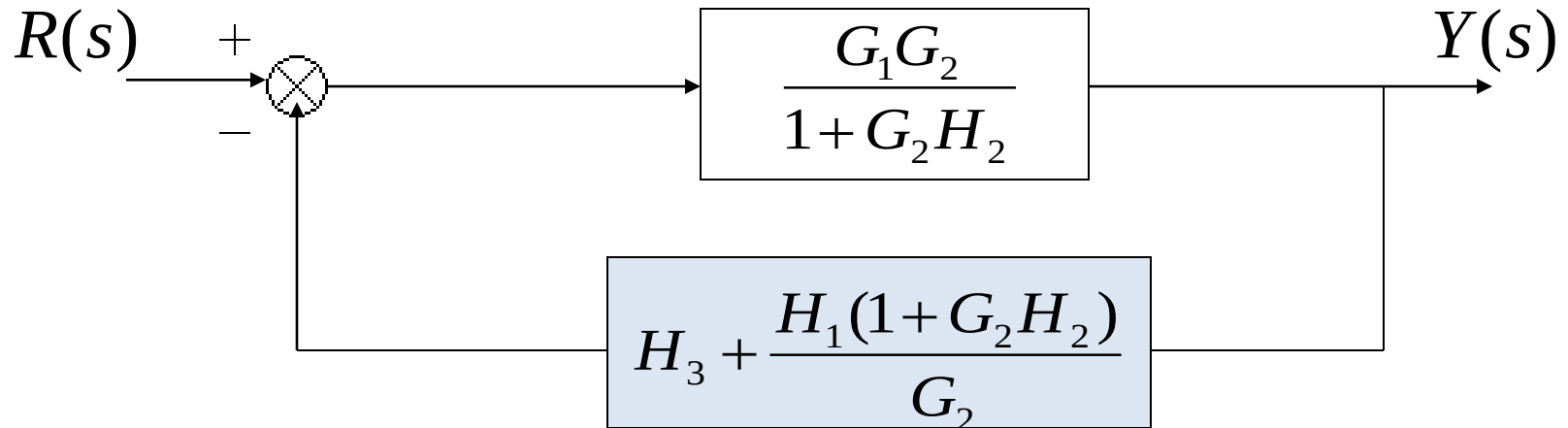
1. Eliminate loop I



2. Moving pickoff point A behind block



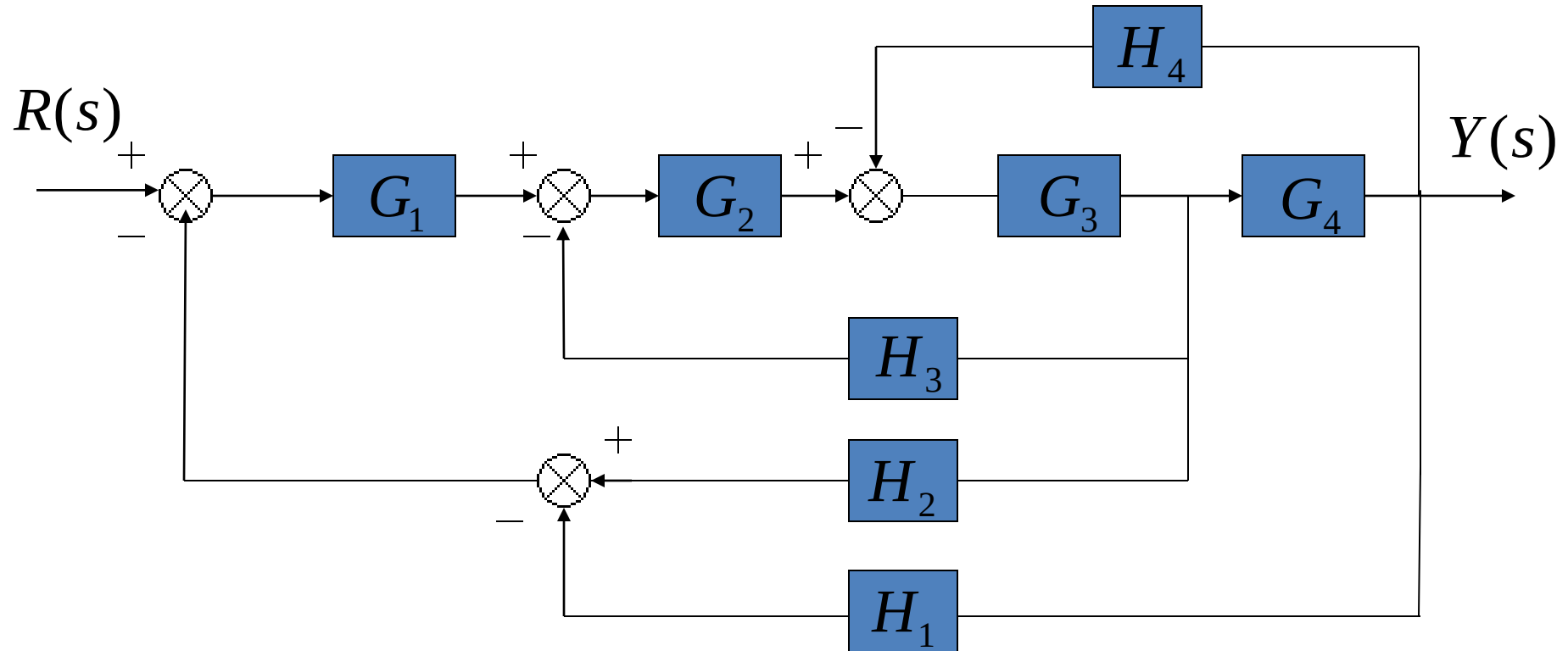
### 3. Eliminate loop II



$$\frac{Y(s)}{R(s)} = \frac{G_1 G_2}{1 + G_2 H_2 + G_1 G_2 H_3 + G_1 H_1 + G_1 G_2 H_1 H_2}$$

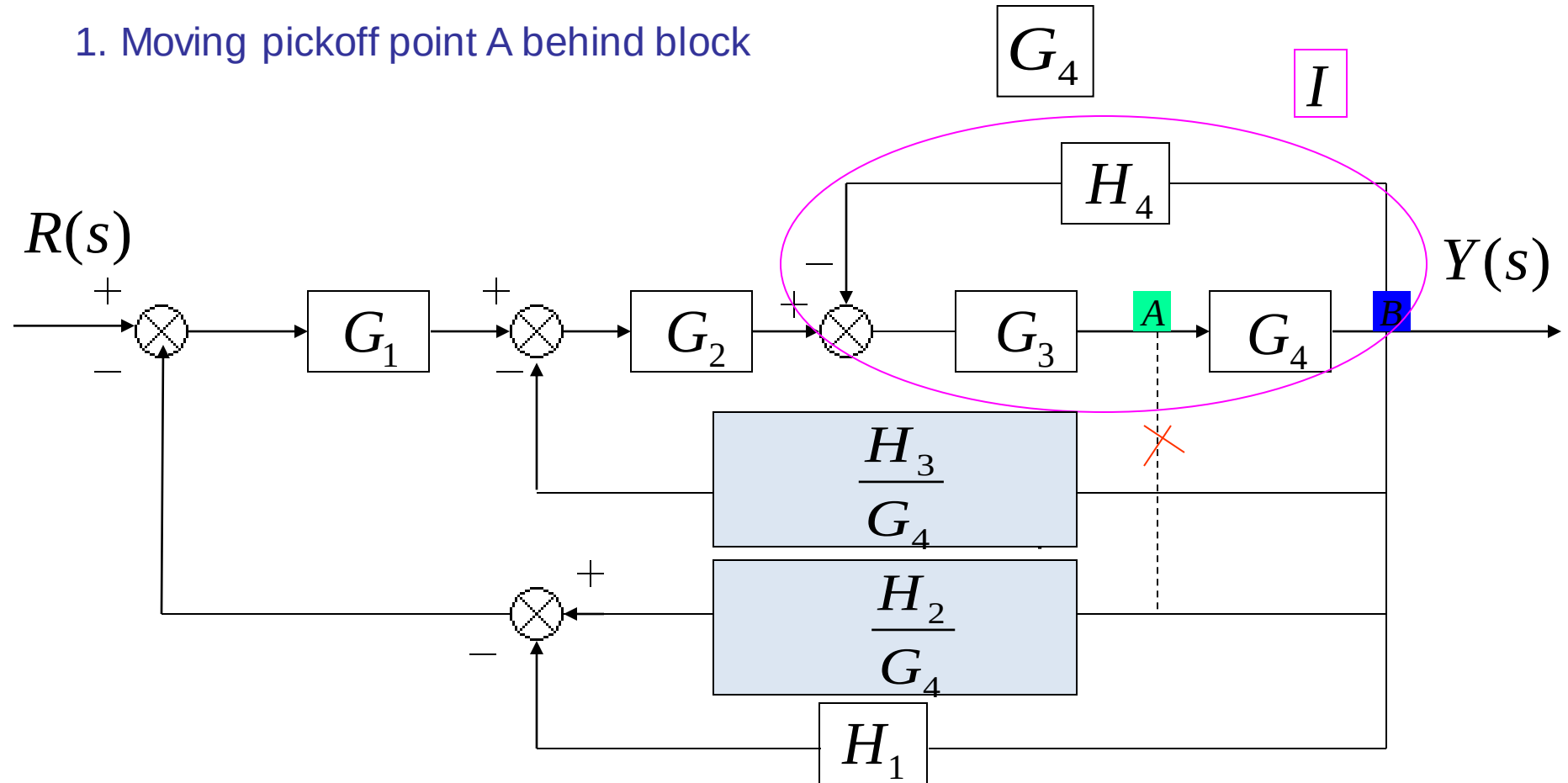
# Example 11

Find the transfer function of the following block diagrams



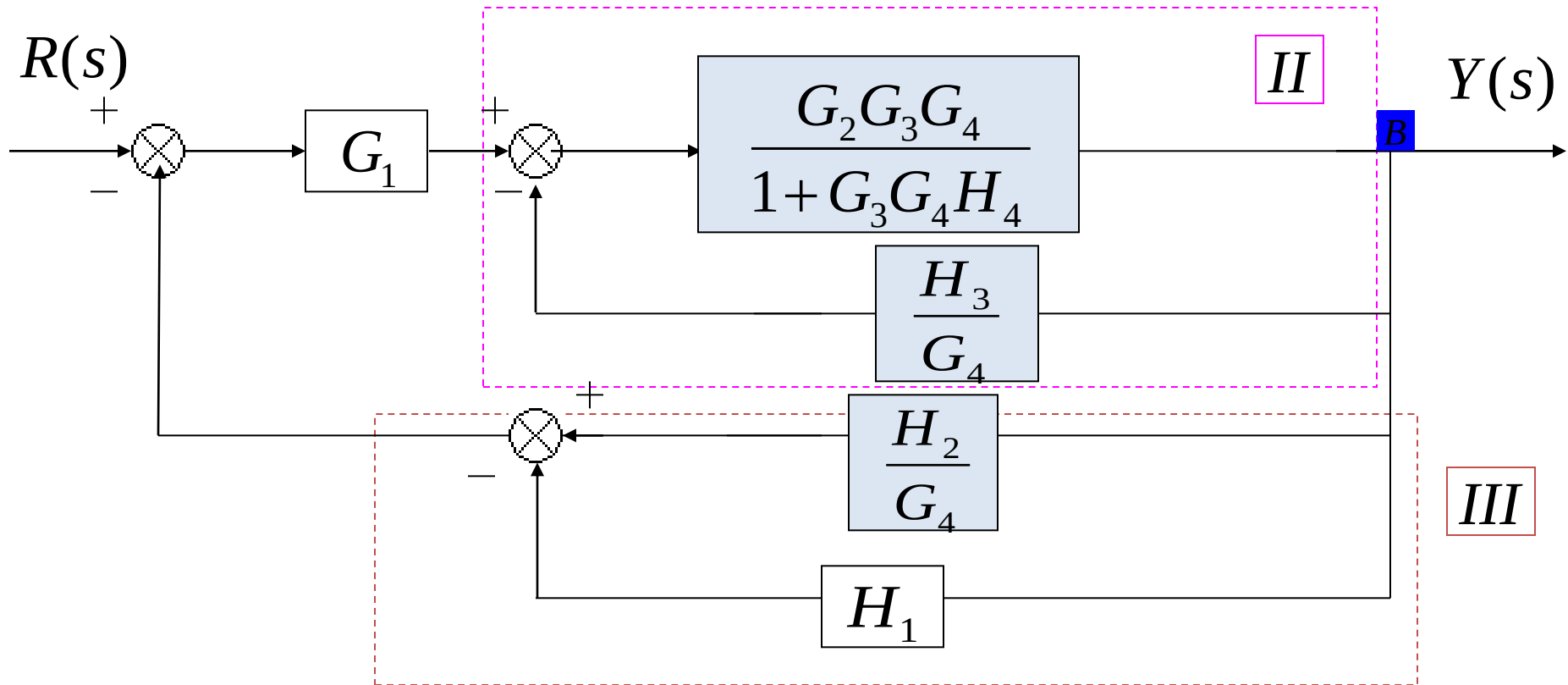
Solution:

1. Moving pickoff point A behind block





## 2. Eliminate loop I and Simplify



**II**



feedback

$$\frac{G_2 G_3 G_4}{1 + G_3 G_4 H_4 + G_2 G_3 H_3}$$

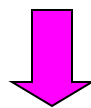
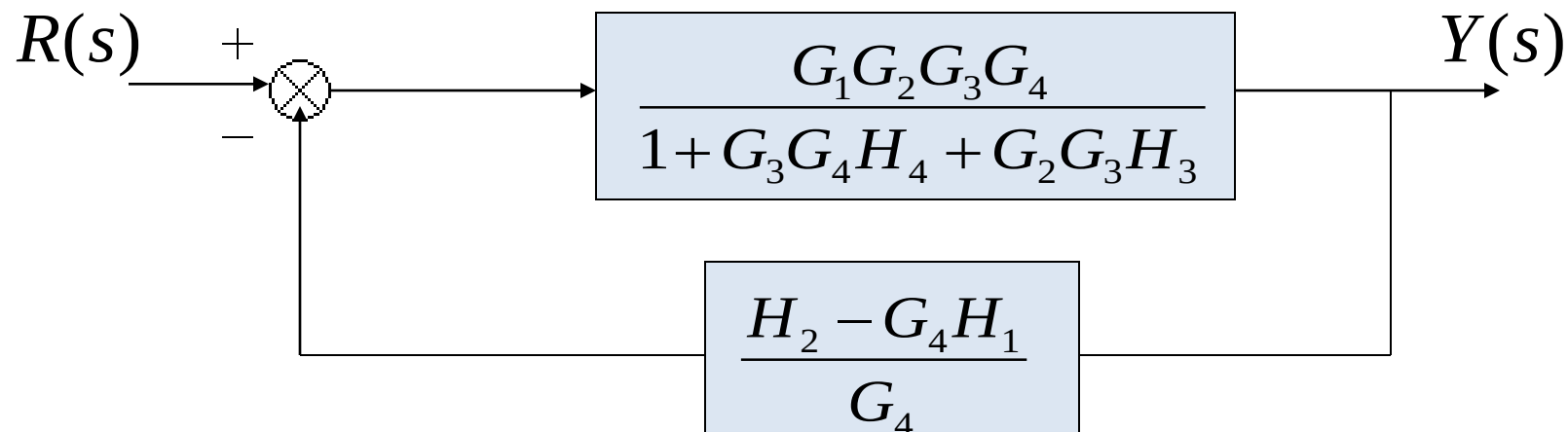
**III**



Not feedback

$$\frac{H_2 - G_4 H_1}{G_4}$$

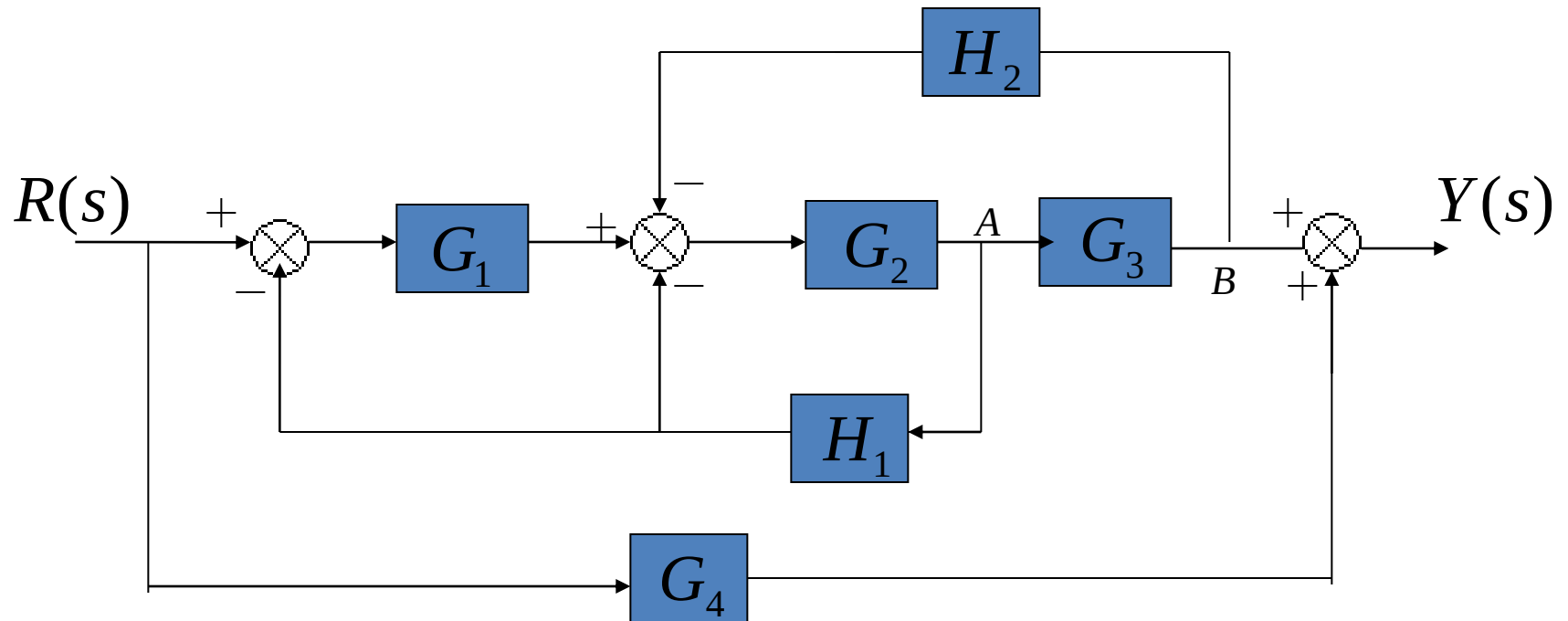
### 3. Eliminate loop II & III



$$\frac{Y(s)}{R(s)} = \frac{G_1 G_2 G_3 G_4}{1 + G_2 G_3 H_3 + G_3 G_4 H_4 + G_1 G_2 G_3 H_2 - G_1 G_2 G_3 G_4 H_1}$$

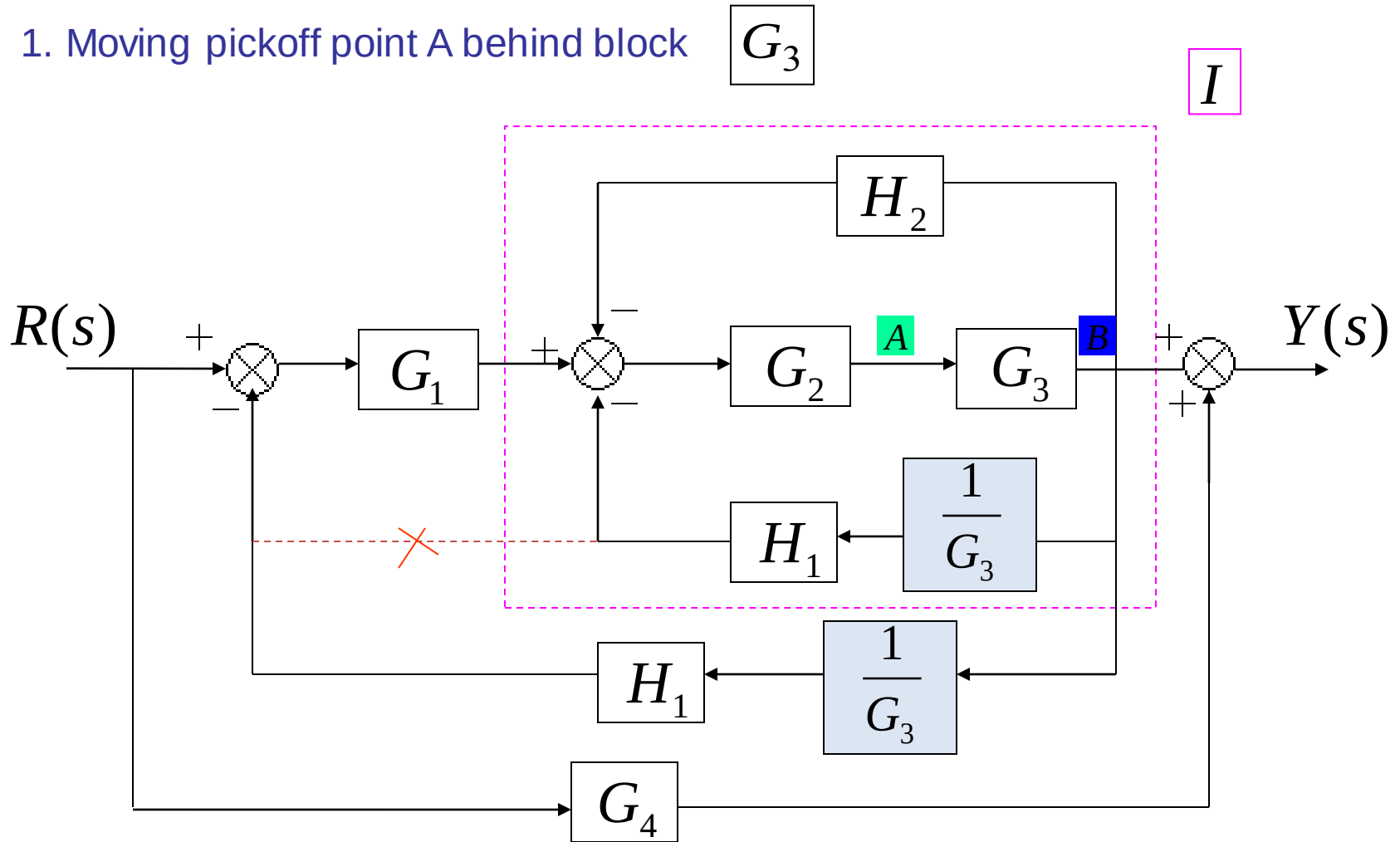
## Example 12

Find the transfer function of the following block diagrams

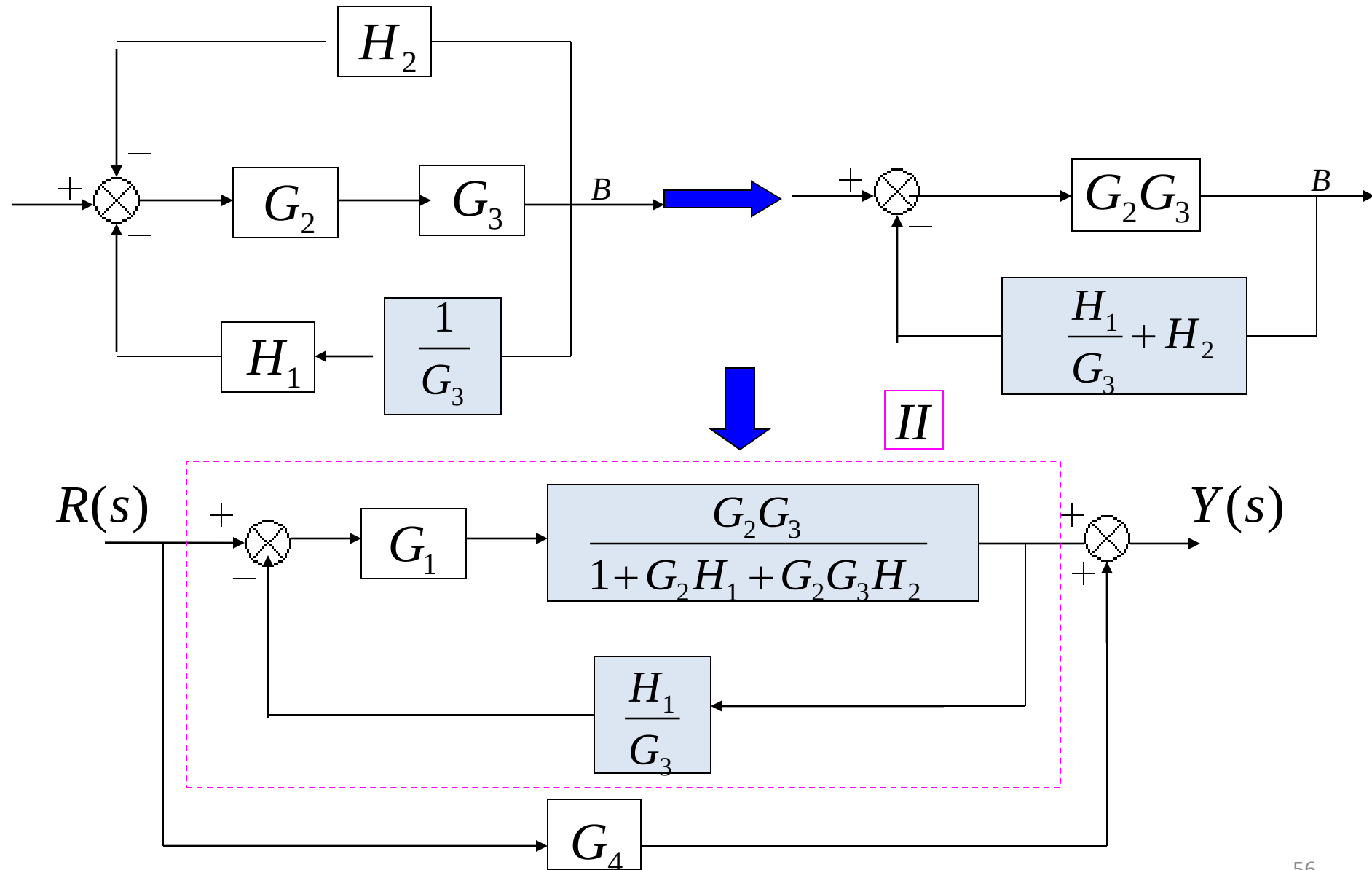


Solution:

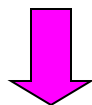
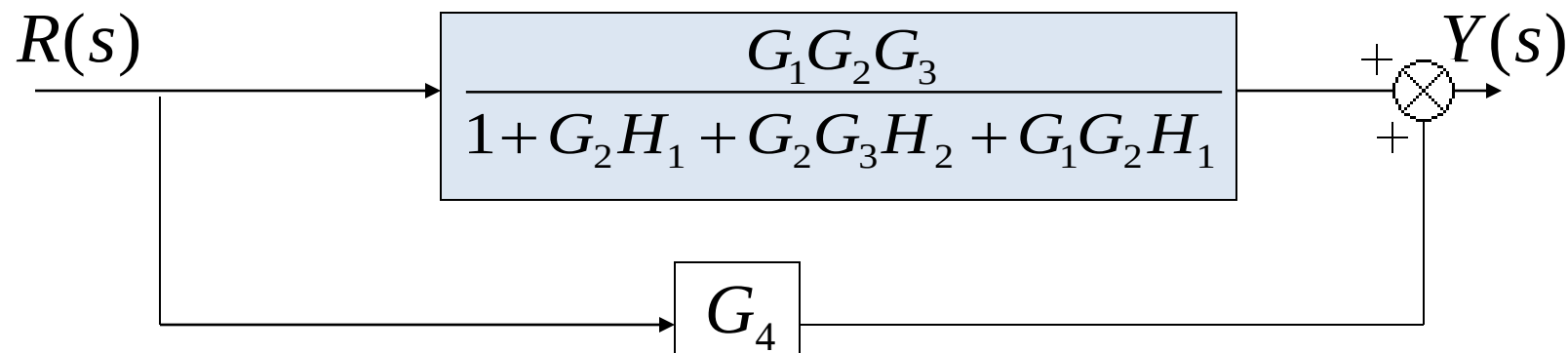
1. Moving pickoff point A behind block  $G_3$



## 2. Eliminate loop I & Simplify

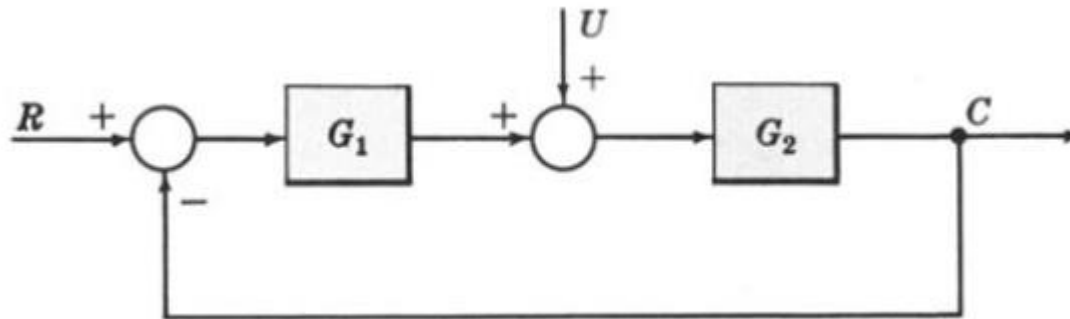


### 3. Eliminate loop II



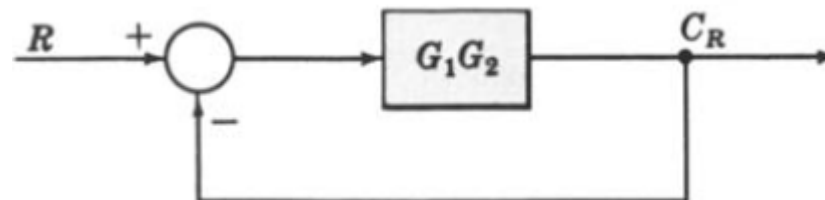
$$T(s) = \frac{Y(s)}{R(s)} = G_4 + \frac{G_1 G_2 G_3}{1 + G_2 H_1 + G_2 G_3 H_2 + G_1 G_2 H_1}$$

Example-13: **Multiple Input System.** Determine the output  $C$  due to inputs  $R$  and  $U$  using the Superposition Method.



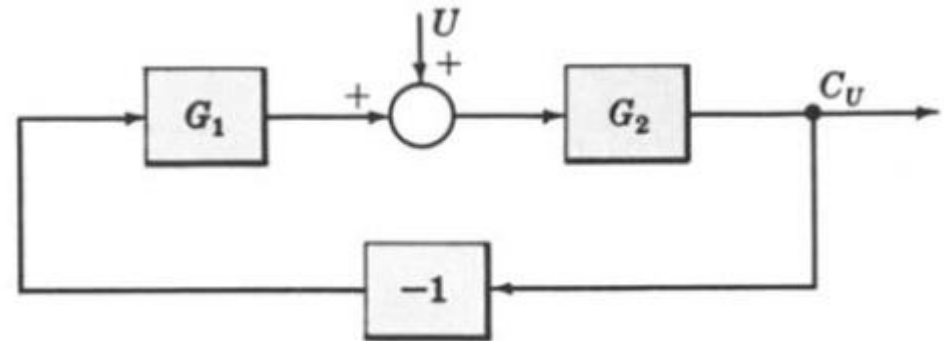
**Step 1:** Put  $U \equiv 0$ .

**Step 2:** The system reduces to



**Step 3:** the output  $C_R$  due to input  $R$  is  $C_R = [G_1G_2/(1 + G_1G_2)]R$ .

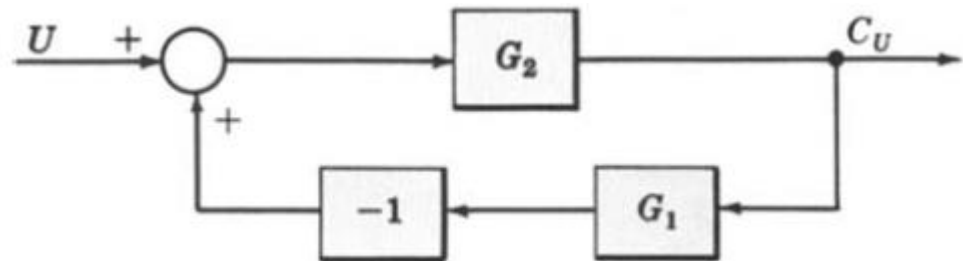
## Example-13: Continue.



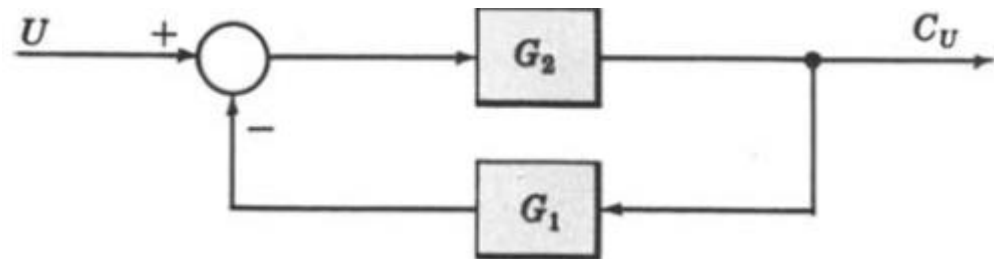
**Step 4a:** Put  $R = 0$ .

**Step 4b:** Put  $-1$  into a block, representing the negative feedback effect:

Rearrange the block diagram:



Let the  $-1$  block be absorbed into the summing point:



**Step 4c:** the output  $C_U$  due to input  $U$  is  $C_U = [G_2/(1 + G_1G_2)]U$ .



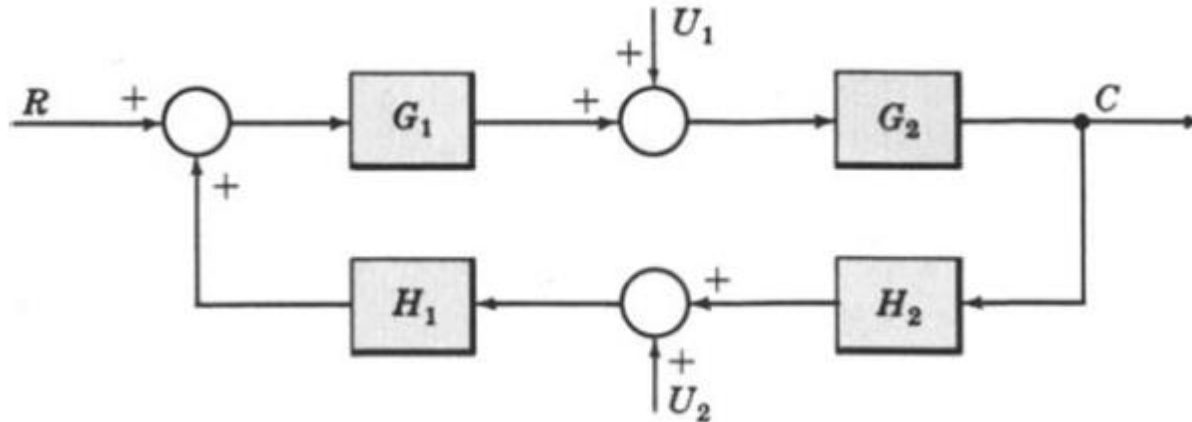
## Example-13: Continue.

**Step 5:** The total output is  $C = C_R + C_U$

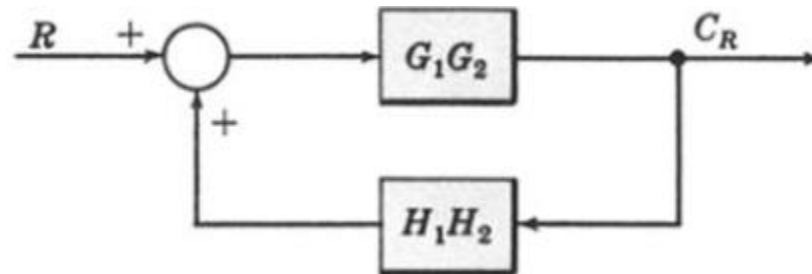
$$= \left[ \frac{G_1 G_2}{1 + G_1 G_2} \right] R + \left[ \frac{G_2}{1 + G_1 G_2} \right] U$$

$$= \left[ \frac{G_2}{1 + G_1 G_2} \right] [G_1 R + U]$$

Example-14: **Multiple-Input System**. Determine the output  $C$  due to inputs  $R$ ,  $U_1$  and  $U_2$  using the Superposition Method.



Let  $U_1 = U_2 = 0$ .

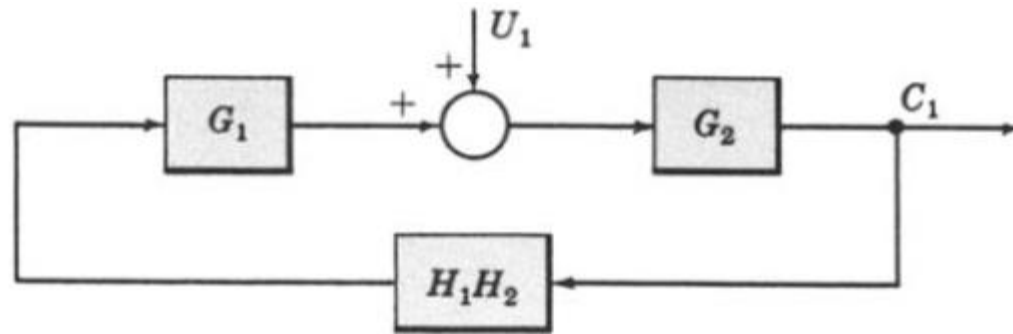


$$C_R = [G_1 G_2 / (1 - G_1 G_2 H_1 H_2)] R$$

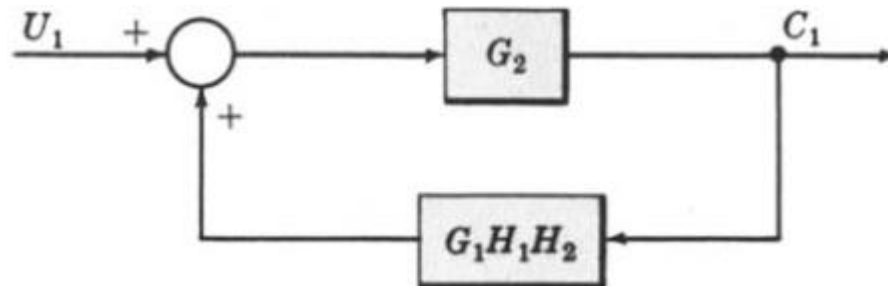
where  $C_R$  is the output due to  $R$  acting alone.

## Example-14: Continue.

Now let  $R = U_2 = 0$ .



Rearranging the blocks, we get

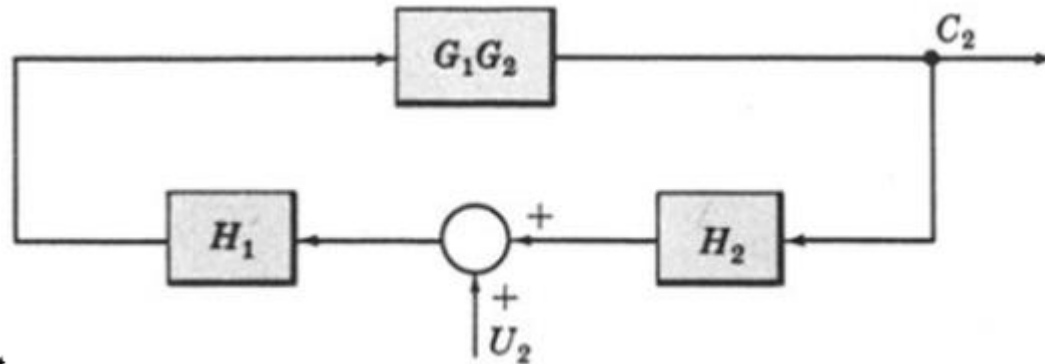


$$C_1 = [G_2 / (1 - G_1 G_2 H_1 H_2)] U_1$$

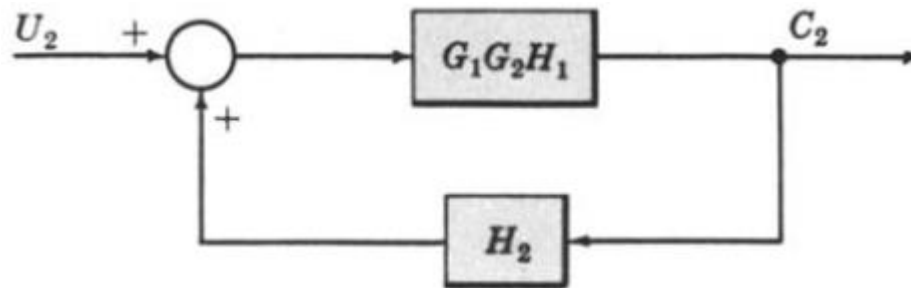
where  $C_1$  is the response due to  $U_1$  acting alone.

## Example-14: Continue.

Finally, let  $R = U_1 = 0$ .



Rearranging the blocks, we get



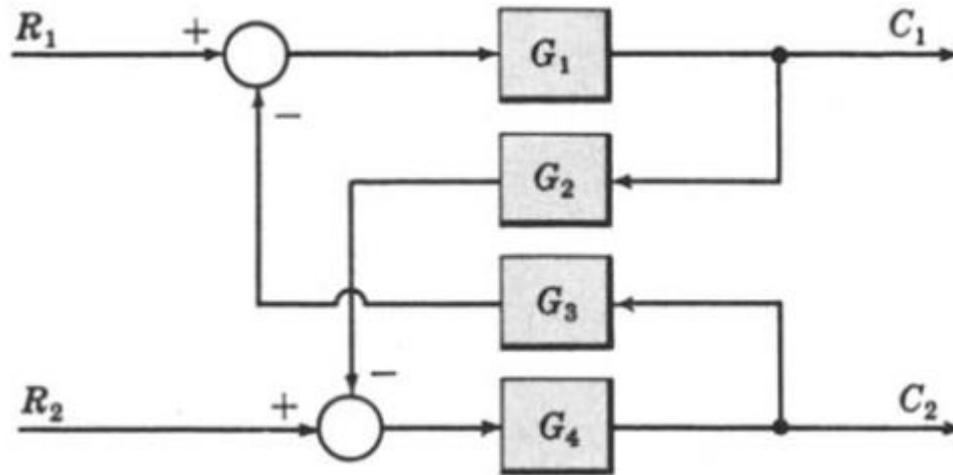
$$C_2 = [G_1 G_2 H_1 / (1 - G_1 G_2 H_1 H_2)] U_2$$

where  $C_2$  is the response due to  $U_2$  acting alone.

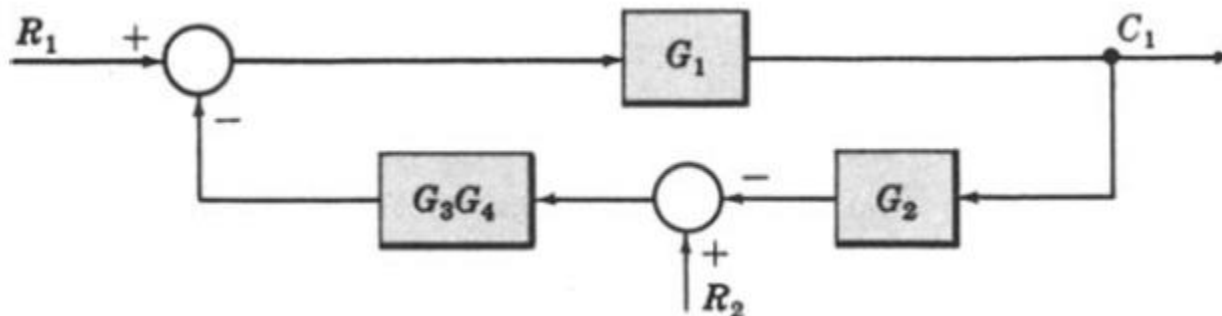
By superposition, the total output is

$$C = C_R + C_1 + C_2 = \frac{G_1 G_2 R + G_2 U_1 + G_1 G_2 H_1 U_2}{1 - G_1 G_2 H_1 H_2}$$

Example-15: **Multi-Input Multi-Output System**. Determine  $C_1$  and  $C_2$  due to  $R_1$  and  $R_2$ .

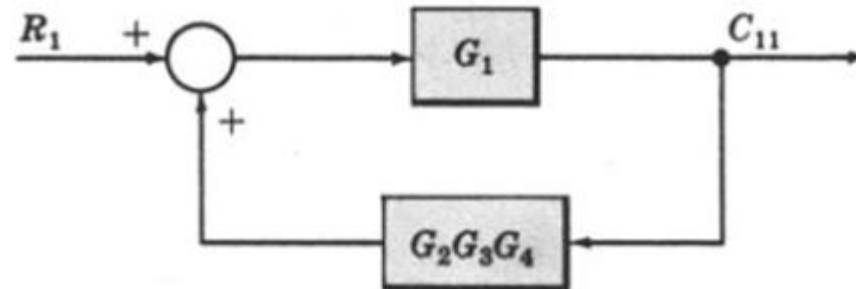


First ignoring the output  $C_2$ .



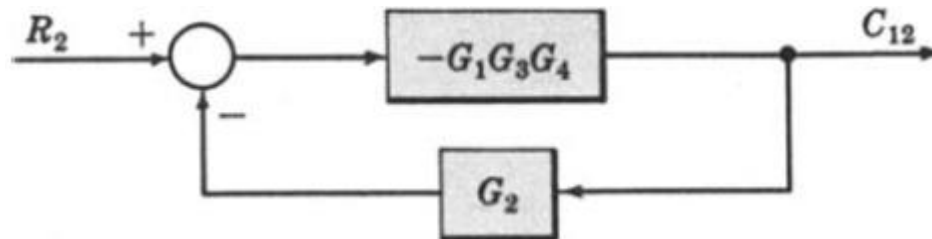
## Example-15: Continue.

Letting  $R_2 = 0$  and combining the summing points,



Hence  $C_{11}$ , the output at  $C_1$  due to  $R_1$  alone, is  $C_{11} = G_1 R_1 / (1 - G_1 G_2 G_3 G_4)$ .

For  $R_1 = 0$ ,

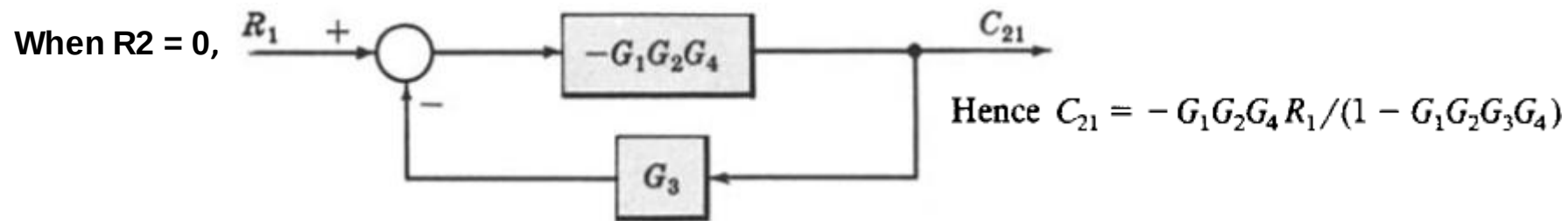
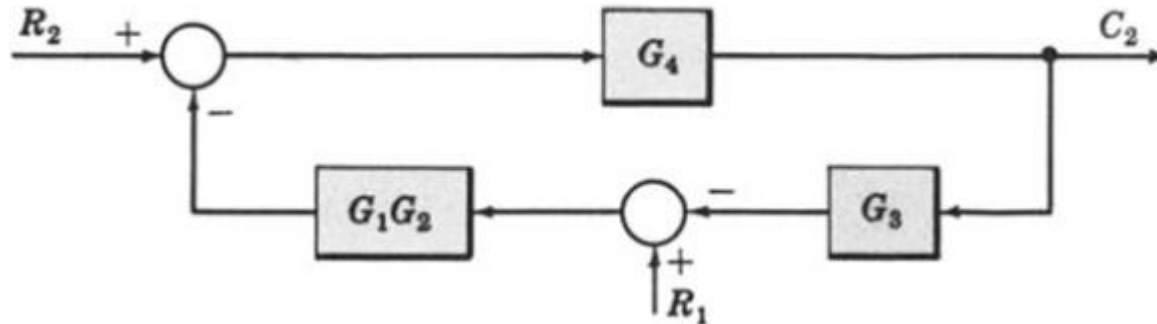


Hence  $C_{12} = -G_1 G_3 G_4 R_2 / (1 - G_1 G_2 G_3 G_4)$  is the output at  $C_1$  due to  $R_2$  alone.

$$\text{Thus } C_1 = C_{11} + C_{12} = (G_1 R_1 - G_1 G_3 G_4 R_2) / (1 - G_1 G_2 G_3 G_4)$$

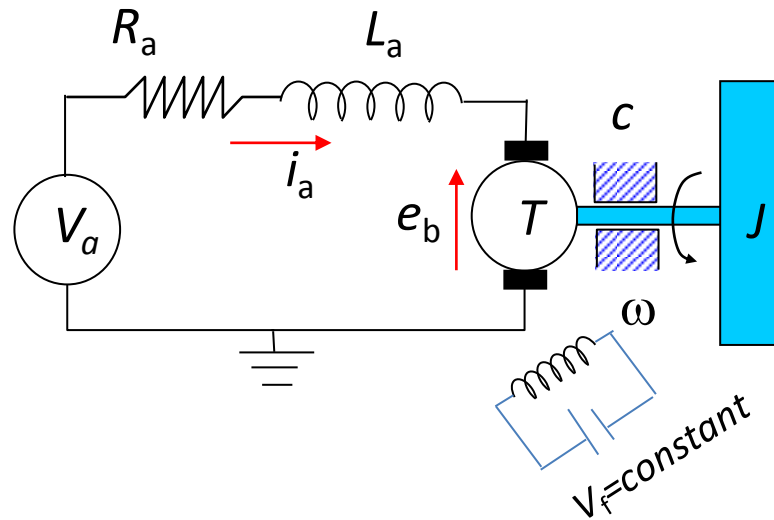
## Example-15: Continue.

Now we reduce the original block diagram, ignoring output  $C_1$ .



Finally,  $C_2 = C_{22} + C_{21} = (G_4 R_2 - G_1 G_2 G_4 R_1) / (1 - G_1 G_2 G_3 G_4)$

# Block Diagram of Armature Controlled D.C Motor



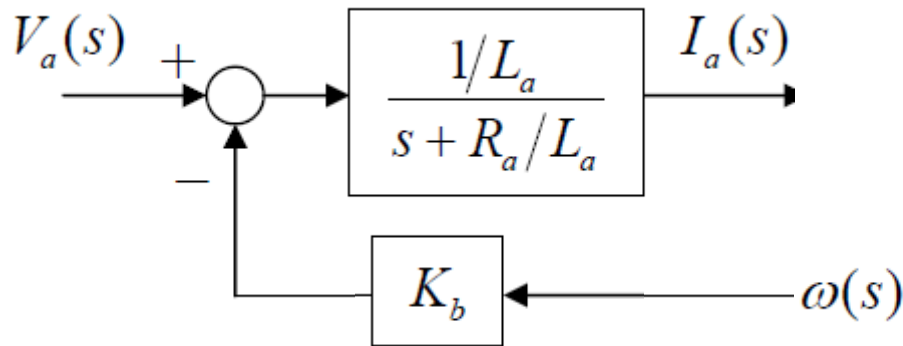
$$(L_a s + R_a)I_a(s) + K_b \omega(s) = V_a(s)$$

$$(Js + c)\omega(s) = K_{ma}I_a(s)$$



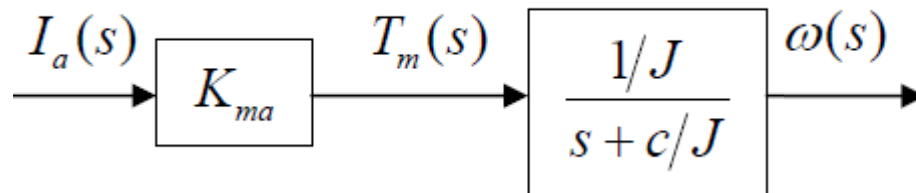
# Block Diagram of Armature Controlled D.C Motor

$$(L_a s + R_a) I_a(s) + K_b \omega(s) = V_a(s)$$

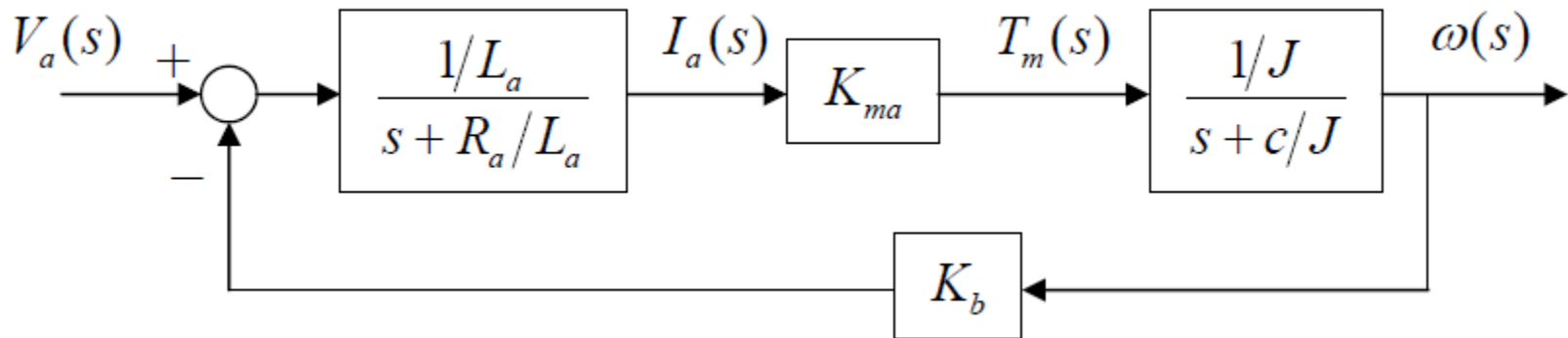
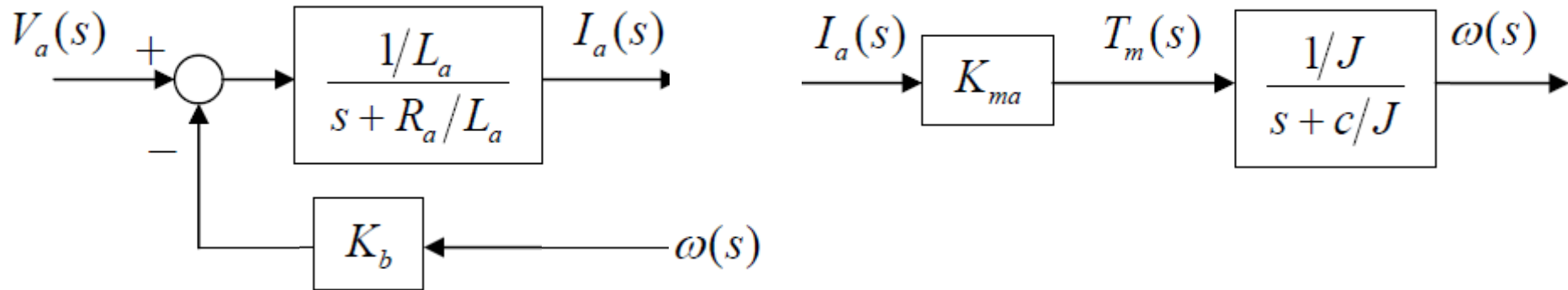


# Block Diagram of Armature Controlled D.C Motor

$$(Js + c)\omega(s) = K_{ma}I_a(s)$$



# Block Diagram of Armature Controlled D.C Motor



# Control Systems (CS)

Lecture-4

مخطط سريان الإشارة

Signal Flow Graphs

# Outline

- Introduction to Signal Flow Graphs
  - Definitions
  - Terminologies
  - Examples
- Mason's Gain Formula
  - Examples
- Signal Flow Graph from Block Diagrams
- Design Examples

# Introduction

- Alternative method to block diagram representation, developed by Samuel Jefferson Mason.
- Advantage: the availability of a flow graph gain formula, also called Mason's gain formula.
- A signal-flow graph consists of a network in which nodes are connected by directed branches.
- It depicts the flow of signals from one point of a system to another and gives the relationships among the signals.

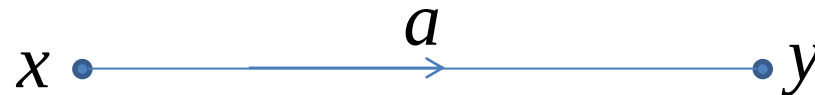
# Fundamentals of Signal Flow Graphs

## المفاهيم الأساسية لسريان الإشارة

- Consider a simple equation below and draw its signal flow graph:

$$y = ax$$

- The signal flow graph of the equation is shown below;

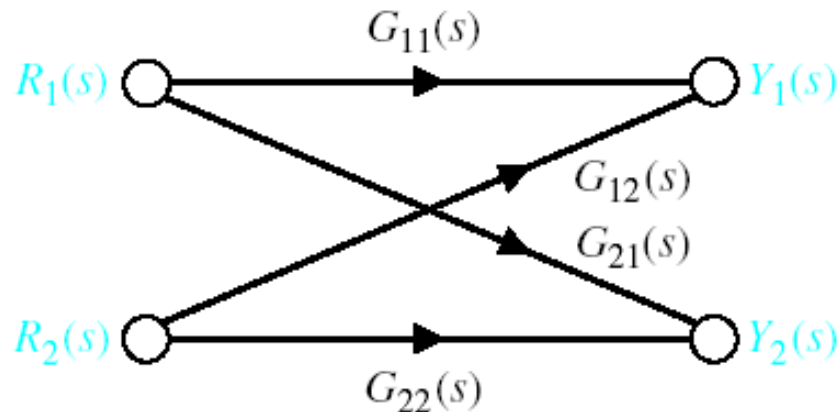


- Every variable in a signal flow graph is designed by a **Node**. عقدة
- Every transmission function in a signal flow graph is designed by a **Branch**. فرع
- Branches are always **unidirectional**. اتجاه واحد
- The arrow in the branch denotes the **direction** of the signal flow.

# Signal-Flow Graph Models

$$Y_1(s) = G_{11}(s) \cdot R_1(s) + G_{12}(s) \cdot R_2(s)$$

$$Y_2(s) = G_{21}(s) \cdot R_1(s) + G_{22}(s) \cdot R_2(s)$$



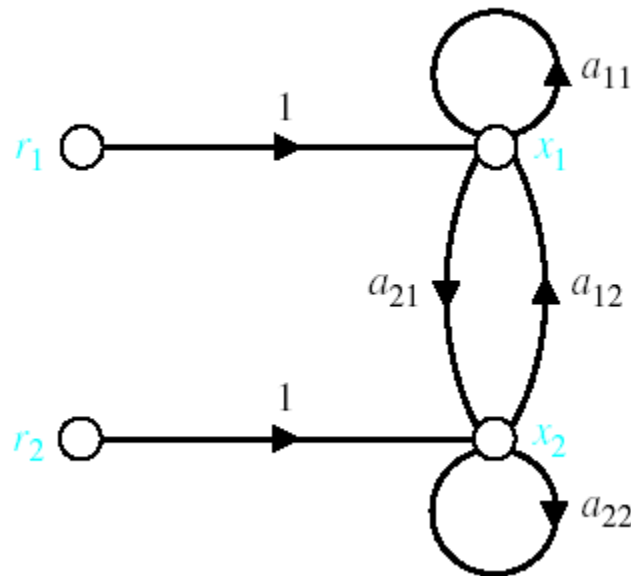


# Signal-Flow Graph Models

$r_1$  and  $r_2$  are inputs and  $x_1$  and  $x_2$  are outputs

$$a_{11} \cdot x_1 + a_{12} \cdot x_2 + r_1 = x_1$$

$$a_{21} \cdot x_1 + a_{22} \cdot x_2 + r_2 = x_2$$



# Signal-Flow Graph Models

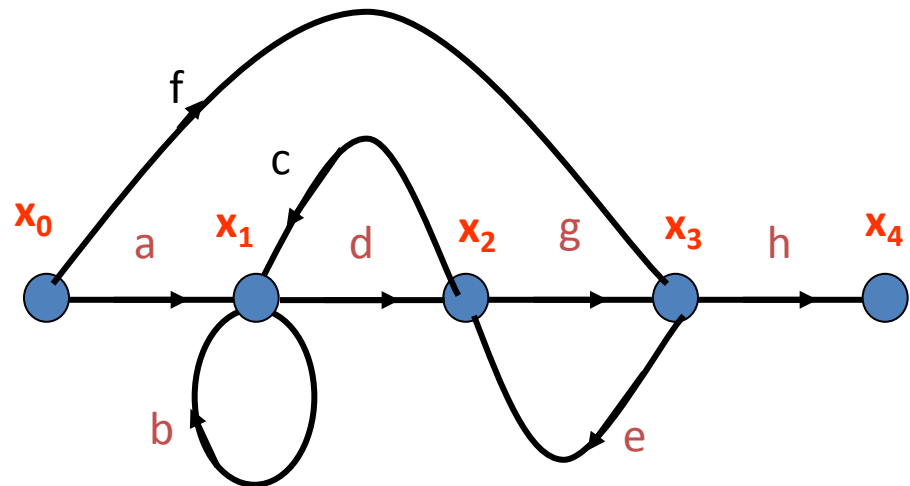
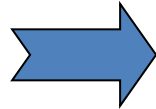
$x_0$  is input and  $x_4$  is output

$$x_1 = ax_0 + bx_1 + cx_2$$

$$x_2 = dx_1 + ex_3$$

$$x_3 = fx_0 + gx_2$$

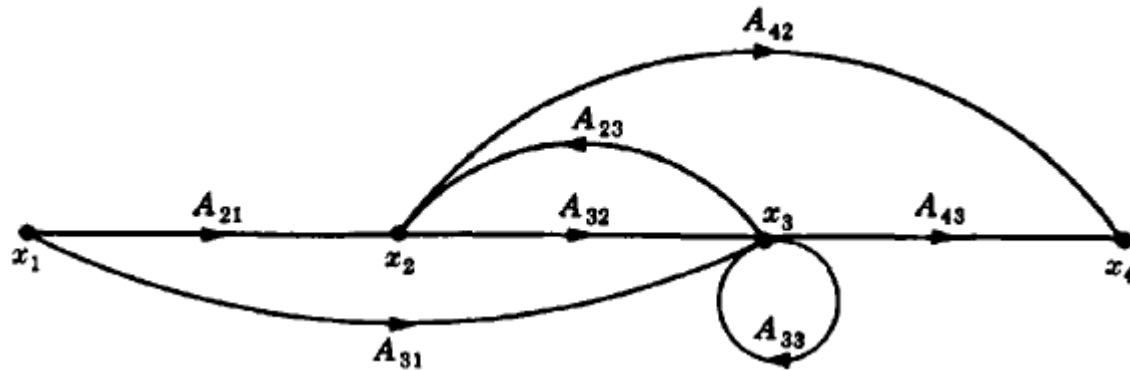
$$x_4 = hx_3$$



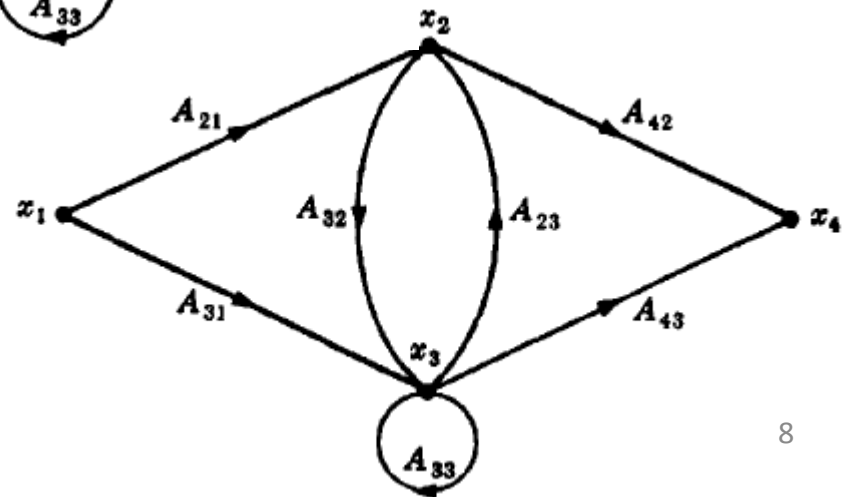
Construct the signal flow graph for the following set of simultaneous equations.

$$x_2 = A_{21}x_1 + A_{23}x_3 \quad x_3 = A_{31}x_1 + A_{32}x_2 + A_{33}x_3 \quad x_4 = A_{42}x_2 + A_{43}x_3$$

- There are four variables in the equations (i.e.,  $x_1, x_2, x_3$ , and  $x_4$ ) therefore four nodes are required to construct the signal flow graph.
- Arrange these four nodes from left to right and connect them with the associated branches.



- Another way to arrange this graph is shown in the figure.



# Terminologies

- An **input node** or source contain only the outgoing branches. i.e.,  $X_1$
- An **output node** or sink contain only the incoming branches. i.e.,  $X_4$
- A **path** is a continuous, unidirectional succession of branches along which no node is passed more than ones. i.e.,

$X_1$  to  $X_2$  to  $X_3$  to  $X_4$

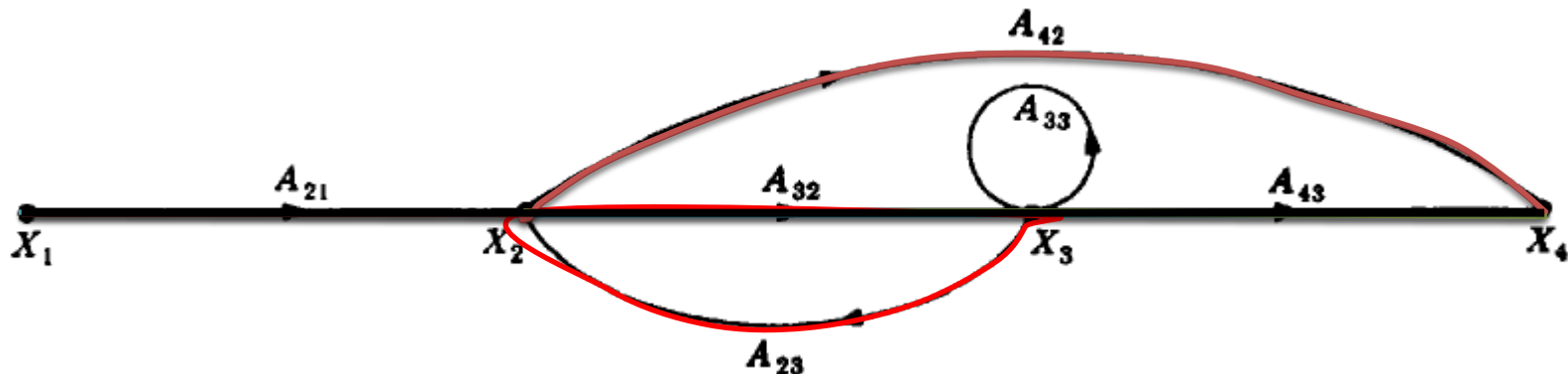
$X_1$  to  $X_2$  to  $X_4$

$X_2$  to  $X_3$  to  $X_4$

- A **forward path** is a path from the input node to the output node. i.e.,

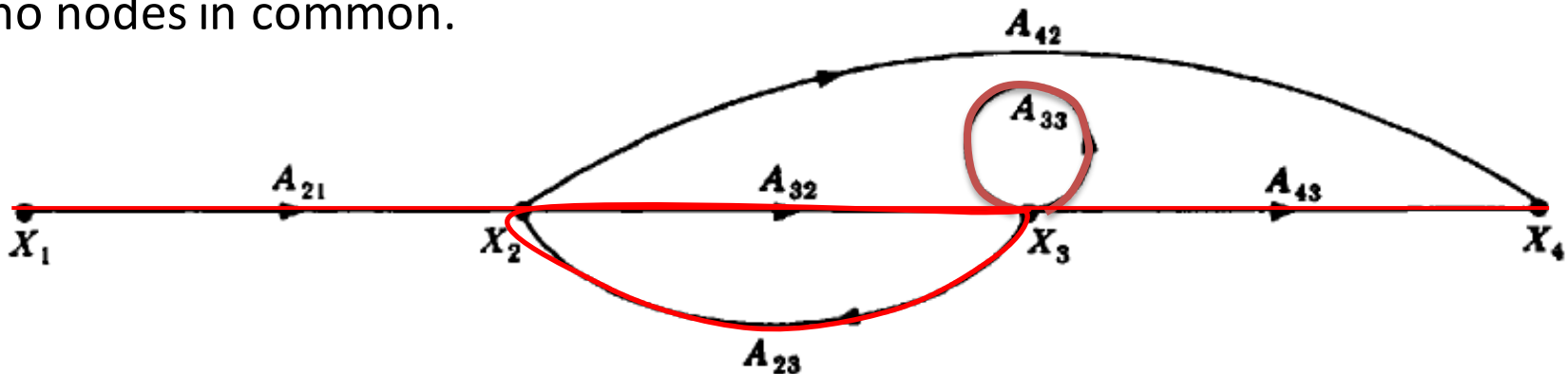
$X_1$  to  $X_2$  to  $X_3$  to  $X_4$ , and  $X_1$  to  $X_2$  to  $X_4$ , are forward paths.

- A **feedback path** or feedback loop is a path which originates and terminates on the same node. i.e.;  $X_2$  to  $X_3$  and back to  $X_2$  is a feedback path.

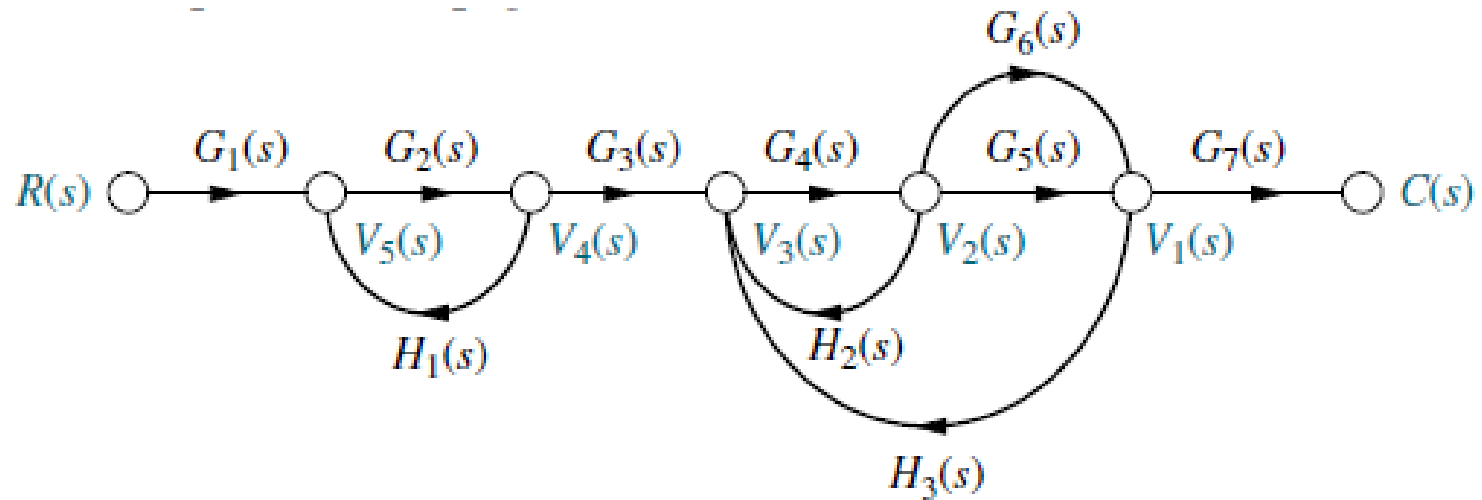


# Terminologies

- A **self-loop** is a feedback loop consisting of a single branch. i.e.;  $A_{33}$  is a self loop.
- The **gain** of a branch is the transmission function of that branch.
- The **path gain** is the product of branch gains encountered in traversing a path. i.e. the gain of forwards path  $X_1$  to  $X_2$  to  $X_3$  to  $X_4$  is  $A_{21}A_{32}A_{43}$
- The **loop gain** is the product of the branch gains of the loop. i.e., the loop gain of the feedback loop from  $X_2$  to  $X_3$  and back to  $X_2$  is  $A_{32}A_{23}$ .
- Two loops, paths, or loop and a path are said to be **non-touching** if they have no nodes in common.

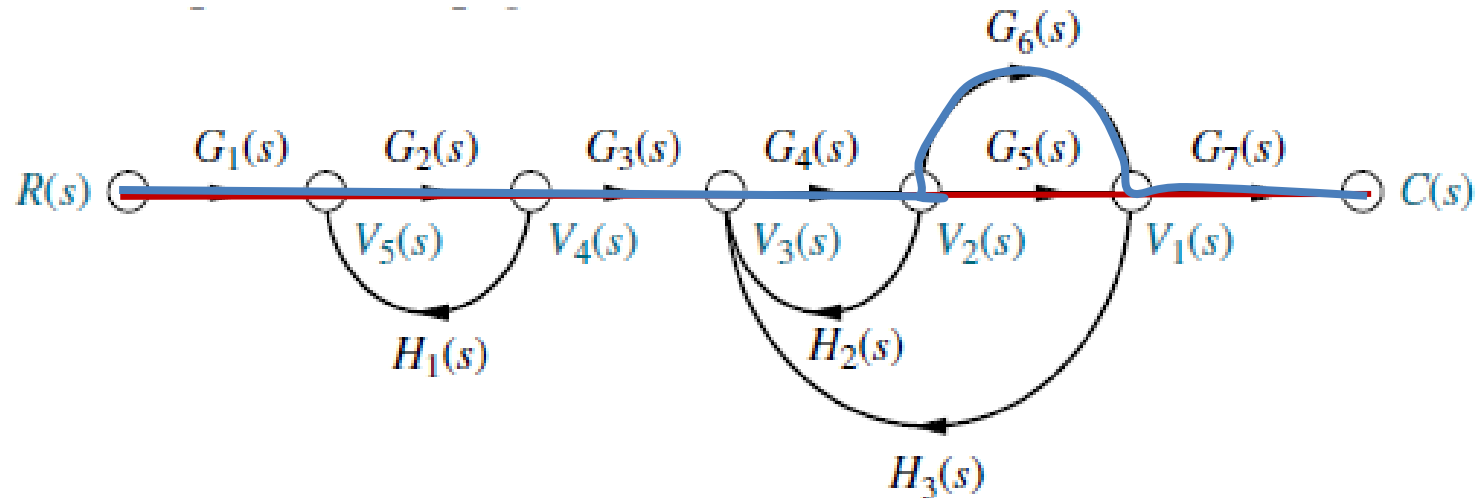


Consider the signal flow graph below and identify the following



- Input node.
- Output node.
- Forward paths.
- Feedback paths (loops).
- Determine the loop gains of the feedback loops.
- Determine the path gains of the forward paths.
- Non-touching loops

Consider the signal flow graph below and identify the following



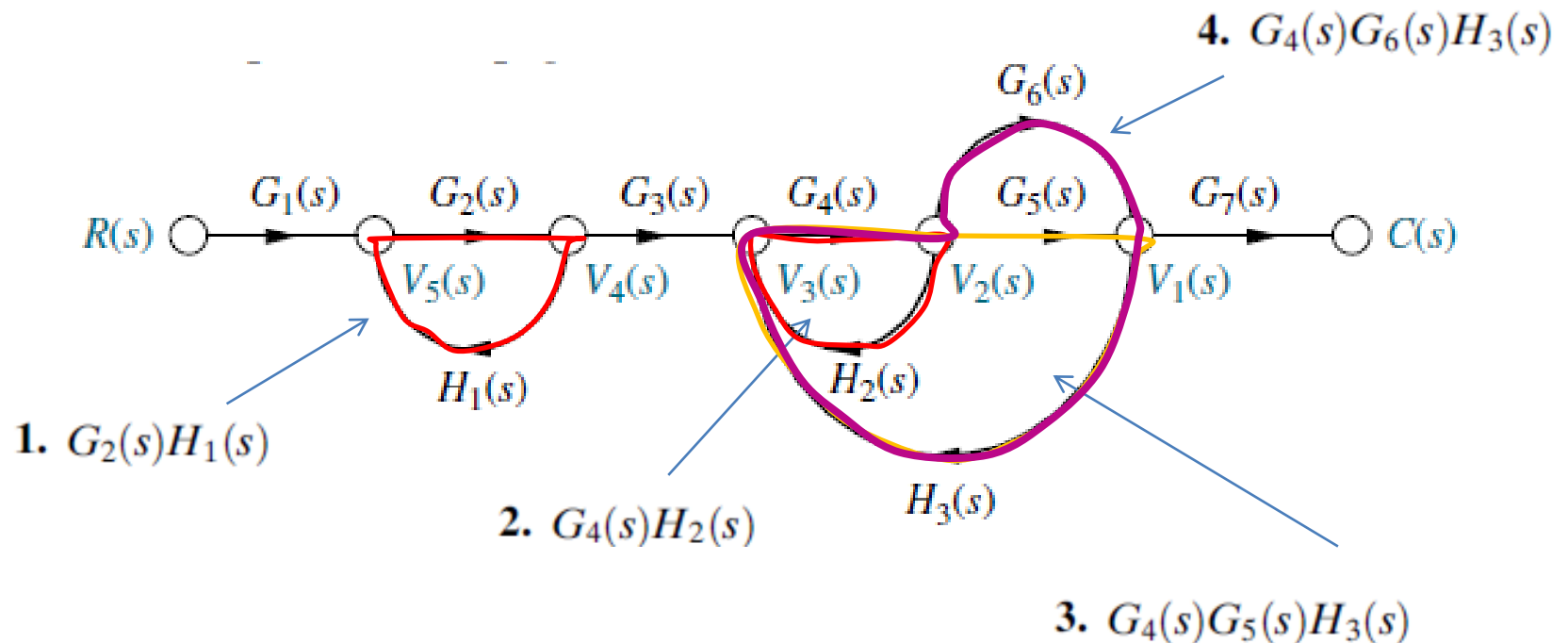
- There are two forward path gains;

1.  $G_1(s)G_2(s)G_3(s)G_4(s)G_5(s)G_7(s)$

2.  $G_1(s)G_2(s)G_3(s)G_4(s)G_6(s)G_7(s)$

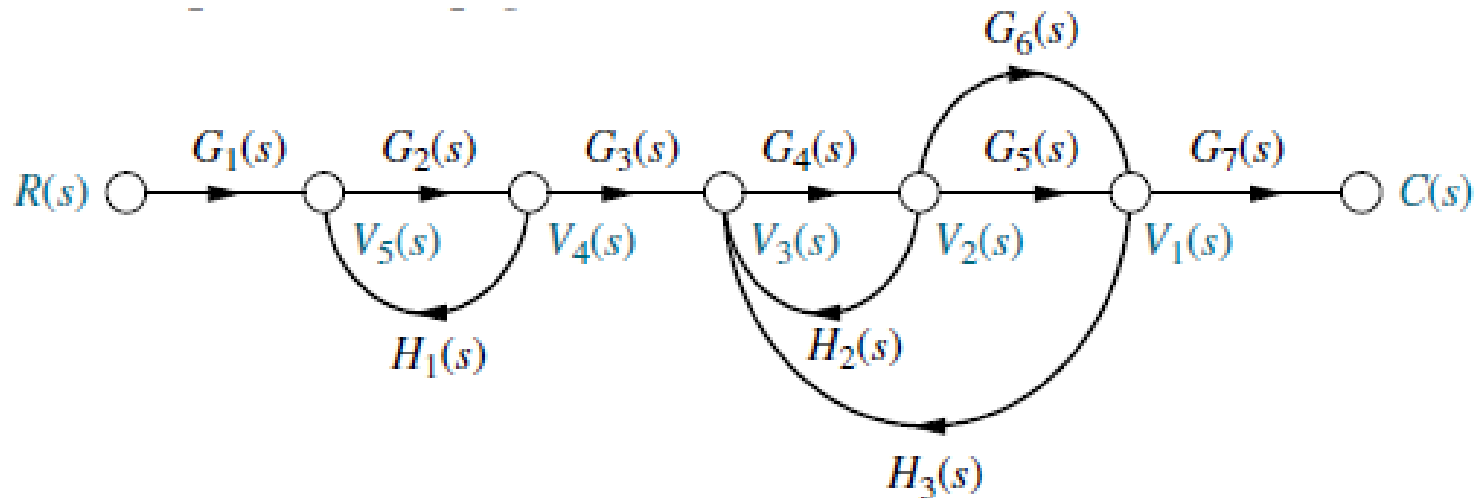
Consider the signal flow graph below and identify the following

- There are four loops





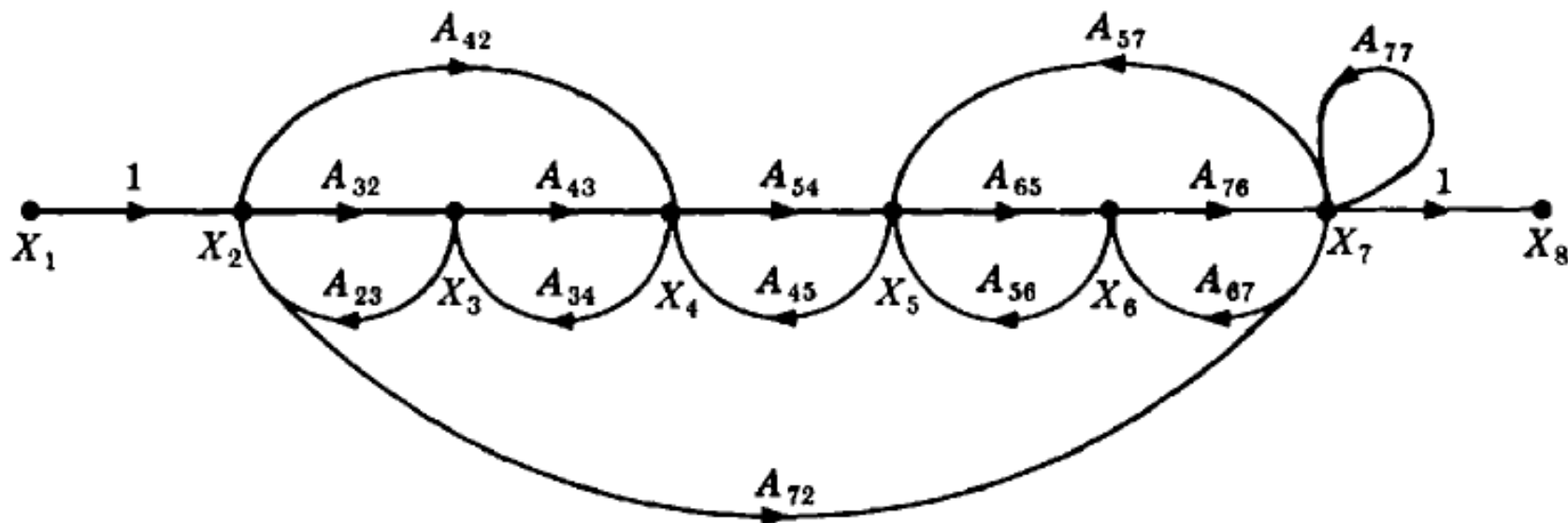
Consider the signal flow graph below and identify the following



- Nontouching loop gains;

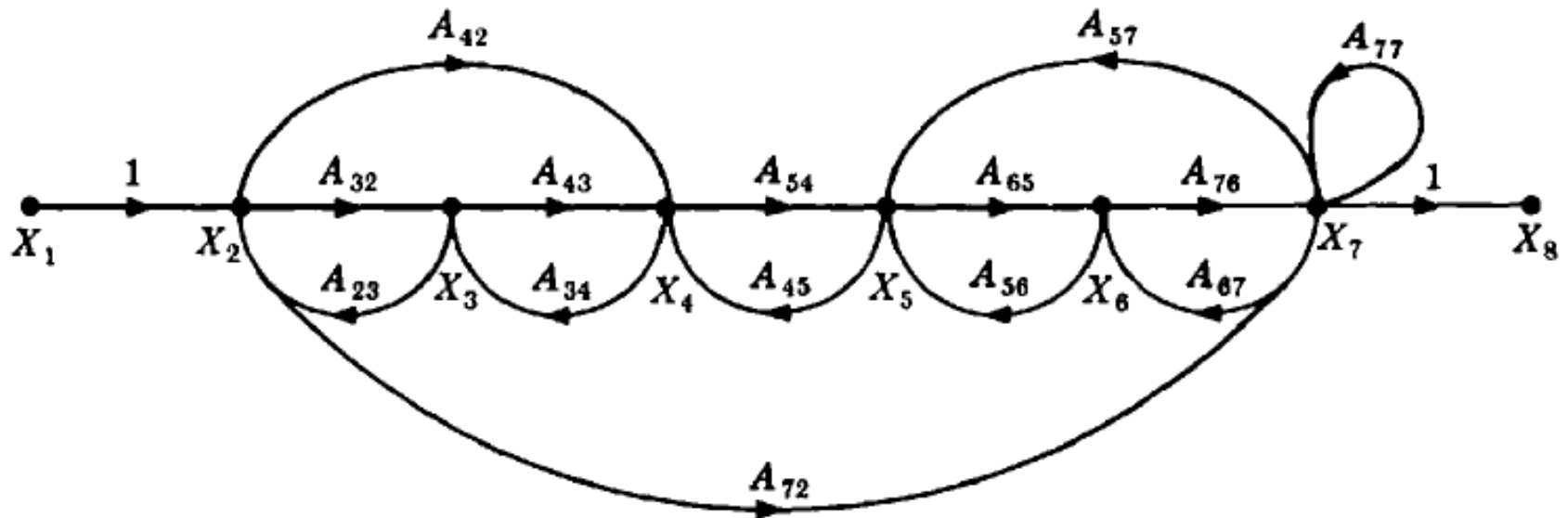
1.  $[G_2(s)H_1(s)][G_4(s)H_2(s)]$
2.  $[G_2(s)H_1(s)][G_4(s)G_5(s)H_3(s)]$
3.  $[G_2(s)H_1(s)][G_4(s)G_6(s)H_3(s)]$

Consider the signal flow graph below and identify the following



- Input node.
- Output node.
- Forward paths.
- Feedback paths.
- Self loop.
- Determine the loop gains of the feedback loops.
- Determine the path gains of the forward paths.

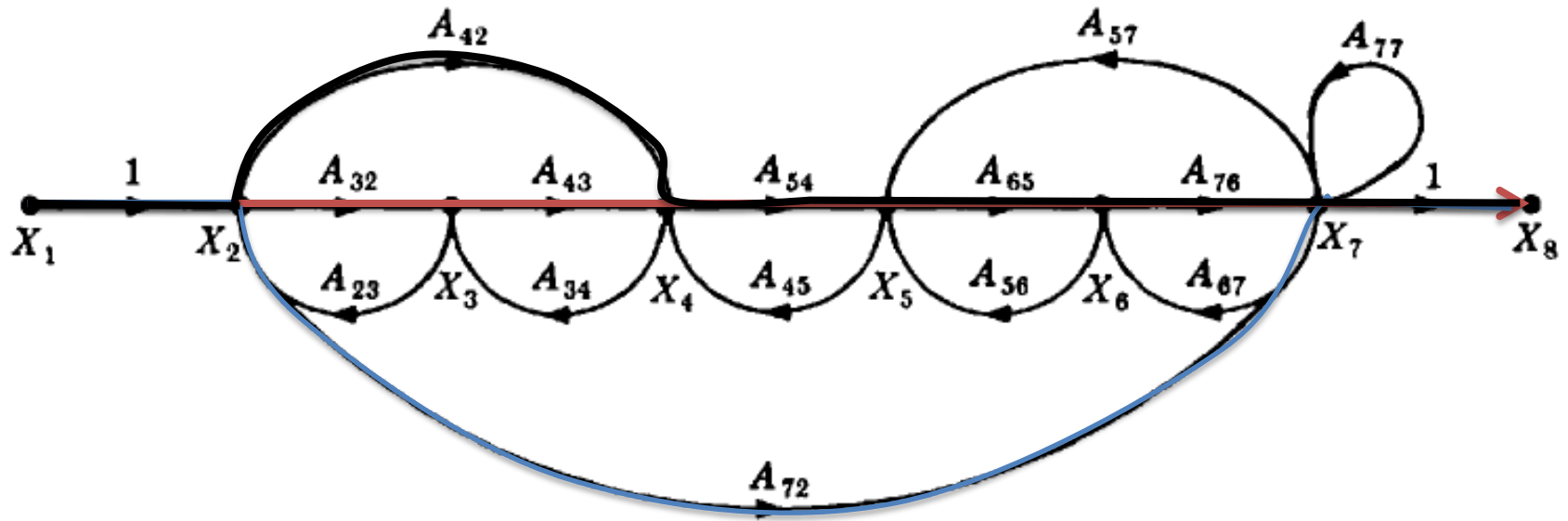
# Input and output Nodes



a) Input node  $X_1$

b) Output node  $X_8$

### (c) Forward Paths

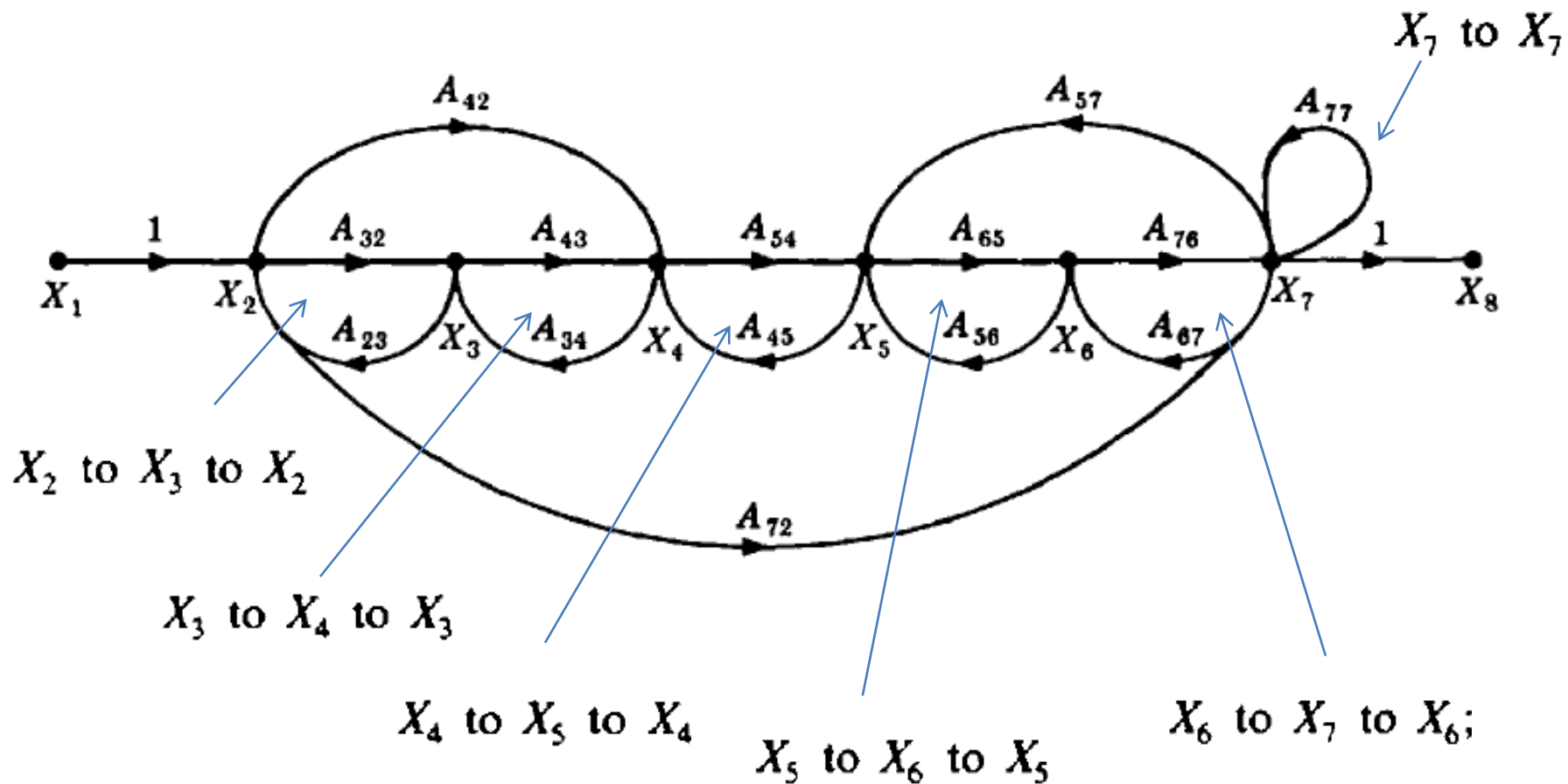


$X_1$  to  $X_2$  to  $X_3$  to  $X_4$  to  $X_5$  to  $X_6$  to  $X_7$  to  $X_8$

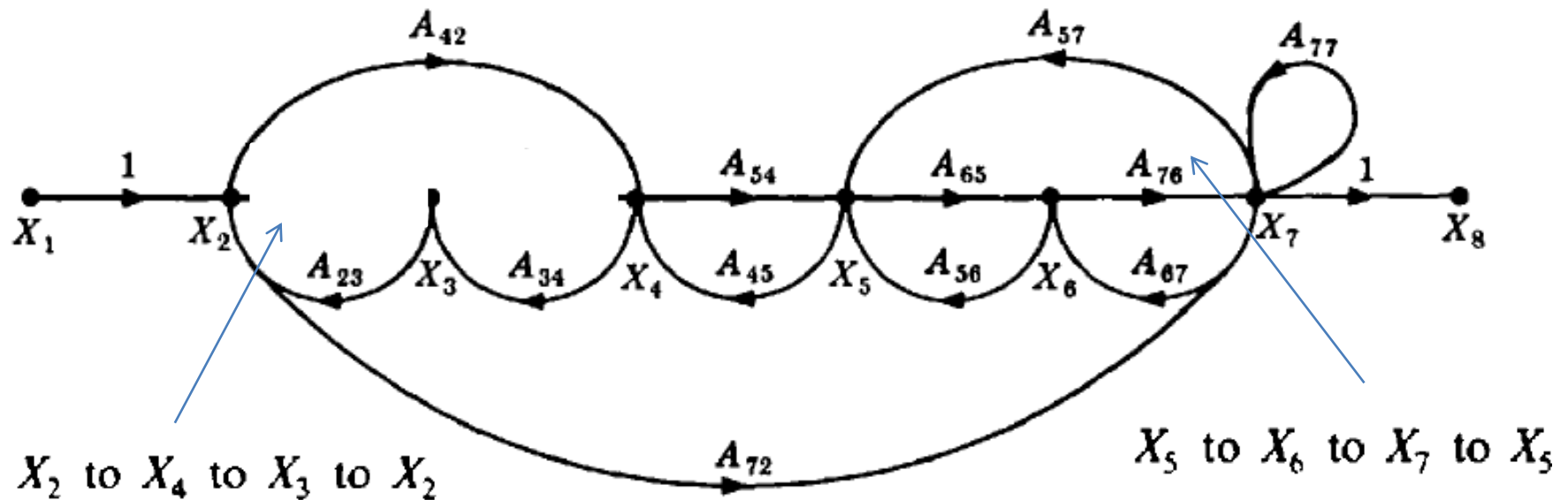
$X_1$  to  $X_2$  to  $X_7$  to  $X_8$

$X_1$  to  $X_2$  to  $X_4$  to  $X_5$  to  $X_6$  to  $X_7$  to  $X_8$

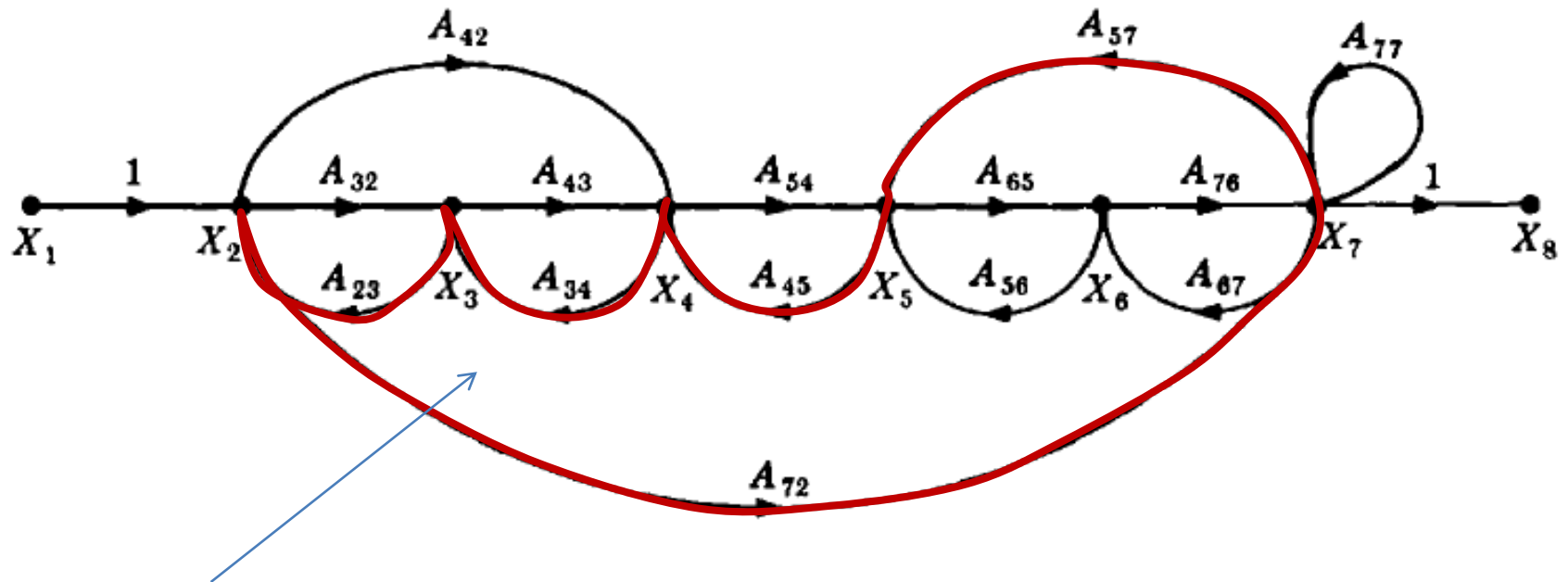
# (d) Feedback Paths or Loops



# (d) Feedback Paths or Loops

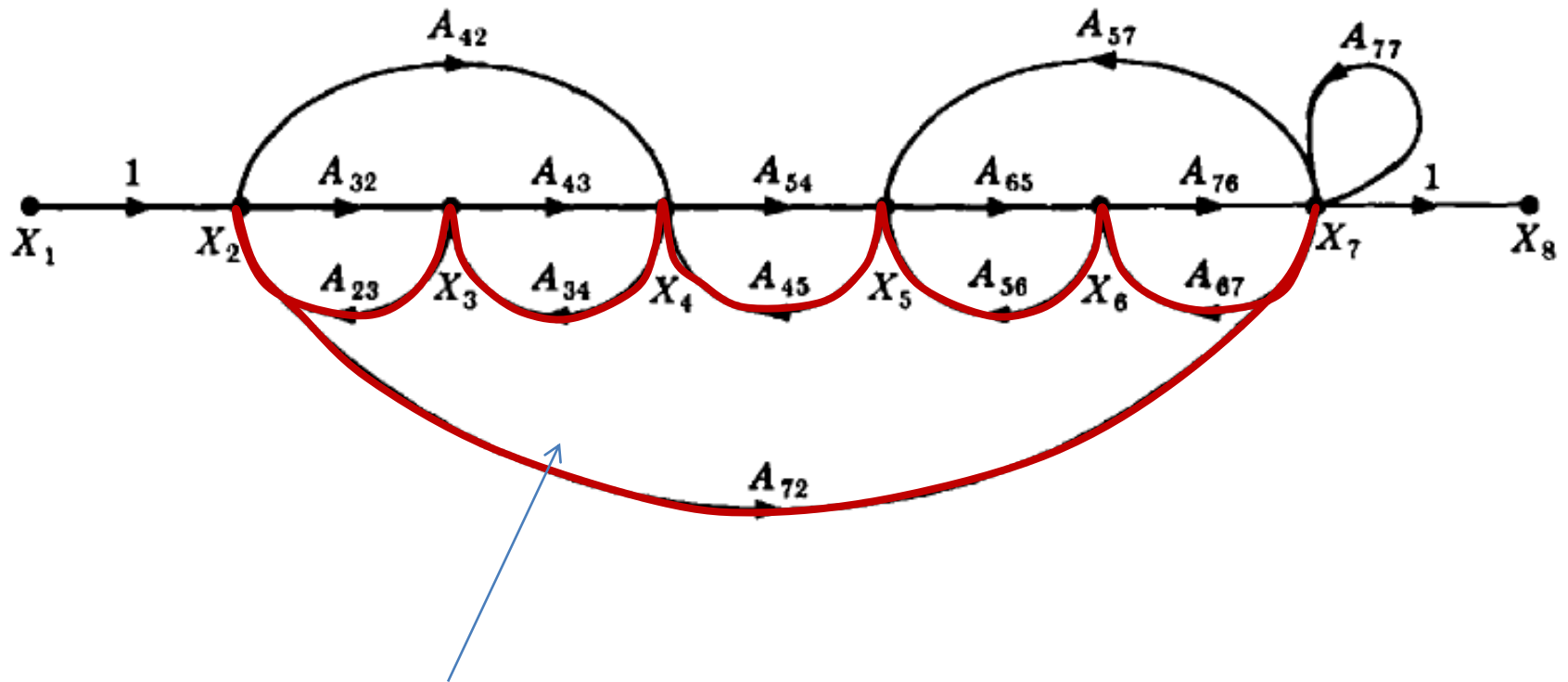


# (d) Feedback Paths or Loops



$X_2$  to  $X_7$  to  $X_5$  to  $X_4$  to  $X_3$  to  $X_2$

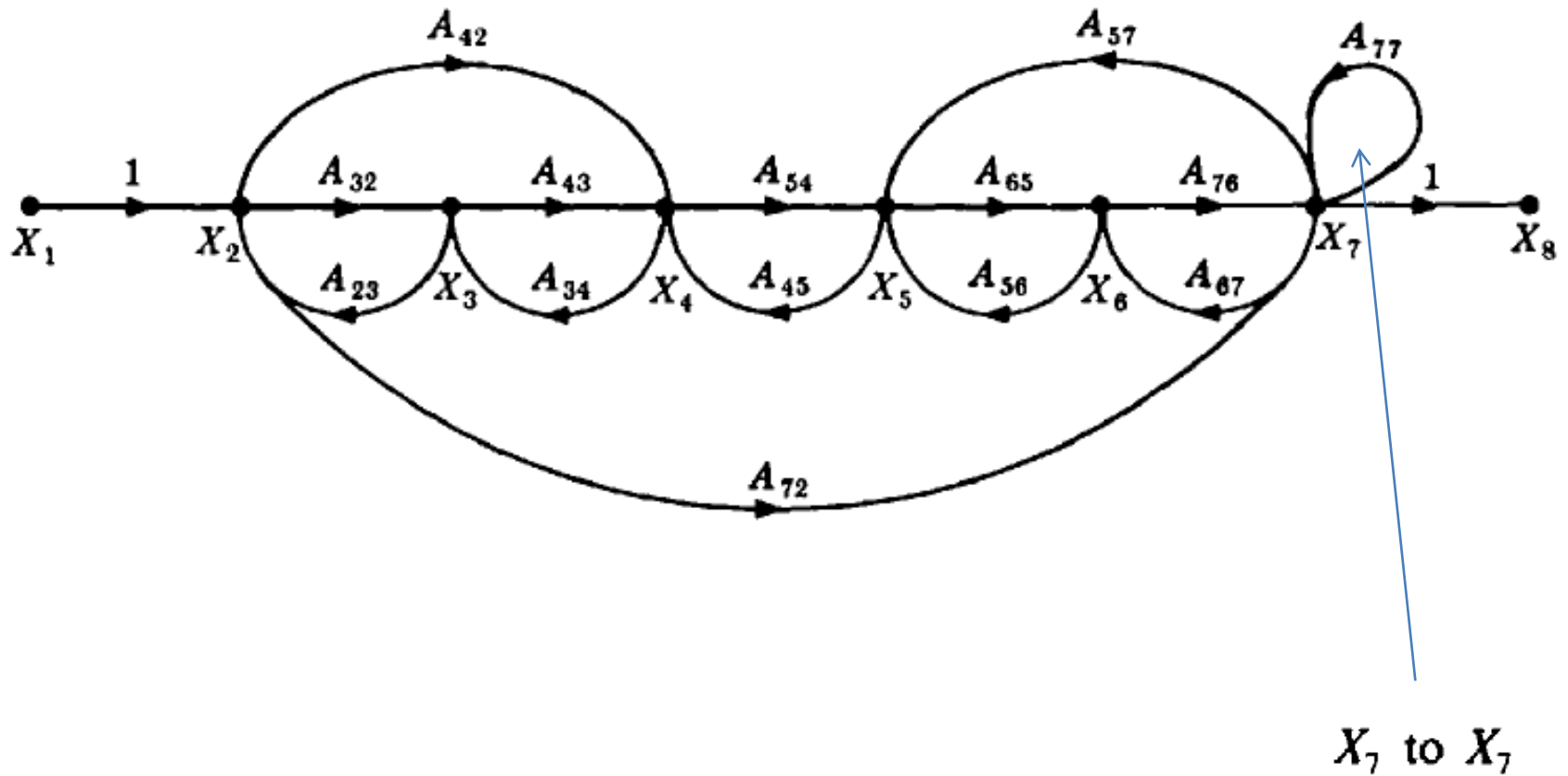
# (d) Feedback Paths or Loops



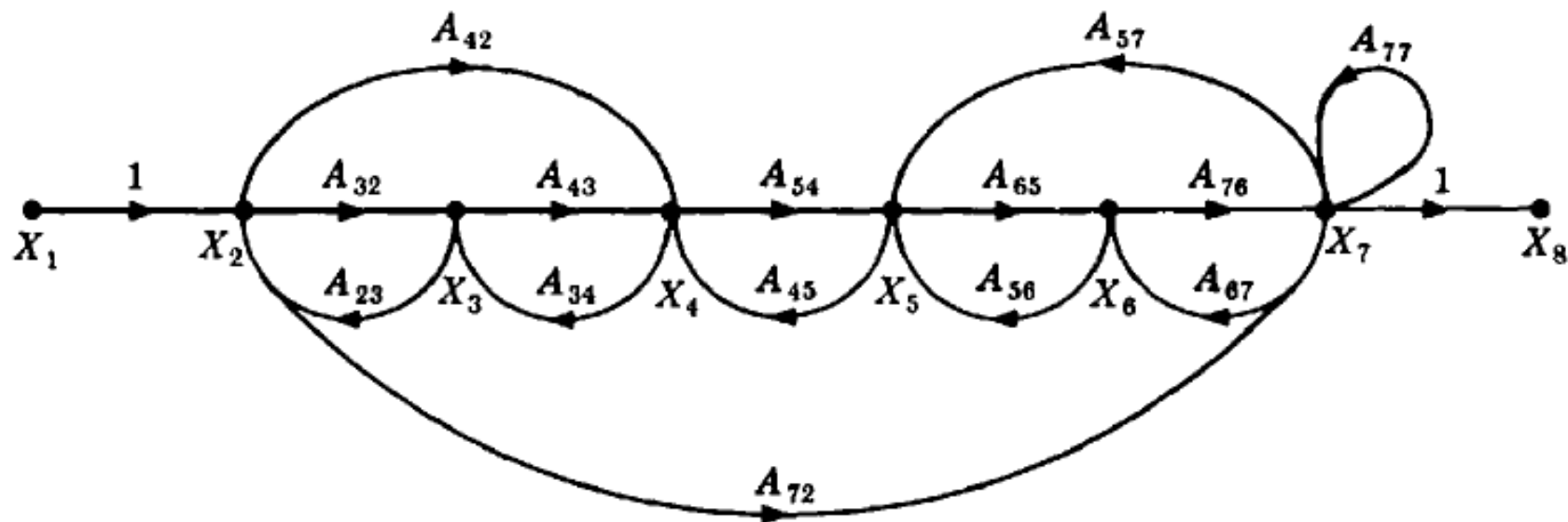
$X_2$  to  $X_7$  to  $X_6$  to  $X_5$  to  $X_4$  to  $X_3$  to  $X_2$



(e) Self Loop(s)



# (f) Loop Gains of the Feedback Loops



$$A_{32} A_{23}$$

$$A_{76} A_{67}$$

$$A_{72} A_{57} A_{45} A_{34} A_{23}$$

$$A_{43} A_{34}$$

$$A_{65} A_{76} A_{57}$$

$$A_{72} A_{67} A_{56} A_{45} A_{34} A_{23}$$

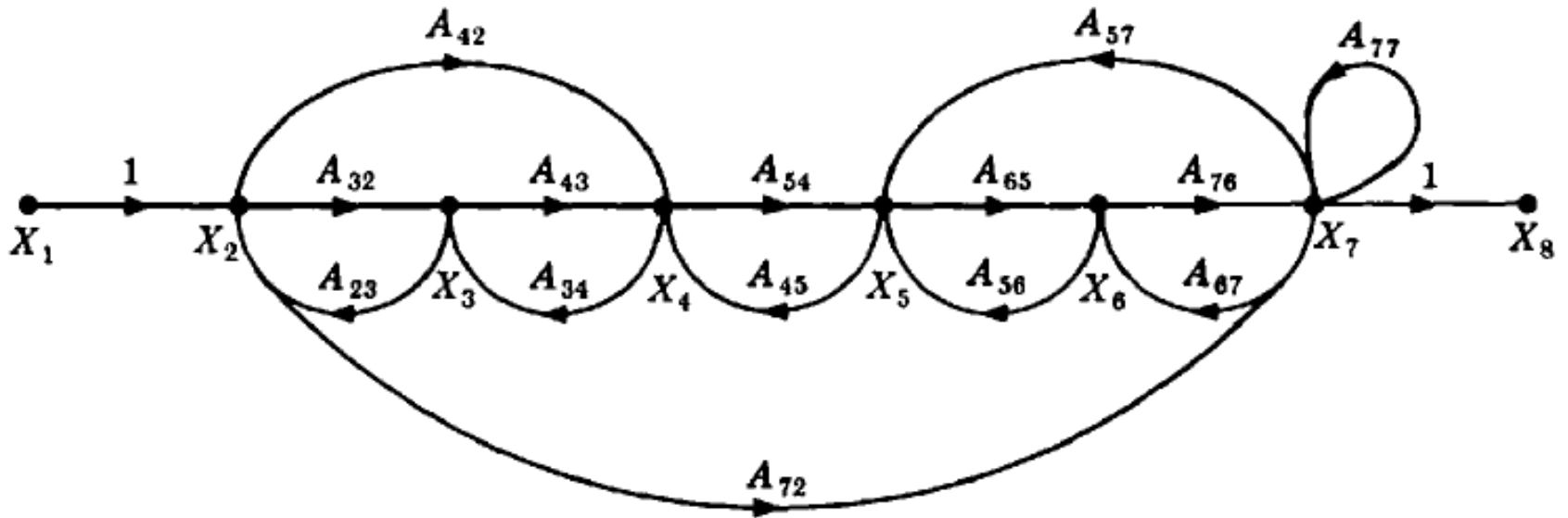
$$A_{54} A_{45}$$

$$A_{77}$$

$$A_{65} A_{56}$$

$$A_{42} A_{34} A_{23}$$

### (g) Path Gains of the Forward Paths



$$A_{32} A_{43} A_{54} A_{65} A_{76}$$

***A*<sub>72</sub>**

$$A_{42} A_{54} A_{65} A_{76}$$

# Mason's Rule (Mason, 1953)

- The block diagram reduction technique requires successive application of fundamental relationships in order to arrive at the system transfer function.
- On the other hand, Mason's rule for reducing a signal-flow graph to a single transfer function requires the application of one formula.
- The formula was derived by S. J. Mason when he related the signal-flow graph to the simultaneous equations that can be written from the graph.

# Mason's Rule:

- The transfer function,  $C(s)/R(s)$ , of a system represented by a signal-flow graph is;

$$\frac{C(s)}{R(s)} = \frac{\sum_{i=1}^n P_i \Delta_i}{\Delta}$$

Where

$n$  = number of forward paths.

$P_i$  = the  $i^{\text{th}}$  forward-path gain.

$\Delta$  = Determinant of the system

$\Delta_i$  = Determinant of the  $i^{\text{th}}$  forward path

- $\Delta$  is called the signal flow graph determinant or characteristic function. Since  $\Delta=0$  is the system characteristic equation.

# Mason's Rule:

$$\frac{C(s)}{R(s)} = \frac{\sum_{i=1}^n P_i \Delta_i}{\Delta}$$

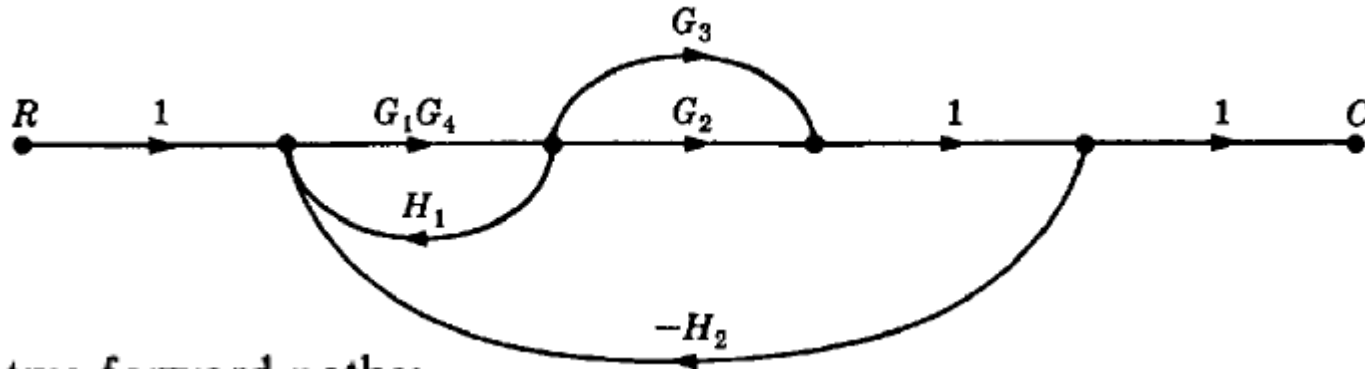
$\Delta = 1 -$  (sum of all individual loop gains) + (sum of the products of the gains of all possible two loops that do not touch each other) – (sum of the products of the gains of all possible three loops that do not touch each other) + ... and so forth with sums of higher number of non-touching loop gains

$\Delta_i$  = value of  $\Delta$  for the part of the block diagram that does not touch the  $i$ -th forward path ( $\Delta_i = 1$  if there are no non-touching loops to the  $i$ -th path.)

# Systematic approach

1. Calculate forward path gain  $P_i$  for each forward path  $i$ .
2. Calculate all loop transfer functions
3. Consider non-touching loops 2 at a time
4. Consider non-touching loops 3 at a time
5. etc
6. Calculate  $\Delta$  from steps 2,3,4 and 5
7. Calculate  $\Delta_i$  as portion of  $\Delta$  not touching forward path  $i$

Example#1: Apply Mason's Rule to calculate the transfer function of the system represented by following Signal Flow Graph



There are two forward paths:

$$P_1 = G_1 G_2 G_4 \quad P_2 = G_1 G_3 G_4$$

Therefore,

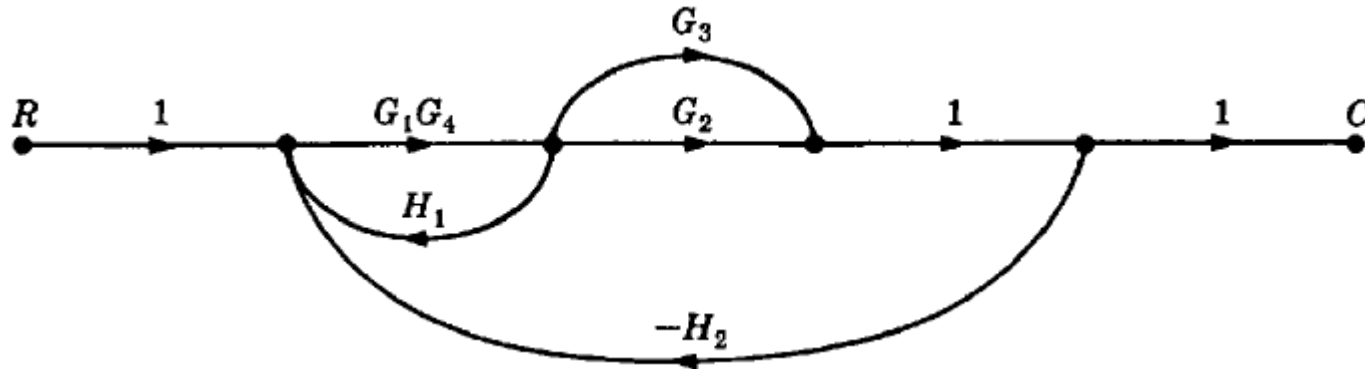
$$\frac{C}{R} = \frac{P_1 \Delta_1 + P_2 \Delta_2}{\Delta}$$

There are three feedback loops

$$L_1 = G_1 G_4 H_1, \quad L_2 = -G_1 G_2 G_4 H_2, \quad L_3 = -G_1 G_3 G_4 H_2$$



Example#1: Apply Mason's Rule to calculate the transfer function of the system represented by following Signal Flow Graph



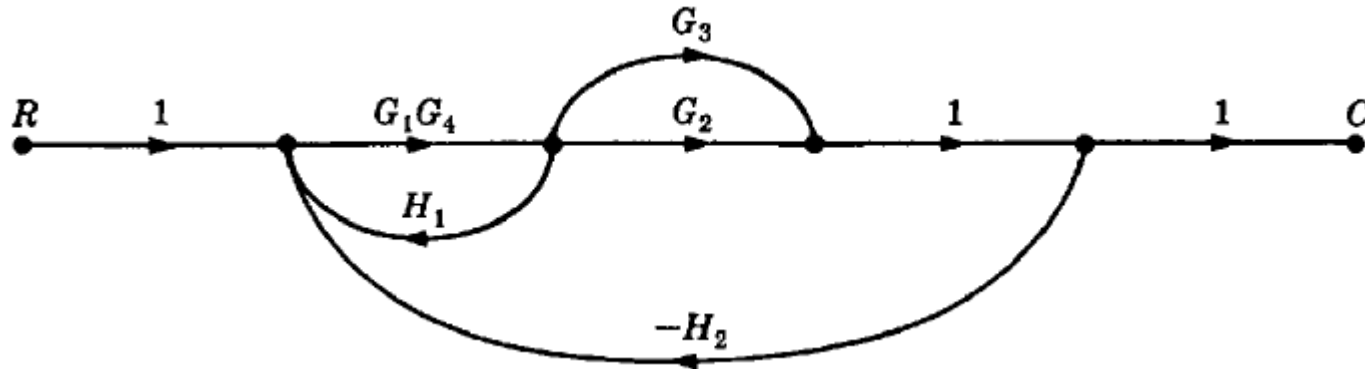
There are no non-touching loops, therefore

$$\Delta = 1 - (\text{sum of all individual loop gains})$$

$$\Delta = 1 - (L_1 + L_2 + L_3)$$

$$\Delta = 1 - (G_1G_4H_1 - G_1G_2G_4H_2 - G_1G_3G_4H_2)$$

Example#1: Apply Mason's Rule to calculate the transfer function of the system represented by following Signal Flow Graph



**Eliminate forward path-1**

$$\Delta_1 = 1 - (\text{sum of all individual loop gains}) + \dots$$

$$\Delta_1 = 1$$

**Eliminate forward path-2**

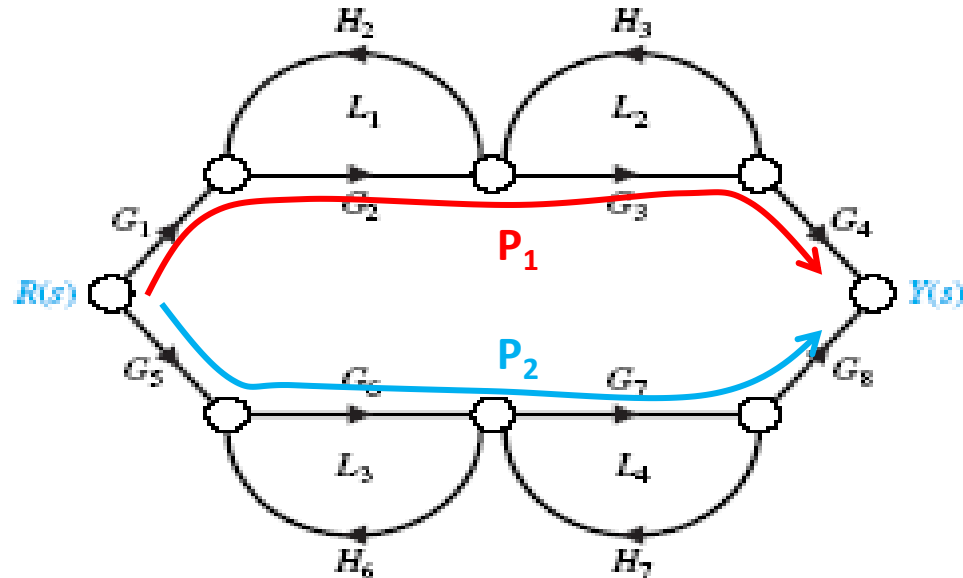
$$\Delta_2 = 1 - (\text{sum of all individual loop gains}) + \dots$$

$$\Delta_2 = 1$$

## Example#1: Continue

$$\begin{aligned}\frac{C}{R} &= \frac{P_1\Delta_1 + P_2\Delta_2}{\Delta} = \frac{G_1G_2G_4 + G_1G_3G_4}{1 - G_1G_4H_1 + G_1G_2G_4H_2 + G_1G_3G_4H_2} \\ &= \frac{G_1G_4(G_2 + G_3)}{1 - G_1G_4H_1 + G_1G_2G_4H_2 + G_1G_3G_4H_2}\end{aligned}$$

## Example#2: Apply Mason's Rule to calculate the transfer function of the system represented by following Signal Flow Graph



1. Calculate forward path gains for each forward path.

$$P_1 = G_1 G_2 G_3 G_4 \text{ (path 1)} \quad \text{and} \quad P_2 = G_5 G_6 G_7 G_8 \text{ (path 2)}$$

2. Calculate all loop gains.

$$L_1 = G_2 H_2, \quad L_2 = H_3 G_3, \quad L_3 = G_6 H_6, \quad L_4 = G_7 H_7$$

3. Consider two non-touching loops.

$$\begin{matrix} L_1 L_3 & L_1 L_4 \\ L_2 L_4 & L_2 L_3 \end{matrix}$$

## Example#2: continue

4. Consider three non-touching loops.

None.

5. Calculate  $\Delta$  from steps 2,3,4.

$$\Delta = 1 - (L_1 + L_2 + L_3 + L_4) + (L_1L_3 + L_1L_4 + L_2L_3 + L_2L_4)$$

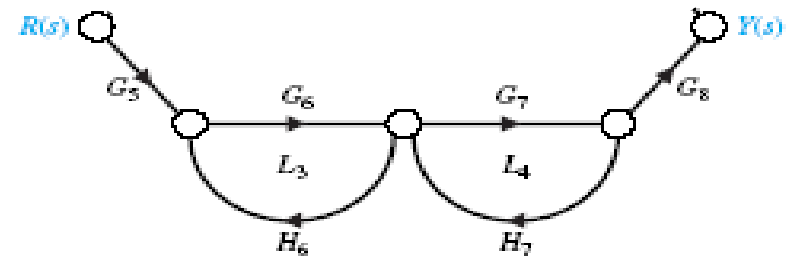
$$\Delta = 1 - (G_2H_2 + H_3G_3 + G_6H_6 + G_7H_7) + \\ (G_2H_2G_6H_6 + G_2H_2G_7H_7 + H_3G_3G_6H_6 + H_3G_3G_7H_7)$$

# Example#2: continue

Eliminate forward path-1

$$\Delta_1 = 1 - (L_3 + L_4)$$

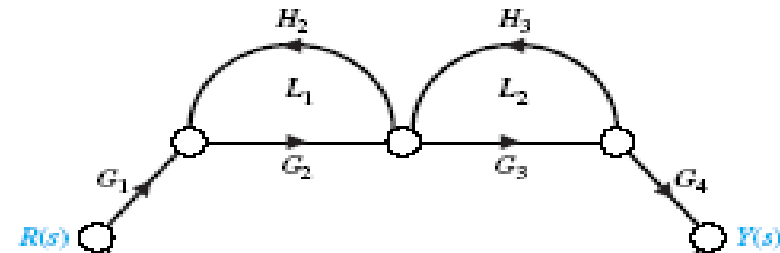
$$\Delta_1 = 1 - (G_6 H_6 + G_7 H_7)$$



Eliminate forward path-2

$$\Delta_2 = 1 - (L_1 + L_2)$$

$$\Delta_2 = 1 - (G_2 H_2 + G_3 H_3)$$



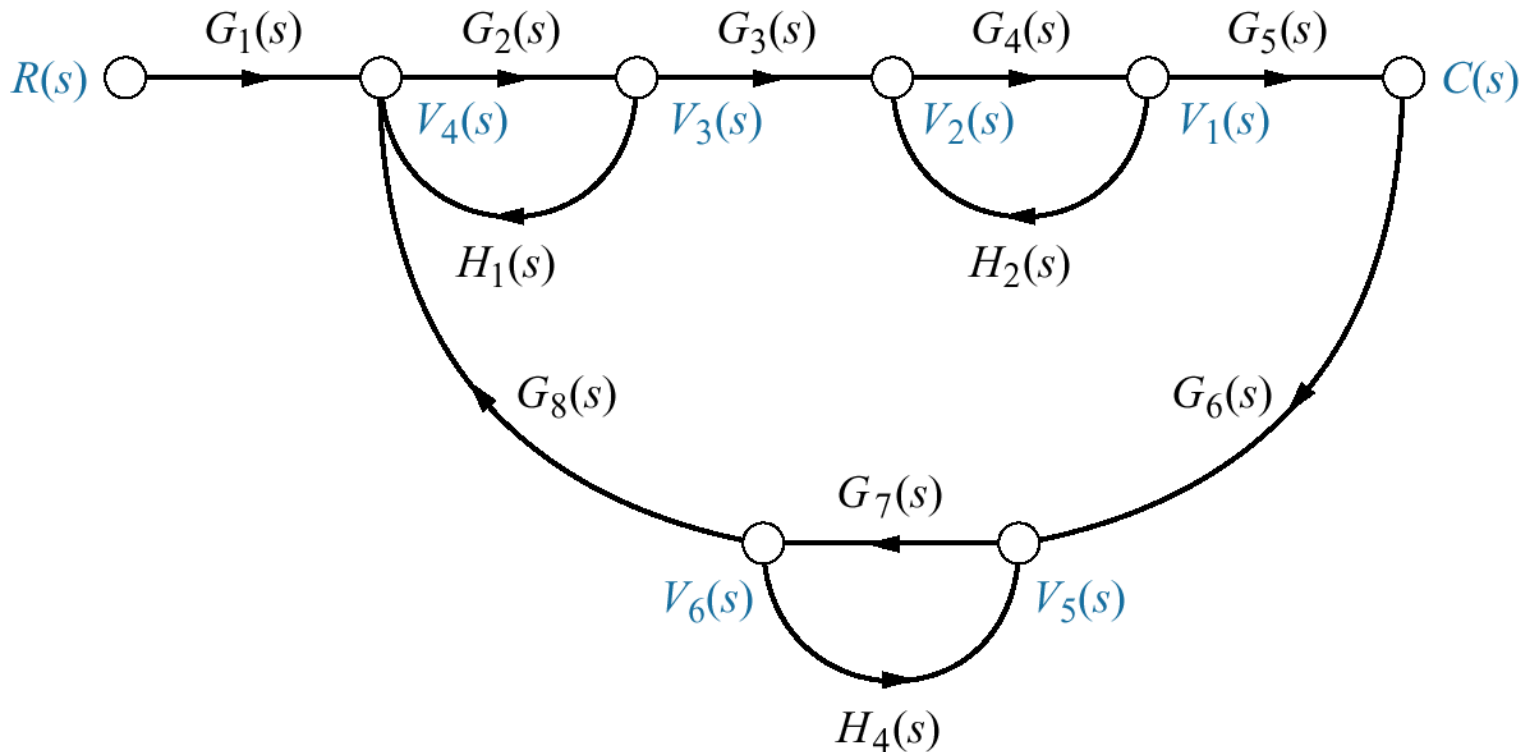
## Example#2: continue

$$\frac{Y(s)}{R(s)} = \frac{P_1\Delta_1 + P_2\Delta_2}{\Delta}$$

$$\frac{Y(s)}{R(s)} = \frac{G_1G_2G_3G_4[1 - (G_6H_6 + G_7H_7)] + G_5G_6G_7G_8[1 - (G_2H_2 + G_3H_3)]}{1 - (G_2H_2 + H_3G_3 + G_6H_6 + G_7H_7) + (G_2H_2G_6H_6 + G_2H_2G_7H_7 + H_3G_3G_6H_6 + H_3G_3G_7H_7)}$$

# Example#3

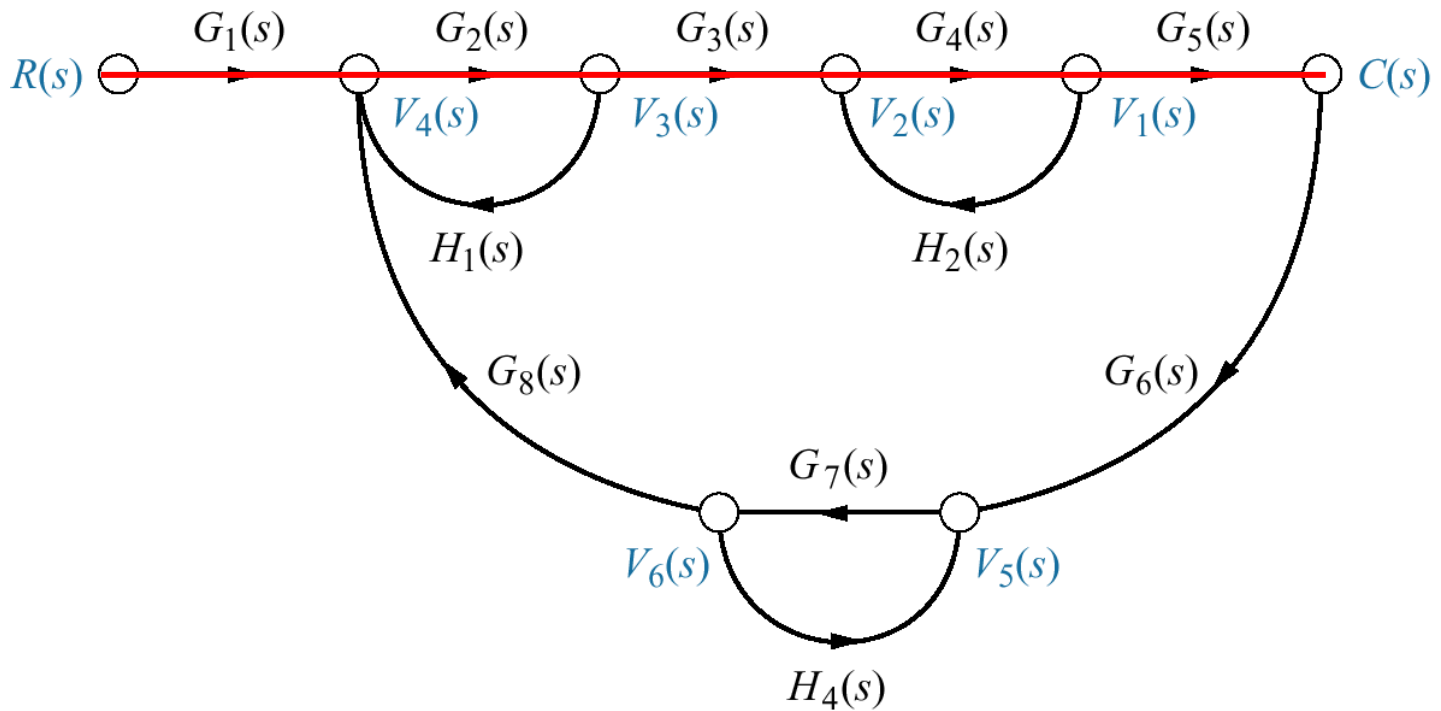
- Find the transfer function,  $C(s)/R(s)$ , for the signal-flow graph in figure below.





# Example#3

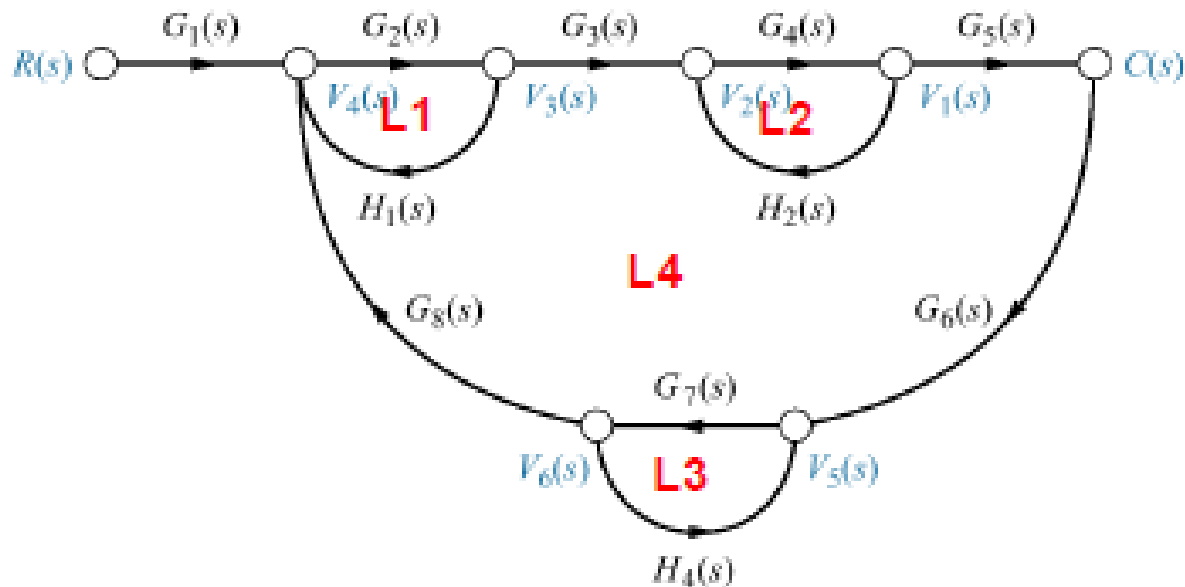
- There is only one forward Path.



$$P_1 = G_1(s)G_2(s)G_3(s)G_4(s)G_5(s)$$

# Example#3

- There are four feedback loops.



$$L1. \quad G_2(s)H_1(s)$$

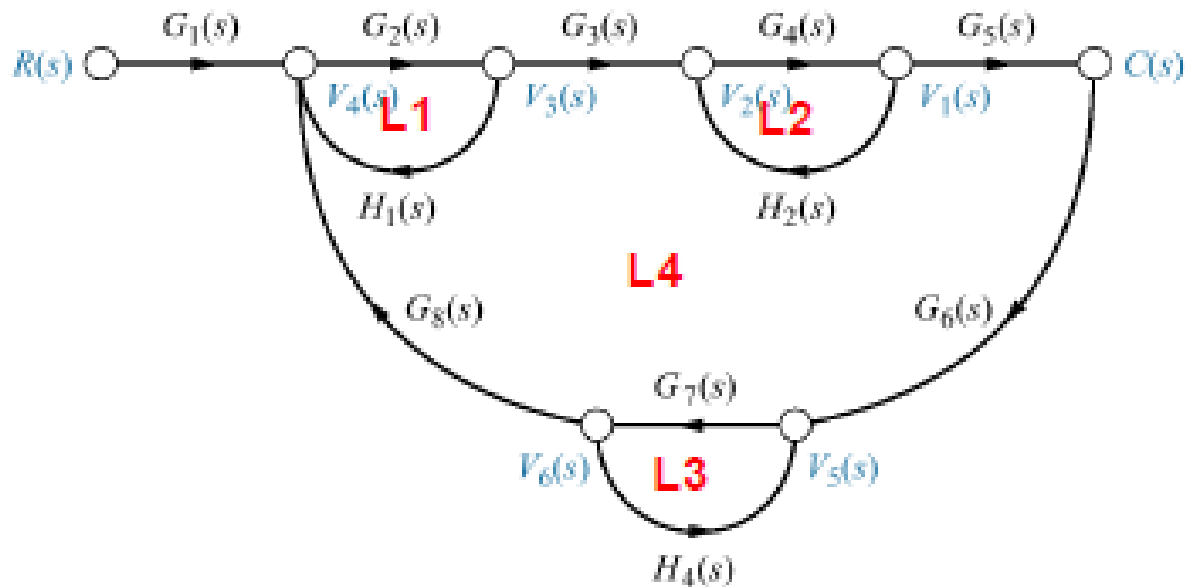
$$L3. \quad G_7(s)H_4(s)$$

$$L2. \quad G_4(s)H_2(s)$$

$$L4. \quad G_2(s)G_3(s)G_4(s)G_5(s)G_6(s)G_7(s)G_8(s)$$

# Example#3

- Non-touching loops taken two at a time.

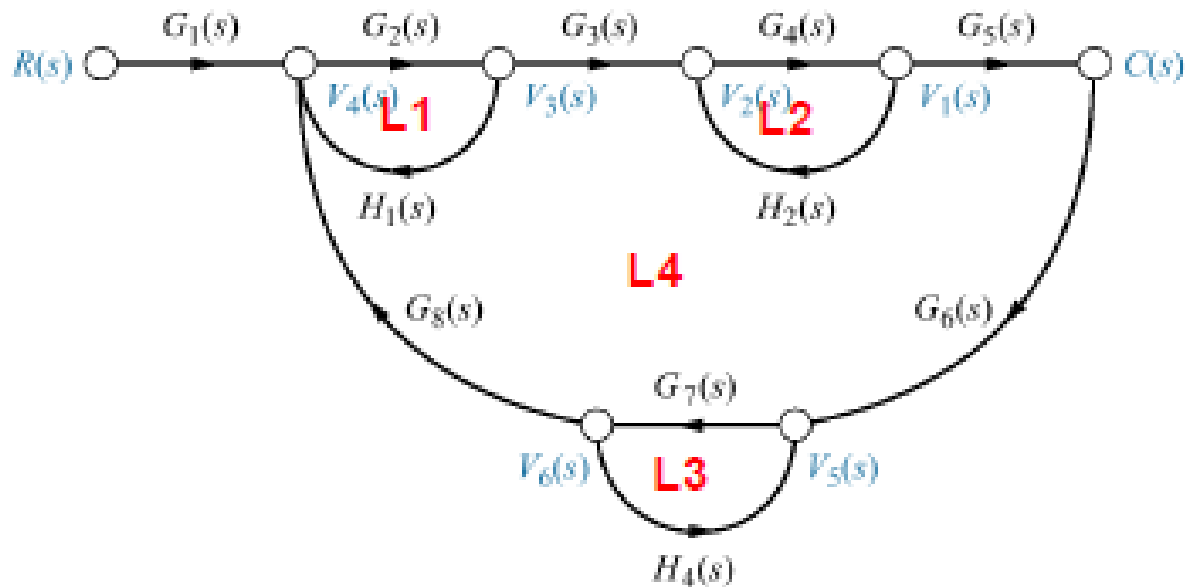


L1 and L2:  $G_2(s)H_1(s)G_4(s)H_2(s)$     L2 and L3:  $G_4(s)H_2(s)G_7(s)H_4(s)$

L1 and L3:  $G_2(s)H_1(s)G_7(s)H_4(s)$

# Example#3

- Non-touching loops taken three at a time.



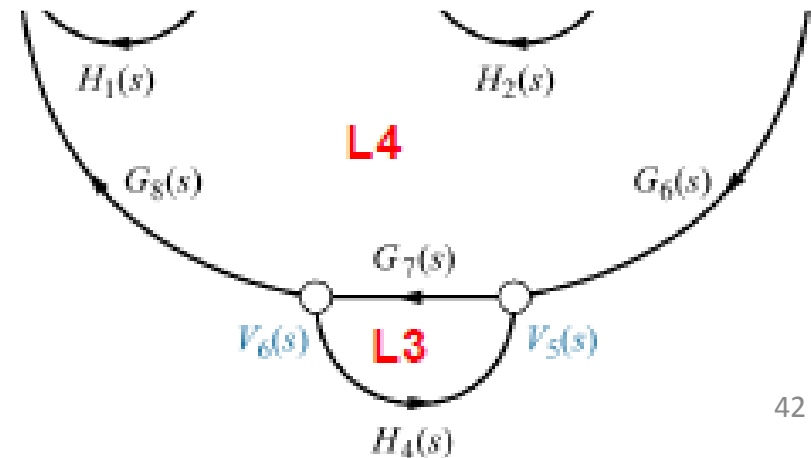
L1, L2, L3:  $G_2(s)H_1(s)G_4(s)H_2(s)G_7(s)H_4(s)$

# Example#3

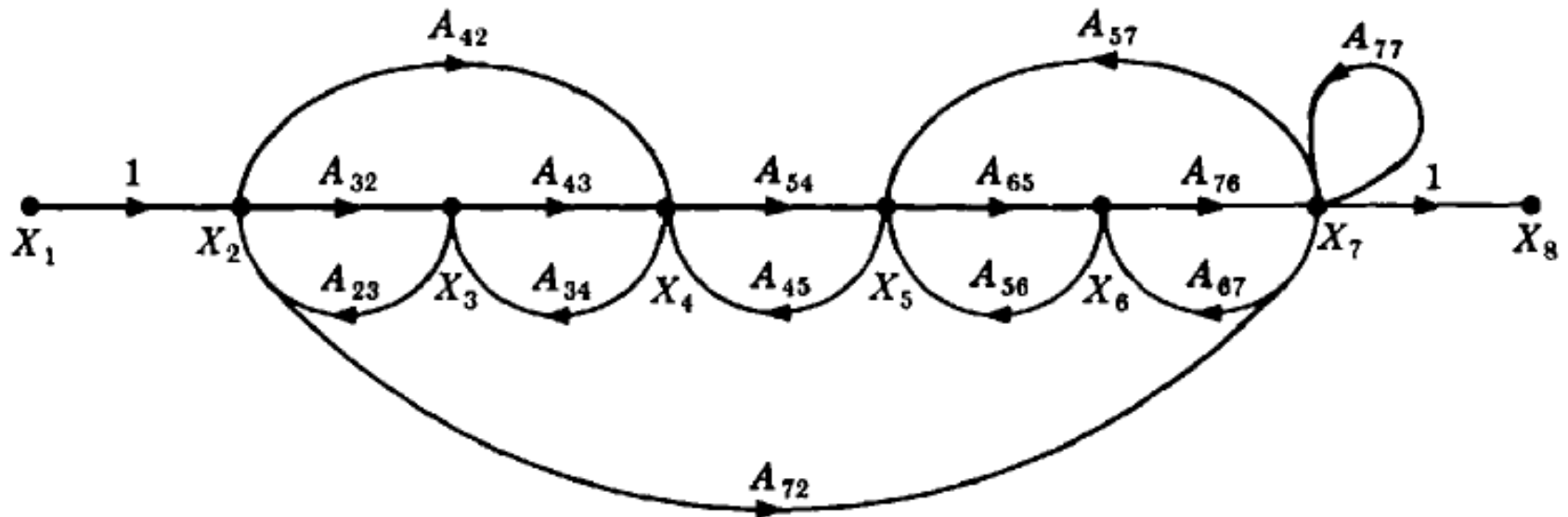
$$\begin{aligned}\Delta = & 1 - [G_2(s)H_1(s) + G_4(s)H_2(s) \\ & + G_7(s)H_4(s) + G_2(s)G_3(s)G_4(s)G_5(s)G_6(s)G_7(s)G_8(s)] \\ & + [G_2(s)H_1(s)G_4(s)H_2(s) + G_2(s)H_1(s)G_7(s)H_4(s) \\ & + G_4(s)H_2(s)G_7(s)H_4(s)] \\ & - [G_2(s)H_1(s)G_4(s)H_2(s)G_7(s)H_4(s)]\end{aligned}$$

Eliminate forward path-1

$$\Delta_1 = 1 - G_7(s)H_4(s)$$



Example#4: Apply Mason's Rule to calculate the transfer function of the system represented by following Signal Flow Graph

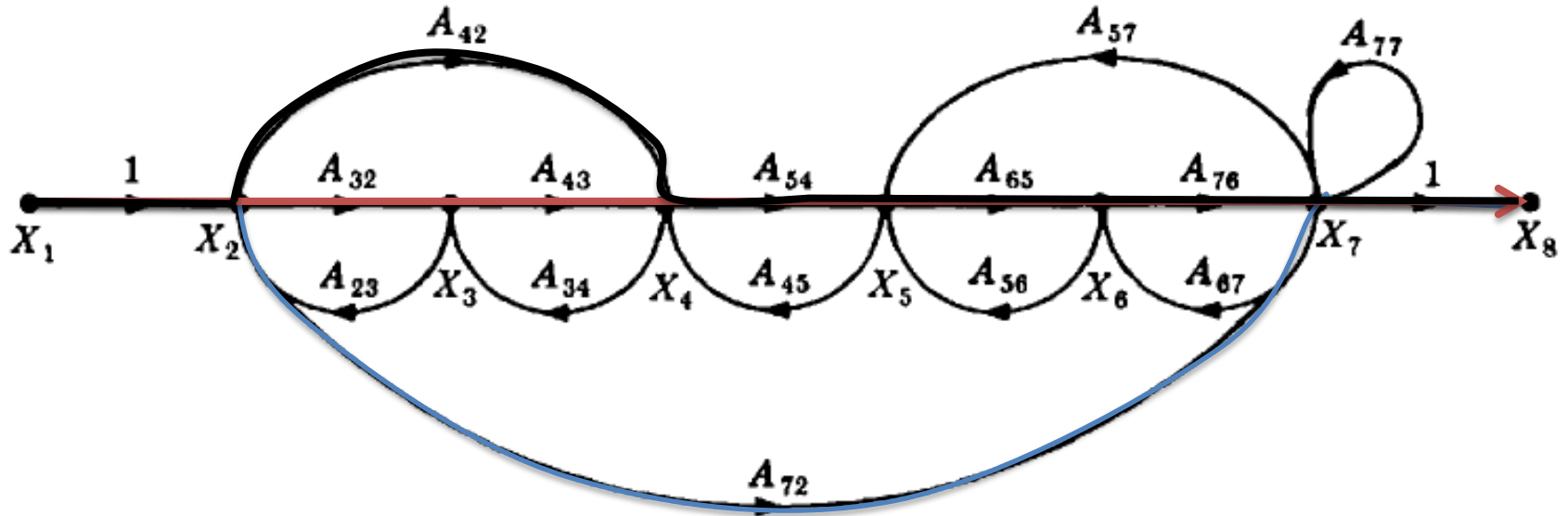


There are three forward paths, therefore  $n=3$ .

$$\frac{C(s)}{R(s)} = \frac{\sum_{i=1}^3 P_i \Delta_i}{\Delta} = \frac{P_1 \Delta_1 + P_2 \Delta_2 + P_3 \Delta_3}{\Delta}$$

## Example#4: Forward Paths

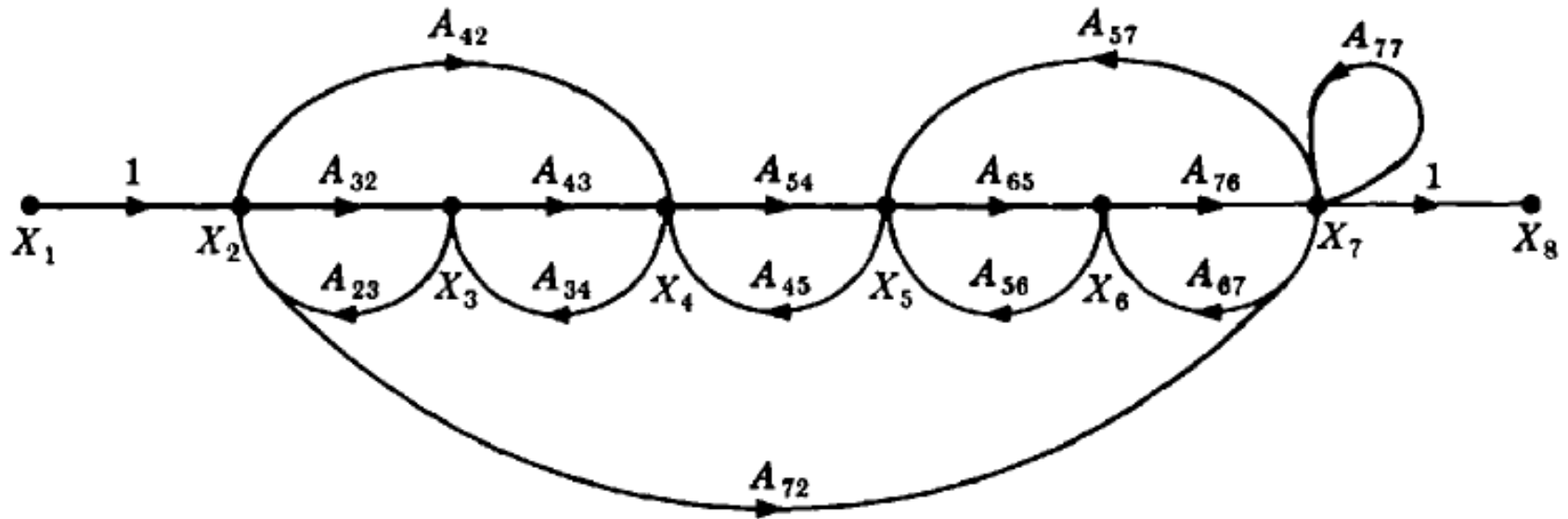
$$P_3 = A_{42}A_{54}A_{65}A_{76}$$



$$P_1 = A_{32}A_{43}A_{54}A_{65}A_{76}$$

$$P_2 = A_{72}$$

# Example#4: Loop Gains of the Feedback Loops



$$L_1 = A_{32}A_{23}$$

$$L_2 = A_{43}A_{34}$$

$$L_3 = A_{54}A_{45}$$

$$L_4 = A_{65}A_{56}$$

$$L_5 = A_{76}A_{67}$$

$$L_6 = A_{77}$$

$$L_7 = A_{42}A_{34}A_{23}$$

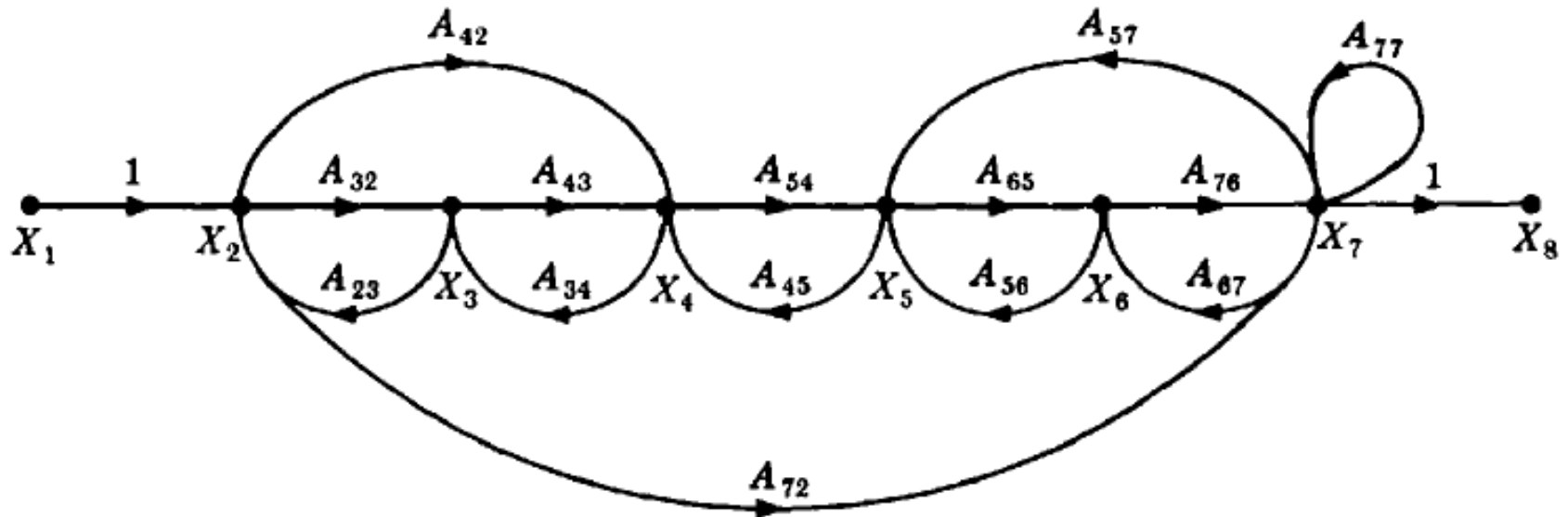
$$L_8 = A_{65}A_{76}A_{67}$$

$$L_9 = A_{72}A_{57}A_{45}A_{34}A_{23}$$

$$L_{10} = A_{72}A_{67}A_{56}A_{45}A_{34}A_{23}$$



# Example#4: two non-touching loops



$L_1 L_3$

$L_2 L_4$

$L_3 L_5$

$L_4 L_6$

$L_5 L_7$

$L_7 L_8$

$L_1 L_4$

$L_2 L_5$

$L_3 L_6$

$L_4 L_7$

$L_1 L_5$

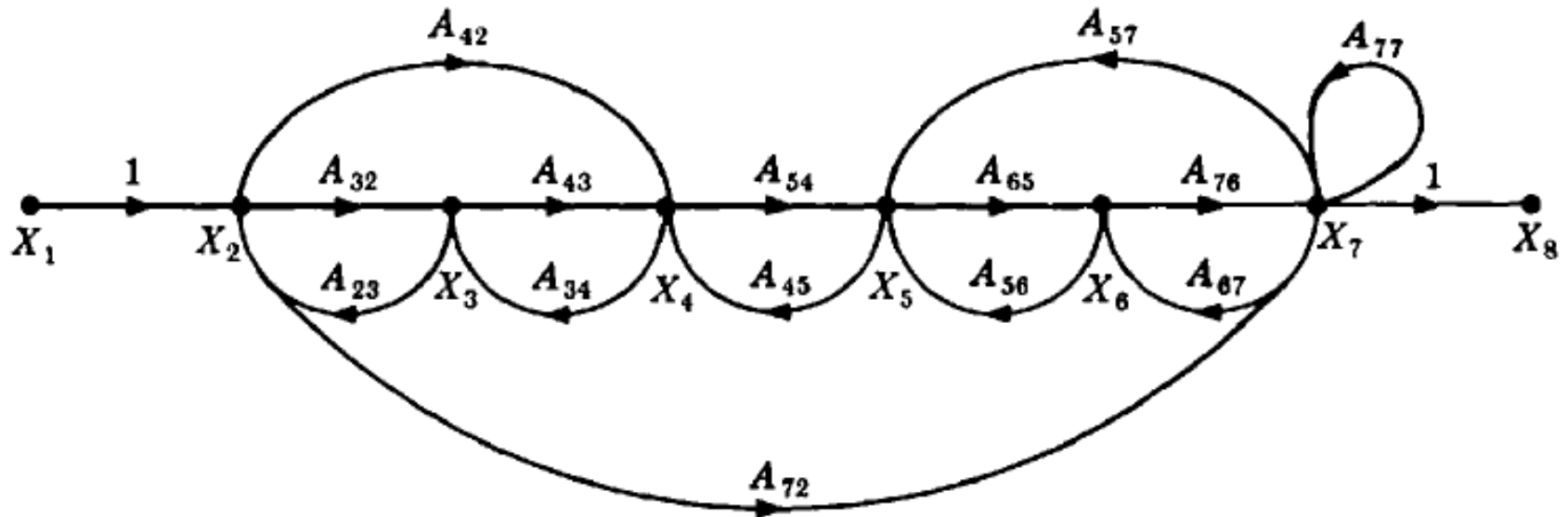
$L_2 L_6$

$L_1 L_6$

$L_2 L_8$

$L_1 L_8$

# Example#4: Three non-touching loops



$L_1 L_3$

$L_2 L_4$

$L_3 L_5$

$L_4 L_6$

$L_5 L_7$

$L_7 L_8$

$L_1 L_4$

$L_2 L_5$

$L_3 L_6$

$L_4 L_7$

$L_1 L_5$

$L_2 L_6$

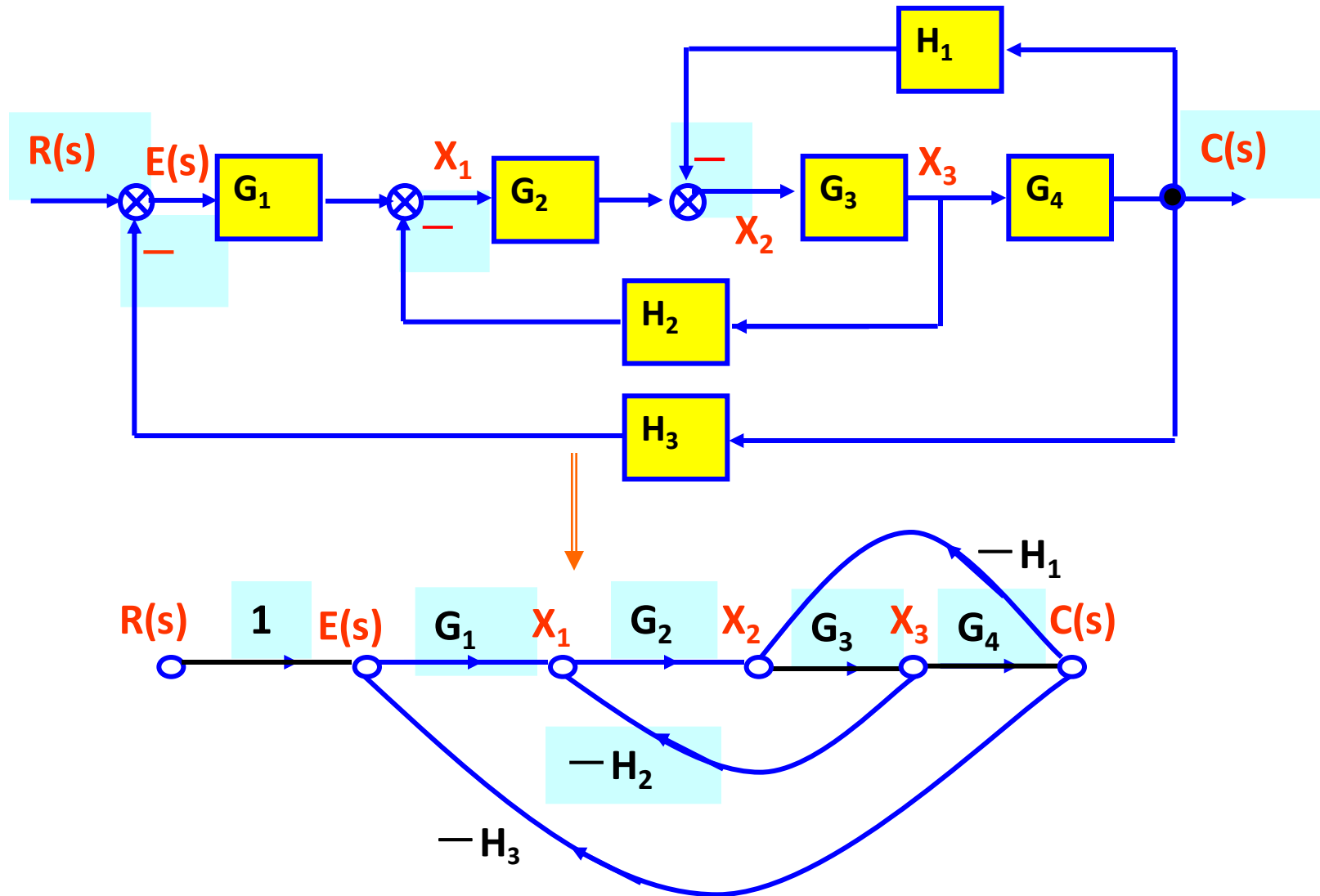
$L_1 L_6$

$L_2 L_8$

$L_1 L_8$

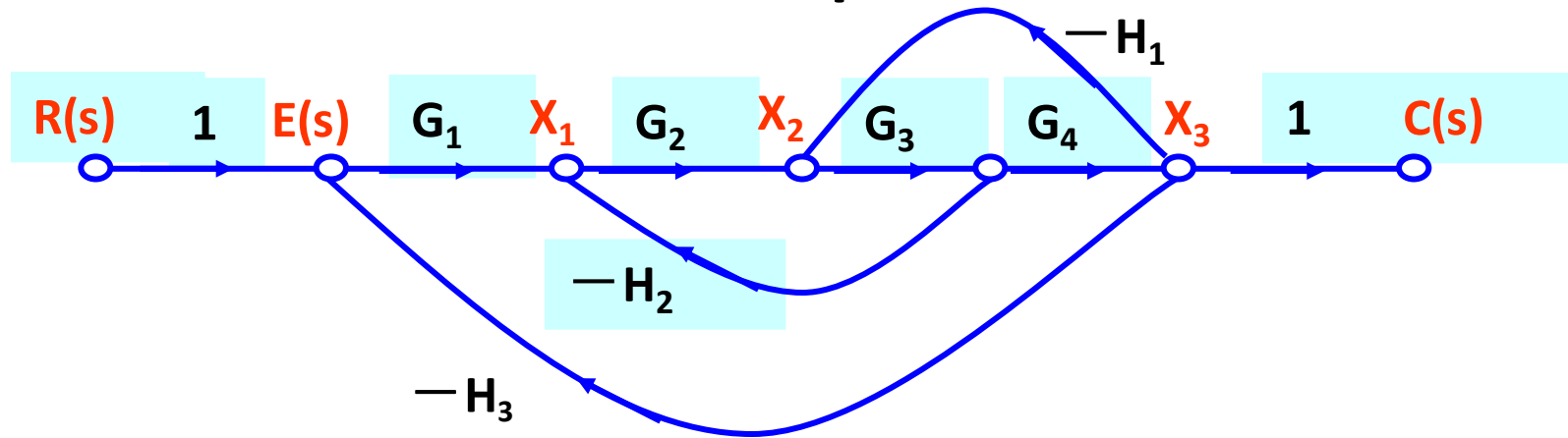
# From Block Diagram to Signal-Flow Graph Models

## Example#5



# From Block Diagram to Signal-Flow Graph Models

## Example#5

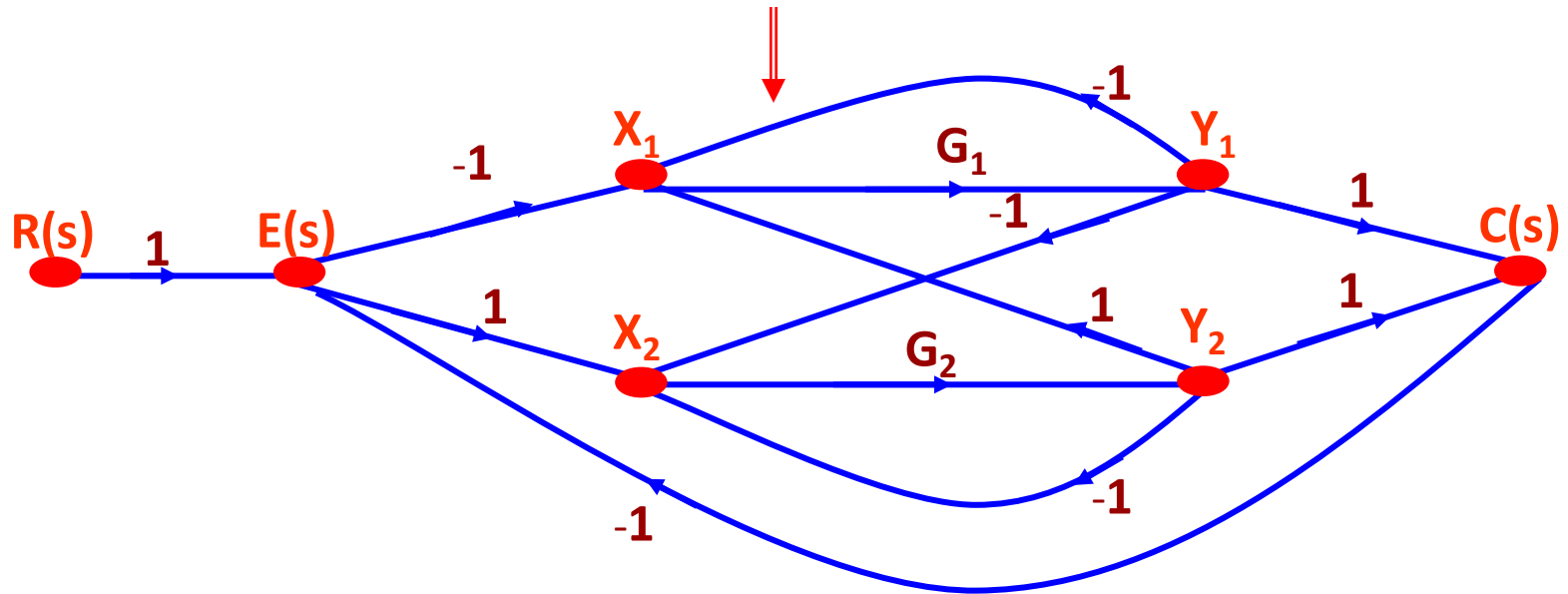
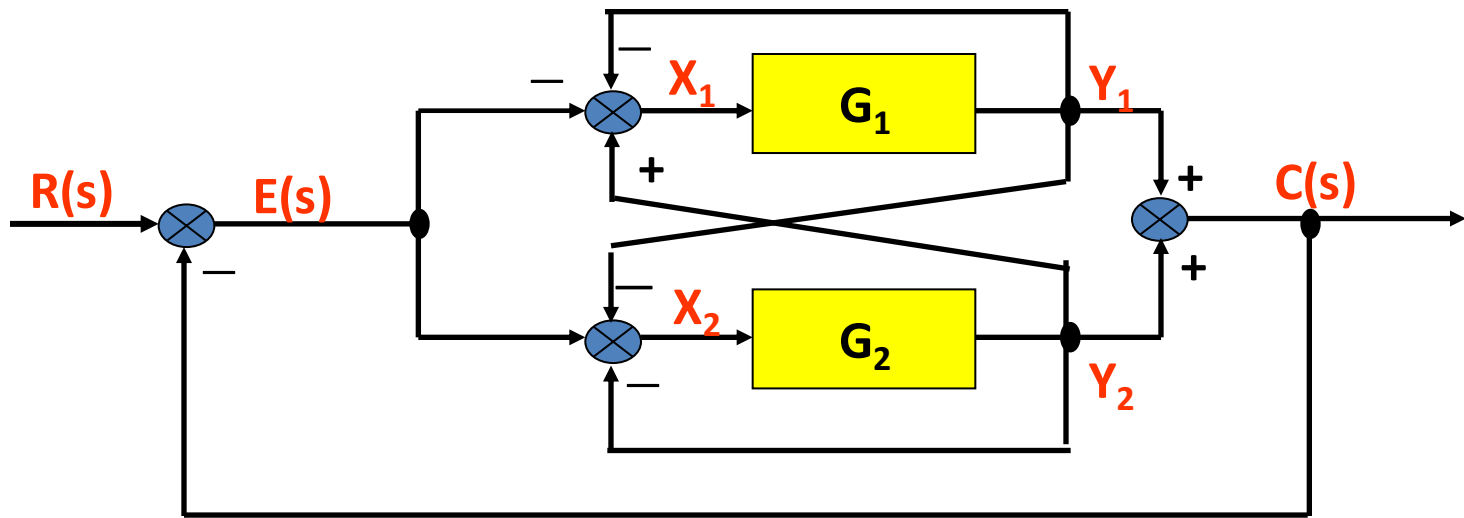


$$\Delta = 1 + (G_1 G_2 G_3 G_4 H_3 + G_2 G_3 H_2 + G_3 G_4 H_1)$$

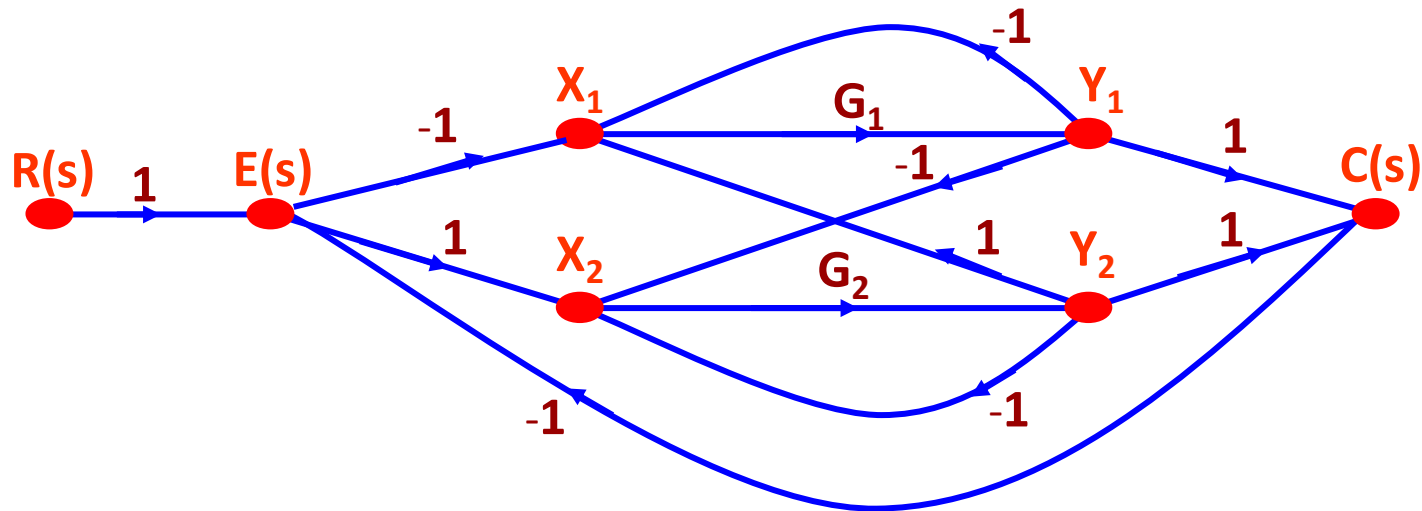
$$P_1 = G_1 G_2 G_3 G_4; \quad \Delta_1 = 1$$

$$G = \frac{C(s)}{R(s)} = \frac{G_1 G_2 G_3 G_4}{1 + G_1 G_2 G_3 G_4 H_3 + G_2 G_3 H_2 + G_3 G_4 H_1}$$

# Example#6



## Example#6



**7 loops:**

$$[G_1 \cdot (-1)]; \quad [G_2 \cdot (-1)]; \quad [G_1 \cdot (-1) \cdot G_2 \cdot 1]; \quad [(-1) \cdot G_1 \cdot 1 \cdot (-1)];$$

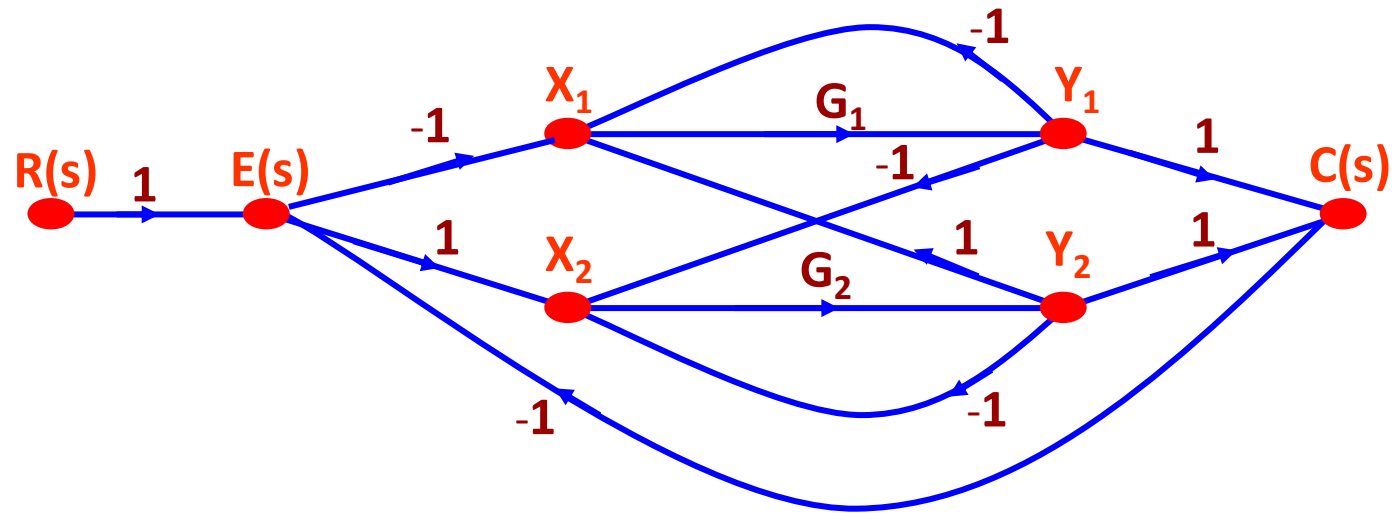
$$[(-1) \cdot G_1 \cdot (-1) \cdot G_2 \cdot 1 \cdot (-1)]; \quad [1 \cdot G_2 \cdot 1 \cdot (-1)]; \quad [1 \cdot G_2 \cdot 1 \cdot G_1 \cdot 1 \cdot (-1)].$$

**3 '2 non-touching loops':**

$$[G_1 \cdot (-1)] \cdot [G_2 \cdot (-1)]; \quad [(-1) \cdot G_1 \cdot 1 \cdot (-1)] \cdot [G_2 \cdot (-1)];$$

$$[1 \cdot G_2 \cdot 1 \cdot (-1)] \cdot [G_1 \cdot (-1)].$$

# Example#6



Then:  $\Delta = 1 + 2G_2 + 4G_1G_2$

4 forward paths:

$$p_1 = (-1) \cdot G_1 \cdot 1 \quad \Delta_1 = 1 + G_2$$

$$p_2 = (-1) \cdot G_1 \cdot (-1) \cdot G_2 \cdot 1 \quad \Delta_2 = 1$$

$$p_3 = 1 \cdot G_2 \cdot 1 \quad \Delta_3 = 1 + G_1$$

$$p_4 = 1 \cdot G_2 \cdot 1 \cdot G_1 \cdot 1 \quad \Delta_4 = 1$$

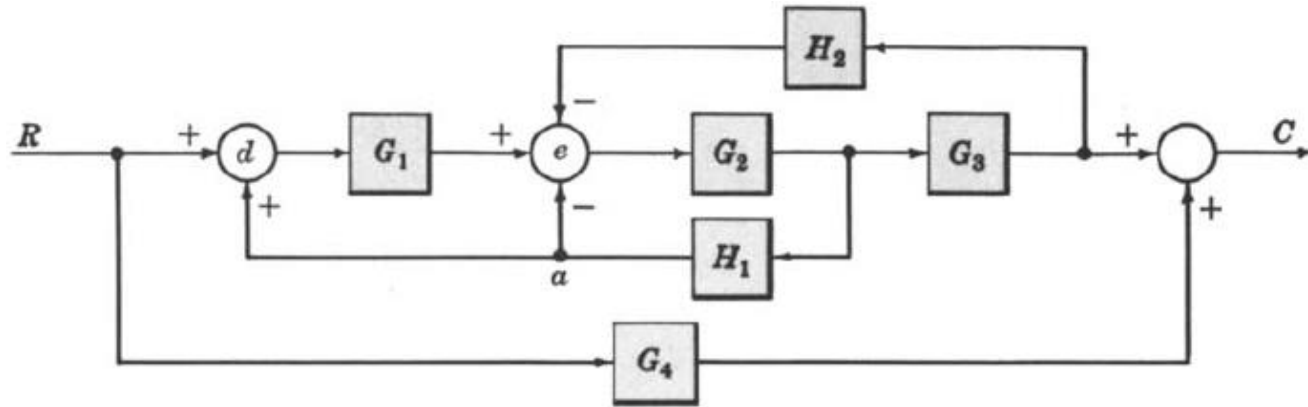
## Example#6

We have

$$\begin{aligned}\frac{C(s)}{R(s)} &= \frac{\sum p_k \Delta_k}{\Delta} \\ &= \frac{G_2 - G_1 + 2G_1G_2}{1 + 2G_2 + 4G_1G_2}\end{aligned}$$



**Example-7:** Determine the transfer function  $C/R$  for the block diagram below by signal flow graph techniques.



- The signal flow graph of the above block diagram is shown below.

- There are two forward paths. The path gains are

$$P_1 = G_1 G_2 G_3 \text{ and } P_2 = G_4$$

- The three feedback loop gains are

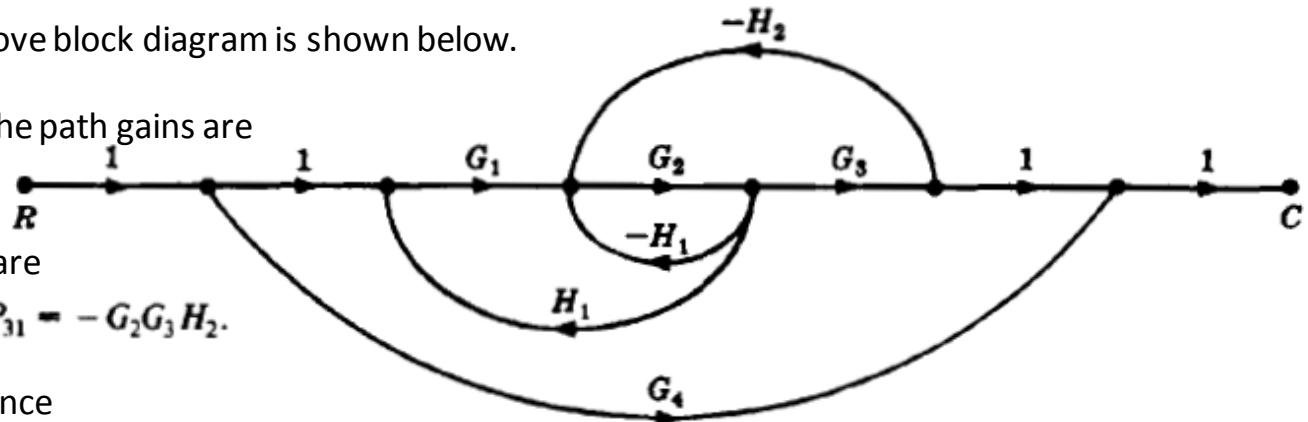
$$P_{11} = -G_2 H_1, P_{21} = G_1 G_2 H_1, P_{31} = -G_2 G_3 H_2.$$

- No loops are non-touching, hence

$$\Delta = 1 - (P_{11} + P_{21} + P_{31})$$

- Because the loops touch the nodes of  $P_1$ , hence  $\Delta_1 = 1$

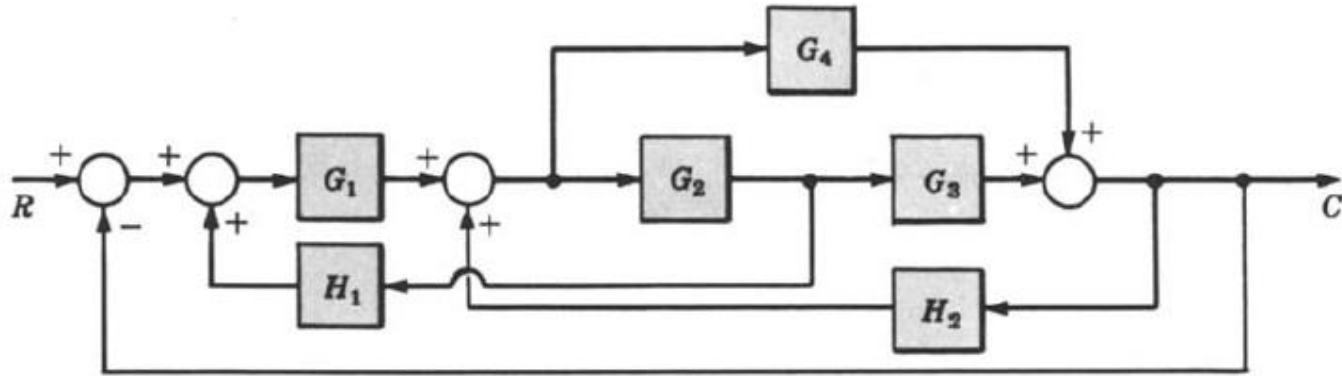
- Since no loops touch the nodes of  $P_2$ , therefore  $\Delta_2 = \Delta$ .



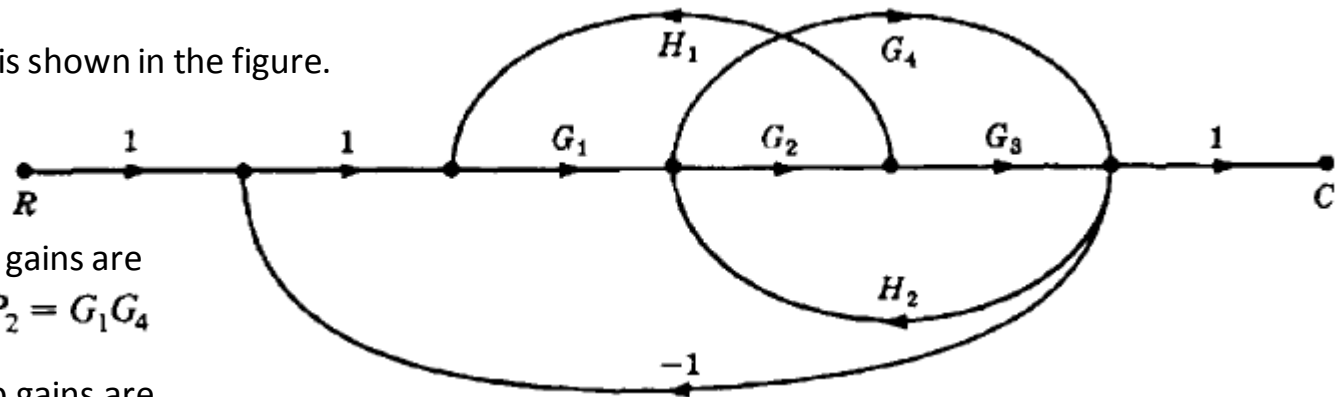
- Hence the control ratio  $T = C/R$  is

$$T = \frac{P_1 \Delta_1 + P_2 \Delta_2}{\Delta} = \frac{G_1 G_2 G_3 + G_4 + G_2 G_4 H_1 - G_1 G_2 G_4 H_1 + G_2 G_3 G_4 H_2}{1 + G_2 H_1 - G_1 G_2 H_1 + G_2 G_3 H_2}$$

**Example-6:** Find the control ratio  $C/R$  for the system given below.



- The signal flow graph is shown in the figure.



- The two forward path gains are  $P_1 = G_1 G_2 G_3$  and  $P_2 = G_1 G_4$
- The five feedback loop gains are  $P_{11} = G_1 G_2 H_1$ ,  $P_{21} = G_2 G_3 H_2$ ,  $P_{31} = -G_1 G_2 G_3$ ,  $P_{41} = G_4 H_2$ , and  $P_{51} = -G_1 G_4$ .
- All feedback loops touches the two forward paths, hence  $\Delta_1 = \Delta_2 = 1$

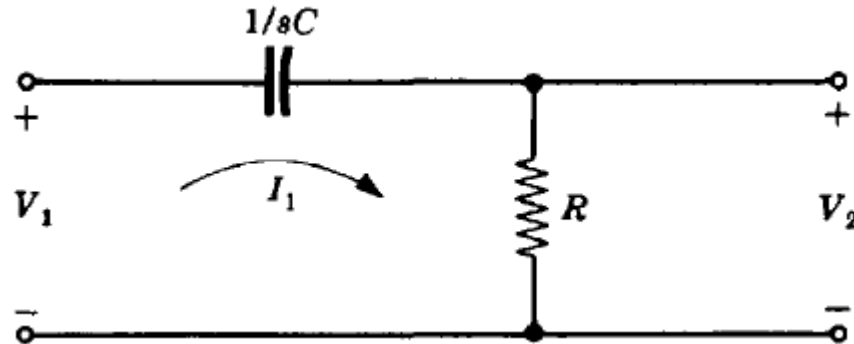
- There are no non-touching loops, hence

$$\Delta = 1 - (P_{11} + P_{21} + P_{31} + P_{41} + P_{51})$$

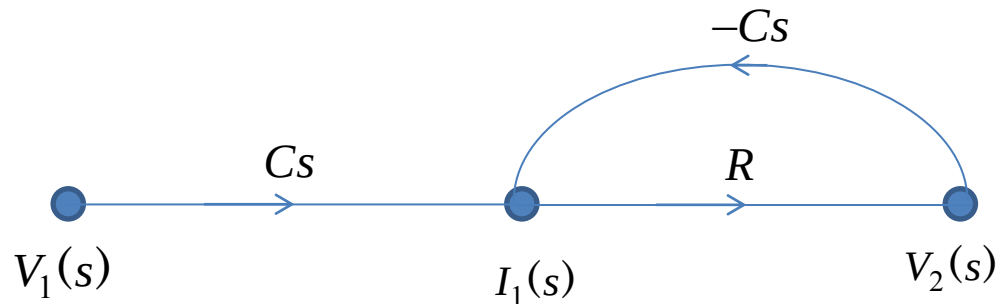
$$= 1 + G_1 G_2 G_3 - G_1 G_2 H_1 - G_2 G_3 H_2 - G_4 H_2 + G_1 G_4$$

- Hence the control ratio  $T = \frac{C}{R} = \frac{P_1 \Delta_1 + P_2 \Delta_2}{\Delta} = \frac{G_1 G_2 G_3 + G_1 G_4}{1 + G_1 G_2 G_3 - G_1 G_2 H_1 - G_2 G_3 H_2 - G_4 H_2 + G_1 G_4}$

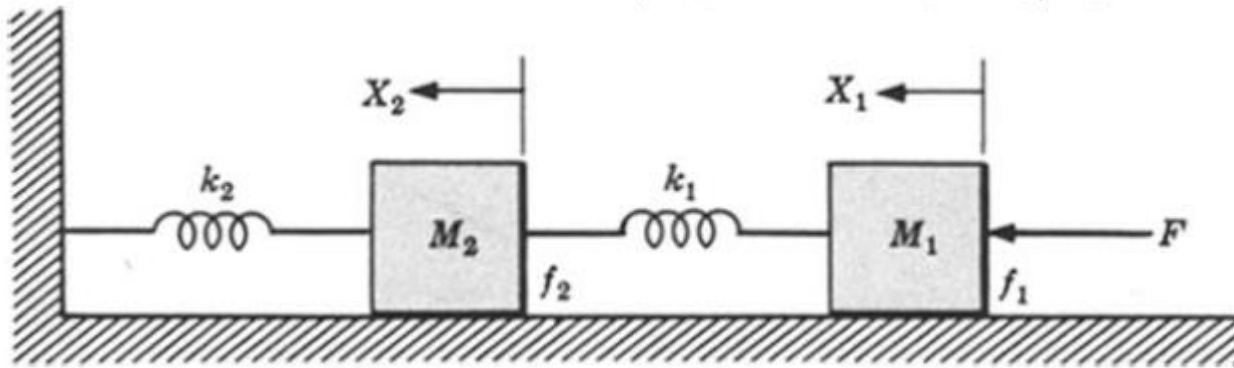
# Design Example#1



$$\left. \begin{aligned} V_1(s) &= \frac{1}{Cs} I_1(s) + I_1(s)R \\ V_2(s) &= I_1(s)R \end{aligned} \right\} \Rightarrow CsV_1(s) - CsV_2(s) = I_1(s)$$



# Design Example#2



$$F = M_1 s^2 X_1 + k_1 (X_1 - X_2) \quad 0 = M_2 s^2 X_2 + k_1 (X_2 - X_1) + k_2 X_2$$

$$(i) \quad F + k_1 X_2 = (M_1 s^2 + f_1 s + k_1) X_1$$

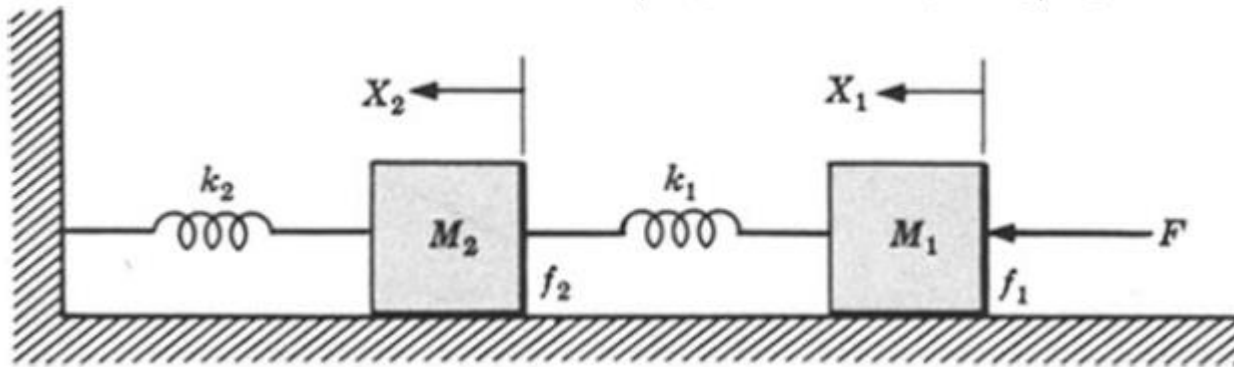
$$(ii) \quad k_1 X_1 = (M_2 s^2 + f_2 s + k_1 + k_2) X_2$$

There are three variables:  $X_1$ ,  $X_2$ , and  $F$ . Therefore we need three nodes.

$$(iii) \quad \left( \frac{1}{A} \right) F + \left( \frac{k_1}{A} \right) X_2 = X_1$$

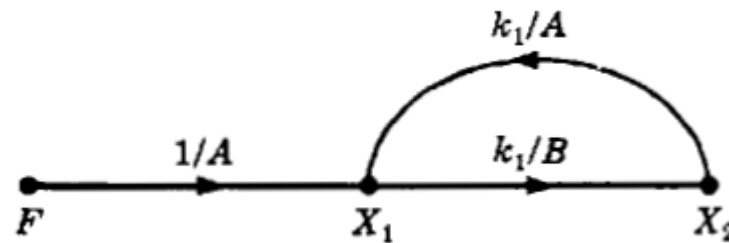
$$(iv) \quad \left( \frac{k_1}{B} \right) X_1 = X_2$$

# Design Example#2

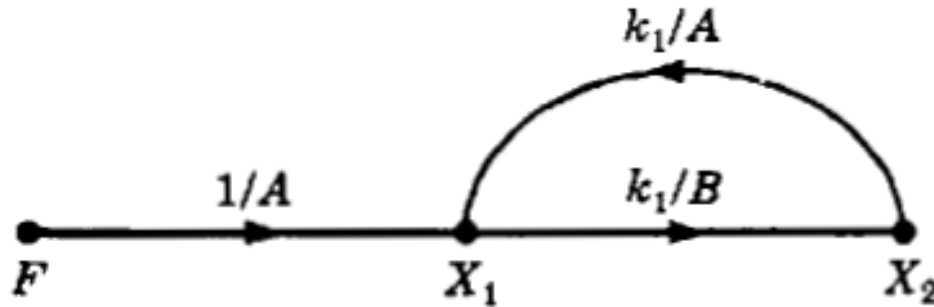


$$(iii) \quad \left( \frac{1}{A} \right) F + \left( \frac{k_1}{A} \right) X_2 = X_1$$

$$(iv) \quad \left( \frac{k_1}{B} \right) X_1 = X_2$$



# Design Example#2



The forward path gain is  $P_1 = k_1/AB$ . The feedback loop gain is  $P_{11} = k_1^2/AB$ . then  $\Delta = 1 - P_{11} = (AB - k_1^2)/AB$  and  $\Delta_1 = 1$ . Finally,

$$\frac{X_2}{F} = \frac{P_1 \Delta_1}{\Delta} = \frac{k_1}{AB - k_1^2} = \frac{k_1}{(M_1 s^2 + f_1 s + k_1)(M_2 s^2 + f_2 s + k_1 + k_2) - k_1^2}$$

# التحليل الزمني لمنظومات السيطرة

الاداء الديناميكي لمنظومات السيطرة

# Introduction

- In time-domain analysis the response of a dynamic system to an input is expressed as a function of time.
- It is possible to compute the time response of a system if the nature of input and the mathematical model of the system are known.
- Usually, the input signals to control systems are not known fully ahead of time.
- For example, in a radar tracking system, the position and the speed of the target to be tracked may vary in a random fashion.
- It is therefore difficult to express the actual input signals mathematically by simple equations.



# Standard Test Signals

## اشارات الفحص

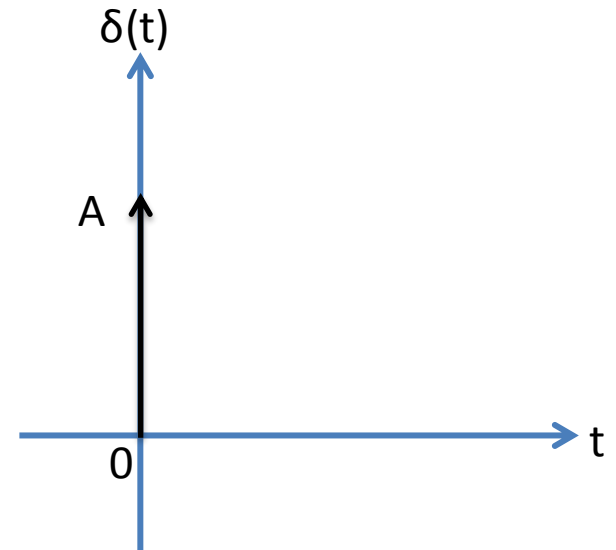
- The characteristics of actual input signals are a sudden shock, a sudden change, a constant velocity, and constant acceleration.
- The dynamic behavior of a system is therefore judged and compared under application of standard test signals – an impulse, a step, a constant velocity, and constant acceleration.
- Another standard signal of great importance is a sinusoidal signal.

# Standard Test Signals

- Impulse signal إشارة النبضة
  - The impulse signal imitate the sudden shock characteristic of actual input signal.

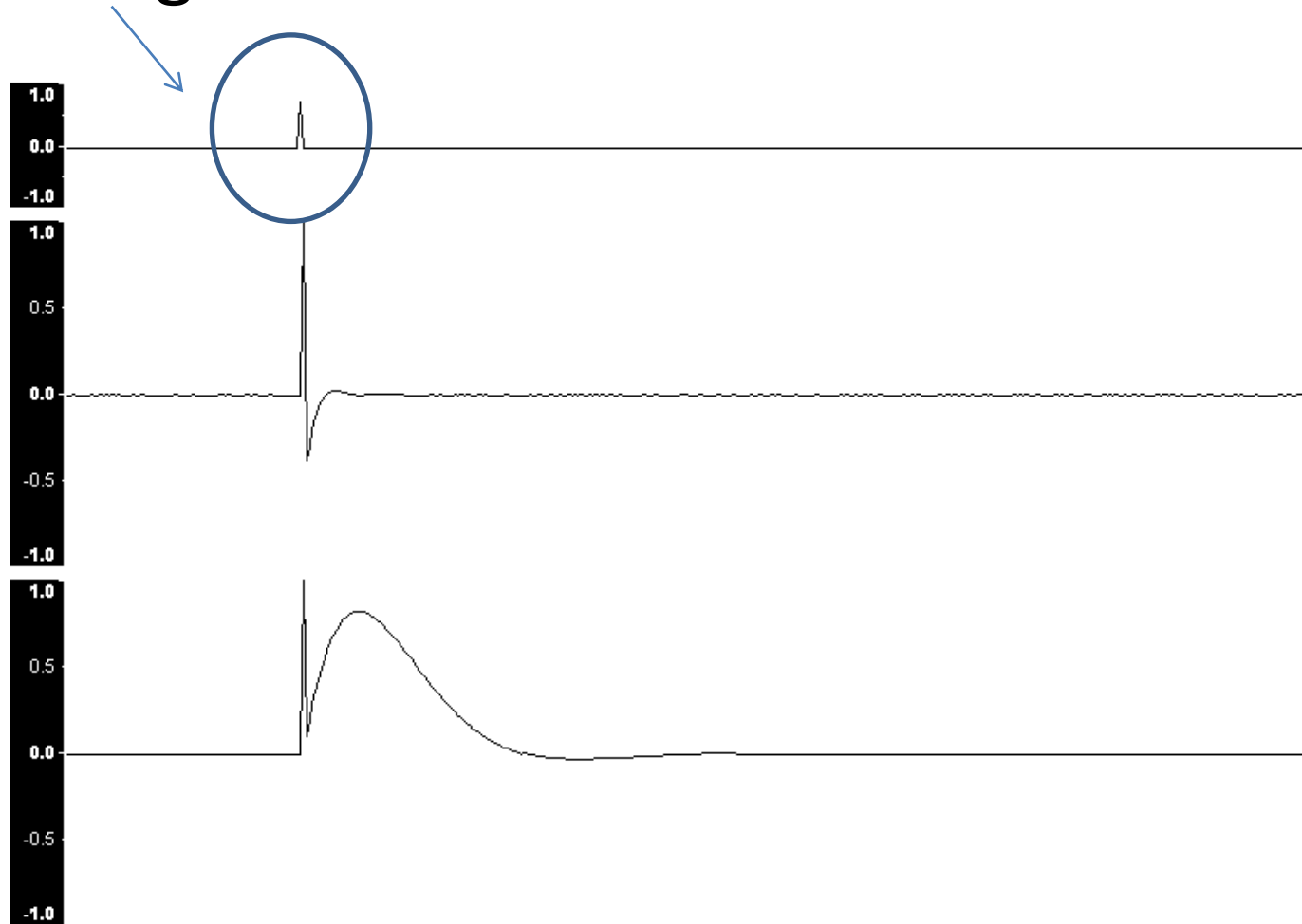
$$\delta(t) = \begin{cases} A & t = 0 \\ 0 & t \neq 0 \end{cases}$$

- If  $A=1$ , the impulse signal is called unit impulse signal.



# Standard Test Signals

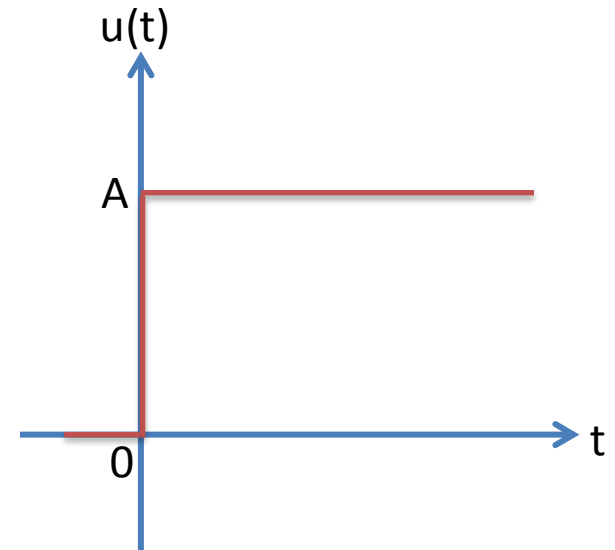
- Impulse signal



# Standard Test Signals

- Step signal إشارة الخطوة
  - The step signal imitate the sudden change characteristic of actual input signal.

$$u(t) = \begin{cases} A & t \geq 0 \\ 0 & t < 0 \end{cases}$$



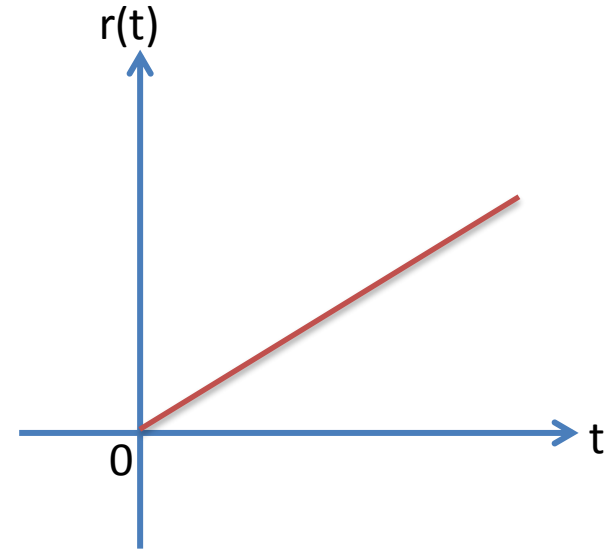
- If  $A=1$ , the step signal is called unit step signal

# Standard Test Signals

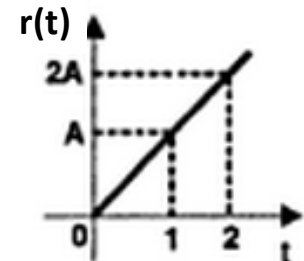
- Ramp signal إشارة المنحدر
  - The ramp signal imitate the constant velocity characteristic of actual input signal.

$$r(t) = \begin{cases} At & t \geq 0 \\ 0 & t < 0 \end{cases}$$

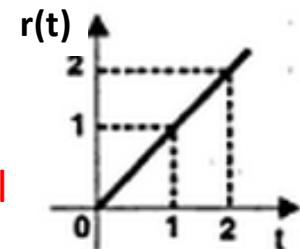
- If  $A=1$ , the ramp signal is called unit ramp signal



ramp signal with slope A



unit ramp signal



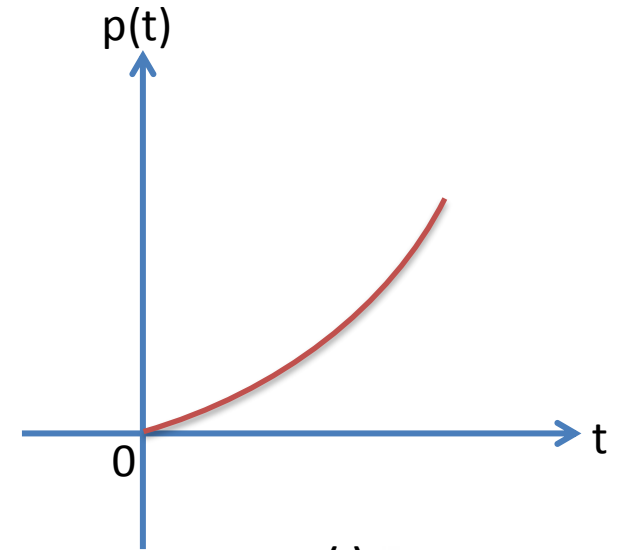
# Standard Test Signals

- Parabolic signal إشارة القطع المكافئ

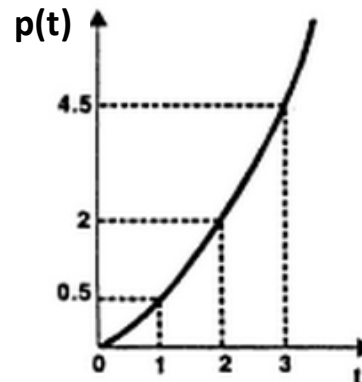
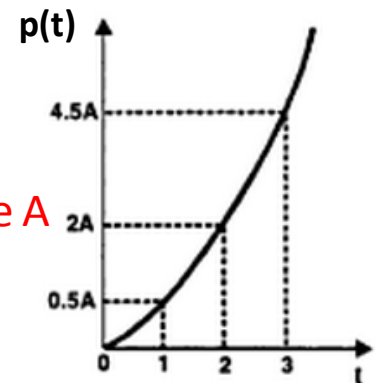
- The parabolic signal imitate the constant acceleration characteristic of actual input signal.

$$p(t) = \begin{cases} \frac{At^2}{2} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

- If  $A=1$ , the parabolic signal is called unit parabolic signal.

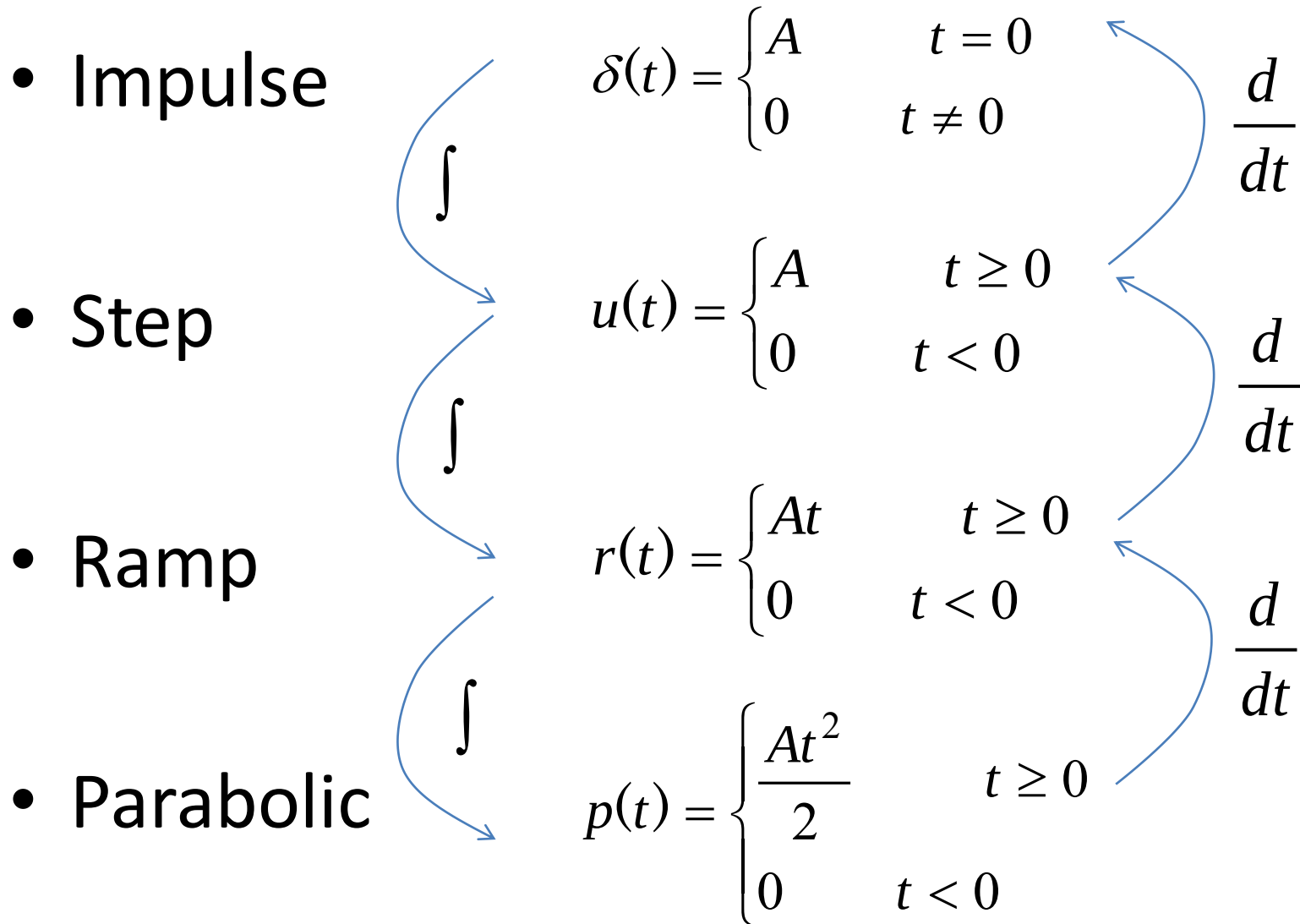


parabolic signal with slope A



Unit parabolic signal

# Relation between standard Test Signals



# Laplace Transform of Test Signals

- Impulse

$$\delta(t) = \begin{cases} A & t = 0 \\ 0 & t \neq 0 \end{cases}$$

$$L\{\delta(t)\} = \delta(s) = A$$

- Step

$$u(t) = \begin{cases} A & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$L\{u(t)\} = U(s) = \frac{A}{s}$$



# Laplace Transform of Test Signals

- Ramp

$$r(t) = \begin{cases} At & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$L\{r(t)\} = R(s) = \frac{A}{s^2}$$

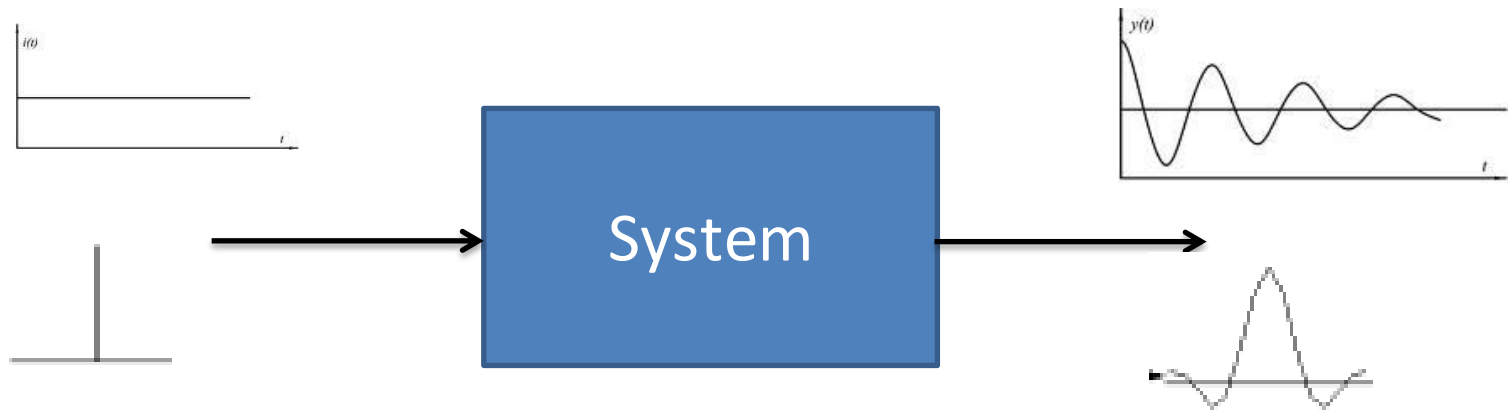
- Parabolic

$$p(t) = \begin{cases} \frac{At^2}{2} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$L\{p(t)\} = P(s) = \frac{2A}{s^3}$$

# Time Response of Control Systems

- Time response of a dynamic system response to an input expressed as a function of time.

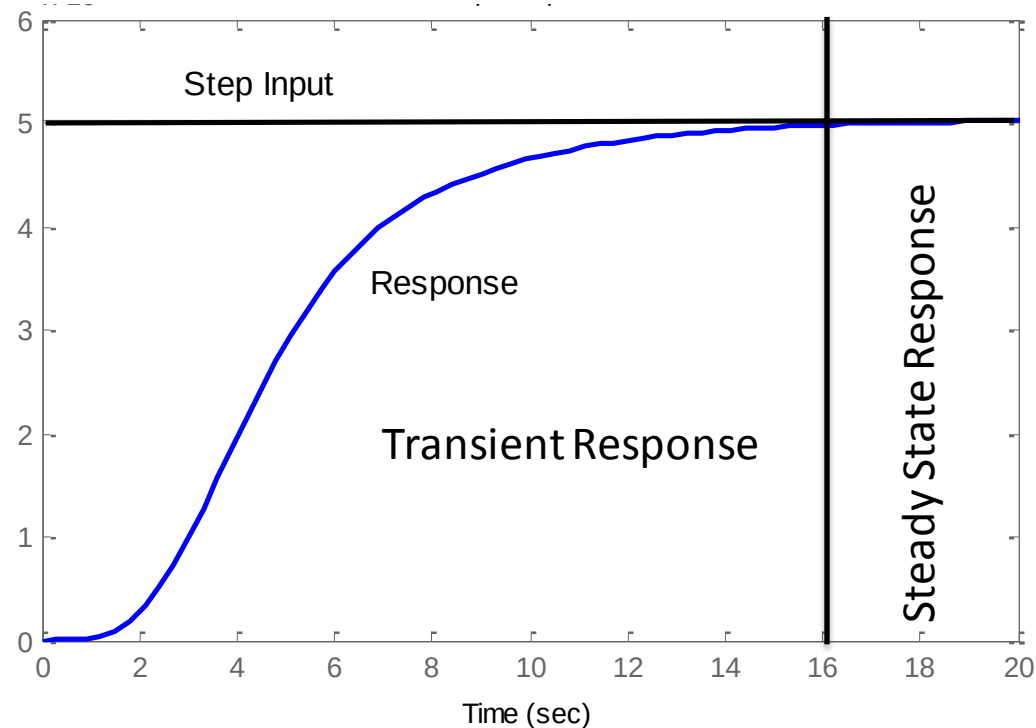


- The time response of any system has two components
  - Transient response
  - Steady-state response.

# Time Response of Control Systems

## الاداء الزمني لمنظومات السيطرة

- When the response of the system is changed from rest or equilibrium it takes some time to settle down.
- Transient response is the response of a system from rest or equilibrium to steady state.
- The response of the system after the transient response is called steady state response.



# Time Response of Control Systems

- Transient response is dependent upon the system poles only and not on the type of input.
- It is therefore sufficient to analyze the transient response using a step input.
- The steady-state response depends on system dynamics and the input quantity.
- It is then examined using different test signals by final value theorem.

# Control Systems (CS)

Lecture-6-b  
Steady State Error

الخطء الثابت للمنظومات المغلقة

# Introduction

- Any physical control system inherently suffers steady-state error in response to certain types of inputs.
- A system may have no steady-state error to a step input, but the same system may exhibit nonzero steady-state error to a ramp input.
- Whether a given system will exhibit steady-state error for a given type of input depends on the type of open-loop transfer function of the system.

# Classification of Control Systems

- Control systems may be classified according to their ability to follow step inputs, ramp inputs, parabolic inputs, and so on.
- The magnitudes of the steady-state errors due to these individual inputs are indicative of the goodness of the system.

# Classification of Control Systems تحديد انواع منظومات السيطرة

- Consider the unity-feedback control system with the following open-loop transfer function

$$G(s) = \frac{K(T_a s + 1)(T_b s + 1) \cdots (T_m s + 1)}{s^N (T_1 s + 1)(T_2 s + 1) \cdots (T_p s + 1)}$$

- It involves the term  $s^N$  in the denominator, representing  $N$  poles at the origin.
- A system is called type 0, type 1, type 2, ... , if  $N=0$ ,  $N=1$ ,  $N=2$ , ... , respectively.



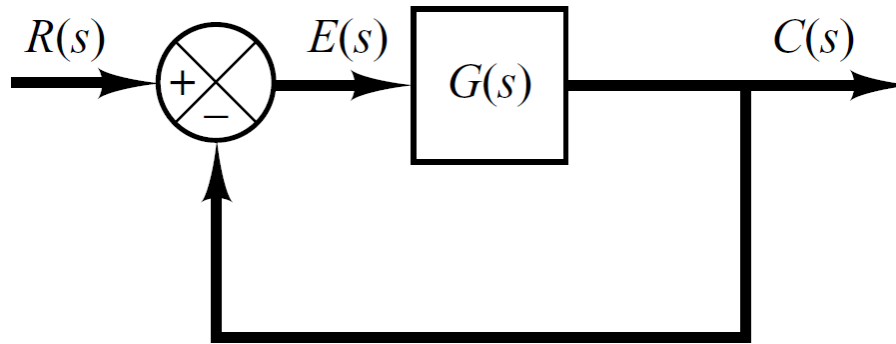
# Classification of Control Systems

- As the type number is increased, accuracy is improved.
- However, increasing the type number aggravates the stability problem.
- A compromise between steady-state accuracy and relative stability is always necessary.

# Steady State Error of Unity Feedback Systems

الخطء المستمر لمنظومة ذات التغذية العكسية الواحدي

- Consider the system shown in following figure.



- The closed-loop transfer function is

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)}$$

# Steady State Error of Unity Feedback Systems

- The transfer function between the error signal  $E(s)$  and the input signal  $R(s)$  is

$$\frac{E(s)}{R(s)} = \frac{1}{1 + G(s)}$$

- The final-value theorem provides a convenient way to find the steady-state performance of a stable system.

- Since  $E(s)$  is 
$$E(s) = \frac{1}{1 + G(s)} R(s)$$

- The steady state error is

$$e_{ss} = \lim_{t \rightarrow \infty} e(t) = \lim_{s \rightarrow 0} sE(s) = \lim_{s \rightarrow 0} \frac{sR(s)}{1 + G(s)}$$

# Static Error Constants

- The static error constants are figures of merit of control systems. The higher the constants, the smaller the steady-state error.
- In a given system, the output may be the position, velocity, pressure, temperature, or the like.
- Therefore, in what follows, we shall call the output “position,” the rate of change of the output “velocity,” and so on.
- This means that in a temperature control system “position” represents the output temperature, “velocity” represents the rate of change of the output temperature, and so on.

# Static Position Error Constant ( $K_p$ ) ثابت

## الخطاء الموضع الاستاتيكي

- The steady-state error of the system for a unit-step input is

$$\begin{aligned} e_{ss} &= \lim_{s \rightarrow 0} \frac{\cancel{s}}{1 + G(s)} \frac{1}{\cancel{s}} \\ &= \frac{1}{1 + G(0)} \end{aligned}$$

- The static position error constant  $K_p$  is defined by

$$K_p = \lim_{s \rightarrow 0} G(s) = G(0)$$

- Thus, the steady-state error in terms of the static position error constant  $K_p$  is given by

$$e_{ss} = \frac{1}{1 + K_p}$$

# Static Position Error Constant ( $K_p$ )

- For a **Type 0** system

$$K_p = \lim_{s \rightarrow 0} \frac{K(T_a s + 1)(T_b s + 1) \cdots}{(T_1 s + 1)(T_2 s + 1) \cdots} = K$$

- For **Type 1** or higher systems

$$K_p = \lim_{s \rightarrow 0} \frac{K(T_a s + 1)(T_b s + 1) \cdots}{s^N (T_1 s + 1)(T_2 s + 1) \cdots} = \infty, \quad \text{for } N \geq 1$$

- For a unit step input the steady state error  $e_{ss}$  is

$$e_{ss} = \frac{1}{1 + K}, \quad \text{for type 0 systems}$$

$$e_{ss} = 0, \quad \text{for type 1 or higher systems}$$

# Static Velocity Error Constant ( $K_v$ ) ثابت

## خطاء السرعة

- The steady-state error of the system for a unit-ramp input is

$$\begin{aligned} e_{ss} &= \lim_{s \rightarrow 0} \frac{s}{1 + G(s)} \frac{1}{s^2} \\ &= \lim_{s \rightarrow 0} \frac{1}{sG(s)} \end{aligned}$$

- The static position error constant  $K_v$  is defined by

$$K_v = \lim_{s \rightarrow 0} sG(s)$$

- Thus, the steady-state error in terms of the static velocity error constant  $K_v$  is given by

$$e_{ss} = \frac{1}{K_v}$$

# Static Velocity Error Constant ( $K_v$ )

- For a **Type 0** system

$$K_v = \lim_{s \rightarrow 0} \frac{sK(T_a s + 1)(T_b s + 1) \cdots}{(T_1 s + 1)(T_2 s + 1) \cdots} = 0$$

- For **Type 1** systems

$$K_v = \lim_{s \rightarrow 0} \frac{sK(T_a s + 1)(T_b s + 1) \cdots}{s(T_1 s + 1)(T_2 s + 1) \cdots} = K$$

- For type 2 or higher systems

$$K_v = \lim_{s \rightarrow 0} \frac{sK(T_a s + 1)(T_b s + 1) \cdots}{s^N(T_1 s + 1)(T_2 s + 1) \cdots} = \infty, \quad \text{for } N \geq 2$$



# Static Velocity Error Constant ( $K_v$ ) ثابت

## خطاء السرعة

- For a ramp input the steady state error  $e_{ss}$  is

$$e_{ss} = \frac{1}{K_v} = \infty, \quad \text{for type 0 systems}$$

$$e_{ss} = \frac{1}{K_v} = \frac{1}{K}, \quad \text{for type 1 systems}$$

$$e_{ss} = \frac{1}{K_v} = 0, \quad \text{for type 2 or higher systems}$$

# Static Acceleration Error Constant ( $K_a$ )

- The steady-state error of the system for parabolic input is

$$\begin{aligned} e_{ss} &= \lim_{s \rightarrow 0} \frac{s}{1 + G(s)} \frac{1}{s^3} \\ &= \frac{1}{\lim_{s \rightarrow 0} s^2 G(s)} \end{aligned}$$

- The static acceleration error constant  $K_a$  is defined by

$$K_a = \lim_{s \rightarrow 0} s^2 G(s)$$

- Thus, the steady-state error in terms of the static acceleration error constant  $K_a$  is given by

$$e_{ss} = \frac{1}{K_a}$$

# Static Acceleration Error Constant

ثابت خطأ التعجيل ( $K_a$ )

- For a **Type 0** system

$$K_a = \lim_{s \rightarrow 0} \frac{s^2 K (T_a s + 1)(T_b s + 1) \cdots}{(T_1 s + 1)(T_2 s + 1) \cdots} = 0$$

- For **Type 1** systems

$$K_a = \lim_{s \rightarrow 0} \frac{s^2 K (T_a s + 1)(T_b s + 1) \cdots}{s (T_1 s + 1)(T_2 s + 1) \cdots} = 0$$

- For **type 2** systems

$$K_a = \lim_{s \rightarrow 0} \frac{s^2 K (T_a s + 1)(T_b s + 1) \cdots}{s^2 (T_1 s + 1)(T_2 s + 1) \cdots} = K$$

- For **type 3** or higher systems

$$K_a = \lim_{s \rightarrow 0} \frac{s^2 K (T_a s + 1)(T_b s + 1) \cdots}{s^N (T_1 s + 1)(T_2 s + 1) \cdots} = \infty, \quad \text{for } N \geq 3$$

# Static Acceleration Error Constant ( $K_a$ )

- For a parabolic input the steady state error  $e_{ss}$  is

$$e_{ss} = \infty, \quad \text{for type 0 and type 1 systems}$$

$$e_{ss} = \frac{1}{K}, \quad \text{for type 2 systems}$$

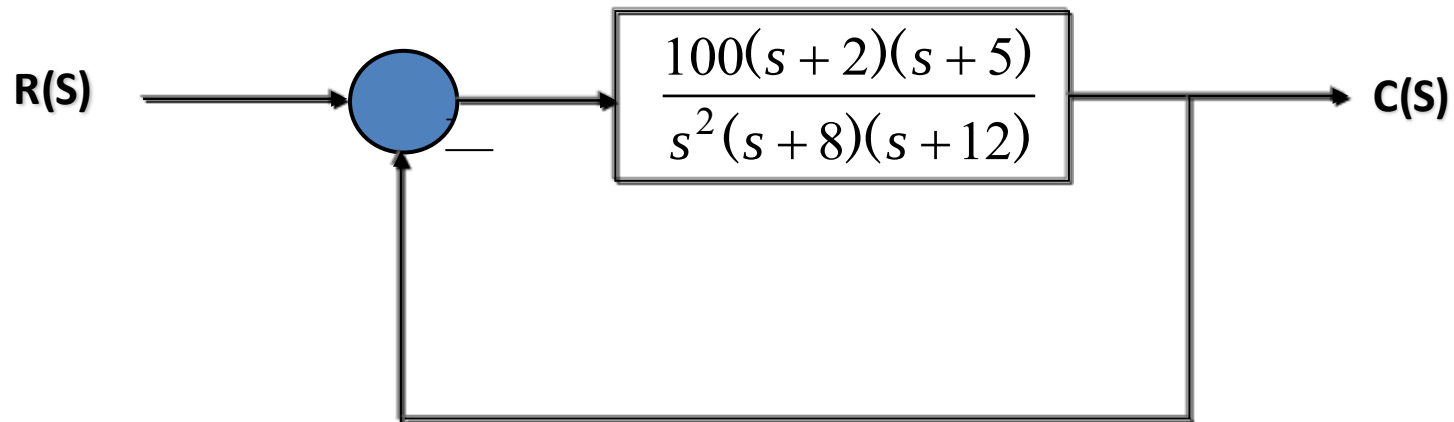
$$e_{ss} = 0, \quad \text{for type 3 or higher systems}$$

# Summary

|               | Step Input<br>$r(t) = 1$ | Ramp Input<br>$r(t) = t$ | Acceleration Input<br>$r(t) = \frac{1}{2}t^2$ |
|---------------|--------------------------|--------------------------|-----------------------------------------------|
| Type 0 system | $\frac{1}{1 + K}$        | $\infty$                 | $\infty$                                      |
| Type 1 system | 0                        | $\frac{1}{K}$            | $\infty$                                      |
| Type 2 system | 0                        | 0                        | $\frac{1}{K}$                                 |

# Example#1

- For the system shown in figure below evaluate the static error constants and find the expected steady state errors for the standard step, ramp and parabolic inputs.



# Example#1 (evaluation of Static Error Constants)

$$G(s) = \frac{100(s+2)(s+5)}{s^2(s+8)(s+12)}$$

$$K_p = \lim_{s \rightarrow 0} G(s)$$

$$K_p = \lim_{s \rightarrow 0} \left( \frac{100(s+2)(s+5)}{s^2(s+8)(s+12)} \right)$$

$$K_p = \infty$$

$$K_v = \lim_{s \rightarrow 0} sG(s)$$

$$K_v = \lim_{s \rightarrow 0} \left( \frac{100s(s+2)(s+5)}{s^2(s+8)(s+12)} \right)$$

$$K_v = \infty$$

$$K_a = \lim_{s \rightarrow 0} s^2 G(s)$$

$$K_a = \lim_{s \rightarrow 0} \left( \frac{100s^2(s+2)(s+5)}{s^2(s+8)(s+12)} \right)$$

$$K_a = \left( \frac{100(0+2)(0+5)}{(0+8)(0+12)} \right) = 10.4$$

# Example#1 (Steady State Errors)

$$K_p = \infty$$

$$K_v = \infty$$

$$K_a = 10.4$$

$$e_{ss} = \frac{1}{1 + K_p} = 0$$

$$e_{ss} = \frac{1}{K_v} = 0$$

$$e_{ss} = \frac{1}{K_a} = 0.09$$



# Feedback Control Systems (FCS)

Lecture-6

Time Domain Analysis of 2<sup>nd</sup> Order Systems

التحليل الزمني لمنظومة الدرجة الثانية

# التحليل الزمني لمنظومة ذات الدرجة الاولى

$$\frac{dy(t)}{dt} + \frac{1}{\tau}y(t) = \frac{1}{\tau}f(t)$$

where  $\tau$  is known as the **time constant** of the system, which is a measure of how fast the system responds to initial conditions of external excitations. Note that the input in Eq. (5-82) is scaled by  $\frac{1}{\tau}$  for cosmetic reasons.

For a unit-step input

$$f(t) = u_s(t) = \begin{cases} 0, & t < 0, \\ 1, & t \geq 0 \end{cases}$$

If  $y(0) = \dot{y}(0) = 0$ ,  $\mathcal{L}(u_s(t)) = \frac{1}{s}$  and  $\mathcal{L}(y(t)) = Y(s)$ , then

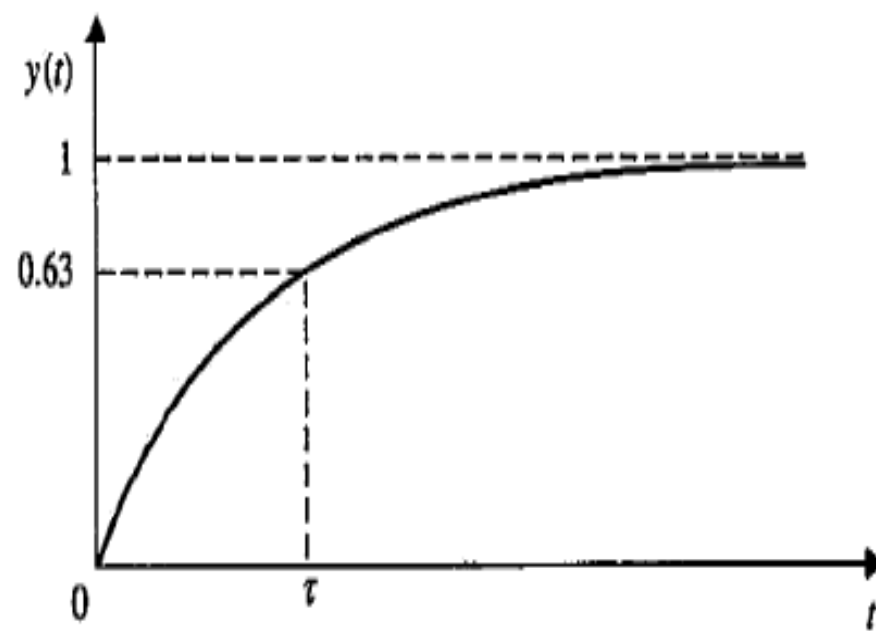
$$Y(s) = \frac{1}{s} \frac{1/\tau}{s + 1/\tau}$$

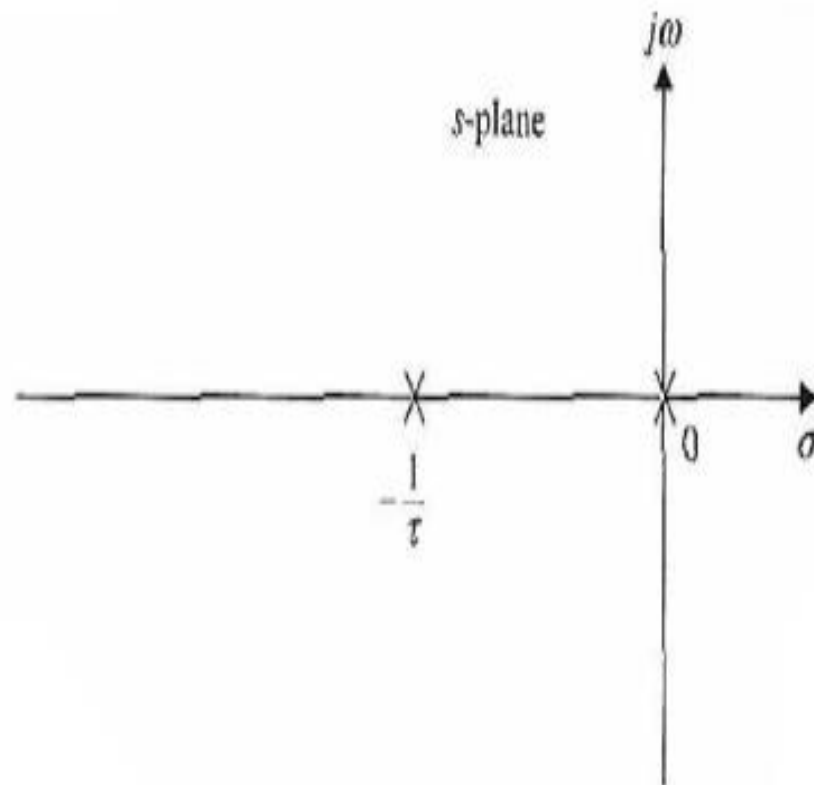
Applying the inverse Laplace transform to Eq. (5-84), we get the time response of Eq.

$$y(t) = 1 - e^{-t/\tau}$$

where  $\tau$  is the time for  $y(t)$  to reach 63% of its final value of  $\lim_{t \rightarrow \infty} y(t) = 1$ .

Fig. 5-11 shows typical unit-step responses of  $y(t)$  for a general value of  $\tau$ . As the value of time constant  $\tau$  decreases, the system response approaches faster to the final value. The step response will not have any overshoot for any combination of system parameters.





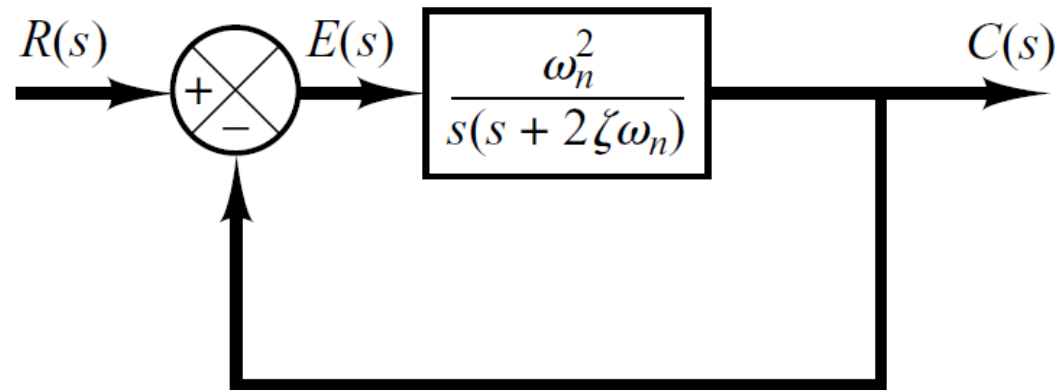
shows the location of the pole at  $s = -\frac{1}{\tau}$  in the  $s$ -plane of the system transfer function. For positive  $\tau$ , the pole at  $s = -\frac{1}{\tau}$  will always stay in the left-half  $s$ -plane, and the system is always stable.

# Introduction منظومة ذات الدرجة الثانية

- We have already discussed the affect of location of poles and zeros on the transient response of 1<sup>st</sup> order systems.
- Compared to the simplicity of a first-order system, a second-order system exhibits a wide range of responses that must be analyzed and described.
- Varying a first-order system's parameter (T, K) simply changes the speed and offset of the response
- Whereas, changes in the parameters of a second-order system can change the *form of* the response.
- A second-order system *منظومة ذات الدرجة الثانية* can display characteristics much like a first-order system or, depending on component values, display damped or *pure oscillations* for its *transient response*.

# الشكل القياسي لمنظومة من الدرجة الثانية

- A general second-order system is characterized by the following transfer function.



$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

# Introduction

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$\omega_n$   $\longrightarrow$  **un-damped natural frequency** of the second order system, which is the frequency of oscillation of the system without damping.

$\zeta$   $\longrightarrow$  **damping ratio** of the second order system, which is a measure of the degree of resistance to change in the system output.

# Example#1

- Determine the un-damped natural frequency and damping ratio of the following second order system.

$$\frac{C(s)}{R(s)} = \frac{4}{s^2 + 2s + 4}$$

- Compare the numerator and denominator of the given transfer function with the general 2<sup>nd</sup> order transfer function.

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$\omega_n^2 = 4 \quad \Rightarrow \quad \omega_n = 2 \text{ rad/sec}$$

$$\Rightarrow 2\zeta\omega_n s = 2s$$

$$\cancel{s^2} + 2\zeta\omega_n s + \cancel{\omega_n^2} = \cancel{s^2} + 2s + \cancel{4}$$

$$\Rightarrow \zeta\omega_n = 1$$

$$\Rightarrow \zeta = 0.5$$



# Introduction

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

- Two poles of the system are

$$-\omega_n\zeta + \omega_n\sqrt{\zeta^2 - 1}$$

$$-\omega_n\zeta - \omega_n\sqrt{\zeta^2 - 1}$$

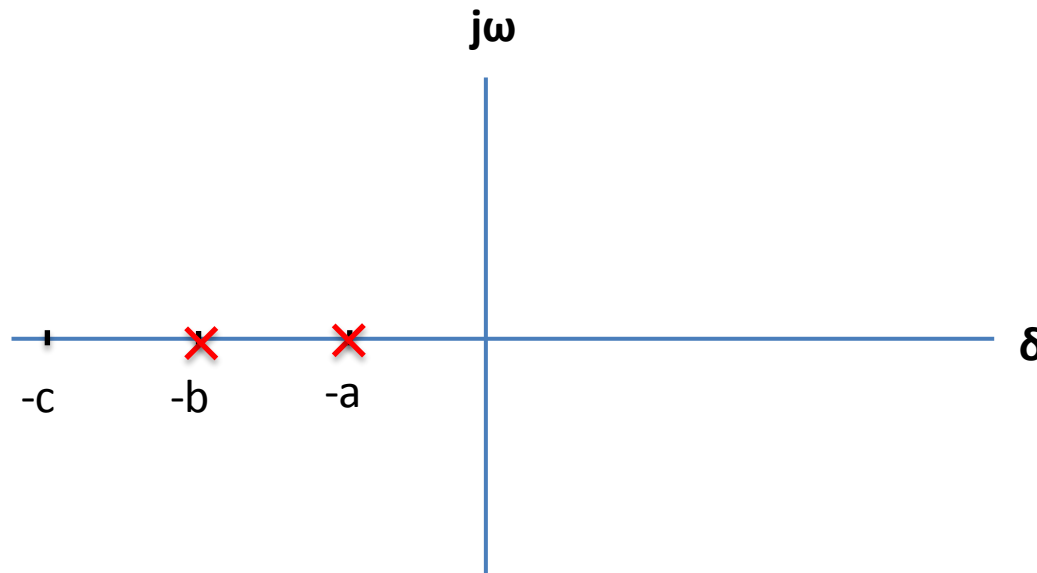
# Introduction

$$-\omega_n\zeta + \omega_n\sqrt{\zeta^2 - 1}$$

$$-\omega_n\zeta - \omega_n\sqrt{\zeta^2 - 1}$$

- According the value of  $\zeta$ , a second-order system can be set into one of the four categories:

*1. Overdamped* - when the system has two real distinct poles ( $\zeta > 1$ ).



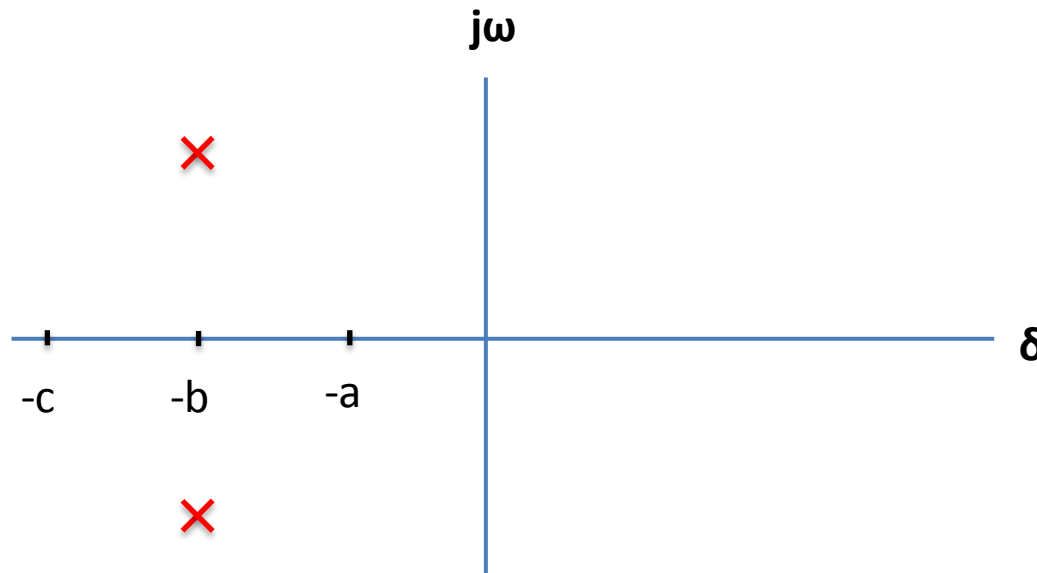
# Introduction

$$-\omega_n \zeta + \omega_n \sqrt{\zeta^2 - 1}$$

$$-\omega_n \zeta - \omega_n \sqrt{\zeta^2 - 1}$$

- According the value of  $\zeta$ , a second-order system can be set into one of the four categories:

2. *Underdamped* - when the system has two complex conjugate poles ( $0 < \zeta < 1$ )



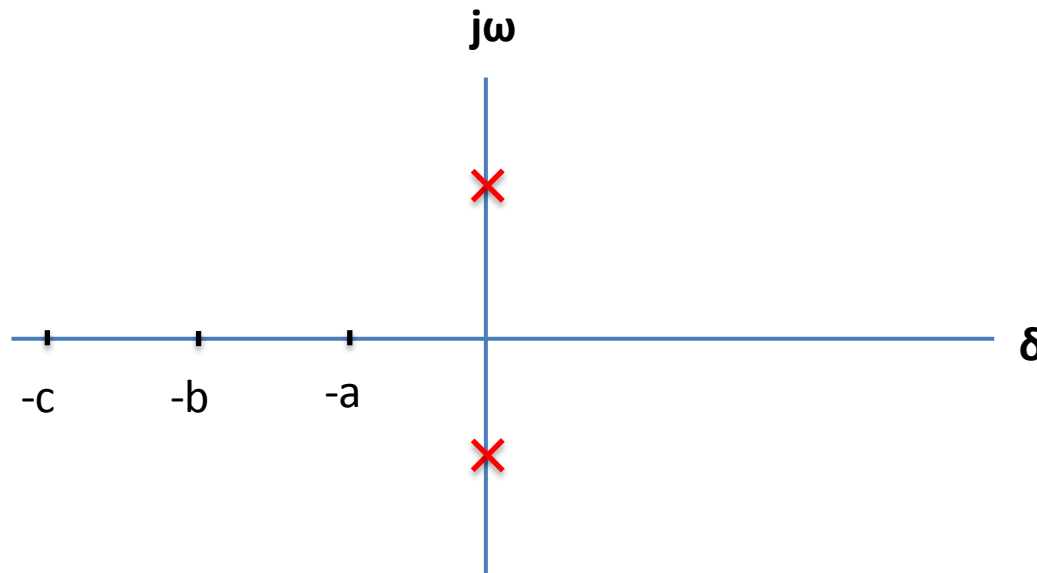
# Introduction

$$-\omega_n \zeta + \omega_n \sqrt{\zeta^2 - 1}$$

$$-\omega_n \zeta - \omega_n \sqrt{\zeta^2 - 1}$$

- According the value of  $\zeta$ , a second-order system can be set into one of the four categories:

3. *Undamped* - when the system has two imaginary poles ( $\zeta = 0$ ).



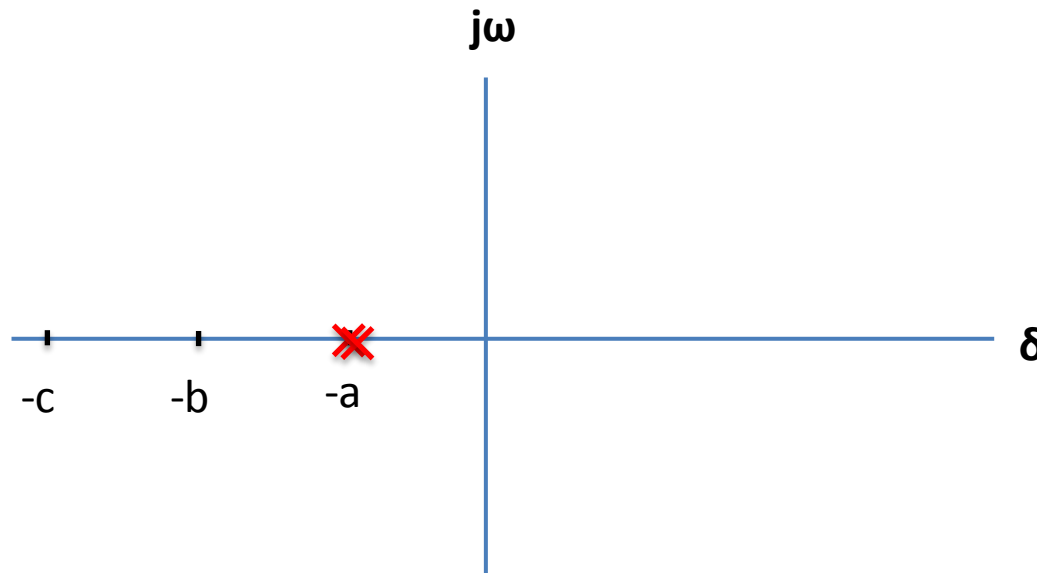
# Introduction

$$-\omega_n \zeta + \omega_n \sqrt{\zeta^2 - 1}$$

$$-\omega_n \zeta - \omega_n \sqrt{\zeta^2 - 1}$$

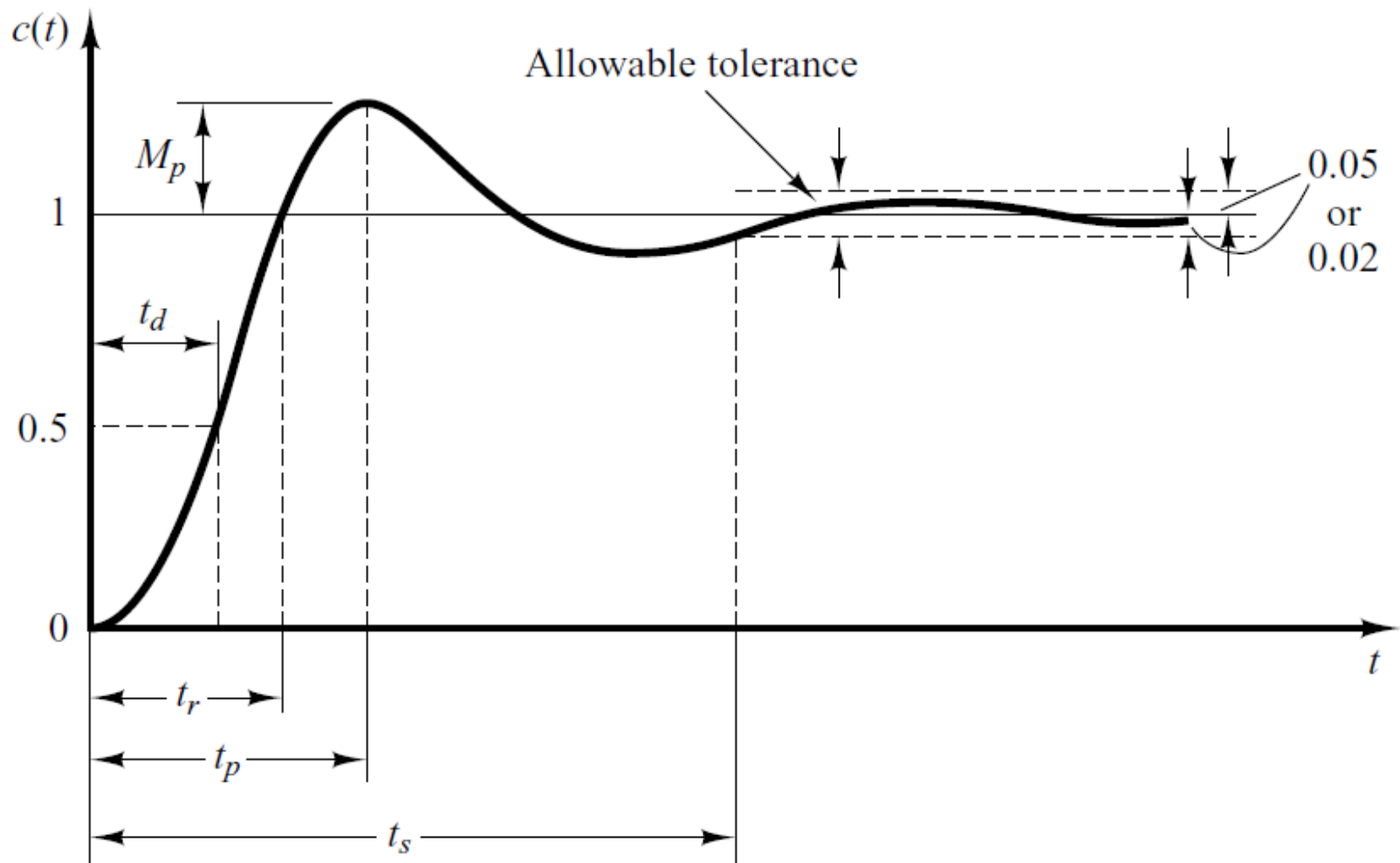
- According the value of  $\zeta$ , a second-order system can be set into one of the four categories:

4. *Critically damped* - when the system has two real but equal poles ( $\zeta = 1$ ).



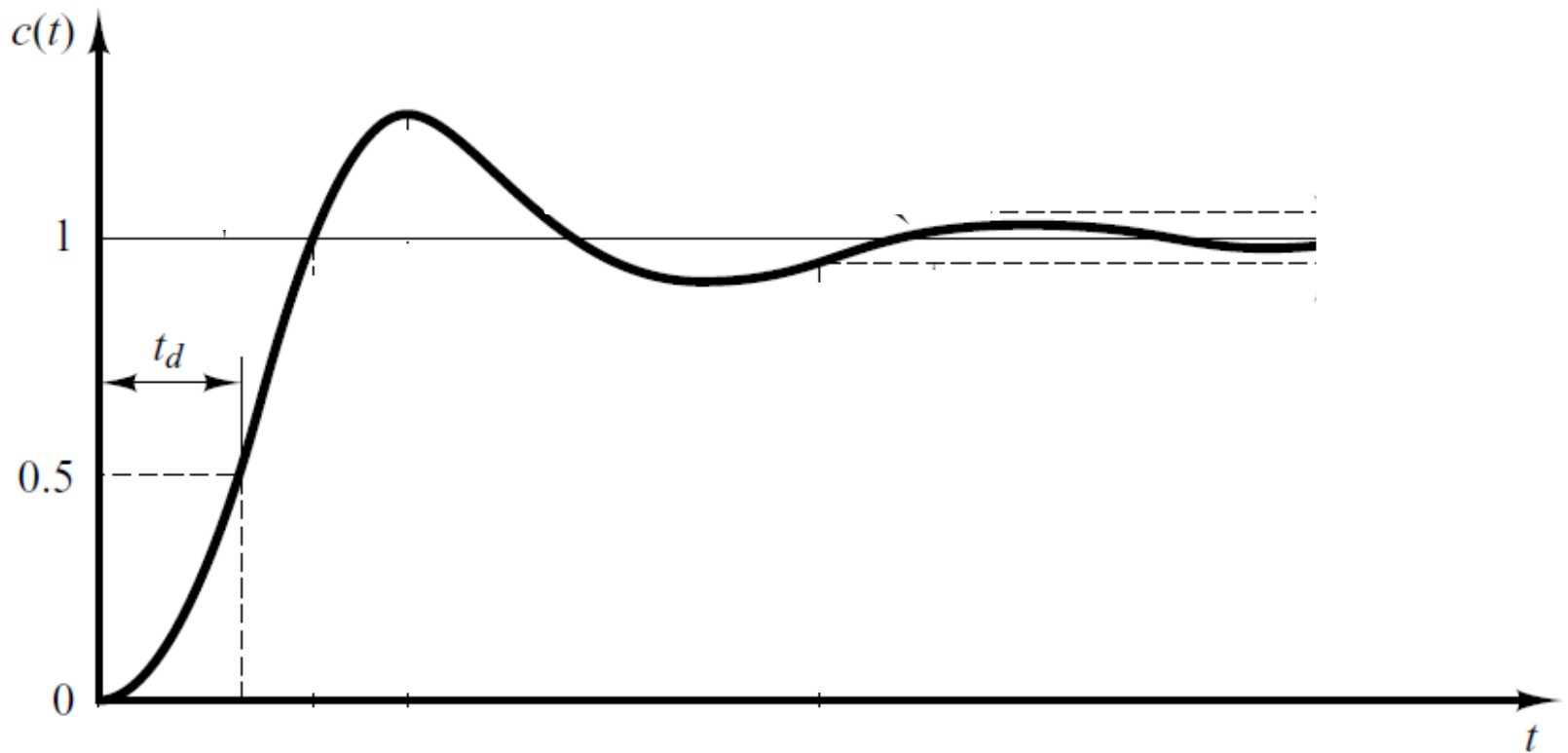
# الموصفات الزمنية Time-Domain Specification

For  $0 < \zeta < 1$  and  $\omega_n > 0$ , the 2<sup>nd</sup> order system's response due to a unit step input looks like



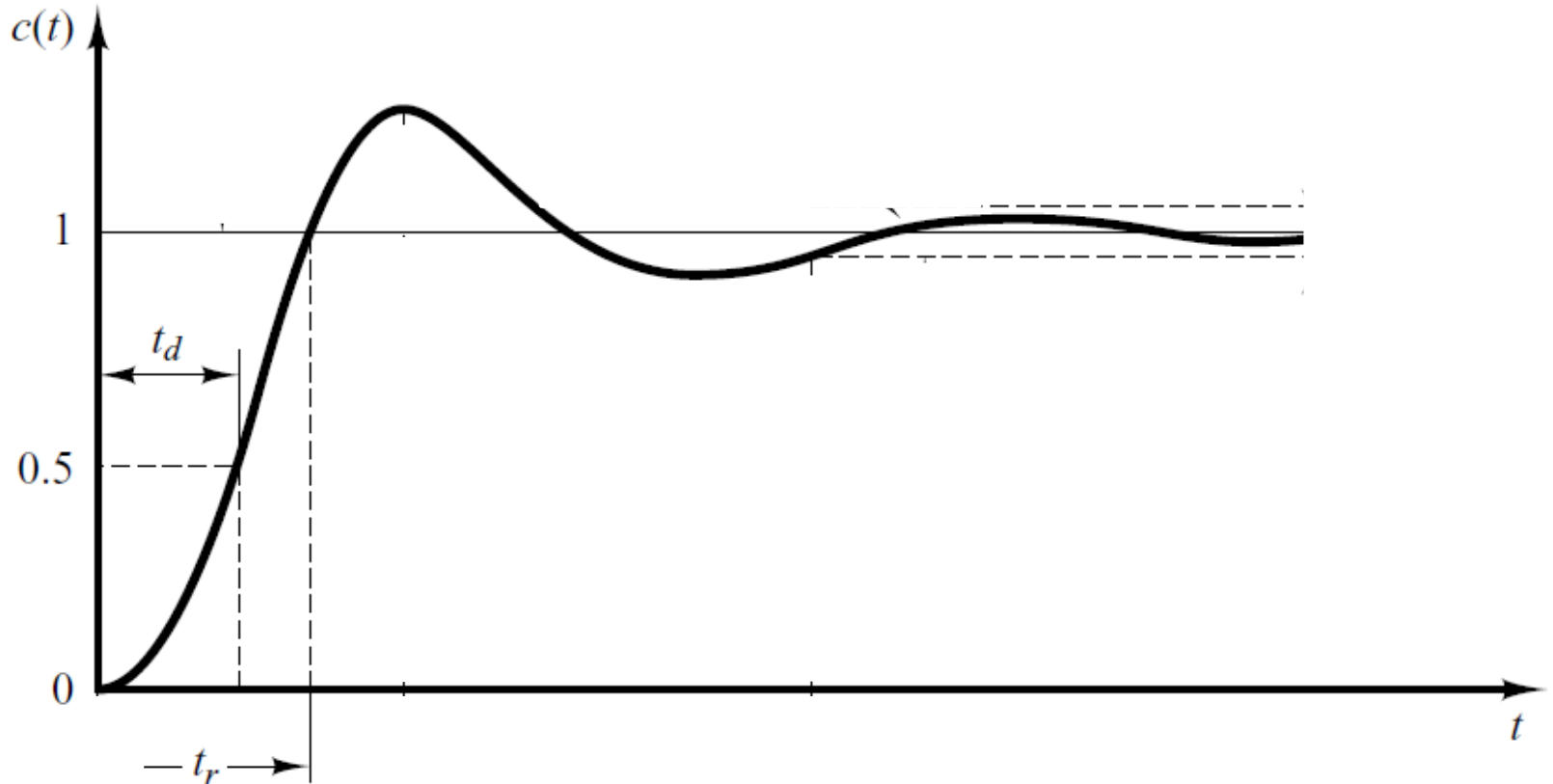
# Time-Domain Specification

- The delay ( $t_d$ ) time is the time required for the response to reach half the final value the very first time.



# Time-Domain Specification

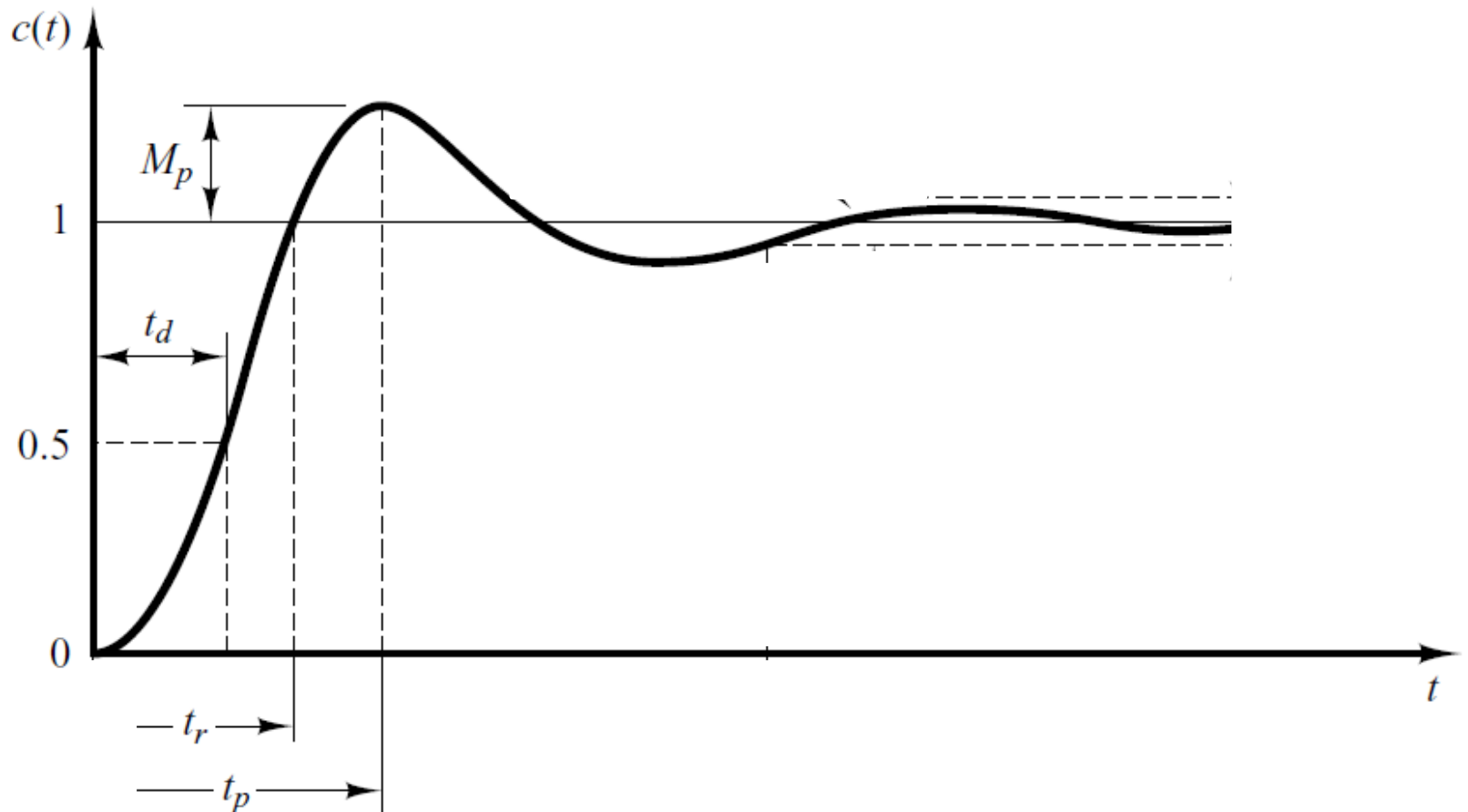
- The rise time is the time required for the response to rise from 10% to 90%, 5% to 95%, or 0% to 100% of its final value.
- For underdamped second order systems, the 0% to 100% rise time is normally used. For overdamped systems, the 10% to 90% rise time is commonly used.





# Time-Domain Specification

- The peak time is the time required for the response to reach the first peak of the overshoot.



# Time-Domain Specification

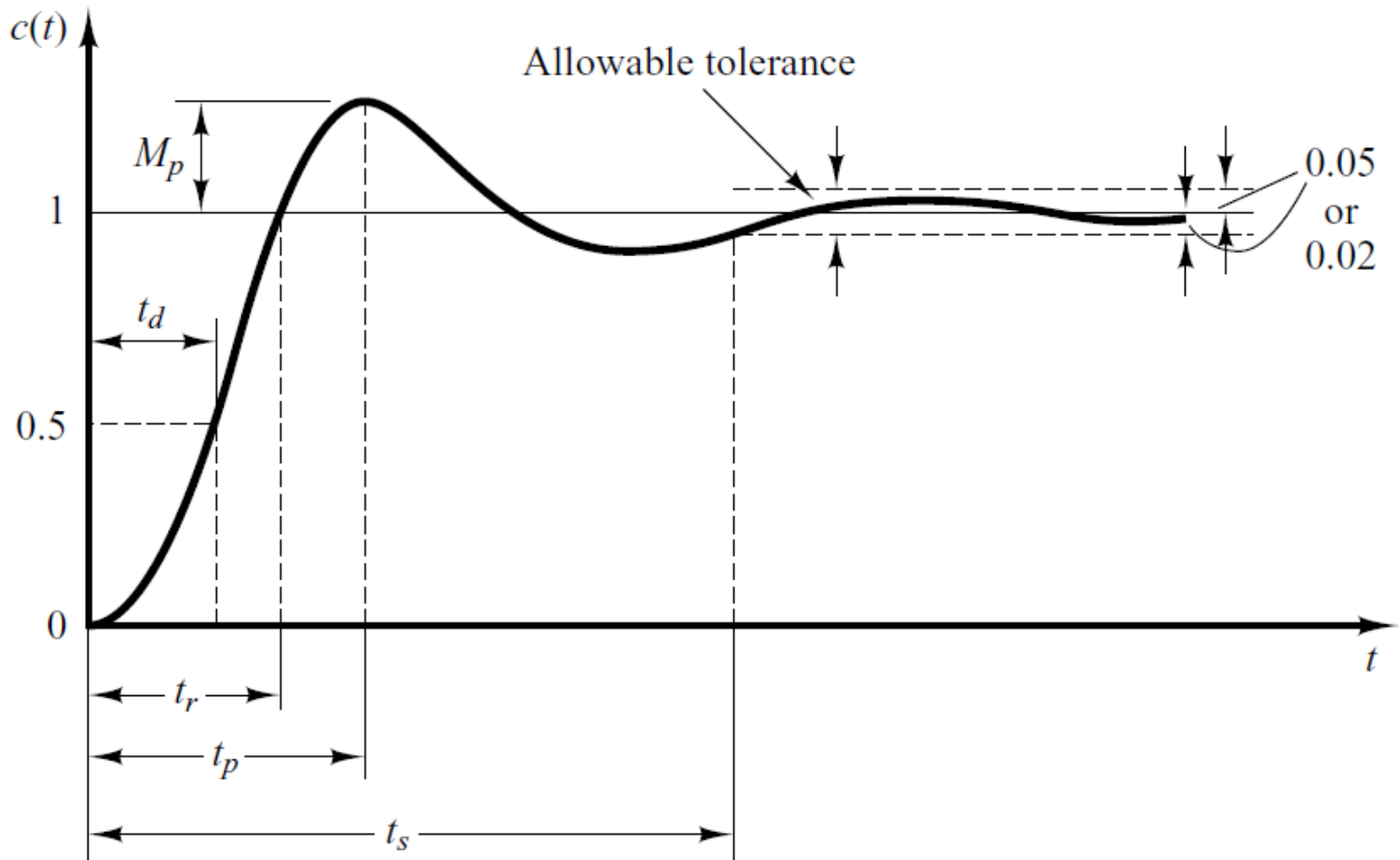
The maximum overshoot is the maximum peak value of the response curve measured from unity. If the final steady-state value of the response differs from unity, then it is common to use the maximum percent overshoot. It is defined by

$$\text{Maximum percent overshoot} = \frac{c(t_p) - c(\infty)}{c(\infty)} \times 100\%$$

The amount of the maximum (percent) overshoot directly indicates the relative stability of the system.

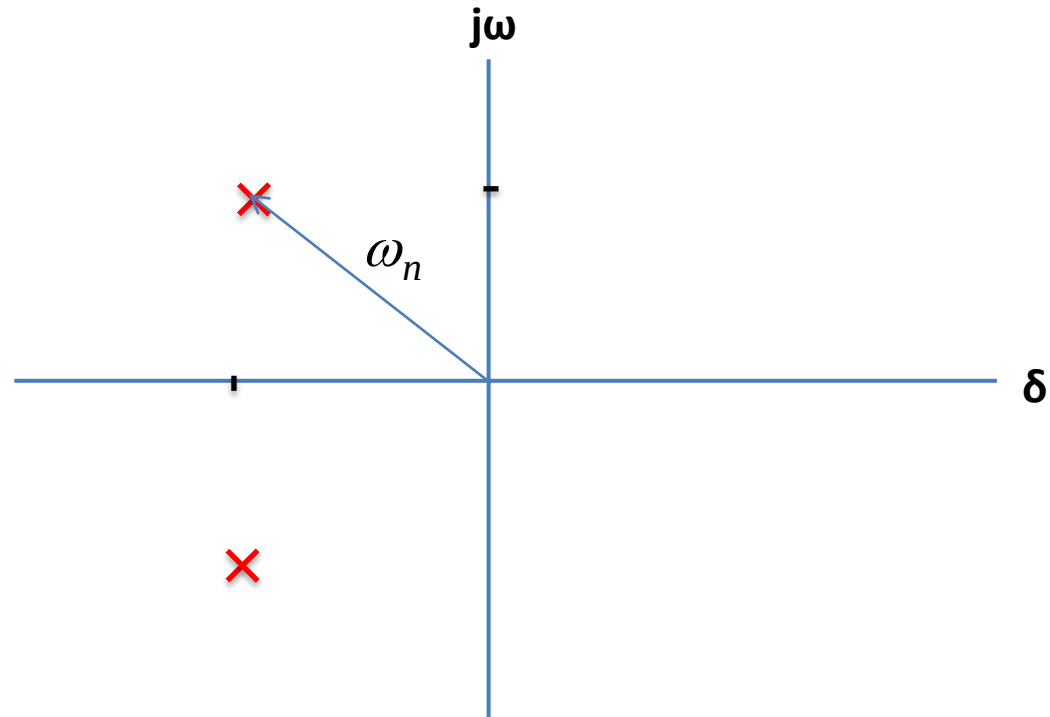
# Time-Domain Specification

- The settling time is the time required for the response curve to reach and stay within a range about the final value of size specified by absolute percentage of the final value (usually 2% or 5%).



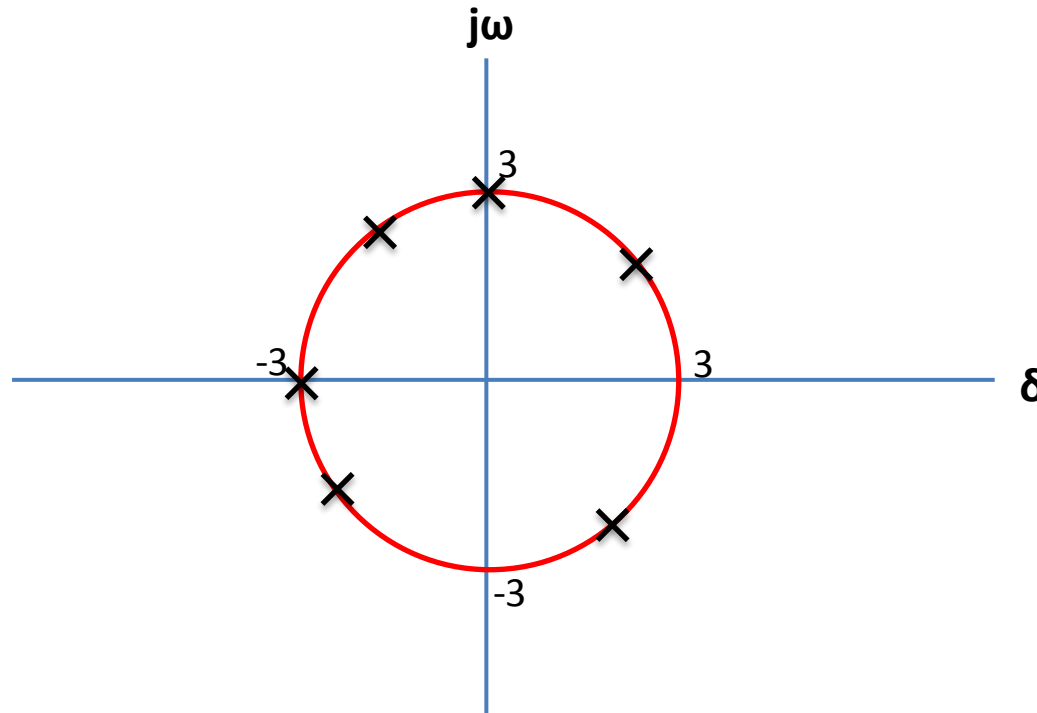
# S-Plane

- Natural Undamped Frequency.
- Distance from the origin of s-plane to pole is natural undamped frequency in rad/sec.



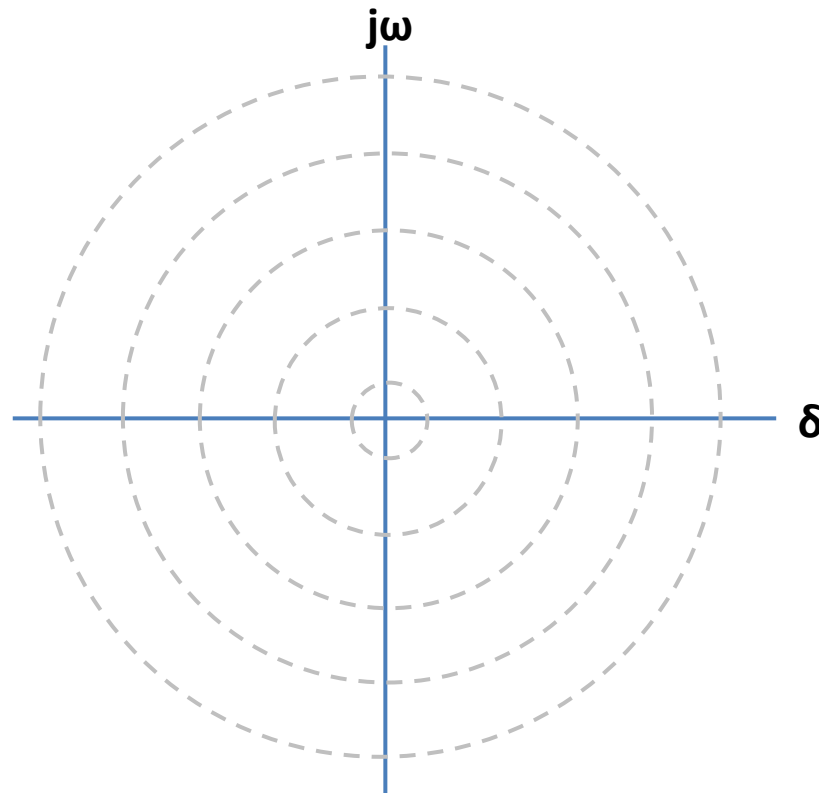
# S-Plane

- Let us draw a circle of radius 3 in s-plane.
- If a pole is located anywhere on the circumference of the circle the natural undamped frequency would be *3 rad/sec*.



# S-Plane

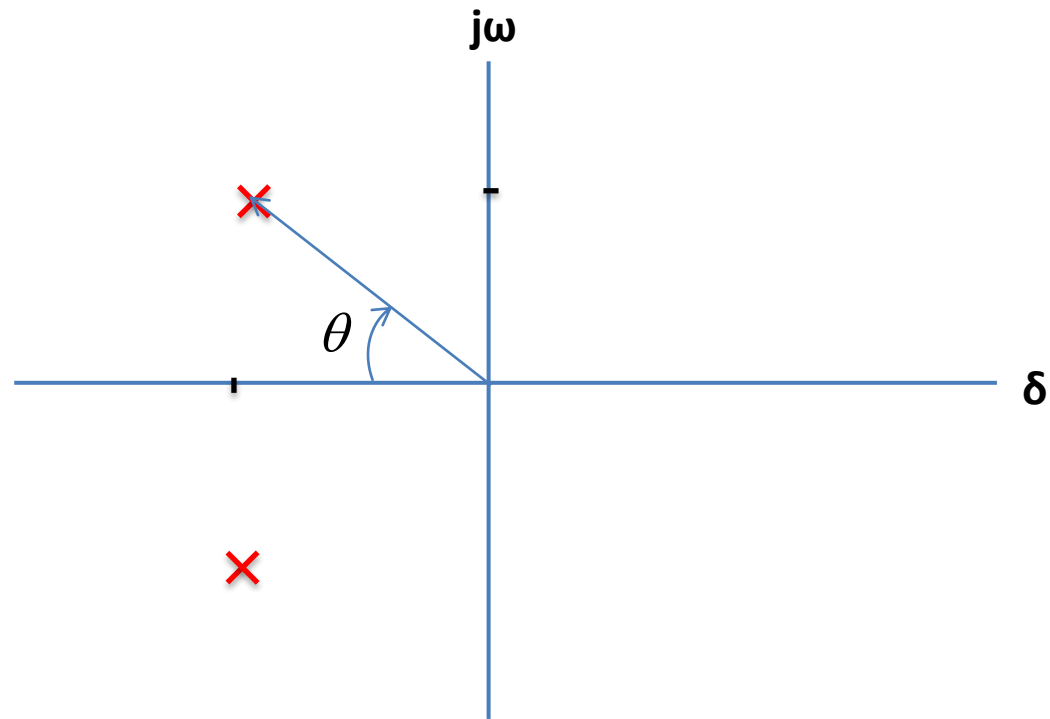
- Therefore the s-plane is divided into Constant Natural Undamped Frequency ( $\omega_n$ ) Circles.



# S-Plane

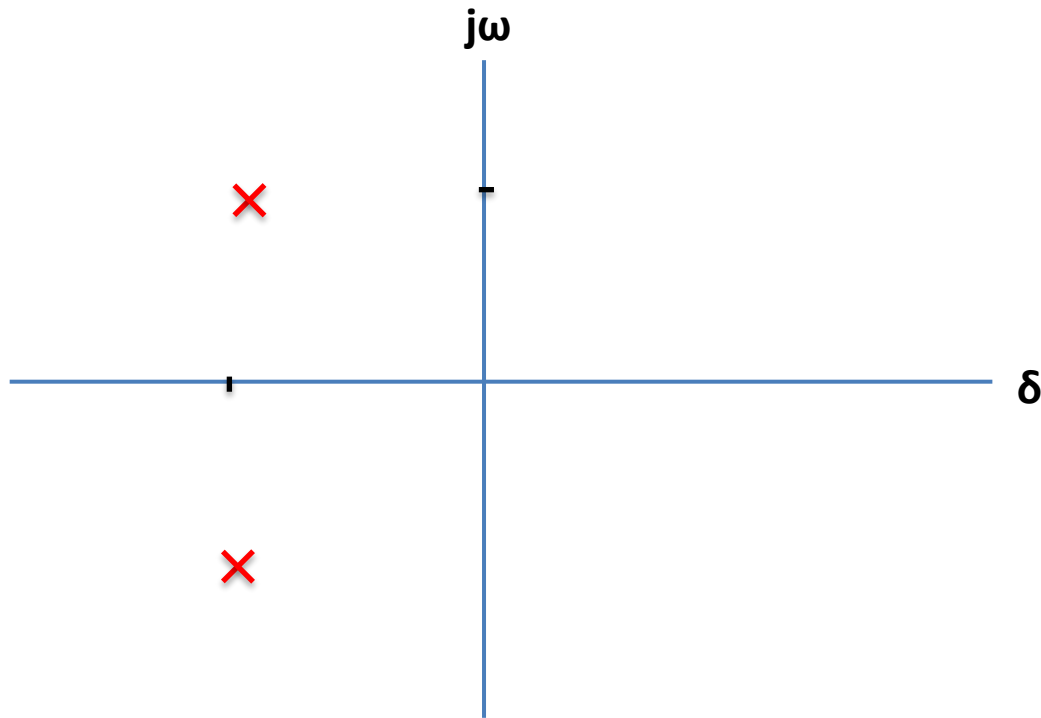
- Damping ratio.
- Cosine of the angle between vector connecting origin and pole and -ve real axis yields damping ratio.

$$\zeta = \cos \theta$$



# S-Plane

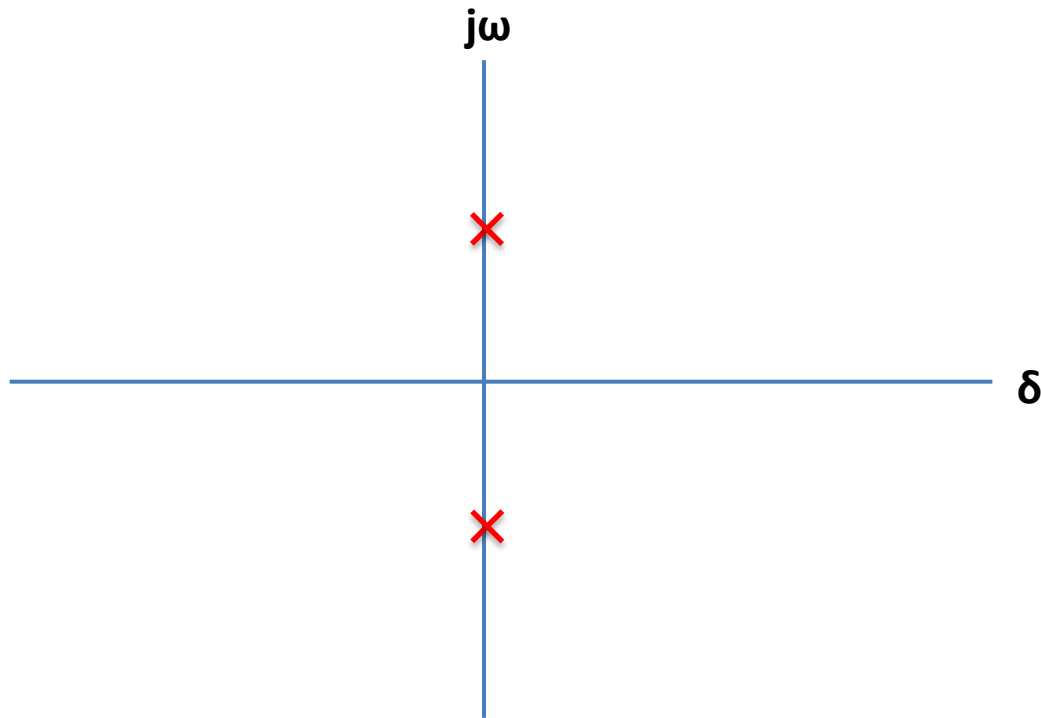
- For Underdamped system  $0^\circ < \theta < 90^\circ$  therefore,  $0 < \zeta < 1$





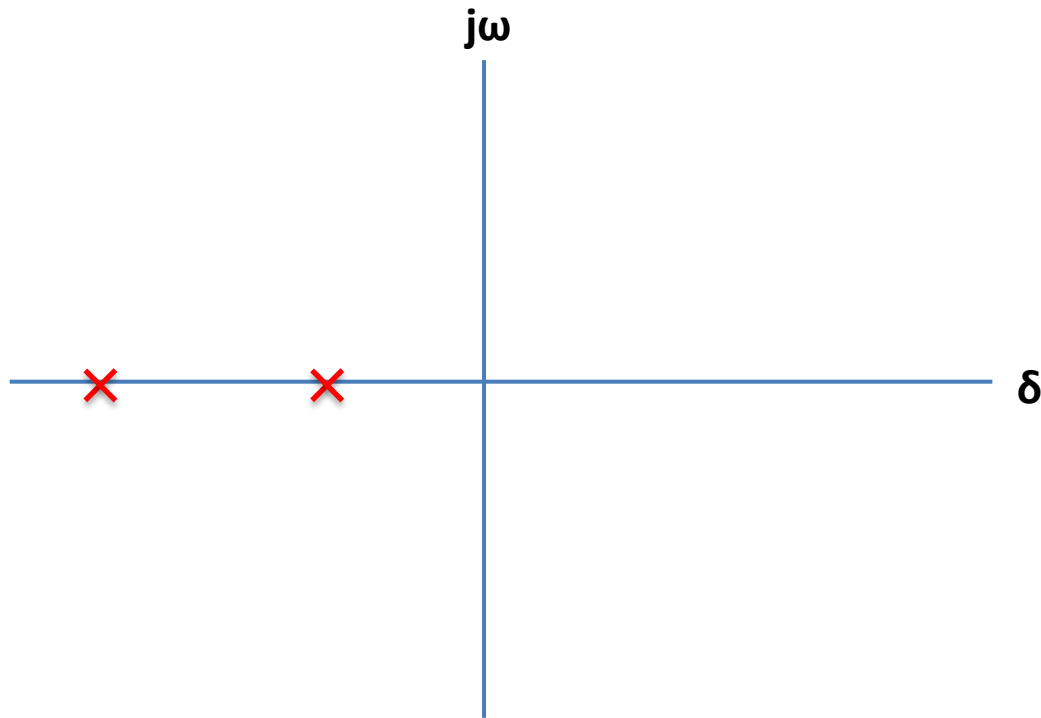
# S-Plane

- For Undamped system  $\theta = 90^\circ$  therefore,  $\zeta = 0$



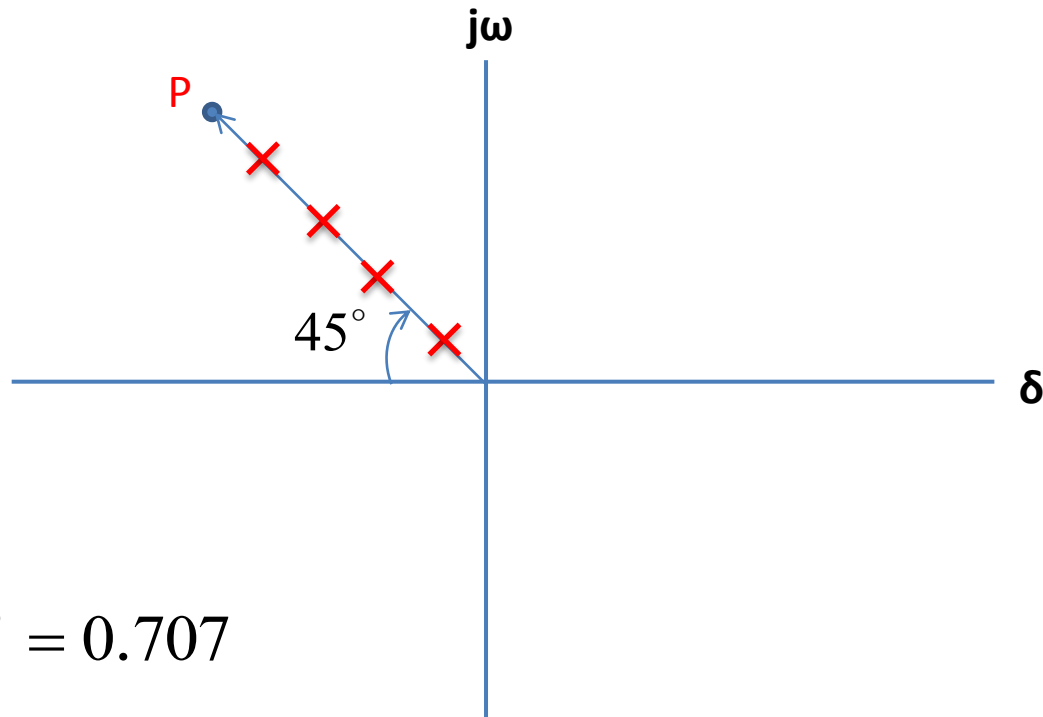
# S-Plane

- For overdamped and critically damped systems  $\theta = 0^\circ$   
therefore,  $\zeta \geq 0$



# S-Plane

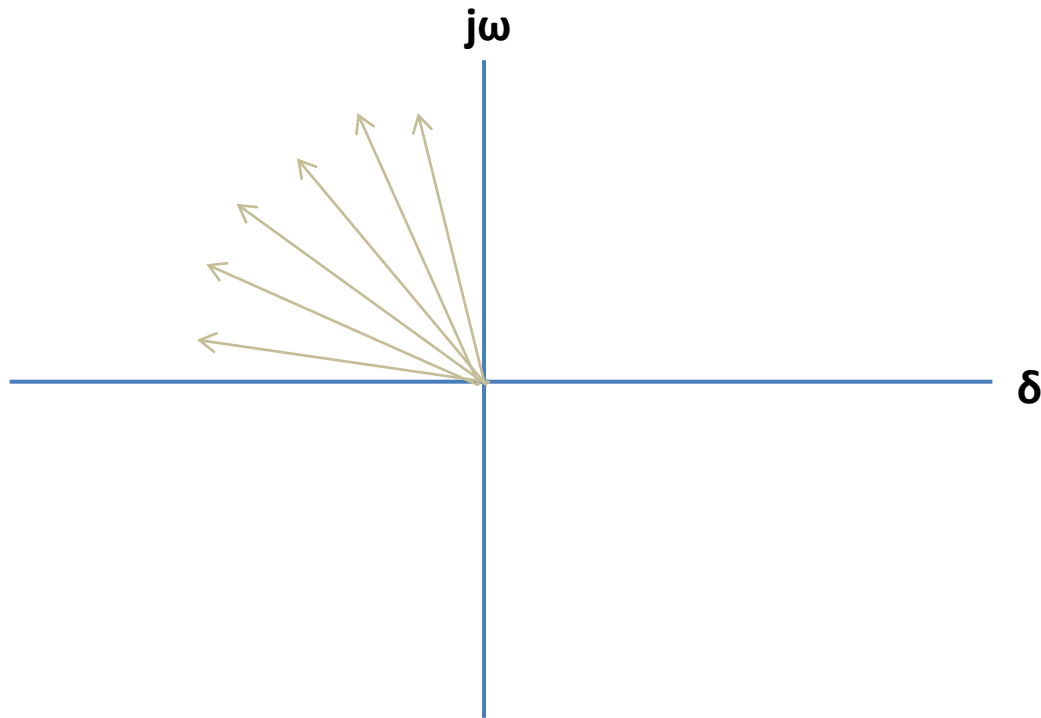
- Draw a vector connecting **origin** of s-plane and some point **P**.



$$\zeta = \cos 45^\circ = 0.707$$

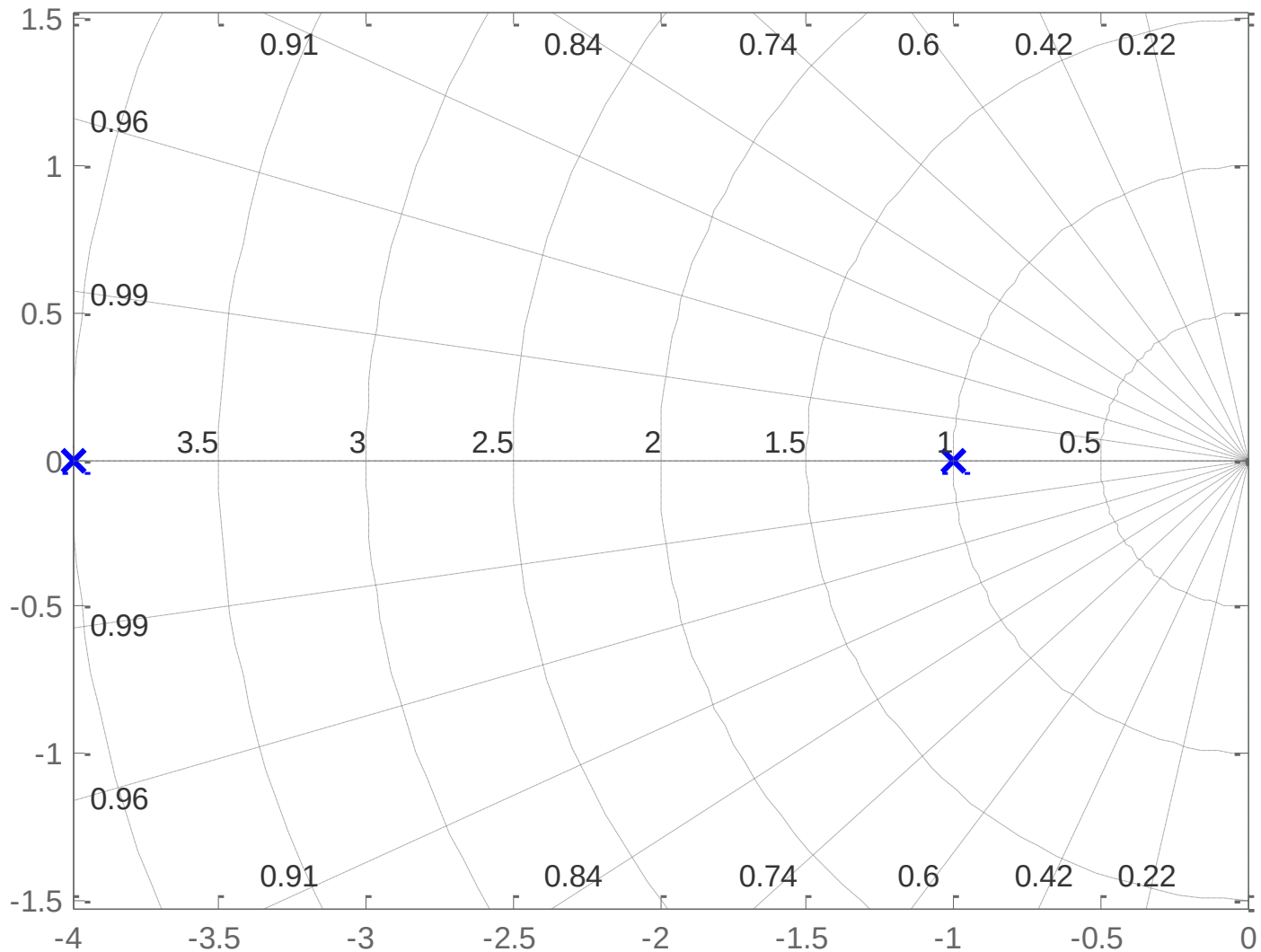
# S-Plane

- Therefore, s-plane is divided into sections of constant damping ratio lines.



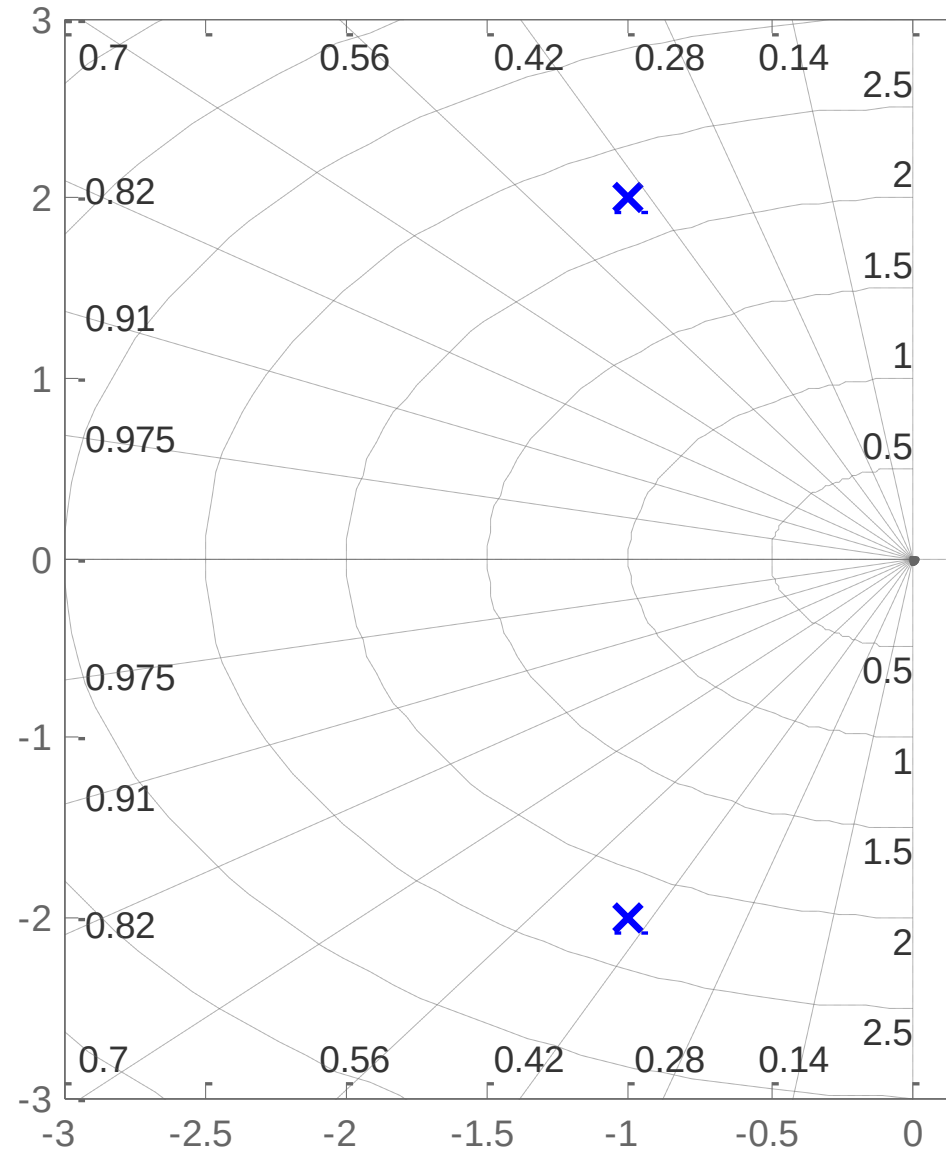
# Example-2

- Determine the natural frequency and damping ratio of the poles from the following pz-map.



# Example-3

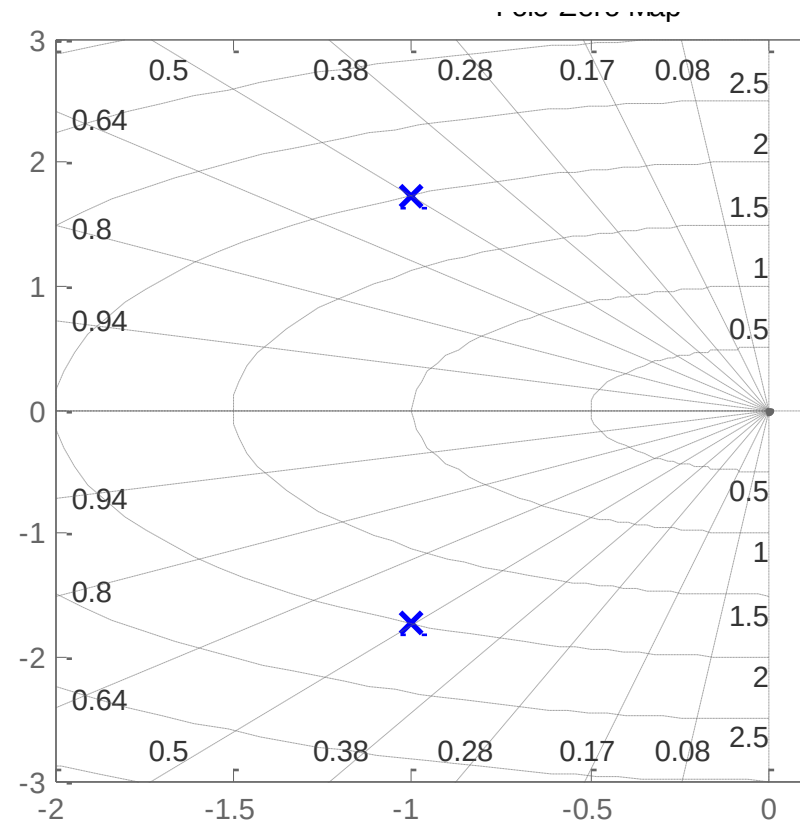
- Determine the natural frequency and damping ratio of the poles from the given pz-map.
- Also determine the transfer function of the system and state whether system is underdamped, overdamped, undamped or critically damped.



# Example-4

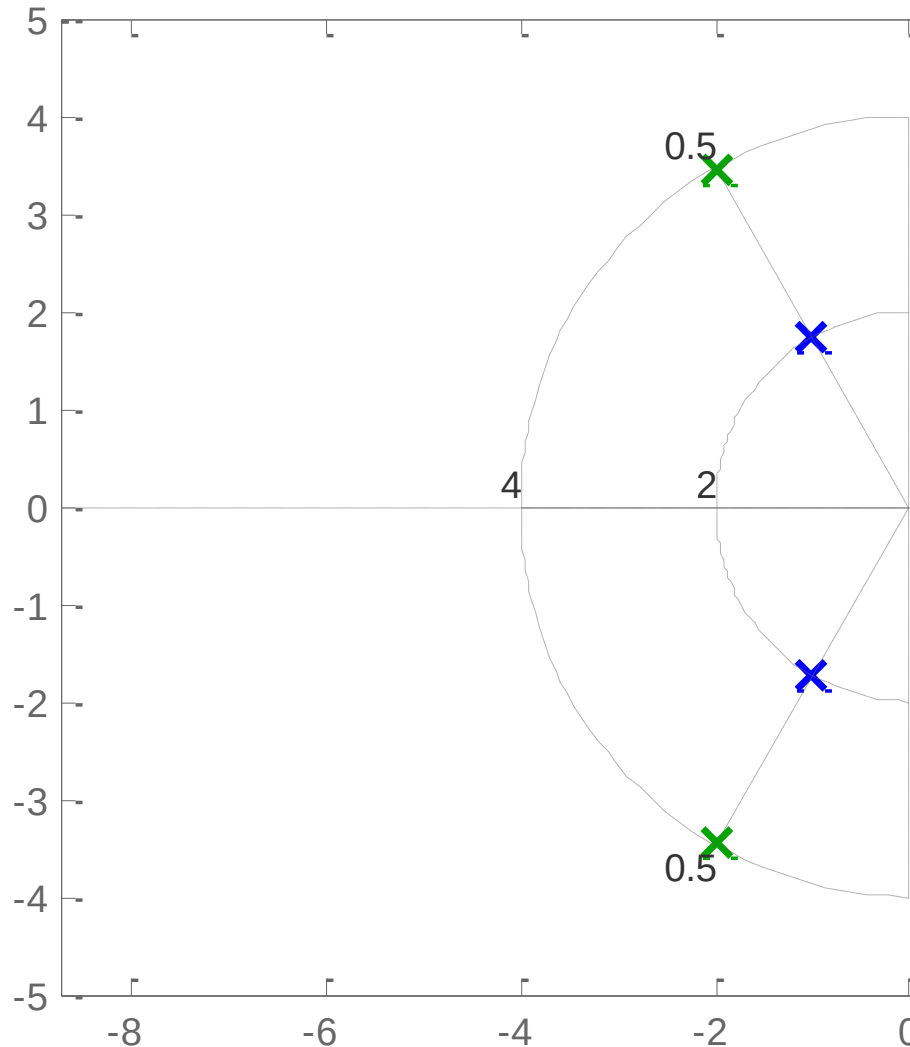
- The natural frequency of closed loop poles of 2<sup>nd</sup> order system is **2 rad/sec** and damping ratio is **0.5**.
- Determine the location of closed loop poles so that the damping ratio remains same but the natural undamped frequency is doubled.

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{4}{s^2 + 2s + 4}$$



# Example-4

Determine the location of closed loop poles so that the damping ratio remains same but the natural undamped frequency is doubled.

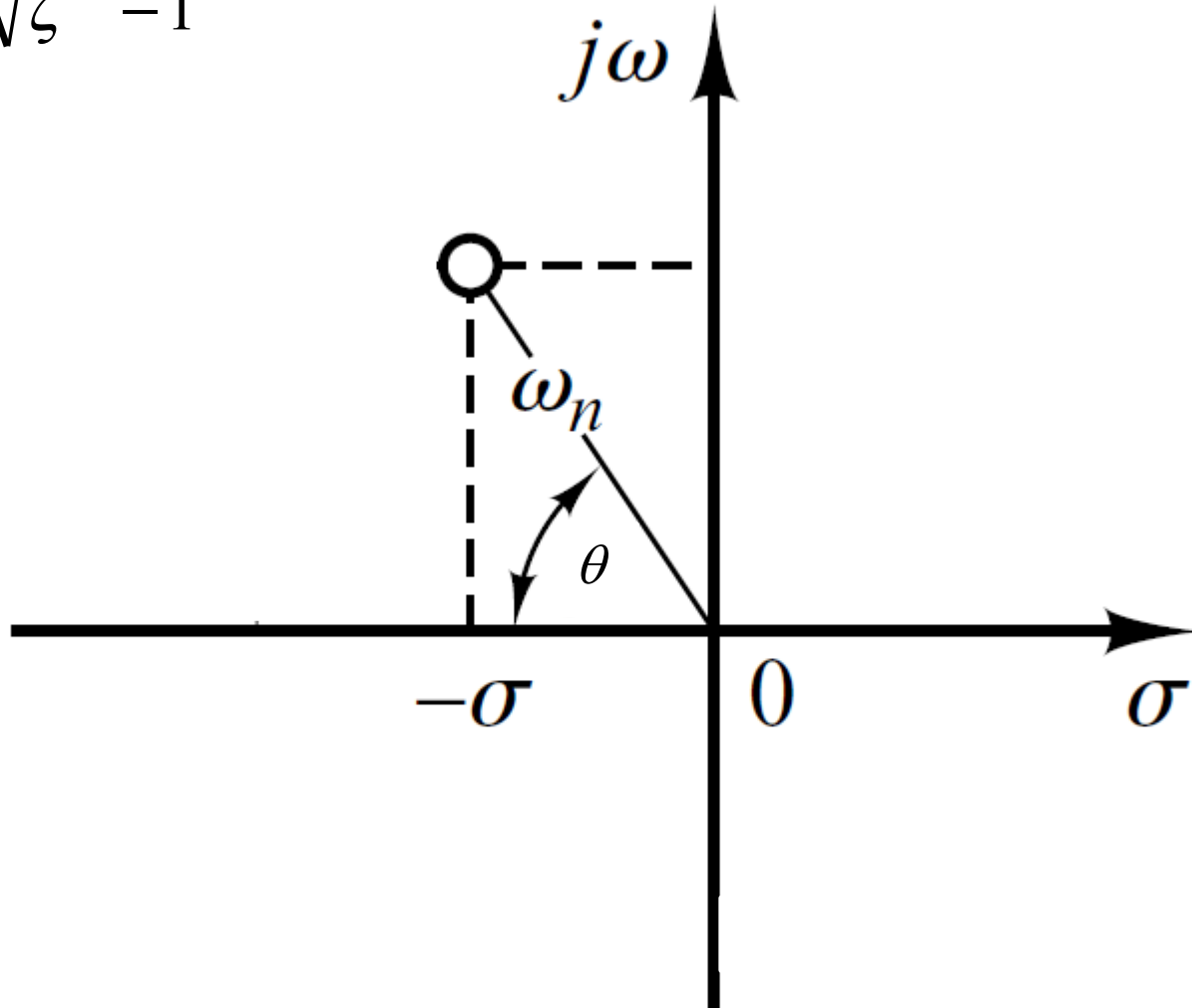




# S-Plane

$$-\omega_n \zeta + \omega_n \sqrt{\zeta^2 - 1}$$

$$-\omega_n \zeta - \omega_n \sqrt{\zeta^2 - 1}$$



# Step Response of underdamped System

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \xrightarrow{\text{Step Response}} C(s) = \frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)}$$

- The partial fraction expansion of above equation is given as

$$C(s) = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$(s + 2\zeta\omega_n)^2 \quad C(s) = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \zeta^2 \omega_n^2 + \omega_n^2 - \zeta^2 \omega_n^2} \quad \omega_n^2(1 - \zeta^2)$$

$$C(s) = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_n^2(1 - \zeta^2)}$$

# Step Response of underdamped System

$$C(s) = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_n^2(1 - \zeta^2)}$$

- Above equation can be written as

$$C(s) = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2}$$

- Where  $\omega_d = \omega_n \sqrt{1 - \zeta^2}$ , is the frequency of transient oscillations and is called **damped natural frequency**.
- The inverse Laplace transform of above equation can be obtained easily if **C(s)** is written in the following form:

$$C(s) = \frac{1}{s} - \frac{s + \zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2} - \frac{\zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2}$$

# Step Response of underdamped System

$$C(s) = \frac{1}{s} - \frac{s + \zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2} - \frac{\zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2}$$

$$C(s) = \frac{1}{s} - \frac{s + \zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2} - \frac{\frac{\zeta}{\sqrt{1-\zeta^2}} \omega_n \sqrt{1-\zeta^2}}{(s + \zeta\omega_n)^2 + \omega_d^2}$$

$$C(s) = \frac{1}{s} - \frac{s + \zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2} - \frac{\zeta}{\sqrt{1-\zeta^2}} \frac{\omega_d}{(s + \zeta\omega_n)^2 + \omega_d^2}$$

$$c(t) = 1 - e^{-\zeta\omega_n t} \cos \omega_d t - \frac{\zeta}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin \omega_d t$$

# Step Response of underdamped System

$$c(t) = 1 - e^{-\zeta\omega_n t} \cos \omega_d t - \frac{\zeta}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin \omega_d t$$

$$c(t) = 1 - e^{-\zeta\omega_n t} \left[ \cos \omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin \omega_d t \right]$$

- When  $\zeta = 0$

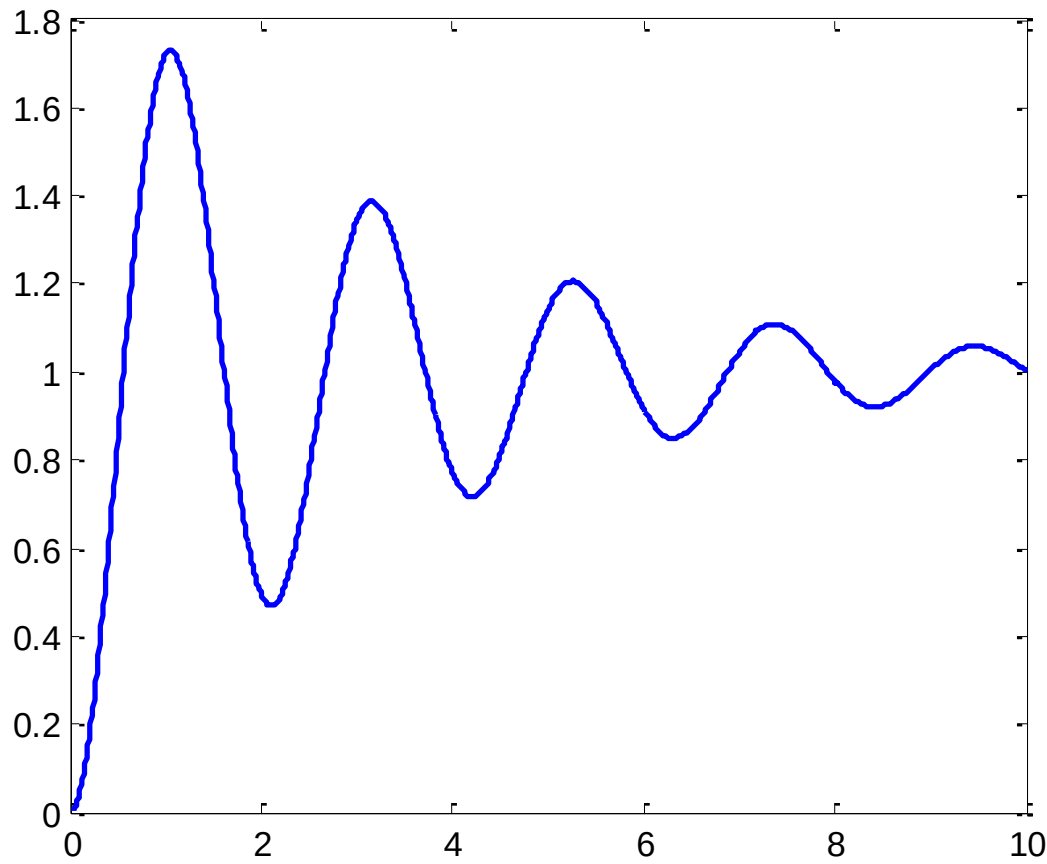
$$\begin{aligned}\omega_d &= \omega_n \sqrt{1-\zeta^2} \\ &= \omega_n\end{aligned}$$

$$c(t) = 1 - \cos \omega_n t$$

# Step Response of underdamped System

$$c(t) = 1 - e^{-\zeta\omega_n t} \left[ \cos \omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin \omega_d t \right]$$

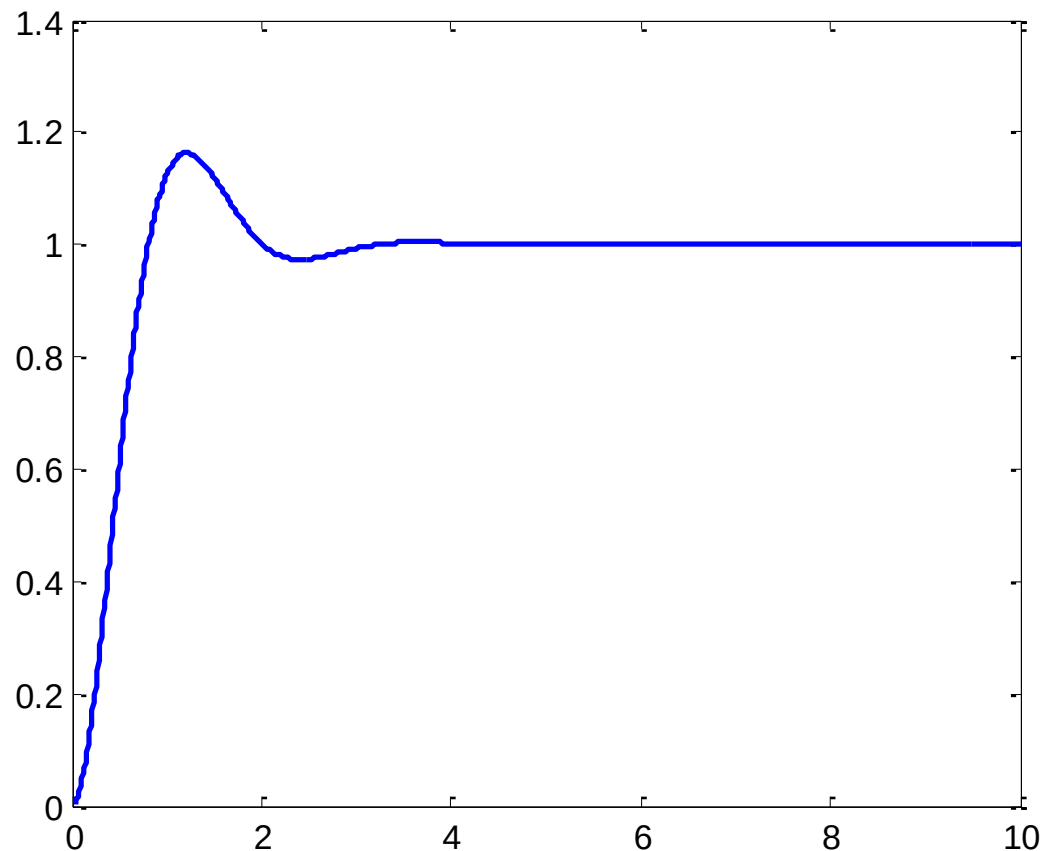
if  $\zeta = 0.1$  and  $\omega_n = 3 \text{ rad/sec}$



# Step Response of underdamped System

$$c(t) = 1 - e^{-\zeta\omega_n t} \left[ \cos \omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin \omega_d t \right]$$

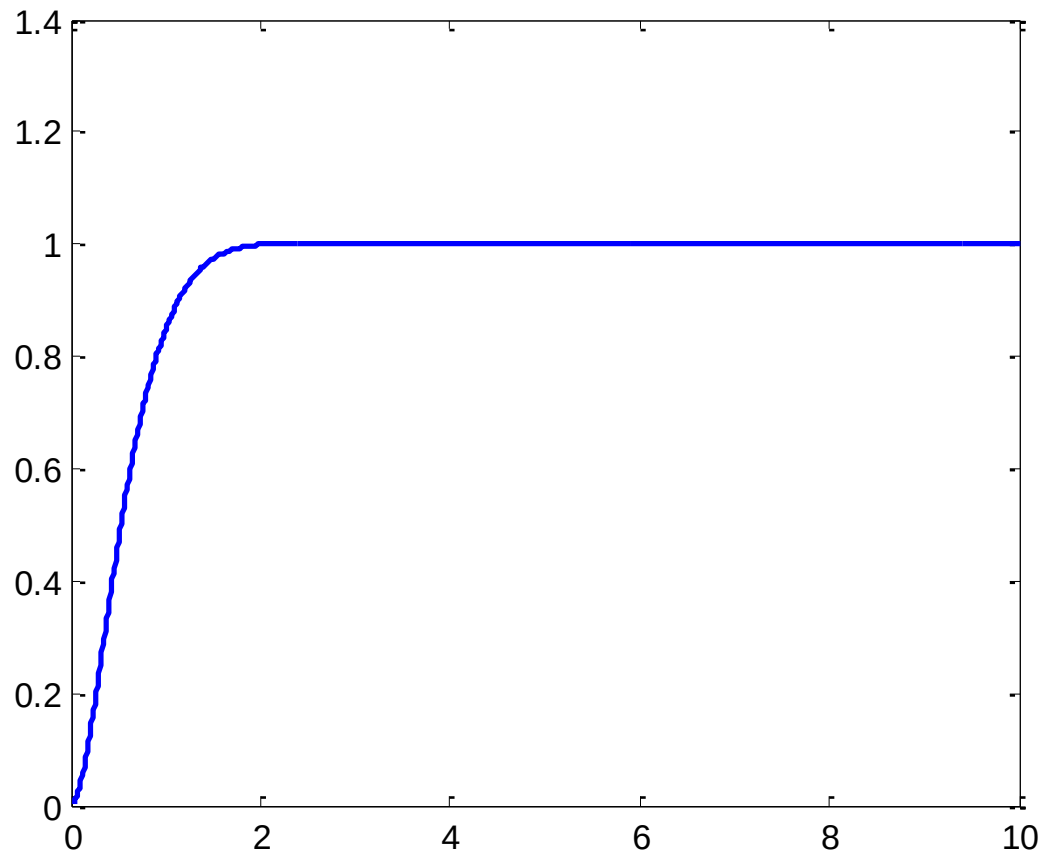
if  $\zeta = 0.5$  and  $\omega_n = 3 \text{ rad/sec}$



# Step Response of underdamped System

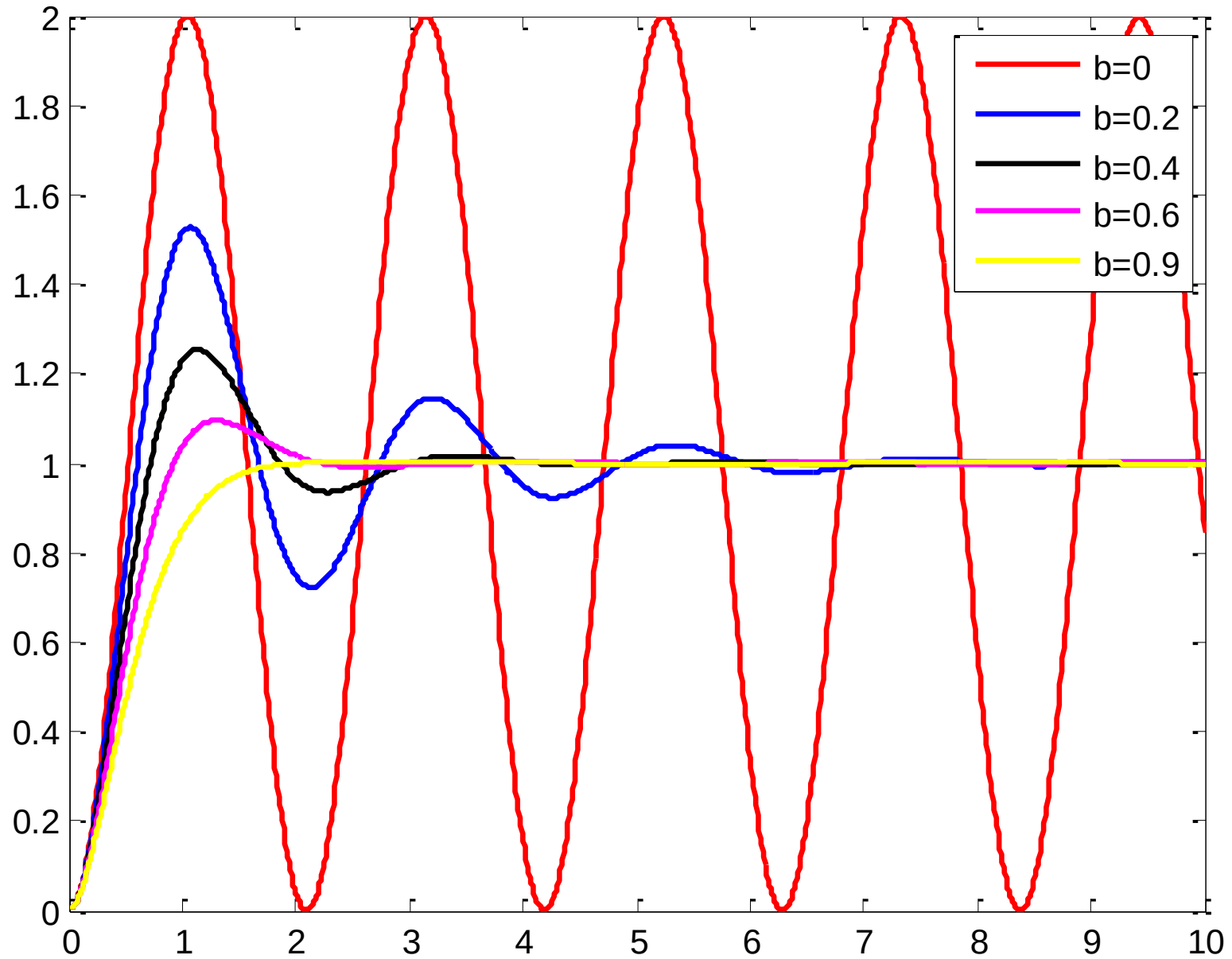
$$c(t) = 1 - e^{-\zeta\omega_n t} \left[ \cos \omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin \omega_d t \right]$$

if  $\zeta = 0.9$  and  $\omega_n = 3 \text{ rad/sec}$

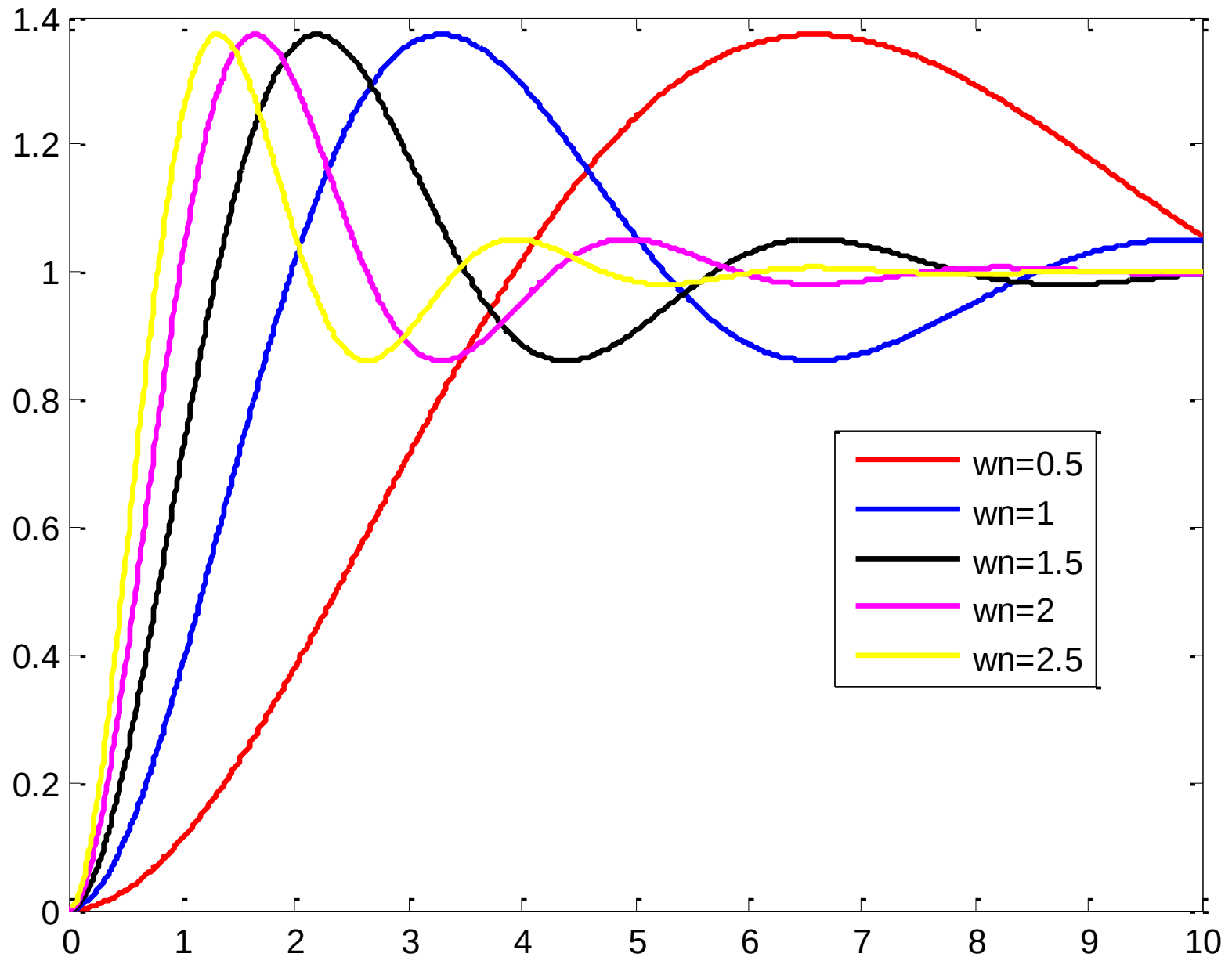




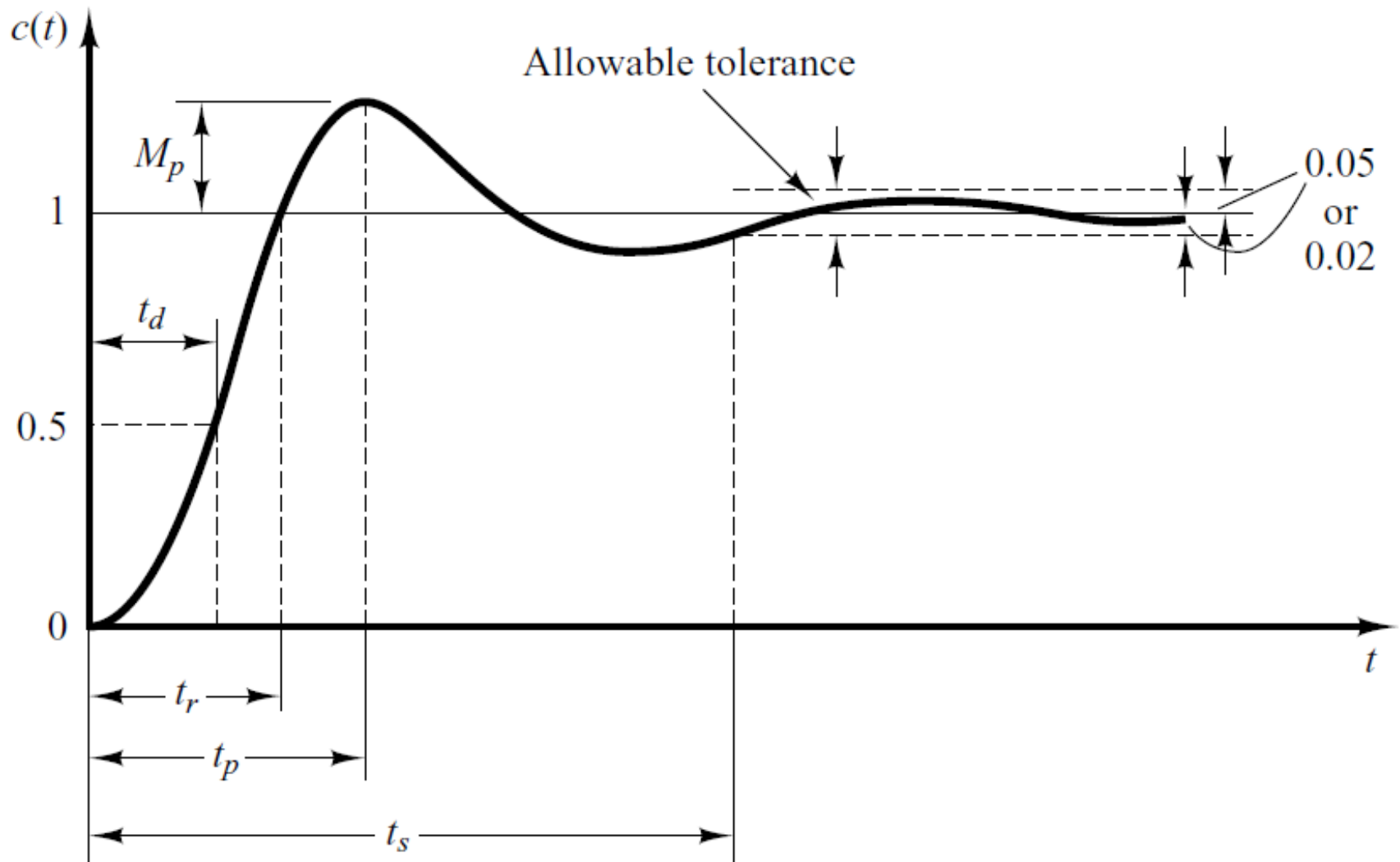
# Step Response of underdamped System



# Step Response of underdamped System



# Time Domain Specifications of Underdamped system



# Time Domain Specifications (Rise Time)

$$c(t) = 1 - e^{-\zeta\omega_n t} \left[ \cos \omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin \omega_d t \right]$$

Put  $t = t_r$  in above equation

$$c(t_r) = 1 - e^{-\zeta\omega_n t_r} \left[ \cos \omega_d t_r + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin \omega_d t_r \right]$$

Where  $c(t_r) = 1$

$$0 = -e^{-\zeta\omega_n t_r} \left[ \cos \omega_d t_r + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin \omega_d t_r \right]$$

$$-e^{-\zeta\omega_n t_r} \neq 0 \quad 0 = \left[ \cos \omega_d t_r + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin \omega_d t_r \right]$$

# Time Domain Specifications (Rise Time)

$$\left[ \cos \omega_d t_r + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin \omega_d t_r \right] = 0$$

above equation can be re-written as

$$\sin \omega_d t_r = -\frac{\sqrt{1-\zeta^2}}{\zeta} \cos \omega_d t_r$$

$$\tan \omega_d t_r = -\frac{\sqrt{1-\zeta^2}}{\zeta}$$

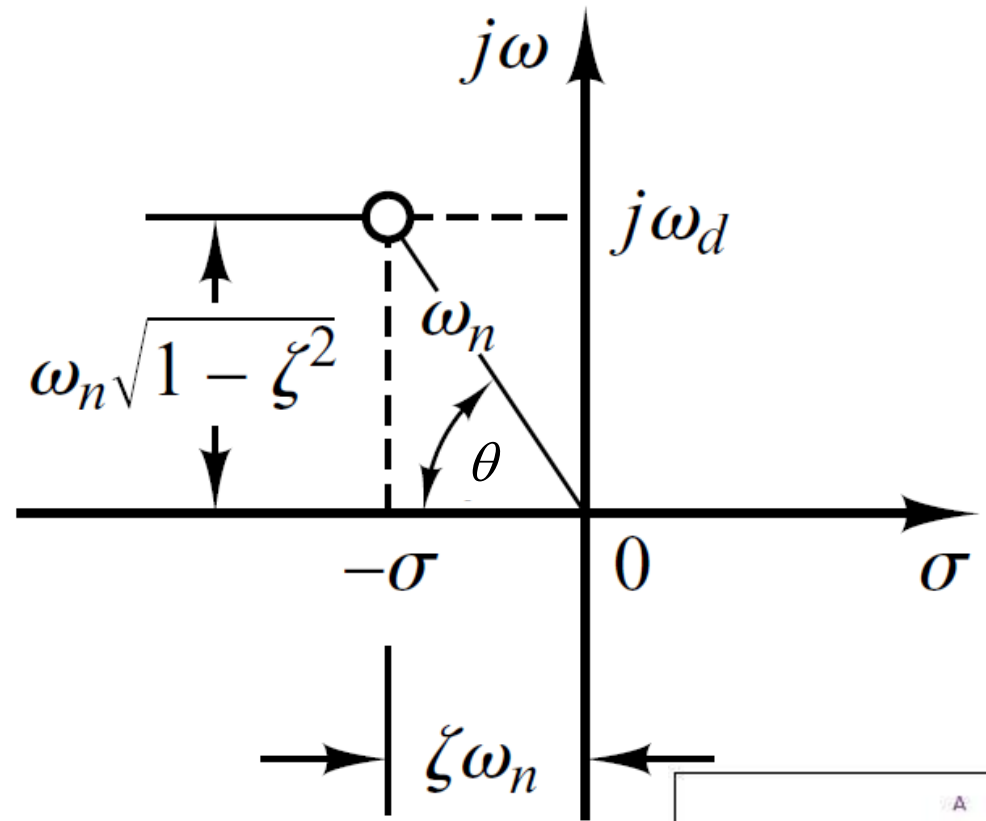
$$\omega_d t_r = \tan^{-1} \left( -\frac{\sqrt{1-\zeta^2}}{\zeta} \right)$$

# Time Domain Specifications (Rise Time)

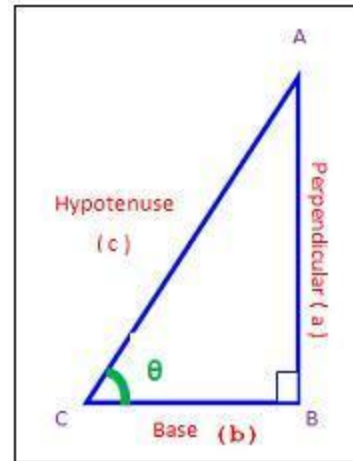
$$\omega_d t_r = \tan^{-1} \left( -\frac{\sqrt{1-\zeta^2}}{\zeta} \right)$$

$$t_r = \frac{1}{\omega_d} \tan^{-1} \left( -\frac{\omega_n \sqrt{1-\zeta^2}}{\omega_n \zeta} \right)$$

$$t_r = \frac{\pi - \theta}{\omega_d}$$



$$\theta = \tan^{-1} \frac{a}{b}$$



# Time Domain Specifications (Peak Time)

$$c(t) = 1 - e^{-\zeta\omega_n t} \left[ \cos \omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin \omega_d t \right]$$

- In order to find peak time let us differentiate above equation w.r.t  $t$ .

$$\frac{dc(t)}{dt} = \zeta\omega_n e^{-\zeta\omega_n t} \left[ \cos \omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin \omega_d t \right] - e^{-\zeta\omega_n t} \left[ -\omega_d \sin \omega_d t + \frac{\zeta\omega_d}{\sqrt{1-\zeta^2}} \cos \omega_d t \right]$$

$$0 = e^{-\zeta\omega_n t} \left[ \zeta\omega_n \cos \omega_d t + \frac{\zeta^2 \omega_n}{\sqrt{1-\zeta^2}} \sin \omega_d t + \omega_d \sin \omega_d t - \frac{\zeta\omega_d}{\sqrt{1-\zeta^2}} \cos \omega_d t \right]$$

$$0 = e^{-\zeta\omega_n t} \left[ \cancel{\zeta\omega_n} \cos \omega_d t + \frac{\zeta^2 \omega_n}{\sqrt{1-\zeta^2}} \sin \omega_d t + \omega_d \sin \omega_d t - \frac{\cancel{\zeta\omega_n} \sqrt{1-\zeta^2}}{\cancel{\sqrt{1-\zeta^2}}} \cos \omega_d t \right]$$

# Time Domain Specifications (Peak Time)

$$0 = e^{-\zeta\omega_n t} \left[ \cancel{\zeta\omega_n} \cos \omega_d t + \frac{\zeta^2 \omega_n}{\sqrt{1-\zeta^2}} \sin \omega_d t + \omega_d \sin \omega_d t - \frac{\cancel{\zeta\omega_n} \sqrt{1-\zeta^2}}{\cancel{\sqrt{1-\zeta^2}}} \cos \omega_d t \right]$$

$$e^{-\zeta\omega_n t} \left[ \frac{\zeta^2 \omega_n}{\sqrt{1-\zeta^2}} \sin \omega_d t + \omega_d \sin \omega_d t \right] = 0$$

$$e^{-\zeta\omega_n t} \neq 0 \quad \left[ \frac{\zeta^2 \omega_n}{\sqrt{1-\zeta^2}} \sin \omega_d t + \omega_d \sin \omega_d t \right] = 0$$

$$\sin \omega_d t \left[ \frac{\zeta^2 \omega_n}{\sqrt{1-\zeta^2}} + \omega_d \right] = 0$$



# Time Domain Specifications (Peak Time)

$$\sin \omega_d t \left[ \frac{\zeta^2 \omega_n}{\sqrt{1 - \zeta^2}} + \omega_d \right] = 0$$
$$\left[ \frac{\zeta^2 \omega_n}{\sqrt{1 - \zeta^2}} + \omega_d \right] \neq 0 \qquad \sin \omega_d t = 0$$

$$\omega_d t = \sin^{-1} 0$$

$$t = \frac{0, \pi, 2\pi, \dots}{\omega_d}$$

- Since for underdamped stable systems first peak is maximum peak therefore,

$$t_p = \frac{\pi}{\omega_d}$$

# Time Domain Specifications (Maximum Overshoot)

$$\text{Maximum percent overshoot} = \frac{c(t_p) - c(\infty)}{c(\infty)} \times 100\%$$

$$c(t_p) = 1 - e^{-\zeta\omega_n t_p} \left[ \cos \omega_d t_p + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin \omega_d t_p \right]$$

$$c(\infty) = 1$$

$$M_p = \left[ 1 - e^{-\zeta\omega_n t_p} \left( \cos \omega_d t_p + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin \omega_d t_p \right) - 1 \right] \times 100$$

Put  $t_p = \frac{\pi}{\omega_d}$  in above equation

$$M_p = \left[ -e^{-\zeta\omega_n \frac{\pi}{\omega_d}} \left( \cos \omega_d \frac{\pi}{\omega_d} + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin \omega_d \frac{\pi}{\omega_d} \right) \right] \times 100$$

# Time Domain Specifications (Maximum Overshoot)

$$M_p = \left[ -e^{-\zeta\omega_n \frac{\pi}{\omega_d}} \left( \cos \frac{\pi}{\omega_d} + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin \frac{\pi}{\omega_d} \right) \right] \times 100$$

Put  $\omega_d = \omega_n \sqrt{1-\zeta^2}$  in above equation

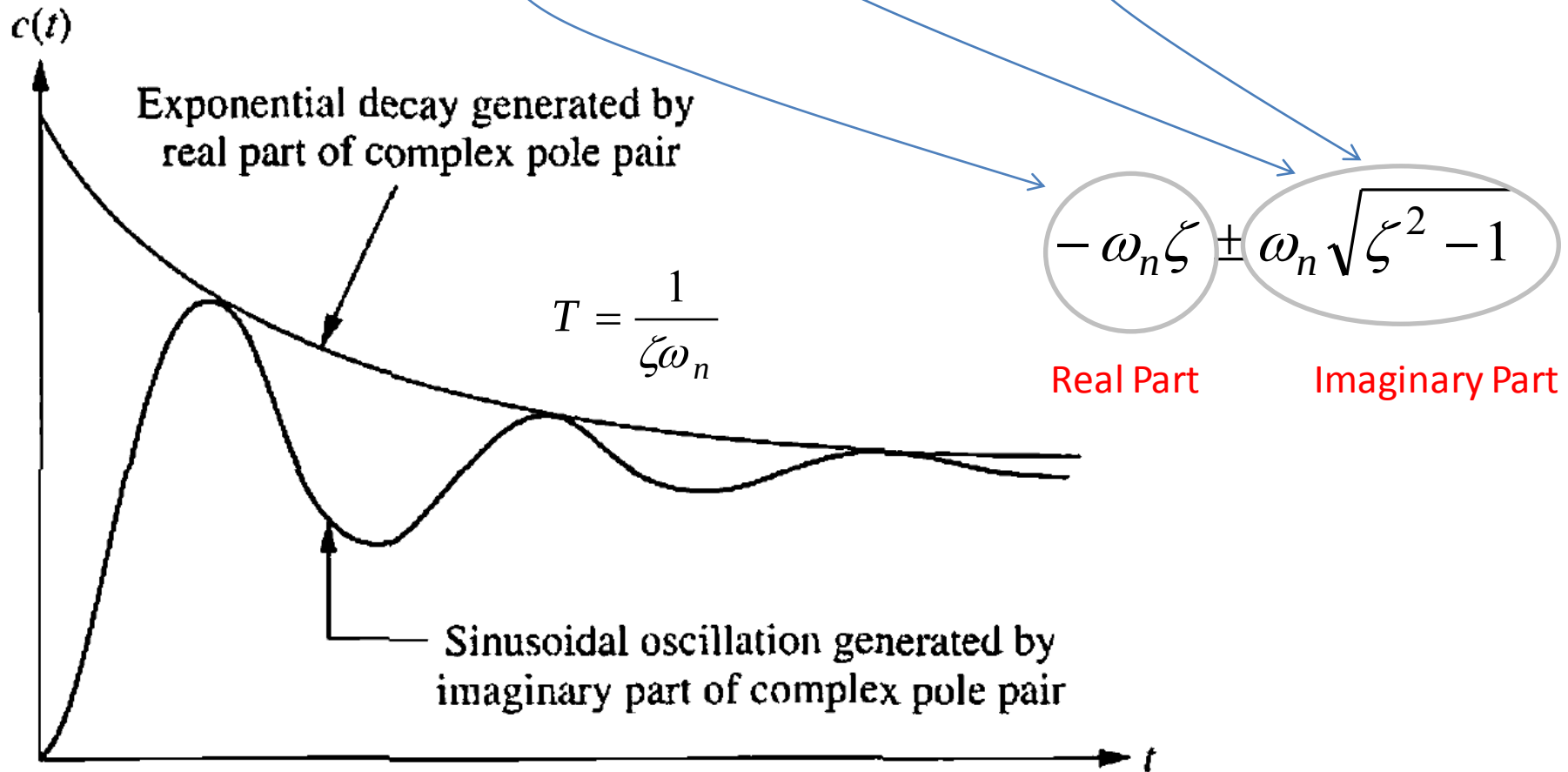
$$M_p = \left[ -e^{-\zeta\omega_n \frac{\pi}{\omega_n \sqrt{1-\zeta^2}}} \left( \cos \pi + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin \pi \right) \right] \times 100$$

$$M_p = \left[ -e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} (-1 + 0) \right] \times 100$$

$$M_p = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} \times 100$$

# Time Domain Specifications (Settling Time)

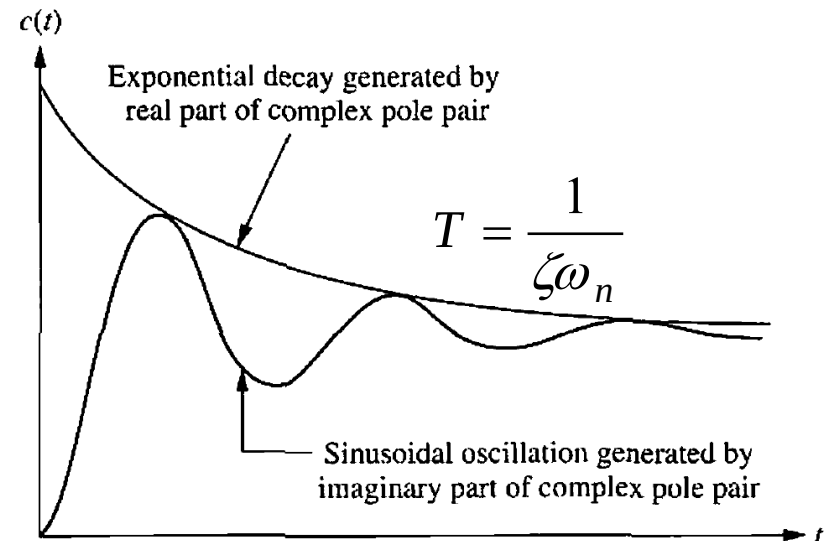
$$c(t) = 1 - e^{-\zeta\omega_n t} \left[ \cos \omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin \omega_d t \right]$$



# Time Domain Specifications (Settling Time)

- Settling time (2%) criterion
  - Time consumed in exponential decay up to 98% of the input.

$$t_s = 4T = \frac{4}{\zeta\omega_n}$$



- Settling time (5%) criterion
  - Time consumed in exponential decay up to 95% of the input.

$$t_s = 3T = \frac{3}{\zeta\omega_n}$$

# Summary of Time Domain Specifications

## Rise Time

$$t_r = \frac{\pi - \theta}{\omega_d} = \frac{\pi - \theta}{\omega_n \sqrt{1 - \zeta^2}}$$

## Peak Time

$$t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n \sqrt{1 - \zeta^2}}$$

## Settling Time (2%)

$$t_s = 4T = \frac{4}{\zeta \omega_n}$$

$$t_s = 3T = \frac{3}{\zeta \omega_n}$$

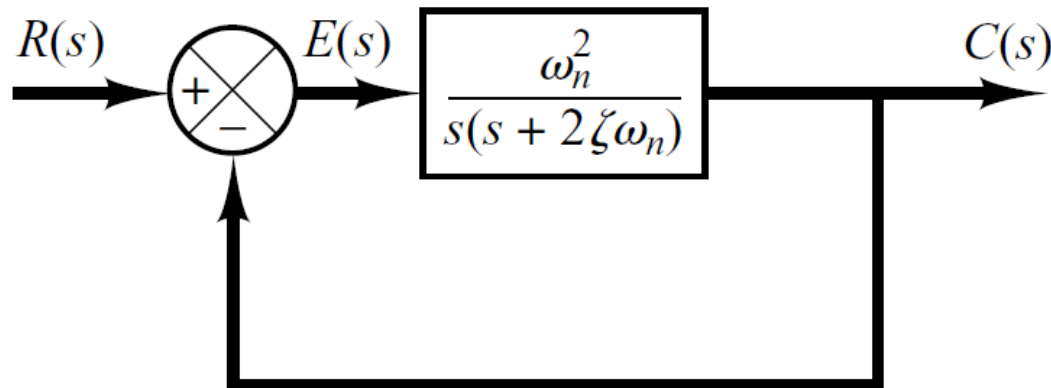
## Settling Time (4%)

## Maximum Overshoot

$$M_p = e^{-\frac{\pi \zeta}{\sqrt{1 - \zeta^2}}} \times 100$$

# Example#5

- Consider the system shown in following figure, where damping ratio is **0.6** and natural undamped frequency is **5 rad/sec**. Obtain the rise time  **$t_r$** , peak time  **$t_p$** , maximum overshoot  **$M_p$** , and settling time 2% and 5% criterion  **$t_s$**  when the system is subjected to a unit-step input.



# Example#5

## Rise Time

$$t_r = \frac{\pi - \theta}{\omega_d}$$

## Peak Time

$$t_p = \frac{\pi}{\omega_d}$$

## Settling Time (2%)

$$t_s = 4T = \frac{4}{\zeta\omega_n}$$

$$t_s = 3T = \frac{3}{\zeta\omega_n}$$

## Settling Time (4%)

## Maximum Overshoot

$$M_p = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} \times 100$$



# Example#5

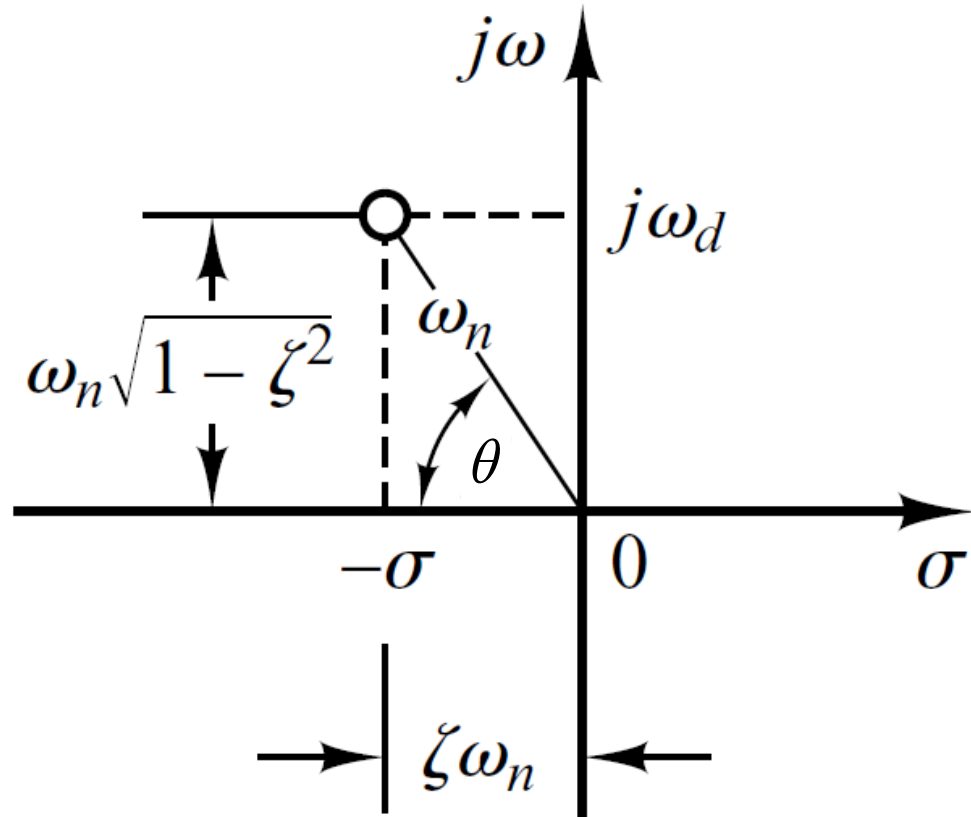
## Rise Time

$$t_r = \frac{\pi - \theta}{\omega_d}$$

$$t_r = \frac{3.141 - \theta}{\omega_n \sqrt{1 - \zeta^2}}$$

$$\theta = \tan^{-1}\left(\frac{\omega_n \sqrt{1 - \zeta^2}}{\zeta \omega_n}\right) = 0.93 \text{ rad}$$

$$t_r = \frac{3.141 - 0.93}{5\sqrt{1 - 0.6^2}} = 0.55 \text{ s}$$



# Example#5

## Peak Time

$$t_p = \frac{\pi}{\omega_d}$$

$$t_p = \frac{3.141}{4} = 0.785s$$

## Settling Time (2%)

$$t_s = \frac{4}{\zeta\omega_n}$$

$$t_s = \frac{4}{0.6 \times 5} = 1.33s$$

## Settling Time (4%)

$$t_s = \frac{3}{\zeta\omega_n}$$

$$t_s = \frac{3}{0.6 \times 5} = 1s$$

# Example#5

## Maximum Overshoot

$$M_p = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} \times 100$$

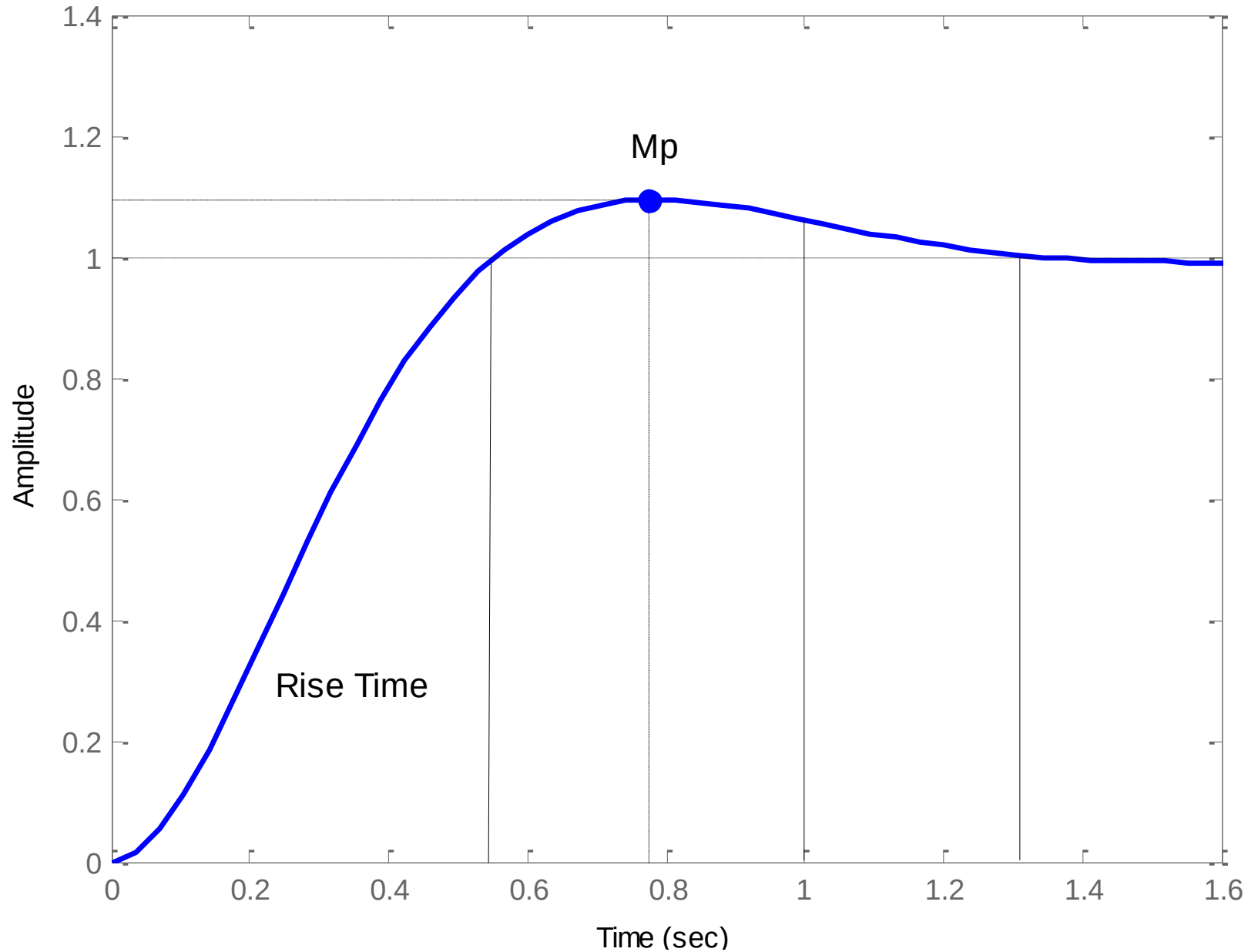
$$M_p = e^{-\frac{3.141 \times 0.6}{\sqrt{1-0.6^2}}} \times 100$$

$$M_p = 0.095 \times 100$$

$$M_p = 9.5\%$$

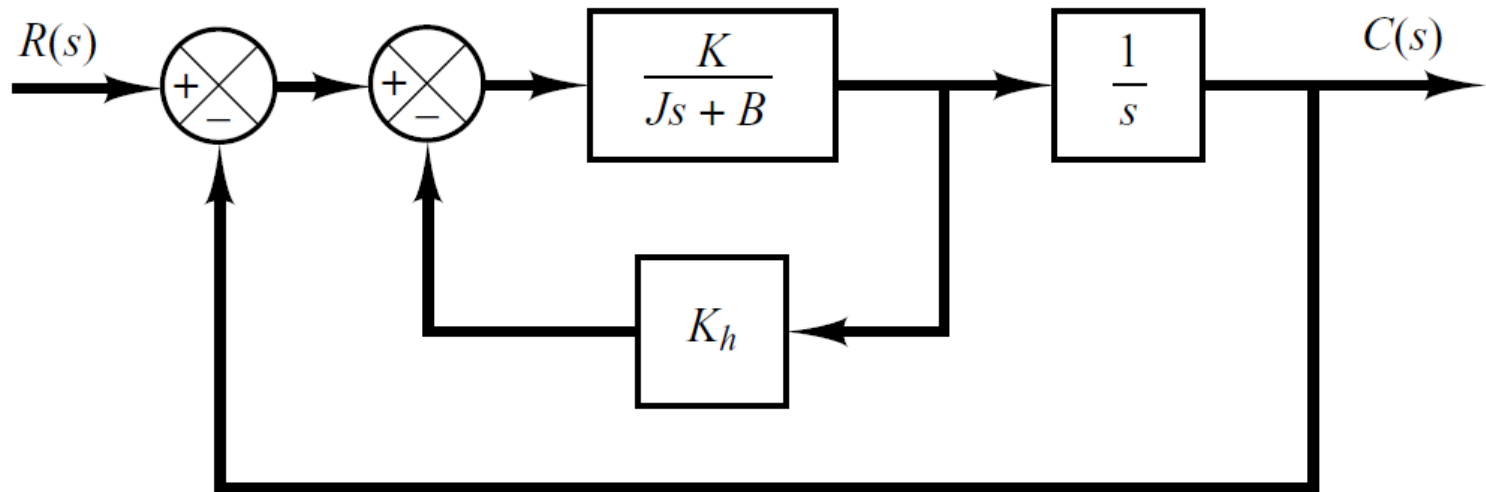
# Example#5

Step Response

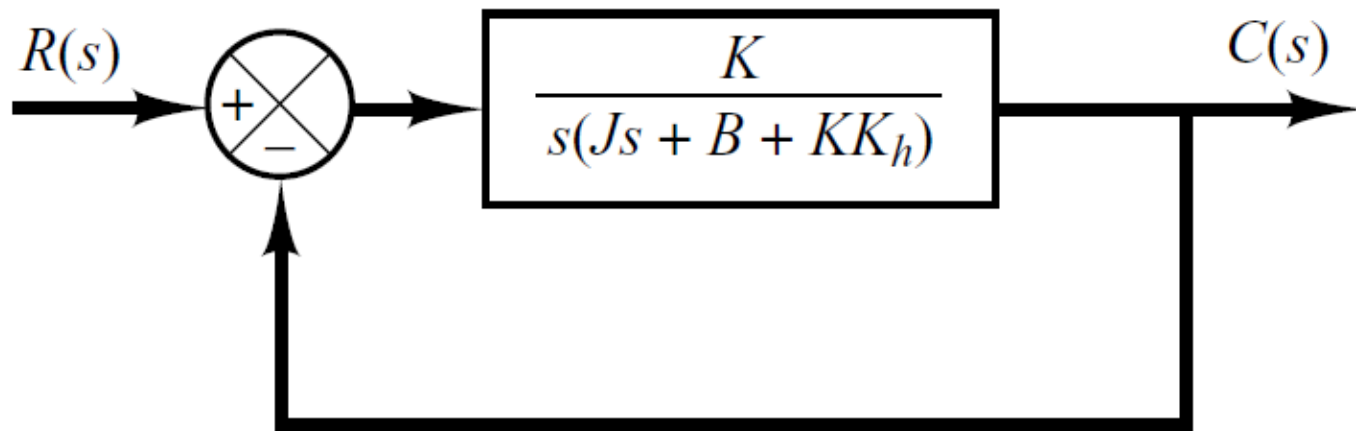
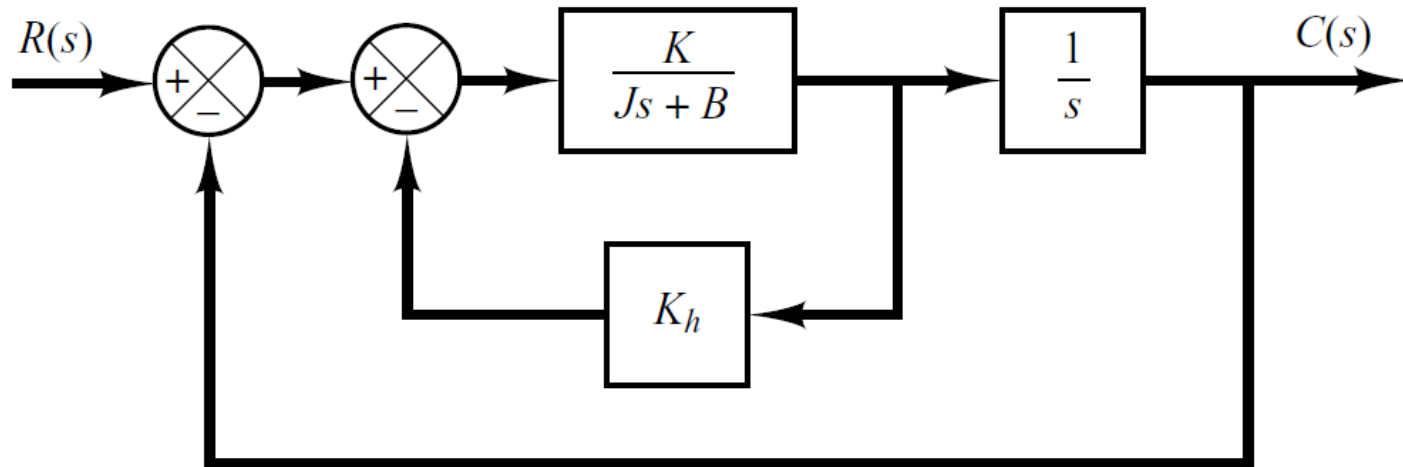


# Example#6

- For the system shown in Figure-(a), determine the values of gain  $K$  and velocity-feedback constant  $K_h$  so that the maximum overshoot in the unit-step response is  $0.2$  and the peak time is  $1$  sec. With these values of  $K$  and  $K_h$ , obtain the rise time and settling time. Assume that  $J=1$  kg-m<sup>2</sup> and  $B=1$  N-m/rad/sec.



# Example#6



$$\frac{C(s)}{R(s)} = \frac{K}{Js^2 + (B + KK_h)s + K}$$

# Example#6

$$\frac{C(s)}{R(s)} = \frac{K}{Js^2 + (B + KK_h)s + K}$$

Since  $J = 1 \text{ kgm}^2$  and  $B = 1 \text{ Nm/rad/sec}$

$$\frac{C(s)}{R(s)} = \frac{K}{s^2 + (1 + KK_h)s + K}$$

- Comparing above T.F with general 2<sup>nd</sup> order T.F

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$\omega_n = \sqrt{K} \quad \zeta = \frac{(1 + KK_h)}{2\sqrt{K}}$$

# Example#6

$$\omega_n = \sqrt{K}$$

$$\zeta = \frac{(1 + KK_h)}{2\sqrt{K}}$$

- Maximum overshoot is **0.2**.

- The peak time is **1 sec**

$$M_p = e^{-(\zeta/\sqrt{1-\zeta^2})\pi}$$

$$e^{-(\zeta/\sqrt{1-\zeta^2})\pi} = 0.2$$

$$\ln(e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}}) = \ln(0.2)$$

$$\frac{\zeta\pi}{\sqrt{1-\zeta^2}} = 1.61$$

$$\zeta = 0.456$$

$$t_p = \frac{\pi}{\omega_d}$$

$$1 = \frac{3.141}{\omega_n \sqrt{1-\zeta^2}}$$

$$\omega_n = \frac{3.141}{\sqrt{1-0.456^2}}$$

$$\omega_n = 3.53$$



# Example#6

$$\zeta = 0.456$$

$$\omega_n = 3.96$$

$$\omega_n = \sqrt{K}$$

$$\zeta = \frac{(1 + KK_h)}{2\sqrt{K}}$$

$$3.53 = \sqrt{K}$$

$$0.456 \times 2\sqrt{12.5} = (1 + 12.5K_h)$$

$$3.53^2 = K$$

$$K_h = 0.178$$

$$K = 12.5$$

# Example#6

$$\zeta = 0.456$$

$$\omega_n = 3.96$$

$$t_r = \frac{\pi - \theta}{\omega_n \sqrt{1 - \zeta^2}}$$

$$t_r = 0.65s$$

$$t_s = \frac{4}{\zeta \omega_n}$$

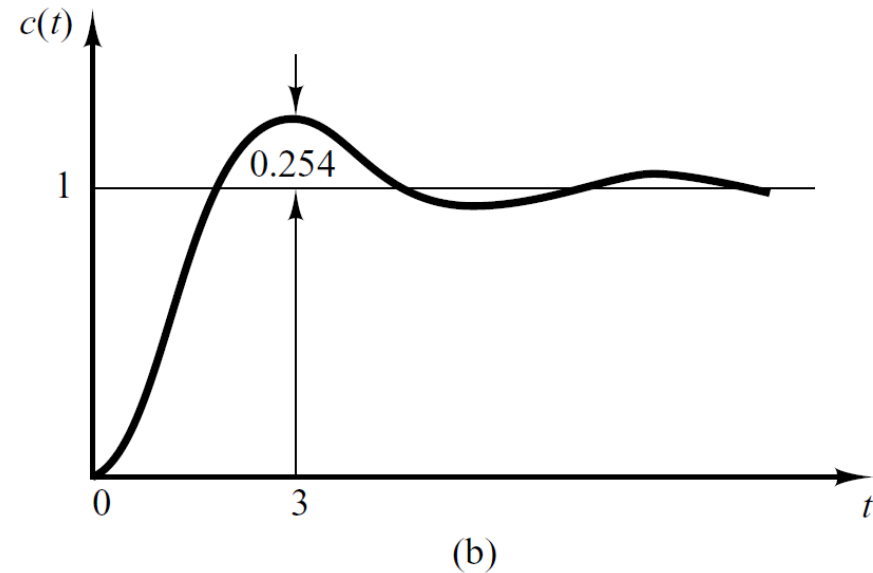
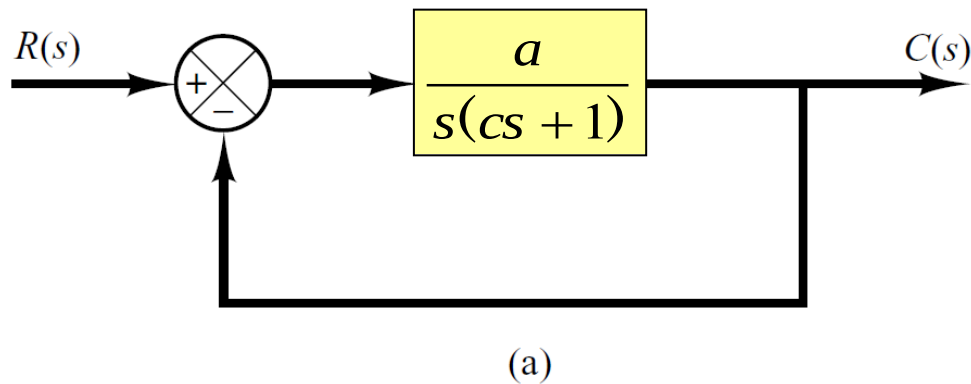
$$t_s = 2.48s$$

$$t_s = \frac{3}{\zeta \omega_n}$$

$$t_s = 1.86s$$

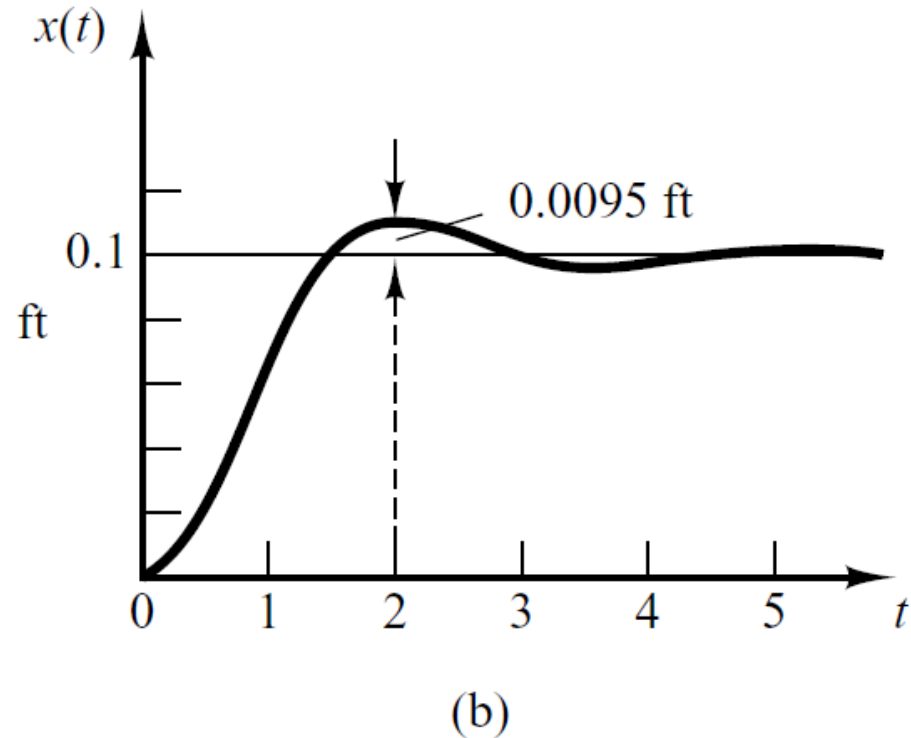
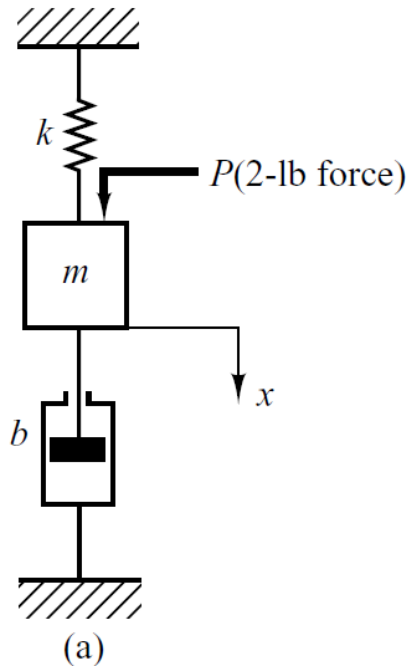
# Example#7

When the system shown in Figure(a) is subjected to a unit-step input, the system output responds as shown in Figure(b). Determine the values of  $a$  and  $c$  from the response curve.



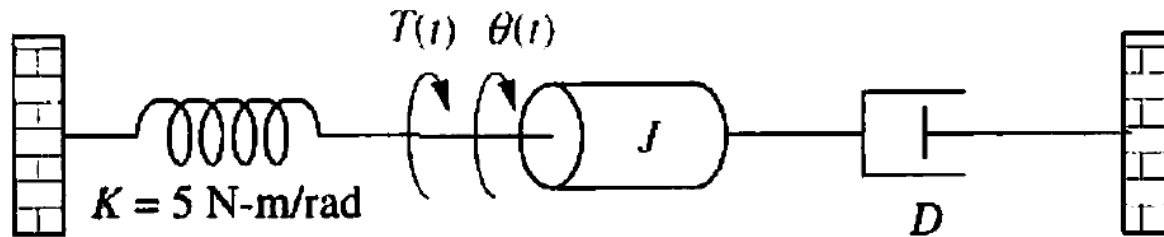
# Example#8

Figure (a) shows a mechanical vibratory system. When **2 lb** of force (step input) is applied to the system, the mass oscillates, as shown in Figure (b). Determine  **$m$** ,  **$b$** , and  **$k$**  of the system from this response curve.



# Example#9

Given the system shown in following figure, find  $J$  and  $D$  to yield 20% overshoot and a settling time of 2 seconds for a step input of torque  $T(t)$ .



$$G(s) = \frac{1/J}{s^2 + \frac{D}{J}s + \frac{K}{J}}$$

$$\omega_n = \sqrt{\frac{K}{J}}$$

$$T_s = 2 = \frac{4}{\zeta\omega_n}$$

$$2\zeta\omega_n = 4$$

$$\zeta = \frac{4}{2\omega_n} = 2\sqrt{\frac{J}{K}}$$

# Example#9

$$\omega_n = \sqrt{\frac{K}{J}}$$

$$\zeta = 2\sqrt{\frac{J}{K}}$$

20% overshoot implies  $\zeta = 0.456$ . Therefore,

$$\zeta = 2\sqrt{\frac{J}{K}} = 0.456$$

Hence,

$$\frac{J}{K} = 0.052$$

From the problem statement,  $K = 5 \text{ N-m/rad}$ .

$$J = 0.26 \text{ kg-m}^2$$

# Example#9

$$G(s) = \frac{1/J}{s^2 + \frac{D}{J}s + \frac{K}{J}}$$

$$2\zeta\omega_n = \frac{D}{J}$$

$$D = 1.04 \text{ N-m-s/rad.}$$

# Step Response of critically damped System ( $\zeta = 1$ )

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{(s + \omega_n)^2} \xrightarrow{\text{Step Response}} C(s) = \frac{\omega_n^2}{s(s + \omega_n)^2}$$

- The partial fraction expansion of above equation is given as

$$\frac{\omega_n^2}{s(s + \omega_n)^2} = \frac{A}{s} + \frac{B}{s + \omega_n} + \frac{C}{(s + \omega_n)^2}$$

$$C(s) = \frac{1}{s} - \frac{1}{s + \omega_n} - \frac{\omega_n}{(s + \omega_n)^2}$$

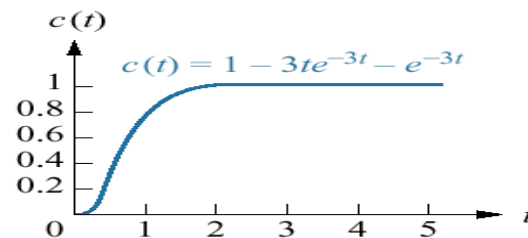
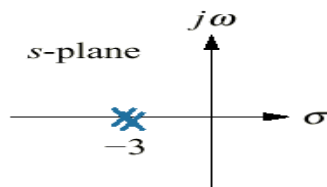
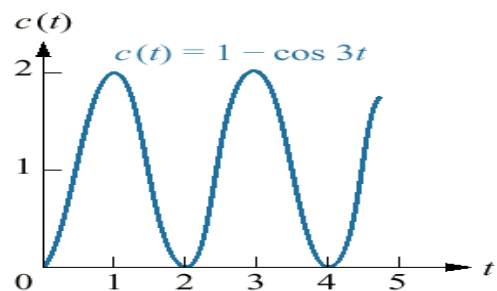
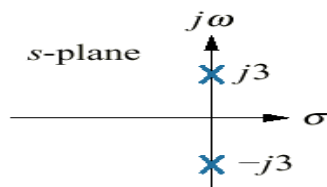
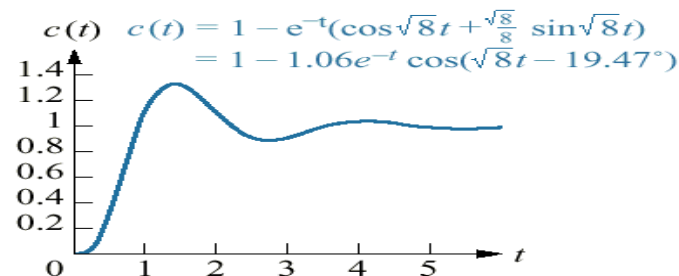
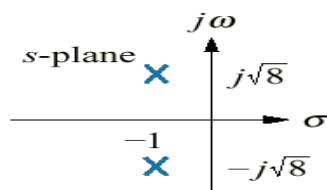
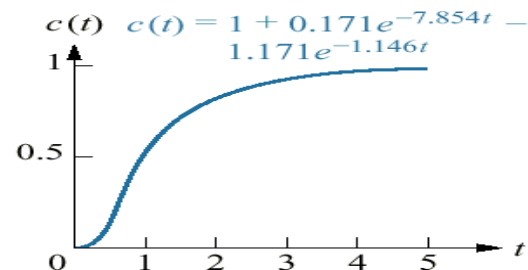
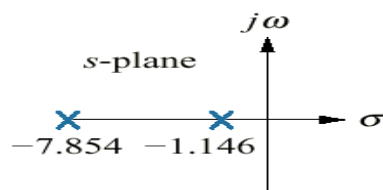
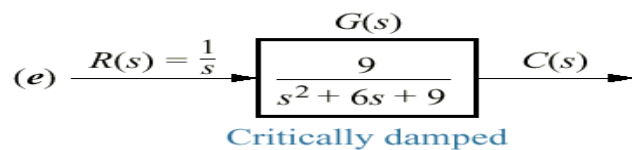
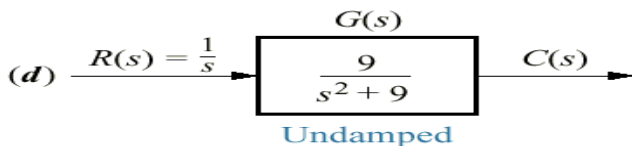
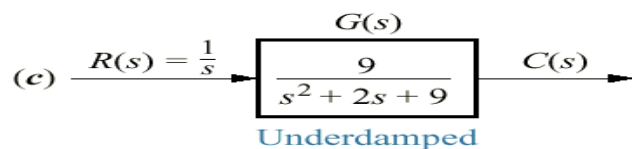
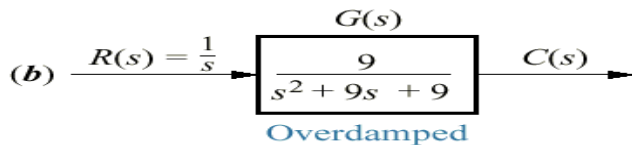
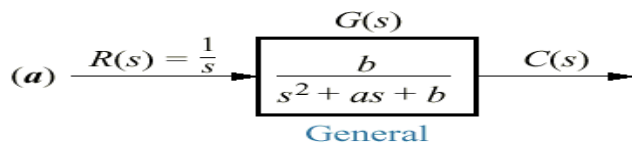
$$c(t) = 1 - e^{-\omega_n t} - \omega_n e^{-\omega_n t} t$$

$$c(t) = 1 - e^{-\omega_n t} (1 + \omega_n t)$$



# Step Response of overdamped and undamped Systems

- Home Work



# Second – Order System

Example 10: Describe the **nature** of the second-order system response via the value of the damping ratio for the systems with transfer function

$$1. \quad G(s) = \frac{12}{s^2 + 8s + 12}$$

$$2. \quad G(s) = \frac{16}{s^2 + 8s + 16}$$

$$3. \quad G(s) = \frac{20}{s^2 + 8s + 20}$$

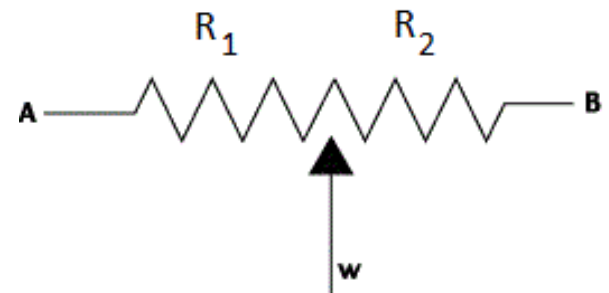
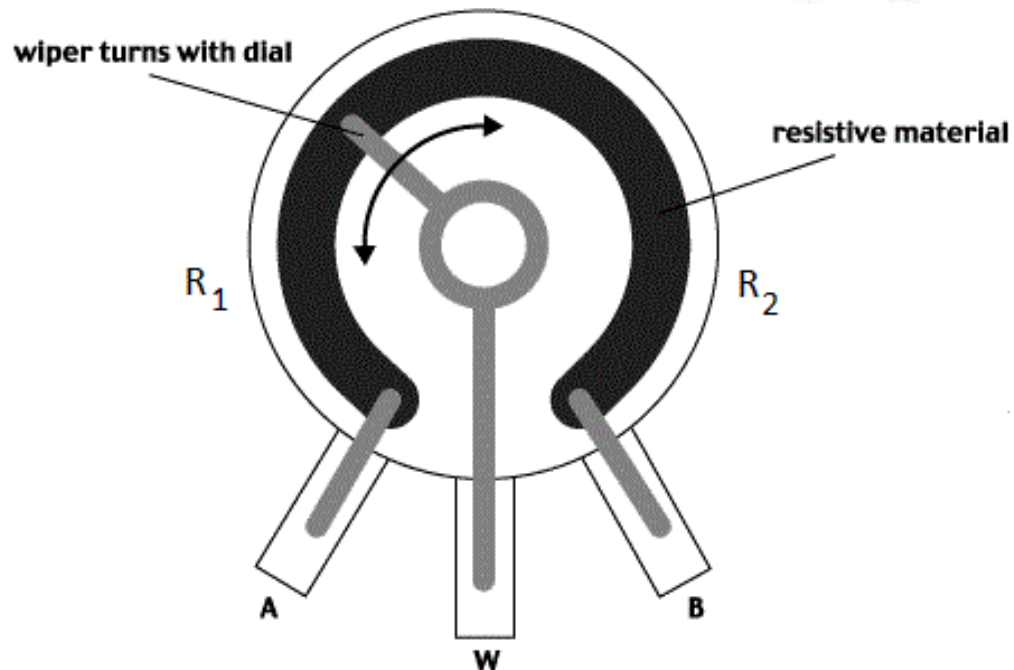
Do them as your own  
revision

# نمذجة وتمثيل منظومات السيطرة

# Electromechanical Systems

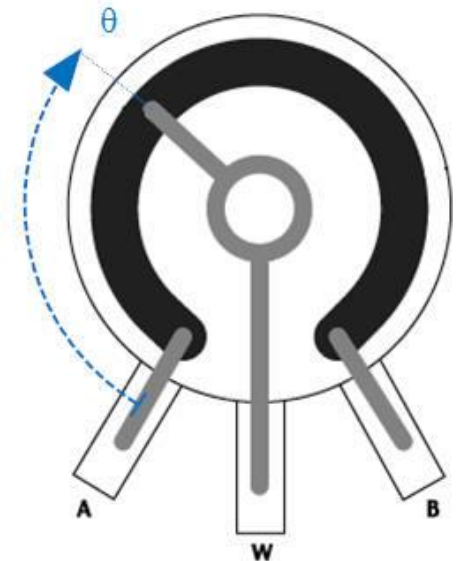
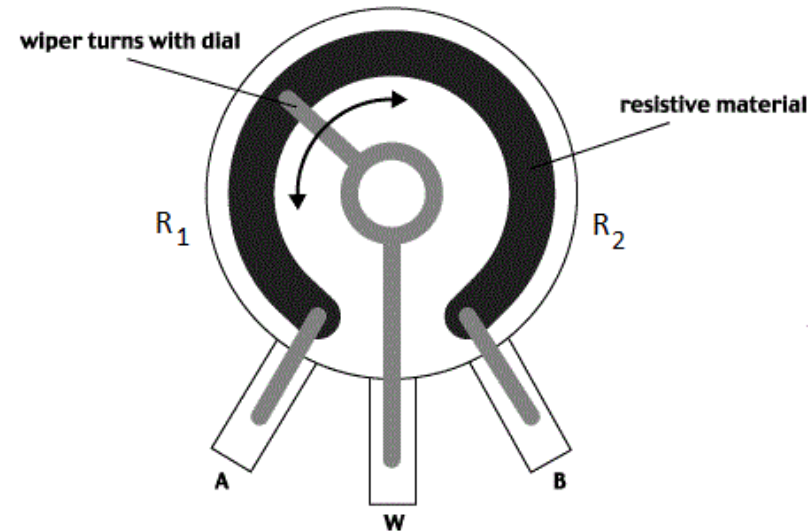
- **Electromechanics** combines electrical and mechanical processes.
- Devices which carry out electrical operations by using moving parts are known as electromechanical.
  - Relays
  - Solenoids
  - Electric Motors
  - Electric Generators
  - Switches and e.t.c

# Example-1: Potentiometer



# Example-1: Potentiometer

- The resistance between the wiper (slider) and "A" is labeled  $R_1$ , the resistance between the wiper and "B" is labeled  $R_2$ .
- The total resistance between "A" and "B" is constant,  $R_1 + R_2 = R_{tot}$ .
- If the potentiometer is turned to the extreme counterclockwise position such that the wiper is touching "A" we will call this  $\theta = 0$ ; in this position  $R_1 = 0$  and  $R_2 = R_{tot}$ .
- If the wiper is in the extreme clockwise position such that it is touching "B" we will call this  $\theta = \theta_{max}$ ; in this position  $R_1 = R_{tot}$  and  $R_2 = 0$ .

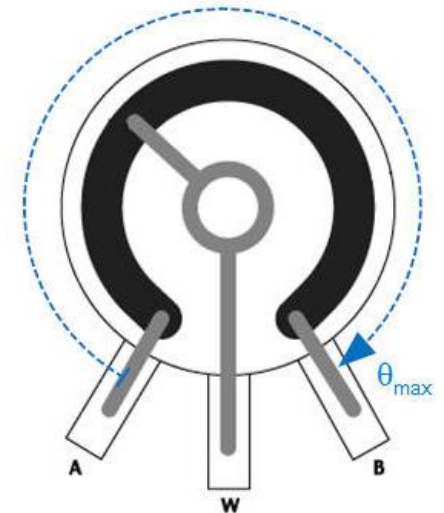
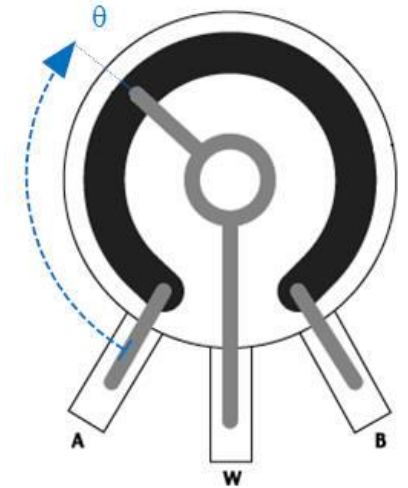


# Example-1: Potentiometer

- $R_1$  and  $R_2$  vary linearly with  $\theta$  between the two extremes:

$$R_1 = \frac{\theta}{\theta_{\max}} R_{\text{tot}}$$

$$R_2 = \frac{\theta_{\max} - \theta}{\theta_{\max}} R_{\text{tot}}$$





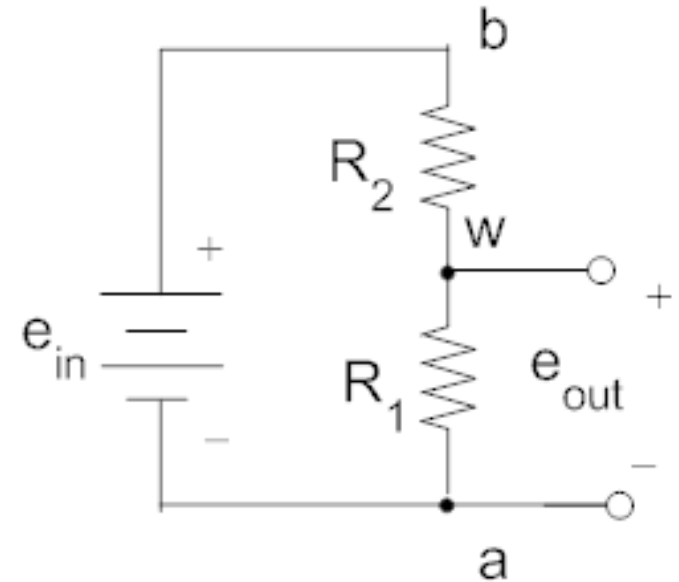
# Example-1: Potentiometer

- Potentiometer can be used to sense angular position, consider the circuit of **figure-1**.
- Using the voltage divider principle we can write:

$$e_{out} = \frac{R_1}{R_1 + R_2} e_{in} = \frac{R_1}{R_{tot}} e_{in}$$

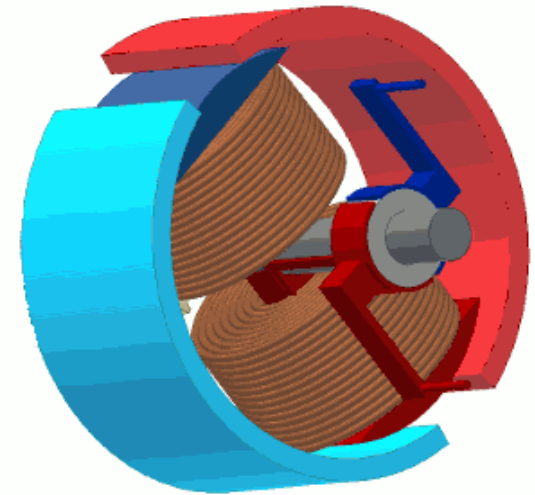
$$e_{out} = \frac{\theta}{\theta_{max}} e_{in}$$

Figure-1



# D.C Drives

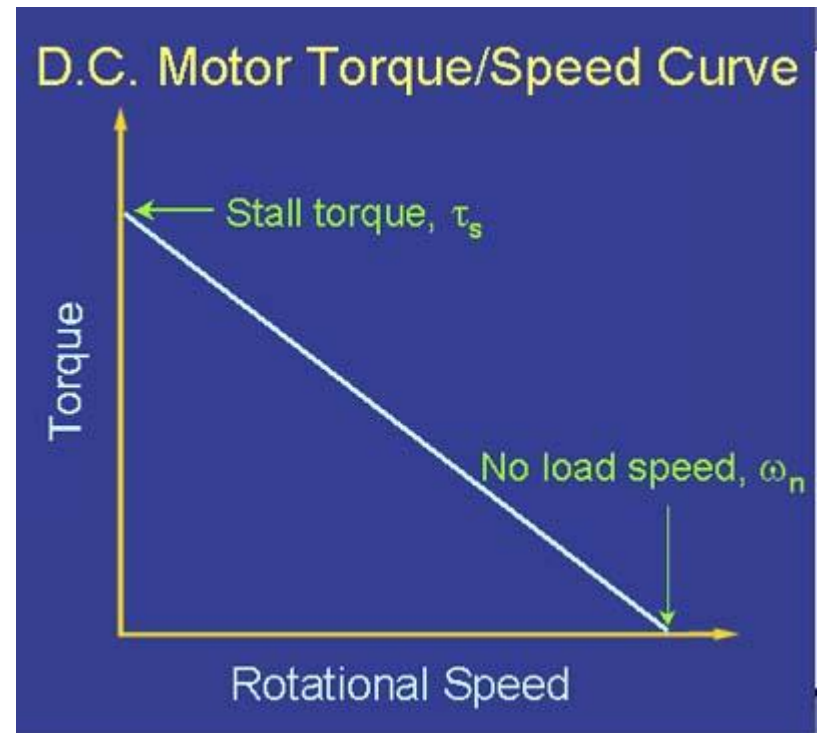
- Speed control can be achieved using DC drives in a number of ways.
- Variable Voltage can be applied to the armature terminals of the DC motor .
- Another method is to vary the flux per pole of the motor.
- The first method involve adjusting the motor's armature while the latter method involves adjusting the motor field. These methods are referred to as “armature control” and “field control.”



# D.C Drives

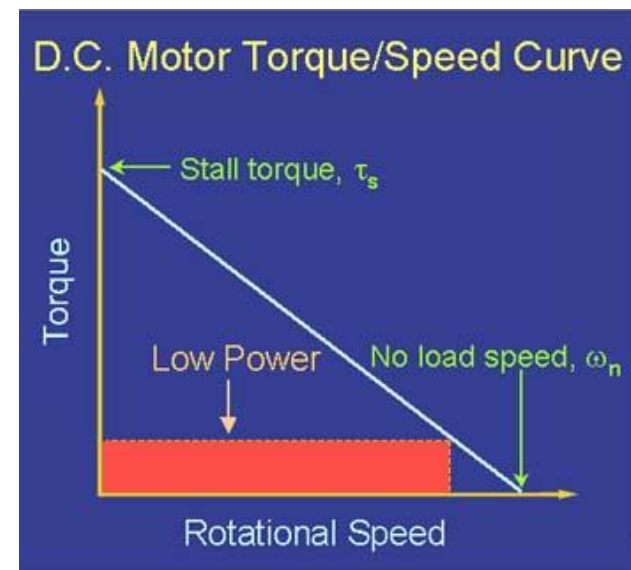
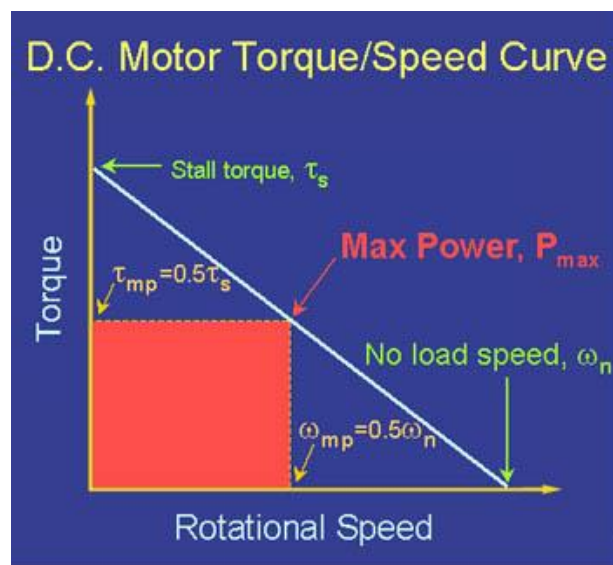
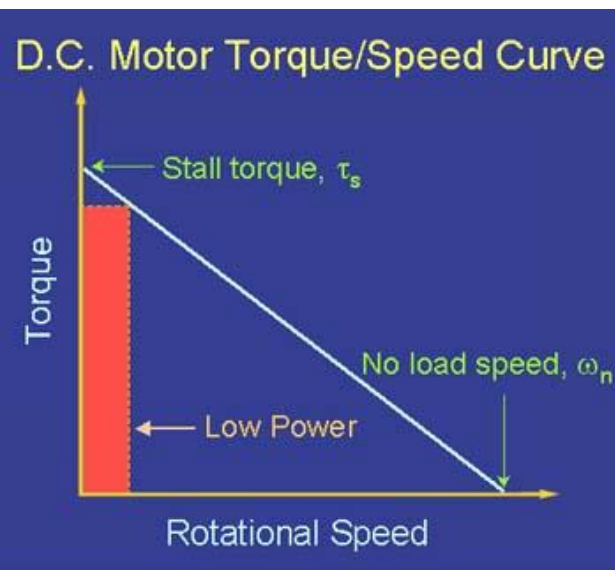
- **Motor Characteristics**

- For every motor, there is a specific Torque/Speed curve and Power curve.
- Torque is inversely proportional to the speed of the output shaft.
- Motor characteristics are frequently given as two points on this graph:
- The stall torque,, represents the point on the graph at which the torque is maximum, but the shaft is not rotating.
- The no load speed is the maximum output speed of the motor.



# D.C Drives

- **Motor Characteristics**
- Power is defined as the product of torque and angular velocity.



# Example-2: Armature Controlled D.C Motor

Input: voltage  $u$

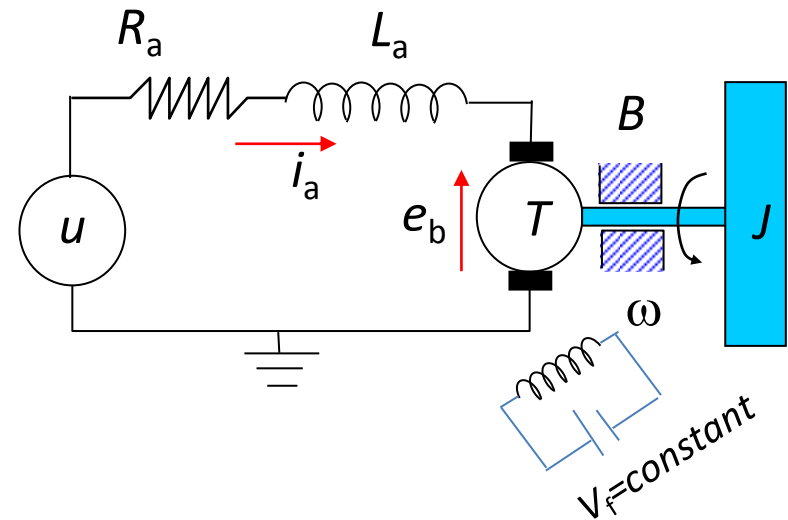
Output: Angular velocity  $\omega$

Electrical Subsystem (loop method):

$$u = R_a i_a + L_a \frac{di_a}{dt} + e_b, \quad \text{where } e_b = \text{back-emf voltage}$$

Mechanical Subsystem

$$T_{motor} = J\dot{\omega} + B\omega$$

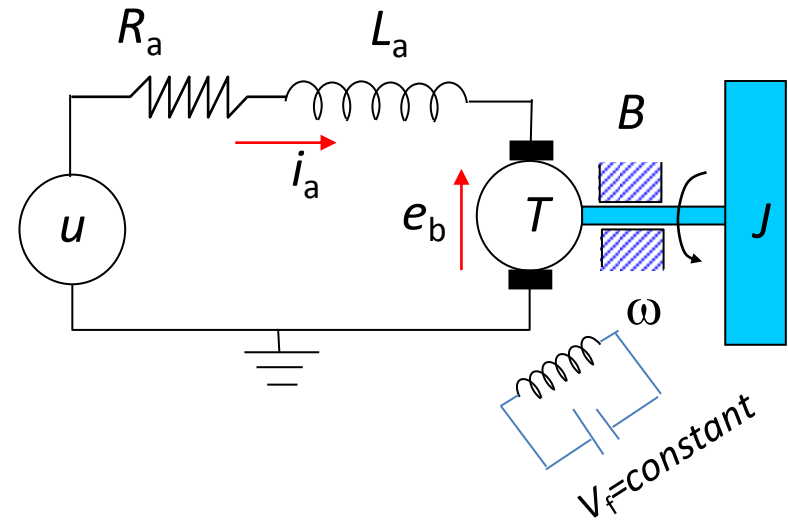


# Example-2: Armature Controlled D.C Motor

## Power Transformation:

Torque-Current:  $T_{motor} = K_t i_a$

Voltage-Speed:  $e_b = K_b \omega$



where  $K_t$ : torque constant,  $K_b$ : velocity constant For an ideal motor

$$K_t = K_b$$

Combing previous equations results in the following mathematical model:

$$\begin{cases} L_a \frac{di_a}{dt} + R_a i_a + K_b \omega = u \\ J \dot{\omega} + B \omega - K_t i_a = 0 \end{cases}$$

# Example-2: Armature Controlled D.C Motor

Taking Laplace transform of the system's differential equations with zero initial conditions gives:

$$\begin{cases} (L_a s + R_a)I_a(s) + K_b \Omega(s) = U(s) \\ (Js + B)\Omega(s) - K_t I_a(s) = 0 \end{cases}$$

Eliminating  $I_a$  yields the input-output transfer function

$$\frac{\Omega(s)}{U(s)} = \frac{K_t}{L_a Js^2 + (JR_a + BL_a)s + BR_a + K_t K_b}$$

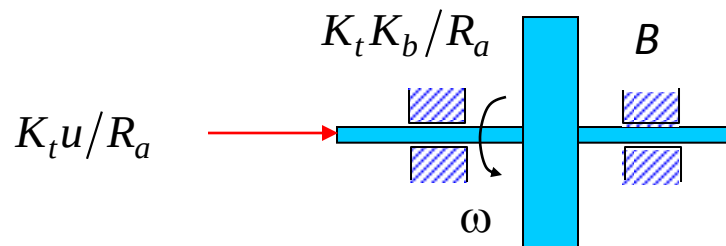
# Example-2: Armature Controlled D.C Motor

## Reduced Order Model

Assuming small inductance,  $L_a \approx 0$

$$\frac{\Omega(s)}{U(s)} = \frac{(K_t/R_a)}{Js + (B + K_t K_b/R_a)}$$

which is equivalent to



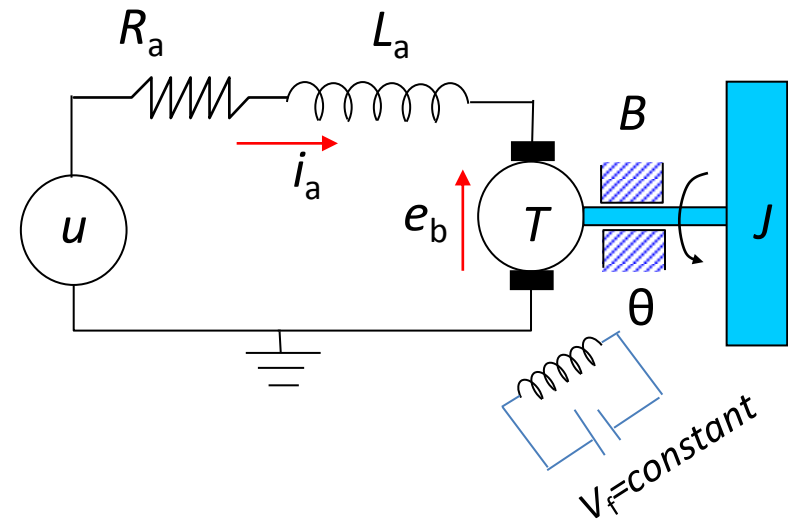
- The D.C. motor provides an *input torque* and an additional *damping* effect known as back-emf damping



# Example-2: Armature Controlled D.C Motor

If output of the D.C motor is angular position  $\theta$  then we know

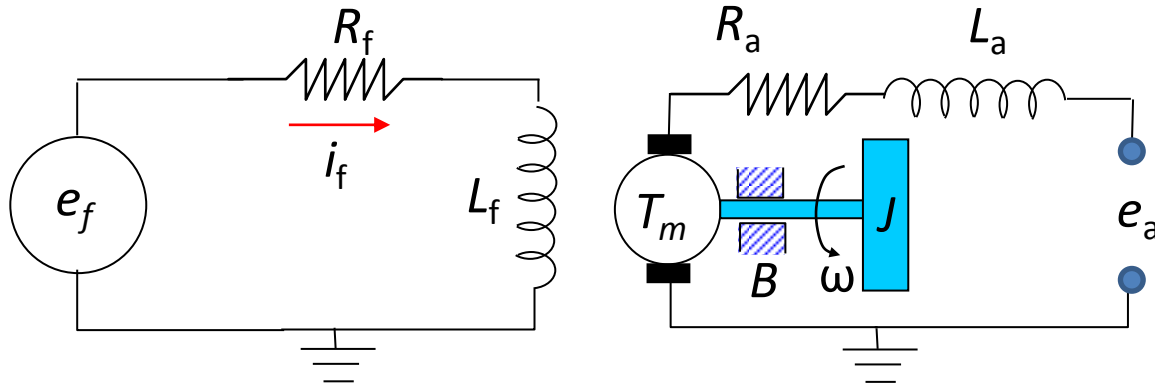
$$\omega = \frac{d\theta}{dt} \quad \text{or} \quad \Omega(s) = s\theta(s)$$



Which yields following transfer function

$$\frac{\theta(s)}{U(s)} = \frac{(K_t/R_a)}{s(Js + (B + K_t K_b/R_a))}$$

# Example-3: Field Controlled D.C Motor



Applying KVL at field circuit

$$e_f = i_f R_f + L_f \frac{di_f}{dt}$$

Mechanical Subsystem

$$T_m = J\dot{\omega} + B\omega$$

# Example-3: Field Controlled D.C Motor

Power Transformation:

Torque-Current:  $T_m = K_f i_f$

where  $K_f$ : torque constant

Combing previous equations and taking Laplace transform (considering initial conditions to zero) results in the following mathematical model:

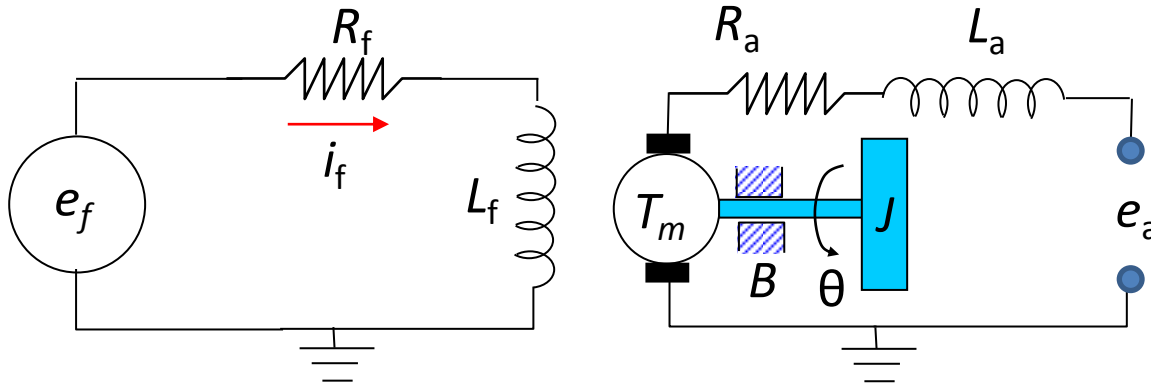
$$\begin{cases} E_f(s) = R_f I_f(s) + sL_f I_f(s) \\ Js\Omega(s) + B\Omega(s) = K_f I_f(s) \end{cases}$$

# Example-3: Field Controlled D.C Motor

Eliminating  $I_f(s)$  yields

$$\frac{\Omega(s)}{E_f(s)} = \frac{K_f}{(Js + B)(L_f s + R_f)}$$

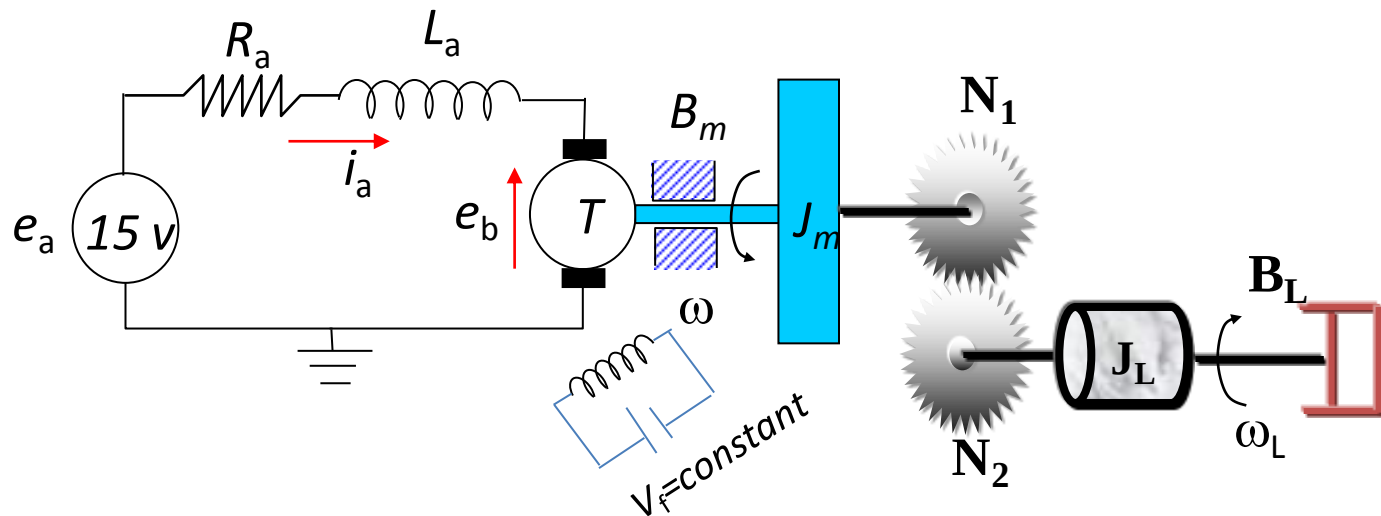
If angular position  $\theta$  is output of the motor



$$\frac{\theta(s)}{E_f(s)} = \frac{K_f}{s(Js + B)(L_f s + R_f)}$$

# Example-4

An armature controlled D.C motor runs at 5000 rpm when 15v applied at the armature circuit. Armature resistance of the motor is  $0.2 \Omega$ , armature inductance is negligible, back emf constant is  $5.5 \times 10^{-2}$  v sec/rad, motor torque constant is  $6 \times 10^{-5}$ , moment of inertia of motor  $10^{-5}$ , viscous friction coefficient is negligible, moment of inertia of load is  $4.4 \times 10^{-3}$ , viscous friction coefficient of load is  $4 \times 10^{-2}$ .



1. Drive the overall transfer function of the system i.e.  $\Omega_L(s)/E_a(s)$
2. Determine the gear ratio such that the rotational speed of the load is reduced to half and torque is doubled.

## System constants

$e_a$  = armature voltage

$e_b$  = back emf

$R_a$  = armature winding resistance =  **$0.2 \Omega$**

$L_a$  = armature winding inductance = **negligible**

$i_a$  = armature winding current

$K_b$  = back emf constant =  **$5.5 \times 10^{-2}$  volt-sec/rad**

$K_t$  = motor torque constant =  **$6 \times 10^{-5}$  N-m/ampere**

$J_m$  = moment of inertia of the motor =  **$1 \times 10^{-5}$  kg-m<sup>2</sup>**

$B_m$  = viscous-friction coefficients of the motor = **negligible**

$J_L$  = moment of inertia of the load =  **$4.4 \times 10^{-3}$  kgm<sup>2</sup>**

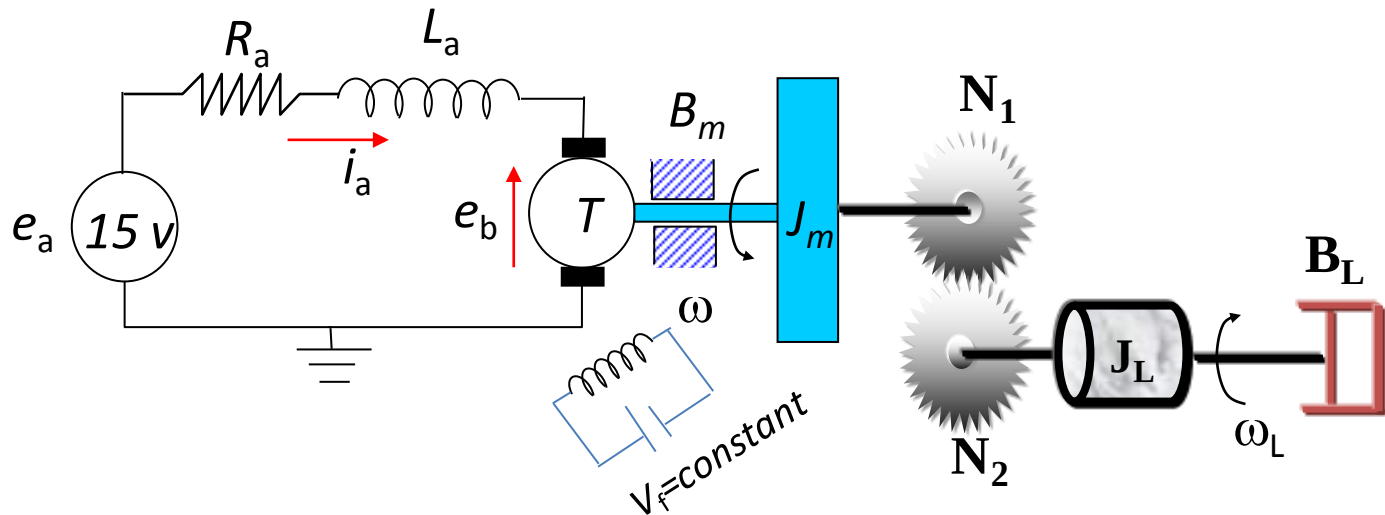
$B_L$  = viscous friction coefficient of the load =  **$4 \times 10^{-2}$  N-m/rad/sec**

gear ratio =  $N_1/N_2$

# Example-4

Since armature inductance is negligible therefore reduced order transfer function of the motor is used.

$$\frac{\Omega_L(s)}{U(s)} = \frac{K_t}{(J_{eq}R_a + B_{eq}L_a)s + B_{eq}R_a + K_tK_b}$$

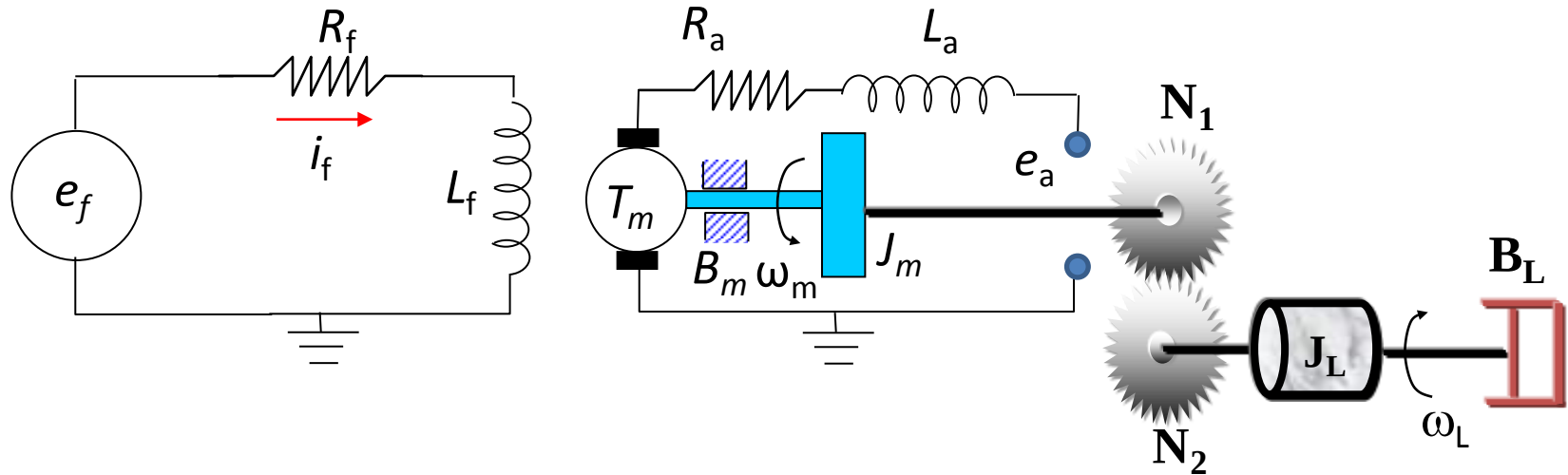


$$J_{eq} = J_m + \left( \frac{N_1}{N_2} \right)^2 J_L$$

$$B_{eq} = B_m + \left( \frac{N_1}{N_2} \right)^2 B_L$$

# Example-5

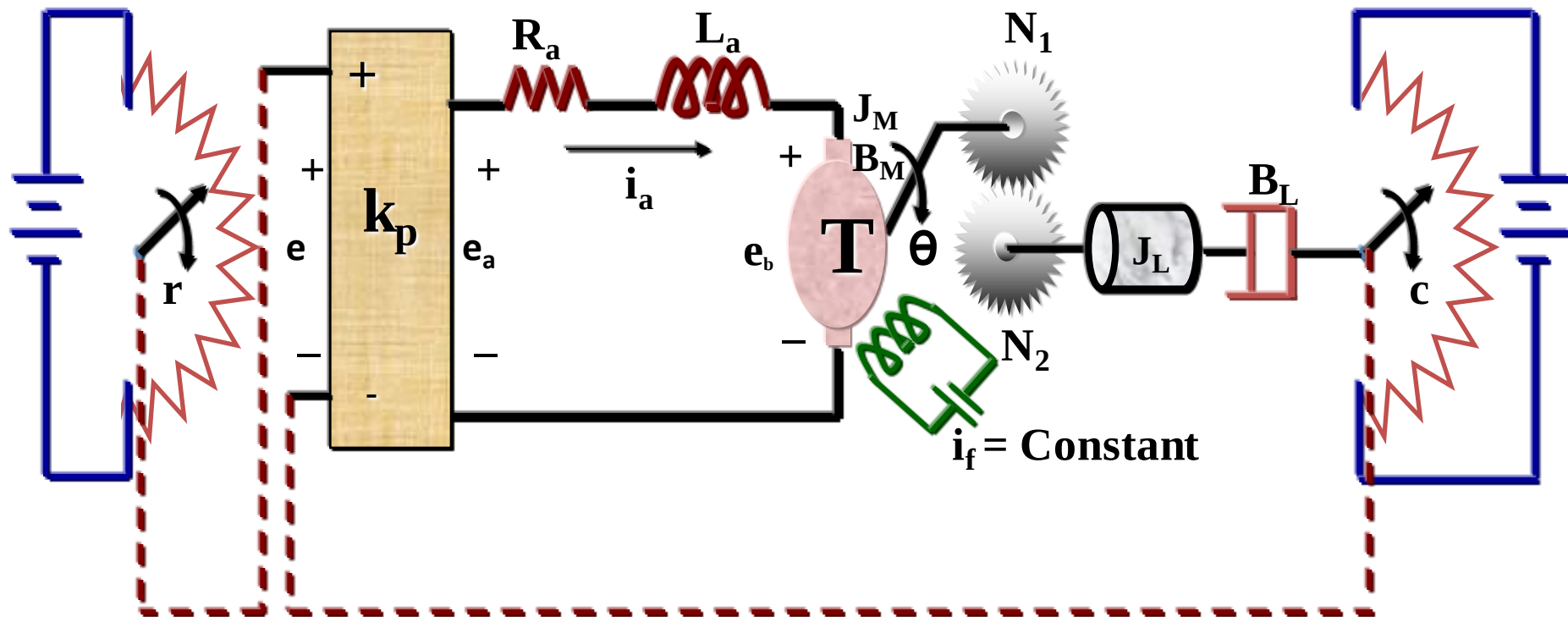
A field controlled D.C motor runs at 10000 rpm when 15v applied at the field circuit. Field resistance of the motor is  $0.25 \Omega$ , Field inductance is  $0.1 \text{ H}$ , motor torque constant is  $1 \times 10^{-4}$ , moment of inertia of motor  $10^{-5}$ , viscous friction coefficient is  $0.003$ , moment of inertia of load is  $4.4 \times 10^{-3}$ , viscous friction coefficient of load is  $4 \times 10^{-2}$ .



1. Drive the overall transfer function of the system i.e.  $\Omega_L(s)/E_f(s)$
2. Determine the gear ratio such that the rotational speed of the load is reduced to 500 rpm.



# Example-6: Angular Position Control System



## Numerical Values for System constants

**$r$**  = angular displacement of the reference input shaft

**$c$**  = angular displacement of the output shaft

**$\theta$**  = angular displacement of the motor shaft

**$K_1$**  = gain of the potentiometer shaft =  **$24/\pi$**

**$K_p$**  = amplifier gain = **10**

**$e_a$**  = armature voltage

**$e_b$**  = back emf

**$R_a$**  = armature winding resistance =  **$0.2 \Omega$**

**$L_a$**  = armature winding inductance = **negligible**

**$i_a$**  = armature winding current

**$K_b$**  = back emf constant =  **$5.5 \times 10^{-2}$  volt-sec/rad**

**$K_t$**  = motor torque constant =  **$6 \times 10^{-5}$  N-m/ampere**

**$J_m$**  = moment of inertia of the motor =  **$1 \times 10^{-5}$  kg-m<sup>2</sup>**

**$B_m$**  = viscous-friction coefficients of the motor = **negligible**

**$J_L$**  = moment of inertia of the load =  **$4.4 \times 10^{-3}$  kgm<sup>2</sup>**

**$B_L$**  = viscous friction coefficient of the load =  **$4 \times 10^{-2}$  N-m/rad/sec**

**$n$**  = gear ratio =  **$N_1/N_2 = 1/10$**

## Example-6: Angular Position Control System

- Transfer function of the armature controlled D.C motor with load connected to it is given by

$$\frac{\theta(s)}{E_a(s)} = \frac{K_t}{s[(J_{eq}R_a + B_{eq}L_a)s + B_{eq}R_a + K_tK_b]}$$

- Where

$$J_{eq} = J_m + \left(\frac{N_1}{N_2}\right)^2 J_L = 1 \times 10^{-5} + \left(\frac{1}{10}\right)^2 \times 4.4 \times 10^{-3} = 5.4 \times 10^{-5}$$

$$B_{eq} = B_m + \left(\frac{N_1}{N_2}\right)^2 B_L = \left(\frac{1}{10}\right)^2 \times 4 \times 10^{-2} = 4 \times 10^{-4}$$

$$\frac{\theta(s)}{E_a(s)} = \frac{6}{s(1.08s + 8.33)}$$

## Example-6: Angular Position Control System

- Error is difference between reference input  $r(t)$  and out calculated  $c(t)$  and can be calculated as

$$e(t) = K_1[r(t) - c(t)]$$

$$E(s) = \frac{24}{\pi}[R(S) - C(S)] = 7.64[R(S) - C(S)] \quad (a)$$

- Output of amplifier is

$$E_a(s) = K_p E(s) = 10E(s) \quad (b)$$

- Merging eq (a) and eq (b) yields

$$E_a(s) = 76.4 [R(S) - C(S)]$$

- Relation between angular position of motor  $\theta$  and angular position of load  $c$  is given as

$$C(s) = \frac{1}{10} \theta(s)$$

- or

$$10C(s) = \theta(s)$$

## Example-6: Angular Position Control System

$$\frac{\theta(s)}{E_a(s)} = \frac{6}{s(1.08s + 8.33)}$$

- **Final Closed loop transfer function of the system can now be written as**

$$\frac{10C(s)}{76.4 [R(s) - C(s)]} = \frac{6}{s(1.08s + 8.33)}$$

$$\frac{C(s)}{R(s)} = \frac{42.3}{s^2 + 7.69s + 42.3}$$

# Control Systems (CS)

Lecture-8

قاعدة راوث هرتز للاستقرارية

Routh-Herwitz Stability Criterion

# Routh-Hurwitz Stability Criterion

- It is a method for determining continuous system stability.
- The Routh-Hurwitz criterion states that “the number of roots of the characteristic equation with positive real parts is equal to the number of changes in sign of the first column of the Routh array”.

# Routh-Hurwitz Stability Criterion

- This method yields stability information without the need to solve for the closed-loop system poles.
- Using this method, we can tell how many closed-loop system poles are in the left half-plane, in the right half-plane, and on the  $j\omega$ -axis. (Notice that we say how many, not where.)
- The method requires two steps:
  1. Generate a data table called a Routh table.
  2. interpret the Routh table to tell how many closed-loop system poles are in the LHP, the RHP, and on *the  $j\omega$ -axis*.



# Routh-Hurwitz Stability Criterion

## جدول قاعدة راوث هرتز

- The characteristic equation of the  $n^{\text{th}}$  order continuous system can be written as:

$$a_n s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0 = 0$$

- The stability criterion is applied using a Routh table which is defined as;

|           |           |           |           |         |
|-----------|-----------|-----------|-----------|---------|
| $s^n$     | $a_n$     | $a_{n-2}$ | $a_{n-4}$ | $\dots$ |
| $s^{n-1}$ | $a_{n-1}$ | $a_{n-3}$ | $a_{n-5}$ | $\dots$ |
| $\cdot$   | $b_1$     | $b_2$     | $b_3$     | $\dots$ |
| $\cdot$   | $c_1$     | $c_2$     | $c_3$     | $\dots$ |
| $\cdot$   | $\dots$   | $\dots$   | $\dots$   | $\dots$ |

- Where  $a_n, a_{n-1}, \dots, a_0$  are coefficients of the characteristic equation.

$$b_1 \equiv \frac{a_{n-1}a_{n-2} - a_n a_{n-3}}{a_{n-1}}$$

$$b_2 \equiv \frac{a_{n-1}a_{n-4} - a_n a_{n-5}}{a_{n-1}}$$

$$c_1 \equiv \frac{b_1 a_{n-3} - a_{n-1} b_2}{b_1}$$

$$c_2 \equiv \frac{b_1 a_{n-5} - a_{n-1} b_3}{b_1}$$

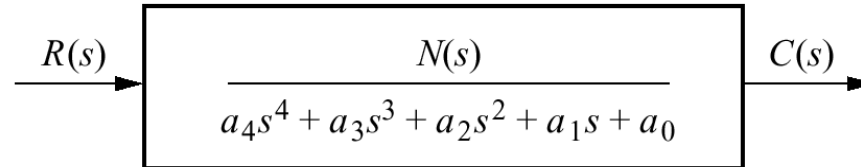
# تكوين جدول راوث Generating a Basic Routh Table

- First label the rows with powers of  $s$  from highest power of  $s$  down to lowest power of  $s$  in a vertical column.
- Next form the first row of the Routh table, using the coefficients of the denominator of the closed-loop transfer function (characteristic equation).
- Start with the coefficient of the highest power and skip every other power of  $s$ .
- Now form the second row with the coefficients of the denominator skipped in the previous step.
- The table is continued horizontally and vertically until zeros are obtained.
- For convenience, any row can be multiplied or divide by a positive constant before the next row is computed without changing the values of the rows below and disturbing the properties of the Routh table.

# Routh's Stability Condition

- If the closed-loop transfer function has all poles in the left half of the  $s$ -plane, the system is stable. Thus, a system is stable if there are no sign changes in the first column of the Routh table.
- The Routh-Hurwitz criterion declares that the number of roots of the polynomial that are lies in the right half-plane is equal to the number of sign changes in the first column. Hence the system is unstable if the poles lies on the right hand side of the  $s$ -plane.

# Example: Generating a basic Routh Table.



- Only the first 2 rows of the array are obtained from the characteristic eq. the remaining are calculated as follows;

|       |                                                                           |                                                                         |                                                                     |
|-------|---------------------------------------------------------------------------|-------------------------------------------------------------------------|---------------------------------------------------------------------|
| $s^4$ | $a_4$                                                                     | $a_2$                                                                   | $a_0$                                                               |
| $s^3$ | $a_3$                                                                     | $a_1$                                                                   | 0                                                                   |
| $s^2$ | $\frac{-\begin{vmatrix} a_4 & a_2 \\ a_3 & a_1 \end{vmatrix}}{a_3} = b_1$ | $\frac{-\begin{vmatrix} a_4 & a_0 \\ a_3 & 0 \end{vmatrix}}{a_3} = b_2$ | $\frac{-\begin{vmatrix} a_4 & 0 \\ a_3 & 0 \end{vmatrix}}{a_3} = 0$ |
| $s^1$ | $\frac{-\begin{vmatrix} a_3 & a_1 \\ b_1 & b_2 \end{vmatrix}}{b_1} = c_1$ | $\frac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$     | $\frac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$ |
| $s^0$ | $\frac{-\begin{vmatrix} b_1 & b_2 \\ c_1 & 0 \end{vmatrix}}{c_1} = d_1$   | $\frac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$     | $\frac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$ |

# Four Special Cases or Configurations in the First Column Array of the Routh's Table:

1. **Case-I:** No element in the first column is zero.
2. **Case-II:** A zero in the first column but some other elements of the row containing the zero in the first column are nonzero.
3. **Case-III:** A zero in the first column and the other elements of the row containing the zero are also zero.
4. **Case-IV:** As in the third case but with repeated roots on the  $j\omega$ -axis.

## Case-I: No element in the first column is zero.

### Second-Order System.

The characteristic polynomial of a second order system is given below

$$q(s) = a_2s^2 + a_1s + a_0$$

The Routh array is written as

$$\begin{array}{c|cc} s^2 & a_2 & a_0 \\ s^1 & a_1 & 0 \\ s^0 & b_1 & 0 \end{array}$$

Where

$$b_1 = \frac{a_1a_0 - (0)a_2}{a_1} = \frac{-1}{a_1} \begin{vmatrix} a_2 & a_0 \\ a_1 & 0 \end{vmatrix} = a_0$$

The requirement for a stable second order system is simply that all the coefficient be positive or all the coefficient s be negative.

### Third-Order System.

The characteristic polynomial of a third order system is given below

$$q(s) = a_3s^3 + a_2s^2 + a_1s + a_0$$

The Routh array is

$$\begin{array}{c|cc} s^3 & a_3 & a_1 \\ s^2 & a_2 & a_0 \\ s^1 & b_1 & 0 \\ s^0 & c_1 & 0 \end{array}$$

Where

$$b_1 = \frac{a_2a_1 - a_0a_3}{a_2} \quad \text{and} \quad c_1 = \frac{b_1a_0}{b_1} = a_0$$

- The requirement for a stable third order system is that the coefficients be positive and  $a_2a_1 > a_0a_3$ .
- The condition when  $a_2a_1 = a_0a_3$  results in a marginally stability case (recognized as Case-3 because there is a zero in the first column) and one pair of roots lies on the imaginary axis in the s-plane.

**Example-1:** Find the stability of the continuous system having the characteristic equation of

$$s^3 + 6s^2 + 12s + 8 = 0$$

The Routh table of the given system is computed and shown is the table below;

|       |                |    |   |
|-------|----------------|----|---|
| $s^3$ | 1              | 12 | 0 |
| $s^2$ | 6              | 8  | 0 |
| $s^1$ | $\frac{64}{6}$ | 0  |   |
| $s^0$ | 8              |    |   |

- Since there is **no changes of the sign** in the first column of the Routh table, it means that all the roots of the characteristic equation have negative real parts and **hence this system is stable**.



**Example-2:** Find the stability of the continuous system having the characteristic polynomial of a third order system is given below

$$s^3 + s^2 + 2s + 24$$

- The Routh array is

$$\begin{array}{c|cc} s^3 & 1 & 2 \\ s^2 & 1 & 24 \\ s^1 & -22 & 0 \\ s^0 & 24 & 0 \end{array}$$

- Because **TWO changes in sign** appear in the first column, we find that two roots of the characteristic equation lie in the right hand side of the s-plane. **Hence the system is unstable.**

**Example-3:** Determine a rang of values of a system parameter  $K$  for which the system is stable.

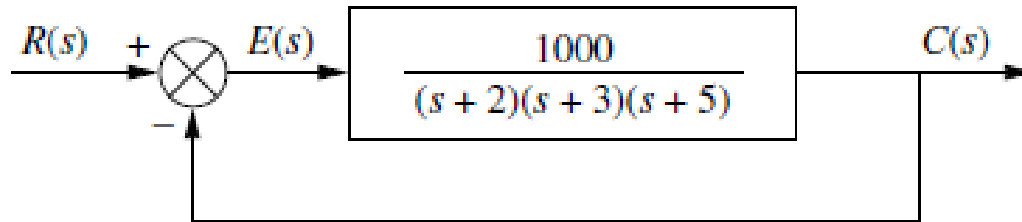
$$s^3 + 3s^2 + 3s + 1 + K = 0$$

- The Routh table of the given system is computed and shown is the table below;

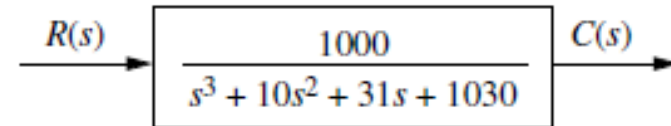
|       |                   |         |   |
|-------|-------------------|---------|---|
| $s^3$ | 1                 | 3       | 0 |
| $s^2$ | 3                 | $1 + K$ | 0 |
| $s^1$ | $\frac{8 - K}{3}$ | 0       |   |
| $s^0$ | $1 + K$           |         |   |

- For system stability, it is necessary that the conditions  $8 - k > 0$ , and  $1 + k > 0$ , must be satisfied. Hence the rang of values of a system parameter  $k$  must be lies between -1 and 8 (i.e.,  $-1 < k < 8$ ).

**Example-4:** Find the stability of the system shown below using Routh criterion.



The close loop transfer function is shown in the figure



The Routh table of the system is shown in the table

|       |                                                                       |                                                                   |                                                                   |
|-------|-----------------------------------------------------------------------|-------------------------------------------------------------------|-------------------------------------------------------------------|
| $s^3$ | 1                                                                     | 31                                                                | 0                                                                 |
| $s^2$ | 10                                                                    | 1030                                                              | 0                                                                 |
| $s^1$ | $-\frac{\begin{vmatrix} 1 & 31 \\ 1 & 103 \end{vmatrix}}{1} = -72$    | $-\frac{\begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix}}{1} = 0$     | $-\frac{\begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix}}{1} = 0$     |
| $s^0$ | $-\frac{\begin{vmatrix} 1 & 103 \\ -72 & 0 \end{vmatrix}}{-72} = 103$ | $-\frac{\begin{vmatrix} 1 & 0 \\ -72 & 0 \end{vmatrix}}{-72} = 0$ | $-\frac{\begin{vmatrix} 1 & 0 \\ -72 & 0 \end{vmatrix}}{-72} = 0$ |

Because **TWO changes in sign** appear in the first column, we find that two roots of the characteristic equation lie in the right hand side of the s-plane. **Hence the system is unstable.**

**Example-5:** Find the stability of the system shown below using Routh criterion.

$$2s^4 + s^3 + 3s^2 + 5s + 10 = 0$$

- The Routh table of the system is

|       |                                       |    |    |
|-------|---------------------------------------|----|----|
| $s^4$ | 2                                     | 3  | 10 |
| $s^3$ | 1                                     | 5  | 0  |
| $s^2$ | $\frac{(1)(3) - (2)(5)}{1} = -7$      | 10 | 0  |
| $s^1$ | $\frac{(-7)(5) - (1)(10)}{-7} = 6.43$ | 0  | 0  |
| $s^0$ | 10                                    | 0  | 0  |

- System is unstable** because there are **two sign changes** in the first column of the Routh's table. Hence the equation has two roots on the right half of the s-plane.

# Case-II: A Zero Only in the First Column

There are TWO methods in case-II.

1. Stability via Epsilon Method.
2. Stability via Reverse Coefficients (Phillips, 1991).

## Case-II: Stability via Epsilon Method

- If the first element of a row is zero, division by zero would be required to form the next row.
- To avoid this phenomenon, an *epsilon*,  $\epsilon$ , (a small positive number) is assigned to replace the zero in the first column.
- The value  $\epsilon$  is then allowed to approach zero from either the positive or the negative side, after which the signs of the entries in the first column can be determined.

## Case-II: Stability via Epsilon Method

**Example-6:** Determine the stability of the system having a characteristic equation given below;

$$q(s) = s^5 + 2s^4 + 2s^3 + 4s^2 + 11s + 10$$

The Routh array is shown in the table;

|       |            |    |    |
|-------|------------|----|----|
| $s^5$ | 1          | 2  | 11 |
| $s^4$ | 2          | 4  | 10 |
| $s^3$ | $\epsilon$ | 6  | 0  |
| $s^2$ | $c_1$      | 10 | 0  |
| $s^1$ | $d_1$      | 0  | 0  |
| $s^0$ | 10         | 0  | 0  |

Where

$$c_1 = \frac{4\epsilon - 12}{\epsilon} = \frac{-12}{\epsilon} \quad \text{and} \quad d_1 = \frac{6c_1 - 10\epsilon}{c_1} \rightarrow 6.$$

There are **TWO sign changes** due to the large negative number in the first column,  $c_1 = -12/\epsilon$ .  
Therefore the **system is unstable**, and two roots of the equation lie in the right half of the s-plane.

**Example-7:** Determine the range of parameter  $K$  for which the system is unstable.

$$q(s) = s^4 + s^3 + s^2 + s + K$$

The Routh array of the above characteristic equation is shown below;

$$\begin{array}{c|ccc} s^4 & 1 & 1 & K \\ s^3 & 1 & 1 & 0 \\ s^2 & \epsilon & K & 0 \\ s^1 & c_1 & 0 & 0 \\ s^0 & K & 0 & 0 \end{array}$$

Where

$$c_1 = \frac{\epsilon - K}{\epsilon} \rightarrow \frac{-K}{\epsilon}$$

- Therefore, for any value of  $K$  greater than zero, the system is unstable.
- Also, because the last term in the first column is equal to  $K$ , a negative value of  $K$  will result in an unstable system.
- Consequently, **the system is unstable for all values of gain  $K$ .**



**Example-8:** Determine the stability of the of the closed-loop transfer function;

$$T(s) = \frac{10}{s^5 + 2s^4 + 3s^3 + 6s^2 + 5s + 3}$$

**Table-1:** The complete Routh table is formed by using the denominator of the characteristic equation T(s).

|       |                                                         |               |   |
|-------|---------------------------------------------------------|---------------|---|
| $s^5$ | 1                                                       | 3             | 5 |
| $s^4$ | 2                                                       | 6             | 3 |
| $s^3$ | <del>0</del> $\epsilon$                                 | $\frac{7}{2}$ | 0 |
| $s^2$ | $\frac{6\epsilon - 7}{\epsilon}$                        | 3             | 0 |
| $s^1$ | $\frac{42\epsilon - 49 - 6\epsilon^2}{12\epsilon - 14}$ | 0             | 0 |
| $s^0$ | 3                                                       | 0             | 0 |

**Table-2:** shows the first column of Table-1 along with the resulting signs for choices of  $\epsilon$  positive and  $\epsilon$  negative.

| Label | First column                                            | $\epsilon = +$ | $\epsilon = -$ |
|-------|---------------------------------------------------------|----------------|----------------|
| $s^5$ | 1                                                       | +              | +              |
| $s^4$ | 2                                                       | +              | +              |
| $s^3$ | <del>0</del> $\epsilon$                                 | +              | -              |
| $s^2$ | $\frac{6\epsilon - 7}{\epsilon}$                        | -              | +              |
| $s^1$ | $\frac{42\epsilon - 49 - 6\epsilon^2}{12\epsilon - 14}$ | +              | +              |
| $s^0$ | 3                                                       | +              | +              |

- A zero appears only in the first column (the  $s^3$  row).
- Next replace the zero by a small number,  $\epsilon$ , and complete the table.
- Assume a sign, positive or negative, for the quantity  $\epsilon$ .
- When quantity  $\epsilon$  is either positive or negative, in both cases the sign in the first column of Routh table is changes twice.
- Hence, **the system is unstable and has two poles in the right half-plane.**

## Case-II: Stability via Reverse Coefficients (Phillips, 1991).

- A polynomial that has the reciprocal roots of the original polynomial has its roots distributed the same—right half-plane, left half plane, or imaginary axis—because taking the reciprocal of the root value does not move it to another region.
- If we can find the polynomial that has the reciprocal roots of the original, it is possible that the Routh table for the new polynomial will not have a zero in the first column.
- The polynomial with reciprocal roots is a polynomial with the coefficients written in reverse order.
- This method is usually computationally easier than the epsilon method.

**Example-9:** Repeated example-8: Determine the stability of the closed-loop transfer function;

$$T(s) = \frac{10}{s^5 + 2s^4 + 3s^3 + 6s^2 + 5s + 3}$$

- First write a polynomial that has the reciprocal roots of the denominator of  $T(s)$ .
- This polynomial is formed by writing the denominator of  $T(s)$  in reverse order. Hence,

$$D(s) = 3s^5 + 5s^4 + 6s^3 + 3s^2 + 2s + 1$$

|                      |       |       |     |   |
|----------------------|-------|-------|-----|---|
| • The Routh table is | $s^5$ | 3     | 6   | 2 |
|                      | $s^4$ | 5     | 3   | 1 |
|                      | $s^3$ | 4.2   | 1.4 |   |
|                      | $s^2$ | 1.33  | 1   |   |
|                      | $s^1$ | -1.75 |     |   |
|                      | $s^0$ | 1     |     |   |

- Since there are **TWO sign changes**, the **system is unstable** and has **TWO right-half-plane poles**.
- This is the same as the result obtained in the previous Example.
- Notice that Table does not have a zero in the first column.

# Case-III: Entire Row is Zero.

- Sometimes while making a Routh table, we find that an entire row consists of zeros.
- This happens because there is an even polynomial that is a factor of the original polynomial.
- This case must be handled differently from the case of a zero in only the first column of a row.

### Example-10: Determine the stability of the system.

The characteristic equation  $q(s)$  of the system is  $q(s) = s^3 + 2s^2 + 4s + K$ .

Where  $K$  is an adjustable loop gain. The Routh array is then;

$$\begin{array}{c|cc} s^3 & 1 & 4 \\ s^2 & 2 & K \\ s^1 & \frac{8-K}{2} & 0 \\ s^0 & K & 0 \end{array}$$

For a stable system, the value of  $K$  must be;  $0 < K < 8$

When  $K = 8$ , the two roots exist on the  $j\omega$  axis and the system will be marginally stable.

- Also, **when  $K = 8$ , we obtain a row of zeros (case-III).**
- The **auxiliary polynomial,  $U(s)$** , is the equation of the row preceding the row of Zeros.
- The  **$U(s)$**  in this case, obtained from the  $s^2$  row.
- The order of the auxiliary polynomial is **always even** and indicates the number of **symmetrical root pairs**.

**Example-10: continue.**

$$q(s) = s^3 + 2s^2 + 4s + K$$

- The auxiliary polynomial,  $U(s)$ , can be obtain as;

$$\begin{aligned} U(s) &= 2s^2 + Ks^0 \\ &= 2s^2 + 8 \\ &= 2(s^2 + 4) \\ &= 2(s + j2)(s - j2) \end{aligned}$$

- To show that the auxiliary polynomial,  $U(s)$ , is indeed a factor of the characteristic polynomial,  $q(s)$ , we divide  $q(s)$  by  $U(s)$  to obtain

$$\begin{array}{r} \frac{1}{2}s + 1 \\ 2s^2 + 8 \overline{) s^3 + 2s^2 + 4s + 8} \\ \underline{s^3 \qquad \qquad + 4s} \phantom{.} \\ 2s^2 \qquad \qquad + 8 \\ \underline{2s^2 \qquad \qquad + 8} \\ 0 \end{array}$$

- When  $K = 8$ , the factors of the characteristic polynomial,  $q(s)$ , are

$$q(s) = (s + 2)(s + j2)(s - j2)$$

## Case-IV: Repeated roots of the characteristic equation on the $j\omega$ -axis.

- If the  $j\omega$ -axis roots are repeated, the system response will be **unstable** with a form  $t\sin(\omega t + \phi)$ . The Routh-Hurwitz criteria will not reveal this form of instability.

**Example-11:** Determine the stability of the system with the characteristic equation of

$$q(s) = s^5 + s^4 + 4s^3 + 24s^2 + 3s + 63$$

The Routh array is;

|       |  |     |     |    |
|-------|--|-----|-----|----|
| $s^5$ |  | 1   | 4   | 3  |
| $s^4$ |  | 1   | 24  | 63 |
| $s^3$ |  | -20 | -60 | 0  |
| $s^2$ |  | 21  | 63  | 0  |
| $s^1$ |  | 0   | 0   | 0  |

### Example-11:Continue.

Therefore, the Auxiliary polynomial,  $U(s)$ , is;

$$U(s) = 21s^2 + 63 = 21(s^2 + 3) = 21(s + j\sqrt{3})(s - j\sqrt{3})$$

Which indicates that **TWO roots are on the imaginary  $j\omega$ -axis**.

To examine the remaining roots, we divide the characteristic equation,  $q(s)$ , by the auxiliary polynomial,  $U(s)$ , to obtain;

$$\frac{q(s)}{s^2 + 3} = s^3 + s^2 + s + 21 \longrightarrow (a)$$

Establishing a Routh table for this equation, we have;

|       |     |    |
|-------|-----|----|
| $s^3$ | 1   | 1  |
| $s^2$ | 1   | 21 |
| $s^1$ | -20 | 0  |
| $s^0$ | 21  | 0  |

- The **TWO** changes in sign in the first column indicate the presence of **TWO** roots in the right-hand plane, and the **system is unstable**.
- There are **THREE** roots of eq. (a). The **ONE** root in left-hand side is  $s = -3$ .
- The **TWO** roots in the right-hand plane are  $s = +1 \pm j\sqrt{6}$ .



# Skill Assessment # 3:

**Problem:** Make a Routh table and tell how many roots of the following polynomial are in the right half-plane and in the left half-plane.

$$P(s) = 3s^7 + 9s^6 + 6s^5 + 4s^4 + 7s^3 + 8s^2 + 2s + 6$$

# Answer of the Skill Assessment # 3:

- **Four roots** of the characteristic equation lie in the right half-plane (rhp), and **three roots** lie of the characteristic equation in the left half-plane (lhp). Therefore the **system is unstable**.

التصميم باستخدام المحل الهندسي

ROOT LOCUSE DESIGN

PROPORTIONAL CONTROL (P CONTROL)

$$1 + KG(s)H(s) = 0$$

Therefore

$$\frac{(s + p_1)(s + p_2) \cdots (s + p_n)}{(s + z_1)(s + z_2) \cdots (s + z_m)} = -K$$

From the above expression, it follows that for a point in the  $s$ -plane to be on the root-locus, when  $0 < K < \infty$ , it must satisfy the following two conditions.

$$K = \frac{|s + p_1| |s + p_2| \cdots |s + p_n|}{|s + z_1| |s + z_2| \cdots |s + z_m|} \quad \text{or} \quad (3)$$

$$K = \frac{\text{product of vector lengths from finite poles}}{\text{product of vector lengths from finite zeros}}$$

and

$$\sum \text{of zeros of } G(s)H(s) - \sum \text{angle of poles of } G(s)H(s) = r(180), \quad r = \pm 1, \pm 3, \dots$$

or

$$\sum_{i=1}^m \theta_{zi} - \sum_{i=1}^n \theta_{pi} = 180r, \quad r = \pm 1, \pm 3, \dots \quad (4)$$

The magnitude and angle criteria given by (3) and (4) are used in the graphical root-locus design.

### Example 1

The open-loop transfer function of a control system is given by

$$KGH(s) = \frac{K}{s(s+1)(s+4)}$$

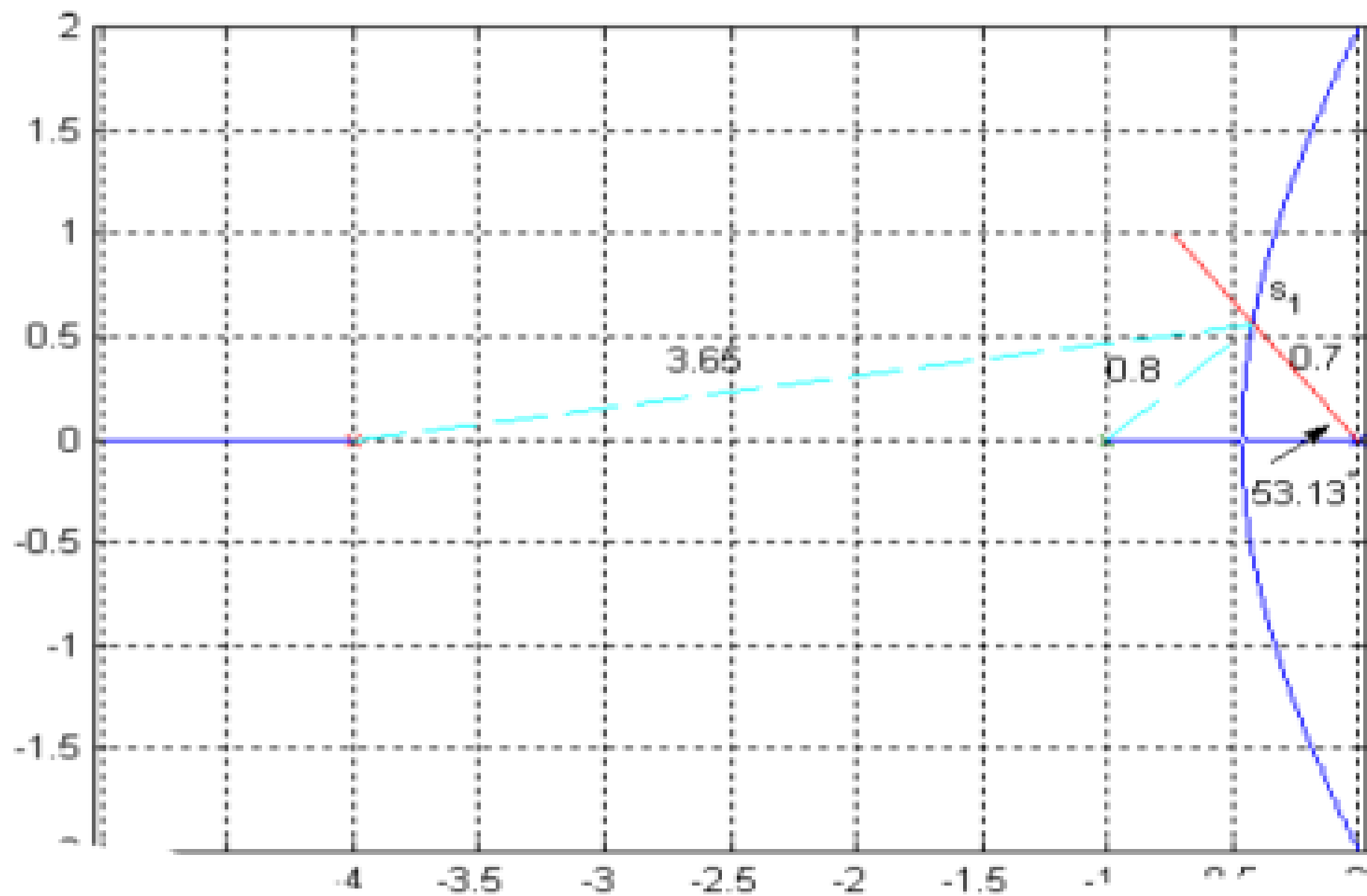
(a) Obtain the gain  $K_0$  of a proportional controller such that the damping ratio of the closed-loop poles will be equal 0.6. Obtain root-locus, step response and the time-domain specifications for the compensated system.

The root-locus plot is shown in Figure 2. For  $\zeta = 0.6$ ,

$$\theta = \cos^{-1} 0.6 = 53.13^\circ$$

The line drawn at this angle intersects the root-locus at approximately,  $s_1 \simeq -0.41 + j0.5$

The vector lengths from  $s_1$  to the poles are marked on the diagram



. From (1), we have

$$K = (0.7)(0.8)(3.65) = 2.04$$

This gain will result in the velocity error constant of  $K_v = \frac{2.04}{4} = 0.51$ . Thus, the steady-

state error due to a ramp input is  $e_{ss} = \frac{1}{K_v} = \frac{1}{0.51} = 1.96$ .

The compensated closed-loop transfer function is

$$\frac{C(s)}{R(s)} = \frac{2.05}{s^3 + 5s^2 + 4s + 2.05}$$

# PD CONTROLLER DESIGN

## 2. *PD* Compensator Design

Here both the error and its derivative are used for control

$$G_c(s) = K_P + K_D s \quad (5)$$

or

$$G_c(s) = K_D \left( s + \frac{K_P}{K_D} \right) = K_D (s + z_0)$$

where

$$Z_0 = \frac{K_P}{K_D} \quad (6)$$

From above, it can be seen that the *PD* controller is equivalent to the addition of a simple zero at  $Z_0 = K_P / K_D$  to the open-loop transfer function, which improves the transient response. From a different point of view, the PD controller may also be used to improve the steady-state error, because it anticipates large errors and attempts corrective action before they occur.



$$Z_0 = \frac{K_P}{K_D} \tag{6}$$

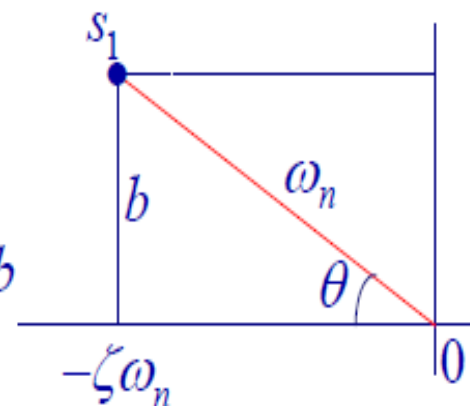
From above, it can be seen that the *PD* controller is equivalent to the addition of a simple zero at  $Z_0 = K_P / K_D$  to the open-loop transfer function, which improves the transient response. From a different point of view, the PD controller may also be used to improve the steady-state error, because it anticipates large errors and attempts corrective action before they occur.

The procedure for the graphical root-locus *PD* compensator design is as follows:

- Construct an accurate root-locus plot
- From the design specifications; the desired damping ratio and time constant of the dominant closed-loop poles, obtain the desired location of the dominant closed-loop poles.

$$\zeta\omega_n = \frac{1}{\tau} \quad \text{and} \quad \theta = \cos^{-1} \zeta$$

$$b = \zeta\omega_n \tan \theta \quad \text{and} \quad s_1 = -\zeta\omega_n + jb$$



- Mark the poles and zeros of the open-loop plant transfer function. Find the location of the compensator zero  $Z_0$  such that the angle criterion as given by (4) is satisfied.

$$\theta_{z0} + (\theta_{z1} + \theta_{z2} + \dots) - (\theta_{p1} + \theta_{p2} + \dots) = -180$$

- Estimate the vector lengths from  $s_1$  to all poles and zeros and apply the magnitude criterion as given by (3) to find  $K_D$ . Find  $K_P$  from (6)

### Example 2

Consider the control system shown in Figure 5.

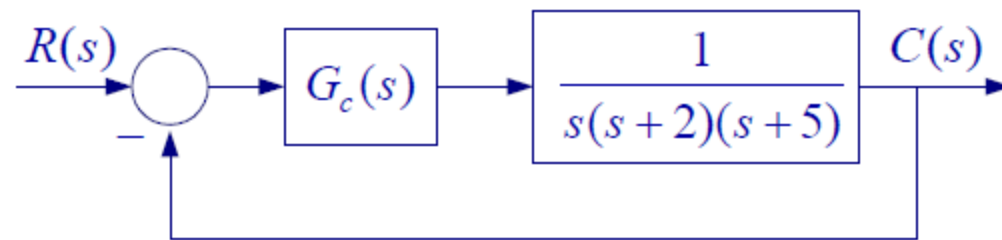


Figure 5

- (a) Assume the compensator is a simple proportional controller  $K$ , obtain all pertinent points for root locus and draw the root-locus. Determine the location of the dominant poles to have critically damped response, and find the time constant corresponding to this location. Also determine the value of  $K$  and the corresponding time constant for dominant poles damping ratio of 0.707. Obtain the compensated system step response.
- (b)  $G_c(s)$  is a  $PD$  compensator. Design the compensator for the following time-domain specifications.

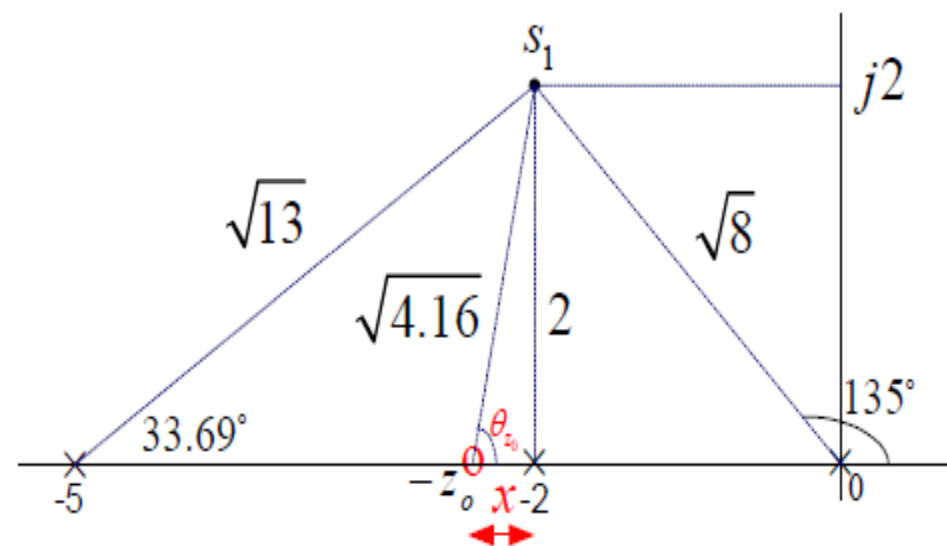
(b) The  $PD$  controller design

$$\zeta\omega_n = \frac{1}{\tau} = \frac{1}{0.5} = 2, \text{ and } \theta = \cos^{-1}(0.707) = 45^\circ$$

Therefore

$$s_1 = -2 + j2$$

The desired location of  $s_1$  requires the root-locus to be shifted towards the left half  $s$ -plane, which requires the addition of zero by the  $PD$  controller as shown in Figure 7.



The position of  $z_0$  is found by applying the angle criterion given by (4)

$$\theta_{z_0} - (135 + 90 + 33.69) = -180 \quad \Rightarrow \quad \theta_{z_0} = 78.69^\circ$$

$$\tan 78.69 = \frac{2}{x} \quad \Rightarrow \quad x = 0.4, \text{ and } z_0 = 2.4 = \frac{K_P}{K_D}$$

The compensated open-loop transfer function is

$$G_c(s)GH(s) = \frac{K_D(s + 2.4)}{s(s + 2)(s + 5)}$$

The vector lengths from  $s_1$  are marked on the diagram as shown. Applying the magnitude criterion, we have

$$K_D = \frac{(\sqrt{8})(2)(\sqrt{13})}{\sqrt{4.16}} = 10$$

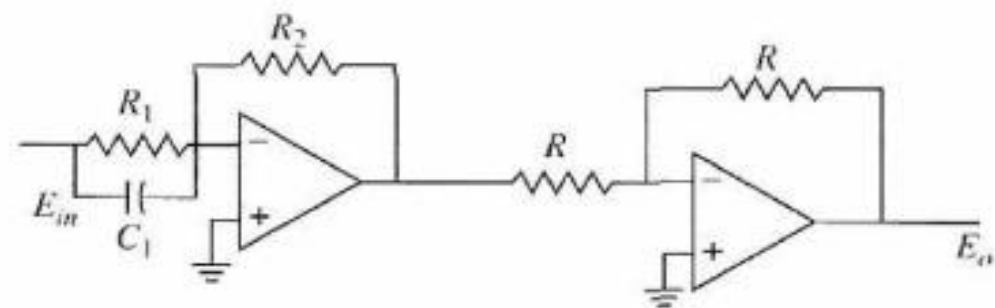
$$\frac{K_P}{K_D} = \frac{K_P}{10} = 2.4 \quad \Rightarrow \quad K_P = 24$$

Therefore, the controller transfer function is

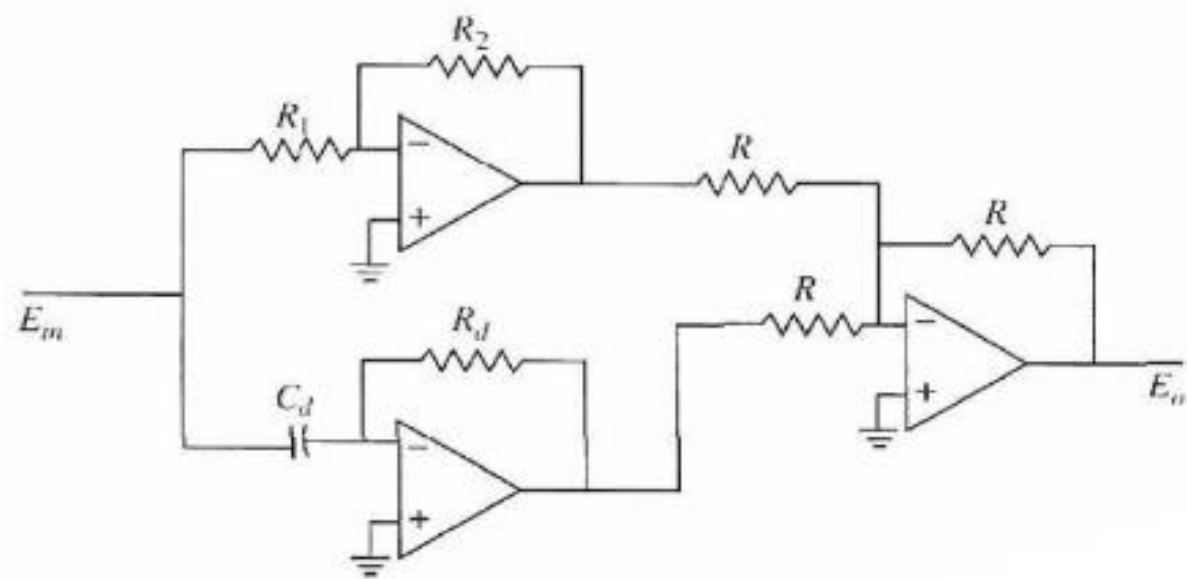
$$G_c(s) = 24 + 10s$$

We use the following commands to obtain the closed-loop transfer function

$$E_{in}(s) \quad R_1$$



(a)



(b)

### 3. PI Compensator Design

The integral of the error as well as the error itself is used for control, and the compensator transfer function is

$$G_c(s) = K_P + \frac{K_I}{s} \quad (7)$$

or

$$G_c(s) = \frac{K_P s + K_I}{s} = \frac{K_P(s + \frac{K_I}{K_P})}{s} = \frac{K_P(s + z_0)}{s}$$

where

$$Z_0 = \frac{K_I}{K_P}$$

Integral control bases its corrective action on the cumulative error integrated over time. The controller increases the type of system by 1 and is used to eliminate the steady-state errors.

# EXAMPLE

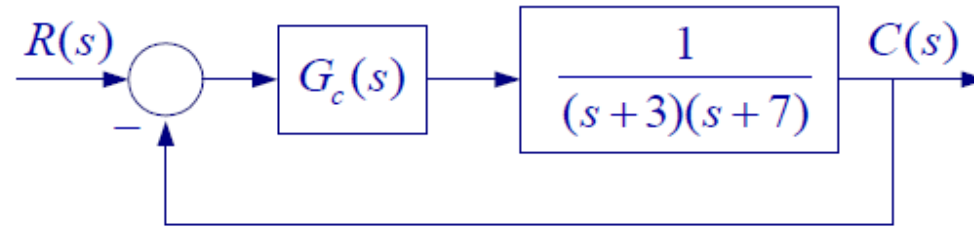


**Figure 9**

- Zero steady-state error due to a step input
- A pair of dominant closed-loop poles with a time constant of 0.25 seconds and a damping ratio of 0.8.

Obtain the compensated system step response.

$$\zeta\omega_n = \frac{1}{\tau} = \frac{1}{0.25} = 4, \text{ and } \theta = \tan^{-1}(0.8) = 36.87^\circ$$

Therefore

$$s_1 = -4 + j4 \tan 36.87^\circ \quad \Rightarrow \quad s_1 = -4 + j3$$

The poles of the open-loop transfer function and the controller pole at origin are marked in Figure 10.



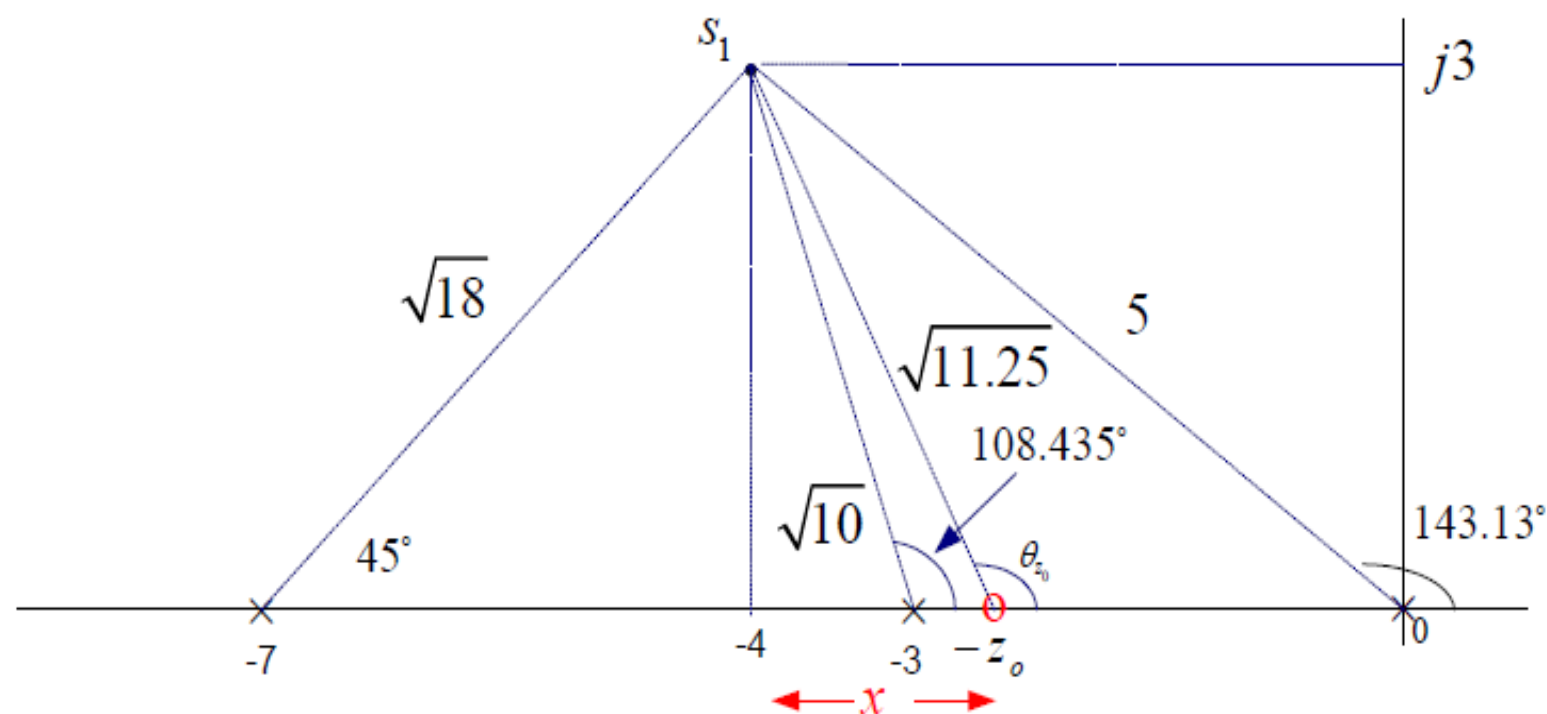


**Figure 10**

The position of controller zero for the desired location of  $s_1$  is obtained by applying the angle criterion given by (4)

$$\theta_{z_0} - (143.13 + 108.435 + 45) = -180 \quad \Rightarrow \quad \theta_{z_0} = 116.565^\circ$$

$$\tan(180 - 116.56) = \frac{3}{x} \quad \Rightarrow \quad x = 1.5, \text{ and } z_0 = 4 - 1.5 = 2.5$$

Therefore

$$\frac{K_I}{K_P} = 2.5$$

The compensated open-loop transfer function is

$$G_c(s)GH(s) = \frac{K_P(s + 2.5)}{s(s + 3)(s + 7)}$$

The vector lengths from  $s_1$  are marked on the diagram as shown. Applying the magnitude criterion, we have

$$K_p = \frac{(5)(\sqrt{10})(\sqrt{18})}{\sqrt{11.25}} = 20$$
$$\frac{K_I}{K_p} = \frac{K_I}{20} = 2.5 \quad \Rightarrow \quad K_I = 50$$

Therefore, the controller transfer function is

$$G_c(s) = 20 + \frac{50}{s}$$

The PI controller increases the system type from zero to 1. That is, we have a type 1 system and the steady-state error due to a step input is zero. We use the following commands to obtain the closed-loop transfer function and the step response.



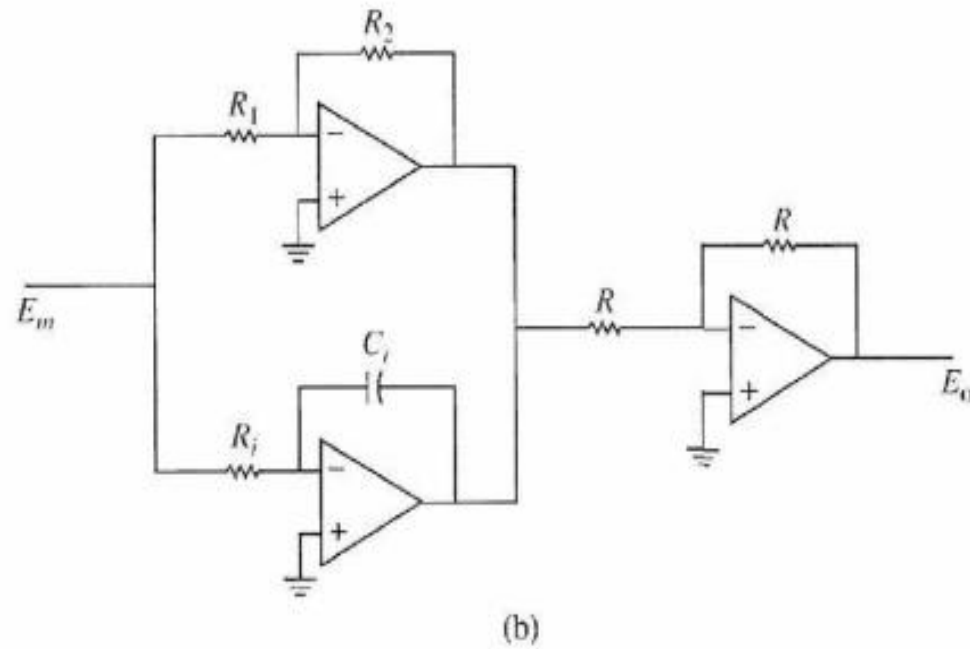
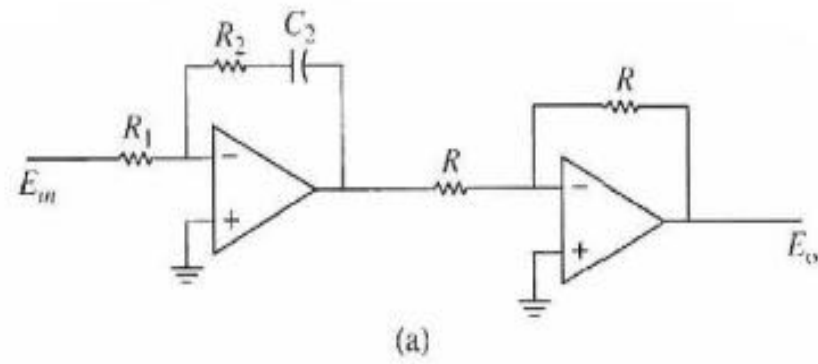


**Figure 9-16** Op-amp-circuit realization of the PI controller,  $G_c(s) = K_P + \frac{K_I}{s}$ .  
 (a) Two-op-amp circuit.  
 (b) Three-op-amp circuit.

$$G_c(s) = K_P + \frac{K_I}{s} + K_D s \quad (1)$$

For *PD* or *PI* controllers, the appropriate gain is set to zero.

Other compensators, are lead, lag, and lead-lag compensators. A first-order compensator having a single zero and pole in its transfer function is

$$G_c(s) = \frac{s + Z_0}{s + P_0} \quad (2)$$

The pole and zero are located in the left half s-plane as shown in Figure 1.

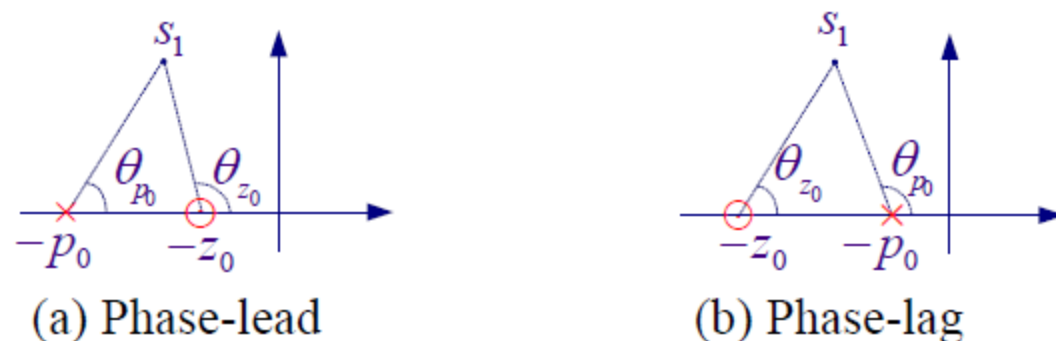


**Figure 1** Compensator phase angle contribution

## 4. PID Compensator

The PID controller is used to improve the dynamic response as well as to reduce or eliminate the steady-state error. With a proportional controller increasing the controller gain will reduce the rise time and the steady-state error. However, in systems of third

order or higher, large gain will make the system unstable. Derivative action contributes phase-lead and will improve the transient response, reducing the overshoot and settling time. The integral action increases the system type by 1 and eliminates the steady-state error, but it may make the transient response worse. When you are designing a PID controller, first set  $K_p$  to a large value to produce a fast response without losing stability. Then add derivative gain  $K_D$  and adjust its value to meet the transient response specifications. If required introduce the Integral gain  $K_I$  to eliminate the steady-state error. Repeat the design and fine-tune the gains to obtain the desired response.

# LEAD COMPENSATOR DESIGN

- From the time-domain specifications obtain the desired location of the closed-loop dominant poles.
- Select the controller zero. Place the zero to the left of the smallest plant's pole (or on the pole for *pole-zero cancellation*)
- Locate the compensator pole so that the angle criterion (3) is satisfied.
- Determine the compensator gain  $K_c$  such that the magnitude criterion (4) is satisfied.
- If the overall response rise time, overshoot and settling time is not satisfactory, place the controller zero at a different location and repeat the design



#### Example 4

For the control system of Example 2 design a phase lead compensator to meet the following time-domain specifications:

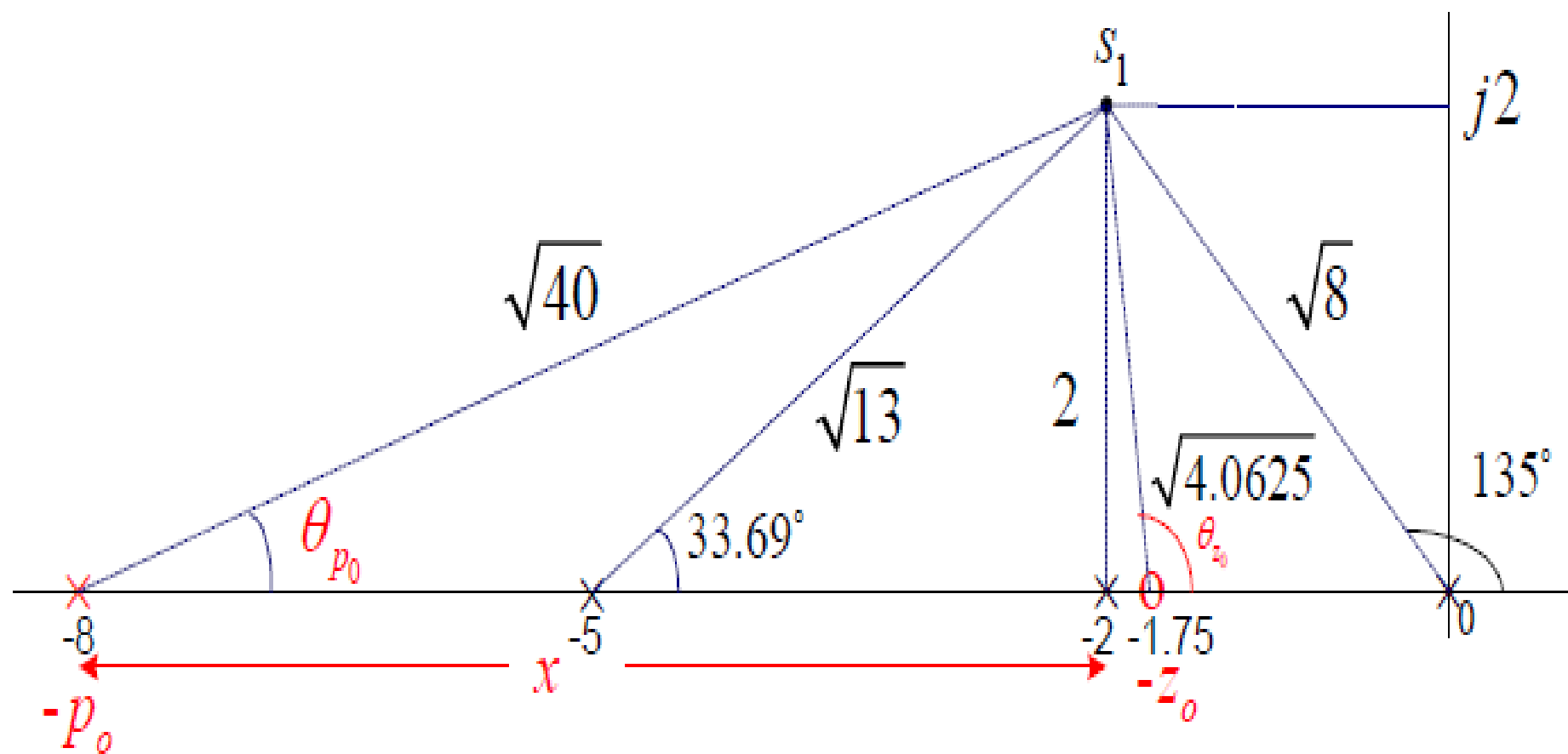
- Dominant poles damping ratio  $\zeta = 0.707$
- Dominant poles time constant  $\tau = 0.5$  second

$$\zeta\omega_n = \frac{1}{\tau} = \frac{1}{0.5} = 2, \text{ and } \theta = \cos^{-1}(0.707) = 45^\circ$$

Therefore

$$s_1 = -2 + j2$$

The desired location of  $s_1$  requires the root-locus to be shifted towards the left half s-plane, which requires the addition of phase lead controller as shown in Figure 12.



Let the controller zero be located at  $z_0 = 1.75$ . The position of the controller  $p_0$  is found by applying the angle criterion given by (4)

$$\theta_0 = 180 - \tan^{-1}(2 / 0.25) = 97.125^\circ$$

$$97.125 - (135 + 90 + 33.69 + \theta_{p0}) = -180 \quad \Rightarrow \quad \theta_{p0} = 18.435^\circ$$

$$\tan 18.435 = \frac{2}{x} \quad \Rightarrow \quad x = 6, \text{ and } z_0 = 2 + 6 = 8$$

The compensated open-loop transfer function is

$$G_c(s)GH(s) = \frac{K_c(s + 1.75)}{s(s + 2)(s + 5)(s + 8)}$$

The vector lengths from  $s_1$  are marked on the diagram as shown. Applying the magnitude criterion, we have

$$K_c = \frac{(\sqrt{8})(2)(\sqrt{13})(\sqrt{40})}{\sqrt{4.0625}} = 64$$

Therefore, the controller transfer function is

$$G_c(s) = \frac{64(s + 1.75)}{(s + 8)}$$

The controller dc gain is  $a_0 = G_c(0) = \frac{(64)(1.75)}{8} = 14$

# Phase-lag compensator approximate design

The lag compensator is an approximate integral control. The phase-lag compensator is used when the system transient response is satisfactory but requires a reduction in the steady-state error. Since  $0 < \alpha < 1$ , the compensator is a low-pass filter. It adds a negative angle to the angle criterion and tends to shift the root-locus to the right in the s-plane.

In the phase-lag control, the controller poles and zeros are placed very close together, and the combination is located relatively close to the origin of the s-plane. Thus, the root-loci in the compensated system are shifted only slightly from the original root-loci. The phase-lag compensator

contributes a magnitude of

$$|G_c(s)| = \frac{K_c |s_1 + z_0|}{|s_1 + p_0|} \approx K_c$$

$$K_0 |GH(s_1)| = 1 \quad \Rightarrow \quad |GH(s_1)| = \frac{1}{K_0}$$

For the compensated system, the magnitude criterion requires that

$$K |GH(s_1)| |G_c(s_1)| = 1 \quad \Rightarrow \quad K \frac{1}{K_0} K_c = 1$$

or

$$K_c = \frac{K_0}{K} = \frac{\text{Gain to satisfy the desired damping ratio}}{\text{Gain to satisfy the desired steady-state error}}$$

- Obtain the root-locus and determine the gain  $K_0$  to satisfy the desired damping ratio.
- Determine the gain  $K$  to satisfy the desired steady-state error.
- Evaluate the controller gain
$$K_c = \frac{\text{Gain to satisfy the desired damping ratio}}{\text{Gain to satisfy the desired steady-state error}} = \frac{K_0}{K}$$
- Select the controller zero  $z_0$  close to origin.
- Based on the compensator DC gain of unity,  $\frac{K_c z_0}{p_0} = 1$ , find the controller pole
$$p_0 = K_c z_0$$

(b) It is required to have approximately the same dominant closed-loop pole locations and the same damping ratio ( $\zeta = 0.8$ ) as in part (a). Design a phase-lag compensator such that the steady-state error due to a unit step input  $e_{ss}$  will be equal to 0.0845. Obtain the step response, and the time-domain specifications for the compensated system.

The gain  $K$ , which results in  $e_{ss} = 0.0845$  is given by

$$e_{ss} = 0.0845 = \frac{1}{1 + K_p} \Rightarrow K_p = 10.8343 = \frac{130K}{(10)(30)}$$

Thus the gain to realize the steady-state error specification is  $K = 25$

Using the approximate method, the controller gain is given by

$$K_c = \frac{\text{Gain to satisfy the desired damping ratio}}{\text{Gain to satisfy the desired steady-state error}} = \frac{2.5}{25} = 0.1$$

Next choose a small value for the compensator zero, e.g.,  $z = 1.5$

Based on the controller dc gain of unity  $K_c z_0 / p_0 = 1$ , the controller pole is found

$p_0 = K_c z_0 = (0.1)(1.5) = 0.15$ . Thus the controller transfer function is

$$G_c(s) = \frac{0.1(s+1.5)}{(s+0.15)}$$

and the compensated open-loop transfer function is

$$G_c(s)KG(s) = \frac{(0.1)(s+1.5)(130)(25)}{(s+0.15)(s+10)(s+30)} = \frac{325s+478.5}{s^3+40.15s^2+306s+45}$$

The compensated closed-loop transfer function is

$$\frac{C(s)}{R(s)} = \frac{325s+478.5}{s^3+40.15s^2+631s+532.5}$$

The compensated characteristic equation roots are  $-19.63 \pm j14.5$ , and  $-0.894$ . The compensated step response is shown in Figure 17. The complex poles are shifted slightly to the left from the specified value of  $-20 \pm j15$ .



# Control Systems (CS)

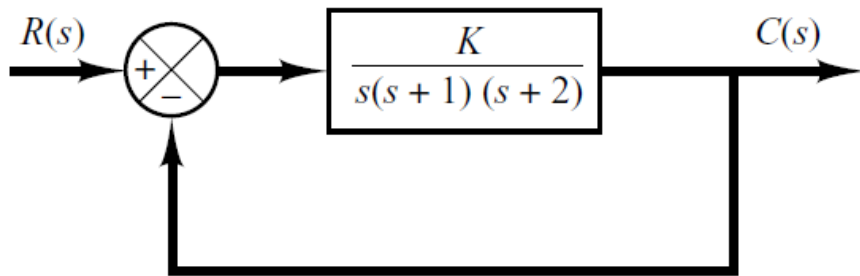
Lecture-9-a

المحل الهندسي للجذور

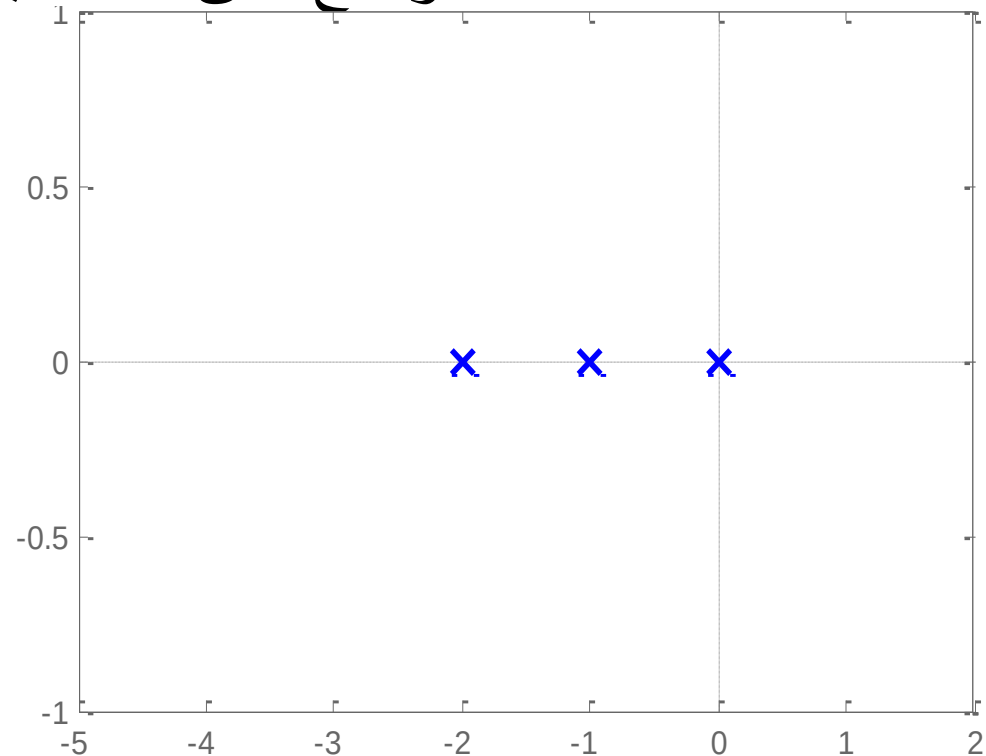
## Construction of Root Loci

# Construction of root loci

- **Step-1:** The first step in constructing a root-locus plot is to locate the open-loop poles and zeros in s-plane. وضع ماق الاقطاب والاصفار في مستقيم اس



$$G(s)H(s) = \frac{K}{s(s+1)(s+2)}$$

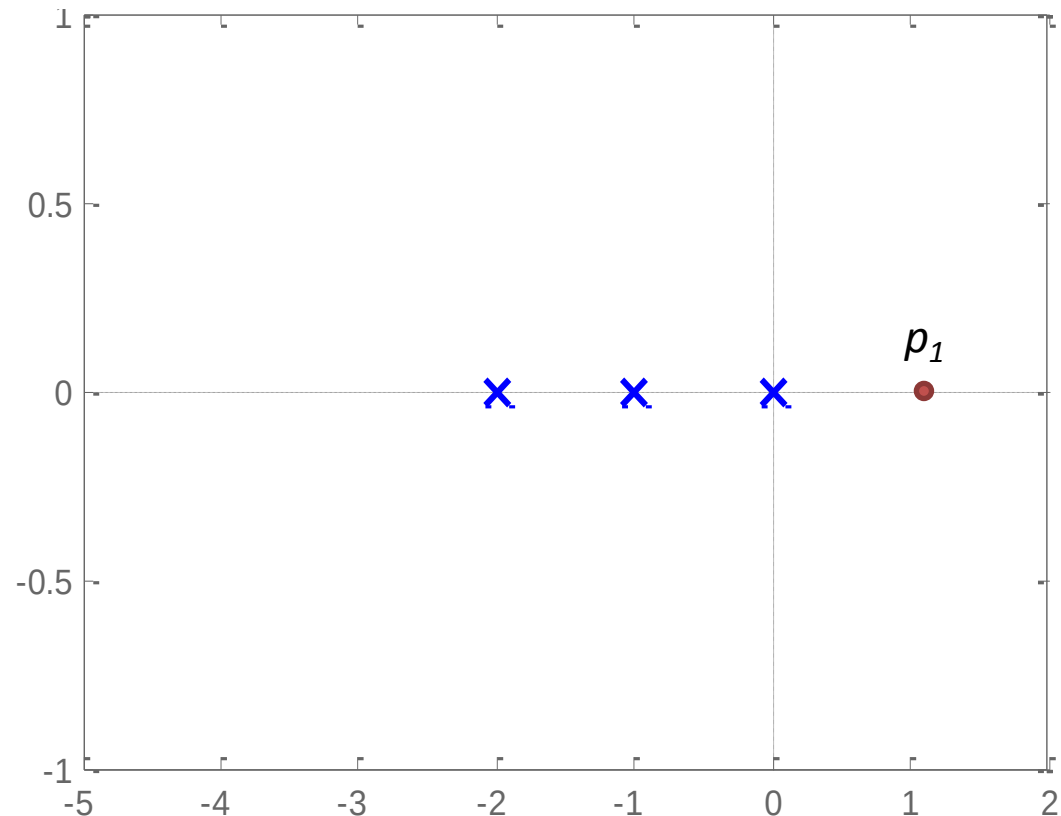


# Construction of root loci

- **Step-2:** Determine the root loci on the real axis. تحديد المحل الهندسي على المحور الحقيقي
- To determine the root loci on real axis we select some test points.
- e.g:  $p_1$  (on positive real axis).

$$\angle s = \angle s + 1 = \angle s + 2 = 0^\circ$$

- The angle condition is not satisfied.
- Hence, there is no root locus on the positive real axis.



# Construction of root loci

- **Step-2:** Determine the root loci on the real axis.

- Next, select a test point on the negative real axis between **0** and **-1**.

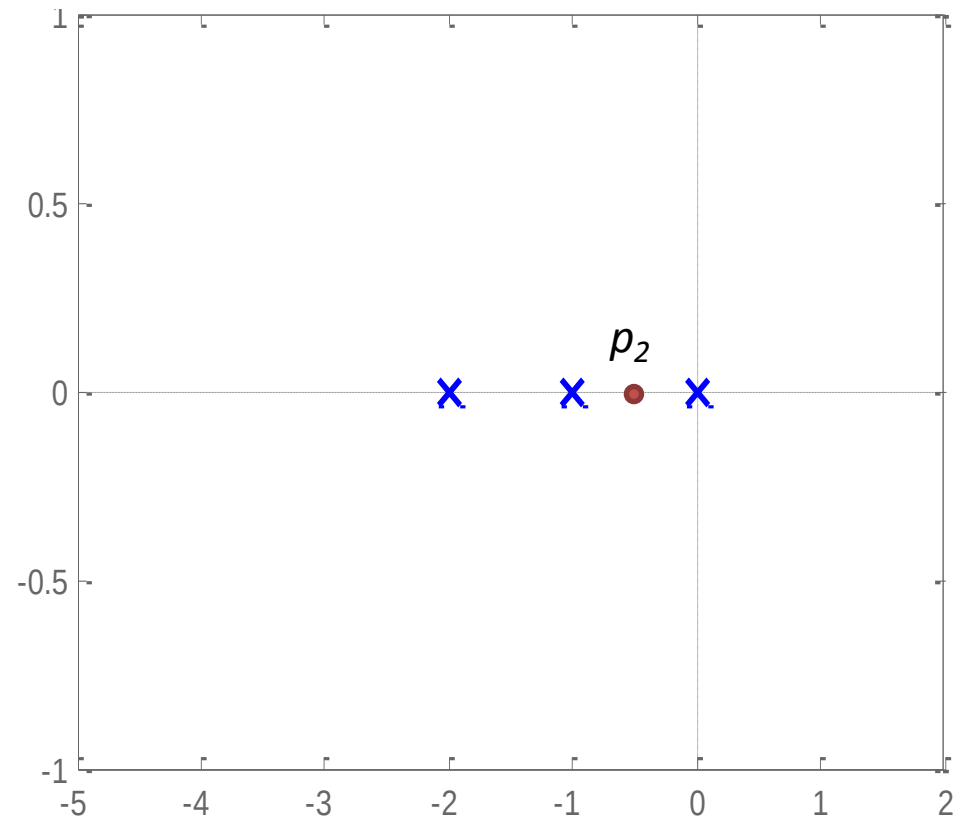
- Then

$$\angle s = 180^\circ, \quad \angle s + 1 = \angle s + 2 = 0^\circ$$

- Thus

$$-\angle s - \angle s + 1 - \angle s + 2 = -180^\circ$$

- The angle condition is satisfied. Therefore, the portion of the negative real axis between **0** and **-1** forms a portion of the root locus.



# Construction of root loci

- **Step-2:** Determine the root loci on the real axis.

- Now, select a test point on the negative real axis between **-1** and **-2**.

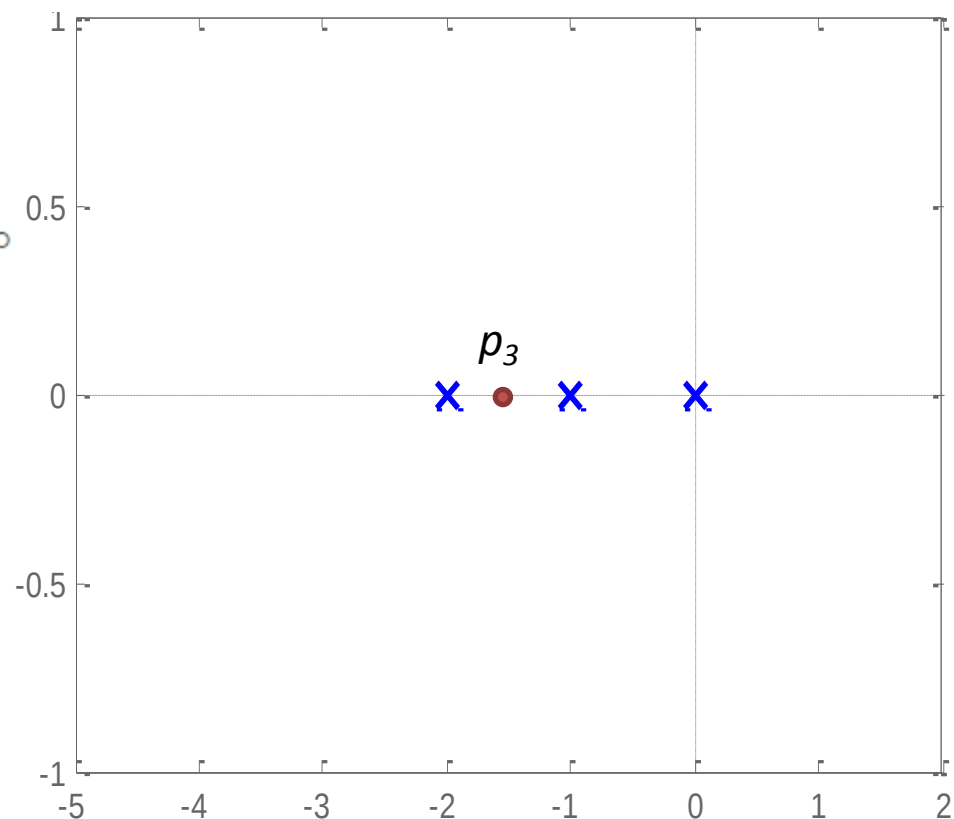
- Then

$$\angle s = \angle s + 1 = 180^\circ, \quad \angle s + 2 = 0^\circ$$

- Thus

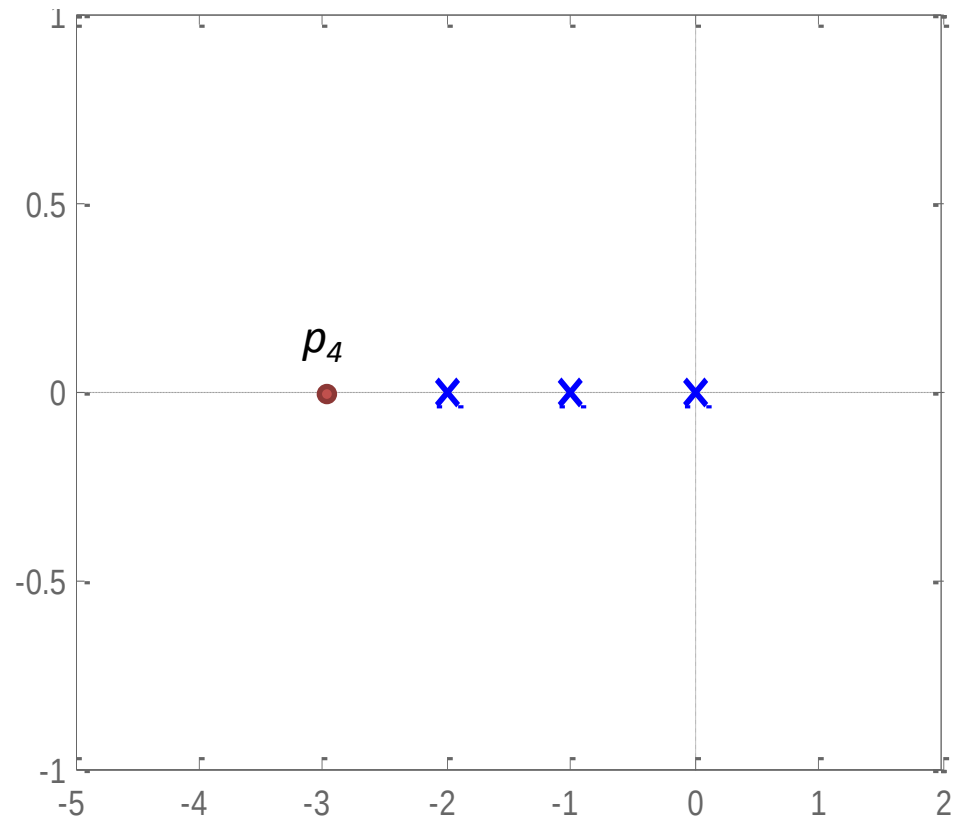
$$-\angle s - \angle s + 1 - \angle s + 2 = -360^\circ$$

- The angle condition is not satisfied. Therefore, the negative real axis between **-1** and **-2** is not a part of the root locus.



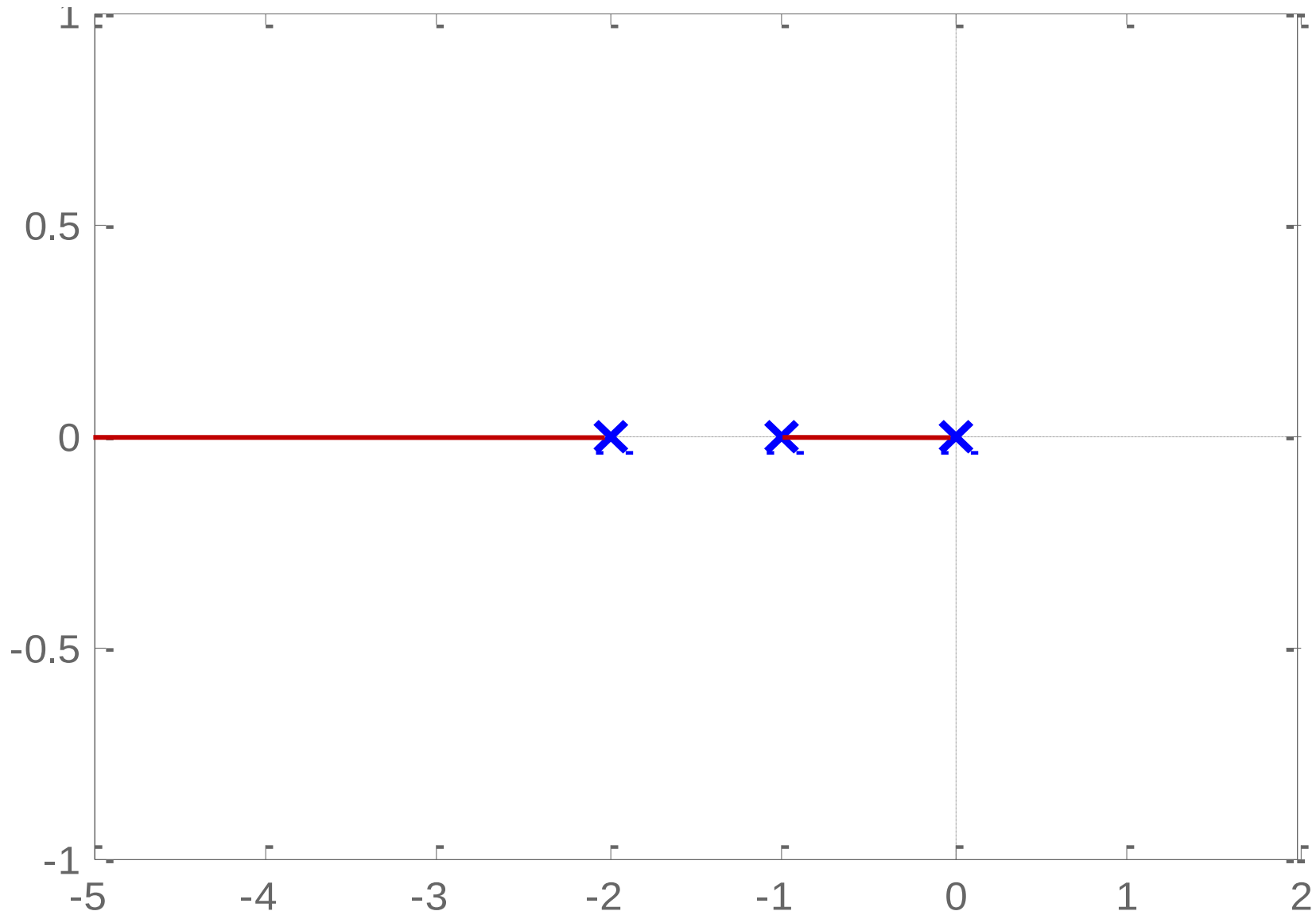
# Construction of root loci

- **Step-2:** Determine the root loci on the real axis.
- Similarly, test point on the negative real axis between **-2** and  $-\infty$  satisfies the angle condition.
- Therefore, the negative real axis between **-2** and  $-\infty$  is part of the root locus.



# Construction of root loci

- **Step-2:** Determine the root loci on the real axis.



# Construction of root loci

- **Step-3:** Determine the *asymptotes* of the root loci. تحديد المحاذيات باستخدام القانون التالي

$$\text{Angle of asymptotes} = \psi = \frac{\pm 180^\circ(2k + 1)}{n - m}$$

- where
- $n$ -----> number of poles
- $m$ -----> number of zeros
- For this Transfer Function

$$G(s)H(s) = \frac{K}{s(s+1)(s+2)}$$

$$\psi = \frac{\pm 180^\circ(2k + 1)}{3 - 0}$$



# Construction of root loci

- **Step-3:** Determine the *asymptotes* of the root loci.

$$\psi = \pm 60^\circ \quad \text{when } k = 0$$

$$= \pm 180^\circ \quad \text{when } k = 1$$

$$= \pm 300^\circ \quad \text{when } k = 2$$

$$= \pm 420^\circ \quad \text{when } k = 3$$

- Since the angle repeats itself as  $k$  is varied, the distinct angles for the asymptotes are determined as  $60^\circ$ ,  $-60^\circ$ ,  $-180^\circ$  and  $180^\circ$ .
- Thus, there are three asymptotes having angles  $60^\circ$ ,  $-60^\circ$ ,  $180^\circ$ .

# Construction of root loci

- **Step-3**: Determine the *asymptotes* of the root loci.
- Before we can draw these asymptotes in the complex plane, we must find the point where they intersect the real axis.
- Point of intersection of asymptotes on real axis (or centroid of asymptotes) can be find as out

$$\sigma = \frac{\sum \text{poles} - \sum \text{zeros}}{n - m}$$

# Construction of root loci

- **Step-3:** Determine the *asymptotes* المحاذيات of the root loci.

$$G(s)H(s) = \frac{K}{s(s+1)(s+2)}$$

- For

$$\sigma = \frac{(0-1-2)-0}{3-0}$$

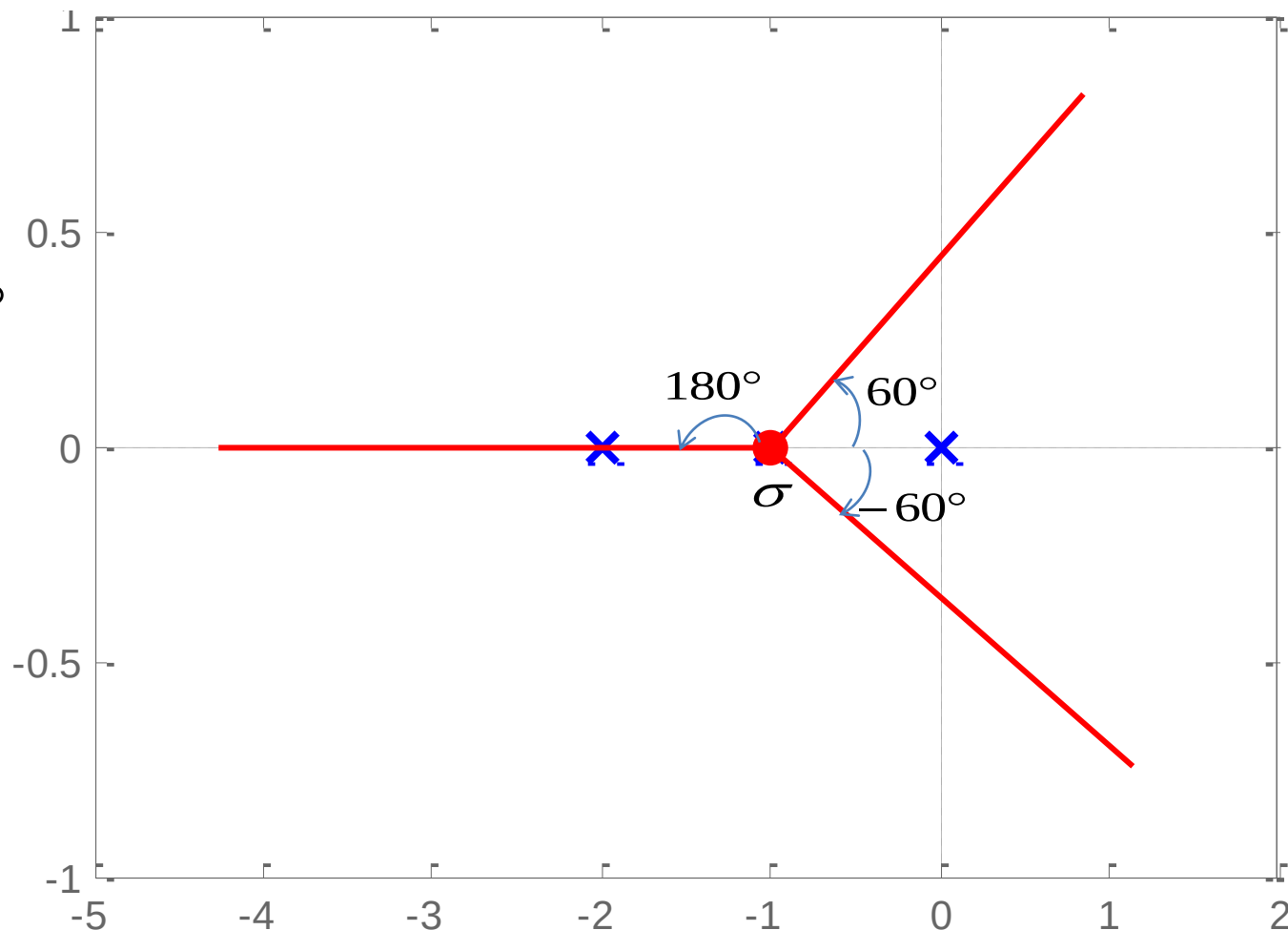
$$\sigma = \frac{-3}{3} = -1$$

# Construction of root loci

- **Step-3:** Determine the *asymptotes* of the root loci.

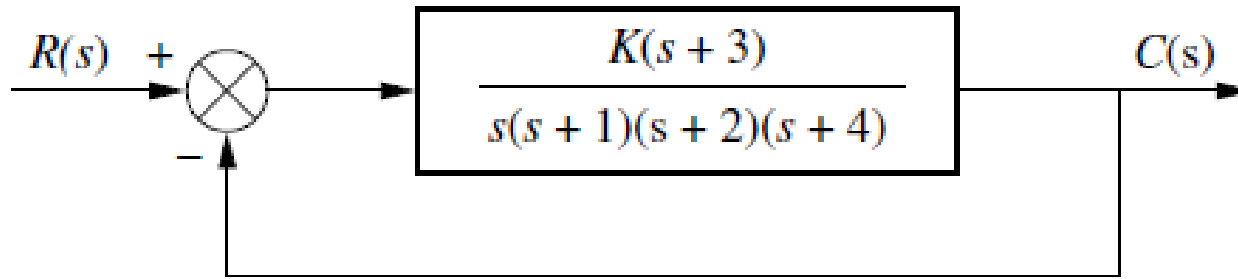
$$\psi = 60^\circ, -60^\circ, 180^\circ$$

$$\sigma = -1$$



# Home Work

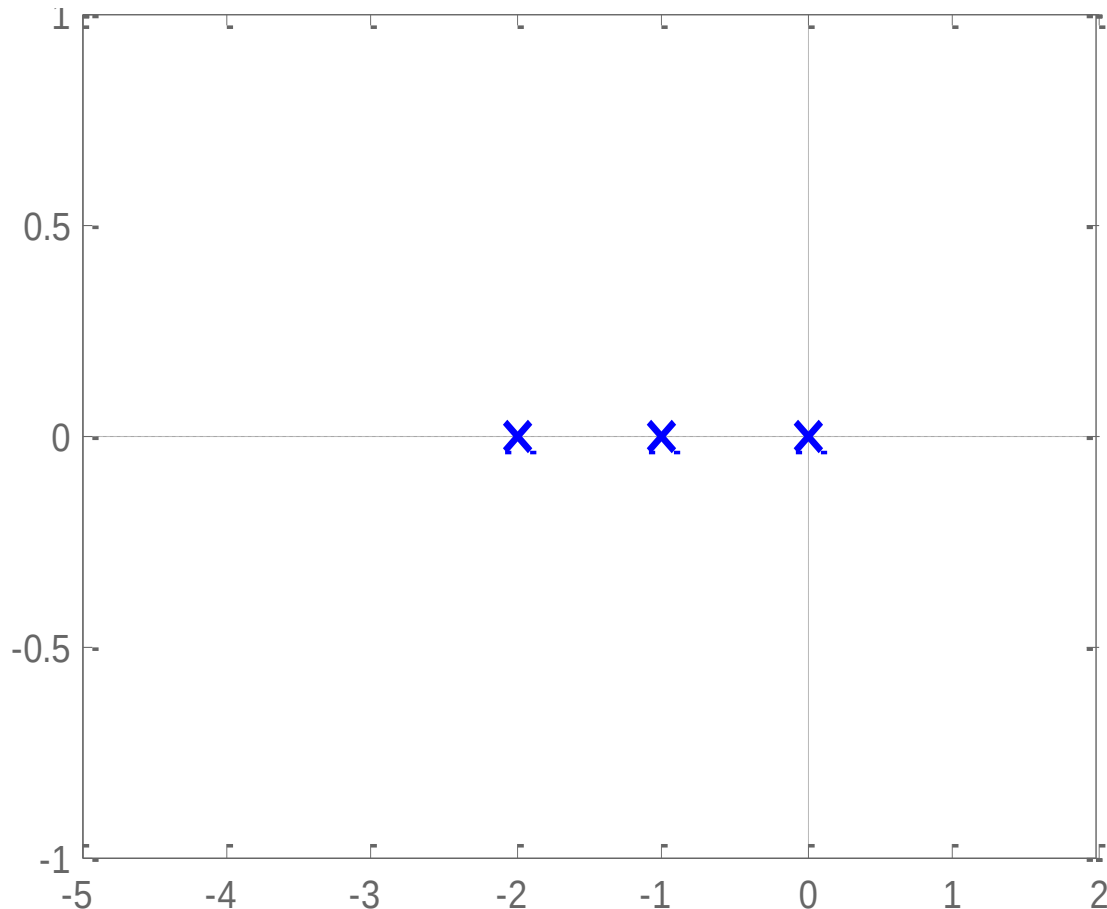
- Consider following unity feedback system.



- Determine
  - Root loci on real axis
  - Angle of asymptotes
  - Centroid of asymptotes

# Construction of root loci

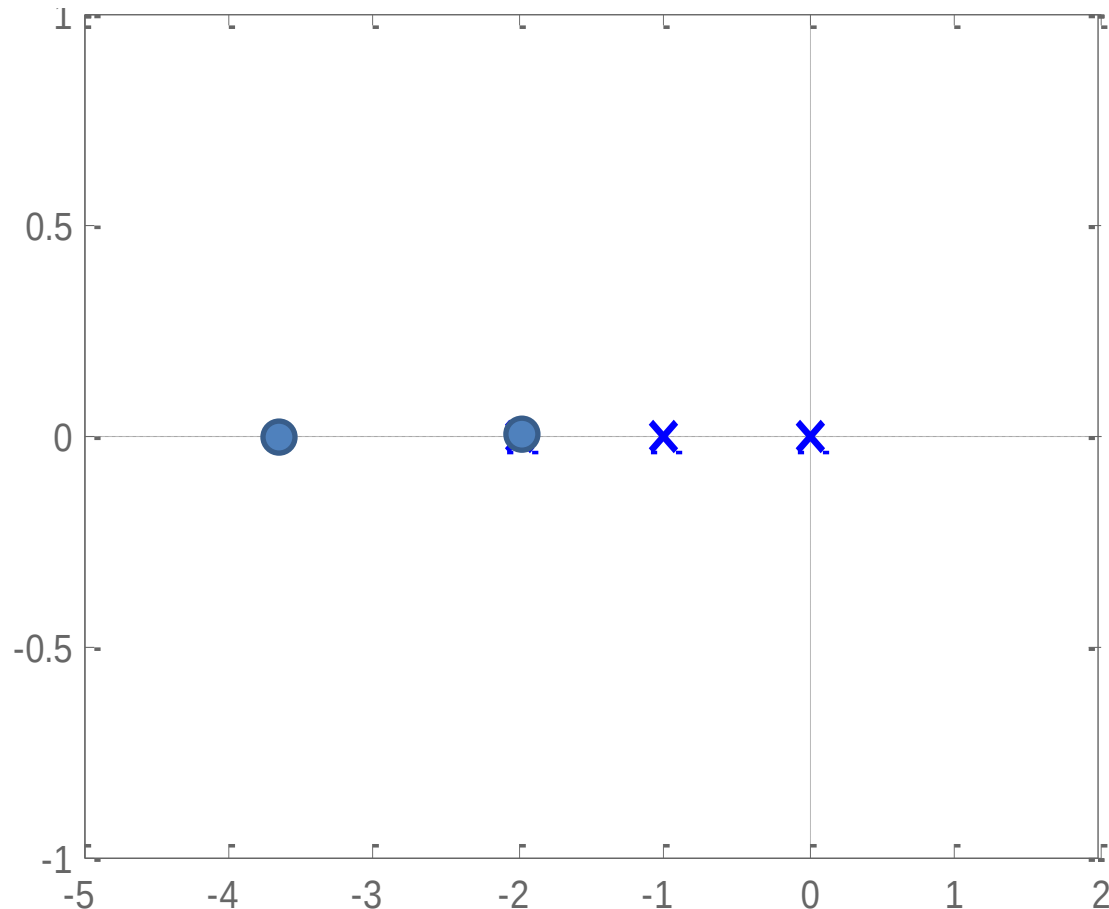
- **Step-4:** تحديد نقاط الافتراق Determine the *breakaway point*.
- The breakaway point corresponds to a point in the  $s$  plane where multiple roots of the characteristic equation occur.
- It is the point from which the root locus branches leave the real axis and enter the complex plane.



# Construction of root loci

- **Step-4:** تحديد نقاط الوصول Determine the *break-in point*.

- The break-in point corresponds to a point in the  $s$  plane where multiple roots of the characteristic equation occur.
- It is the point where the root locus branches arrive at real axis.



# Construction of root loci

- **Step-4:** Determine the *breakaway point* or *break-in point*. يتم تحديد النقاط باستخدام القانون التالي.

- The breakaway or break-in points can be determined from the roots of

$$\frac{dK}{ds} = 0$$

- It should be noted that not all the solutions of  $dK/ds=0$  correspond to actual breakaway points.
- If a point at which  $dK/ds=0$  is on a root locus, it is an actual breakaway or break-in point.
- Stated differently, if at a point at which  $dK/ds=0$  the value of  $K$  takes a real positive value, then that point is an actual breakaway or break-in point.



# Construction of root loci

- **Step-4:** Determine the *breakaway point* or *break-in point*.

$$G(s)H(s) = \frac{K}{s(s+1)(s+2)}$$

- The characteristic equation of the system is

$$1 + G(s)H(s) = 1 + \frac{K}{s(s+1)(s+2)} = 0$$

$$\frac{K}{s(s+1)(s+2)} = -1$$

$$K = -[s(s+1)(s+2)]$$

- The breakaway point can now be determined as

$$\frac{dK}{ds} = -\frac{d}{ds}[s(s+1)(s+2)]$$

# Construction of root loci

- **Step-4:** Determine the *breakaway point* or *break-in point*.

$$\frac{dK}{ds} = -\frac{d}{ds}[s(s+1)(s+2)]$$

$$\frac{dK}{ds} = -\frac{d}{ds}[s^3 + 3s^2 + 2s]$$

$$\frac{dK}{ds} = -3s^2 - 6s - 2$$

- Set  $dK/ds=0$  in order to determine breakaway point.

$$-3s^2 - 6s - 2 = 0$$

$$3s^2 + 6s + 2 = 0$$

$$s = -0.4226$$

$$= -1.5774$$

# Construction of root loci

- **Step-4:** Determine the *breakaway point* or *break-in point*.

$$s = -0.4226$$

$$= -1.5774$$

- Since the breakaway point must lie on a root locus between 0 and  $-1$ , it is clear that  $s=-0.4226$  corresponds to the actual breakaway point.
- Point  $s=-1.5774$  is not on the root locus. Hence, this point is not an actual breakaway or break-in point.
- In fact, evaluation of the values of  $K$  corresponding to  $s=-0.4226$  and  $s=-1.5774$  yields

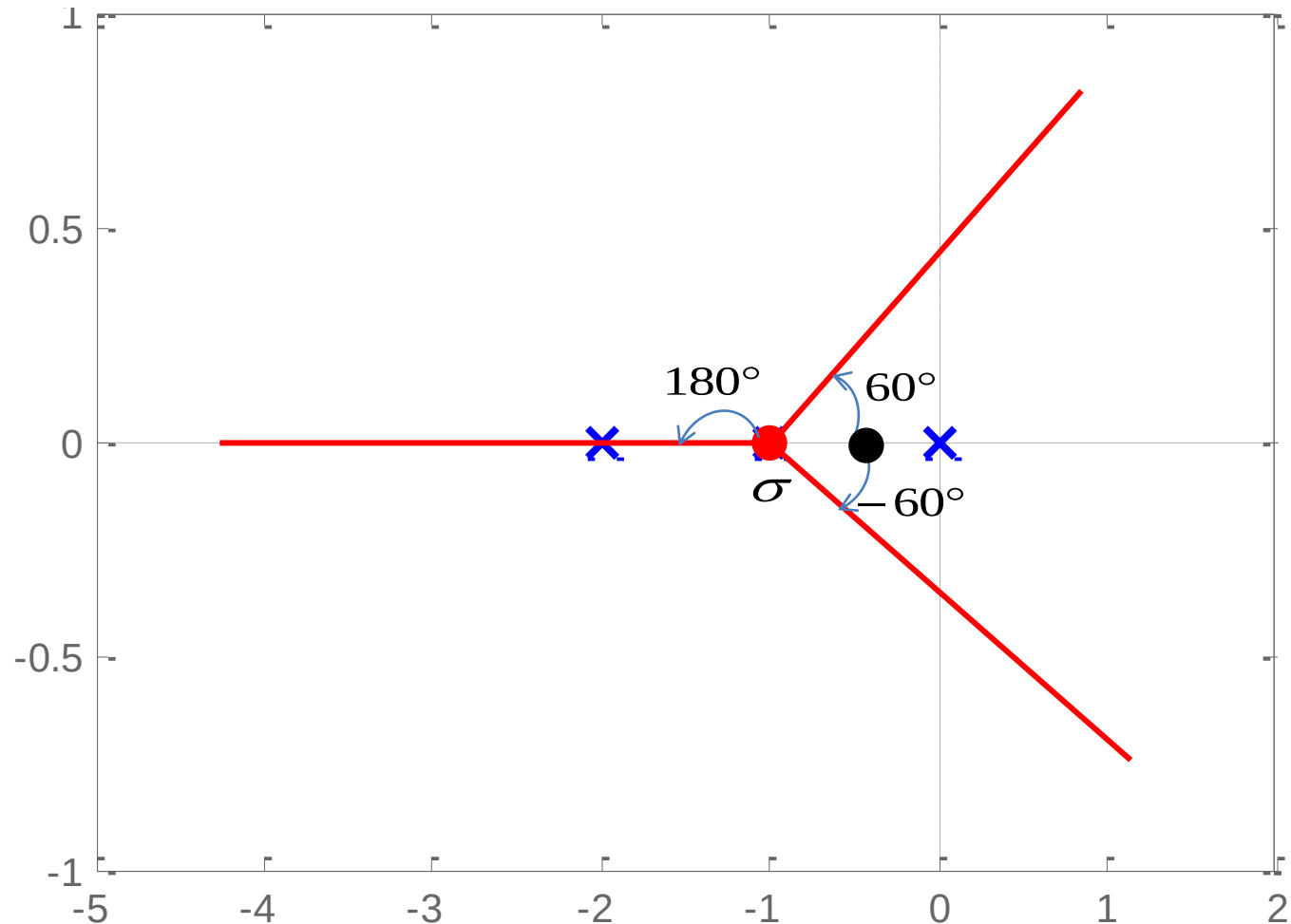
$$K = 0.3849, \quad \text{for } s = -0.4226$$

$$K = -0.3849, \quad \text{for } s = -1.5774$$

# Construction of root loci

- **Step-4:** Determine the *breakaway point*.

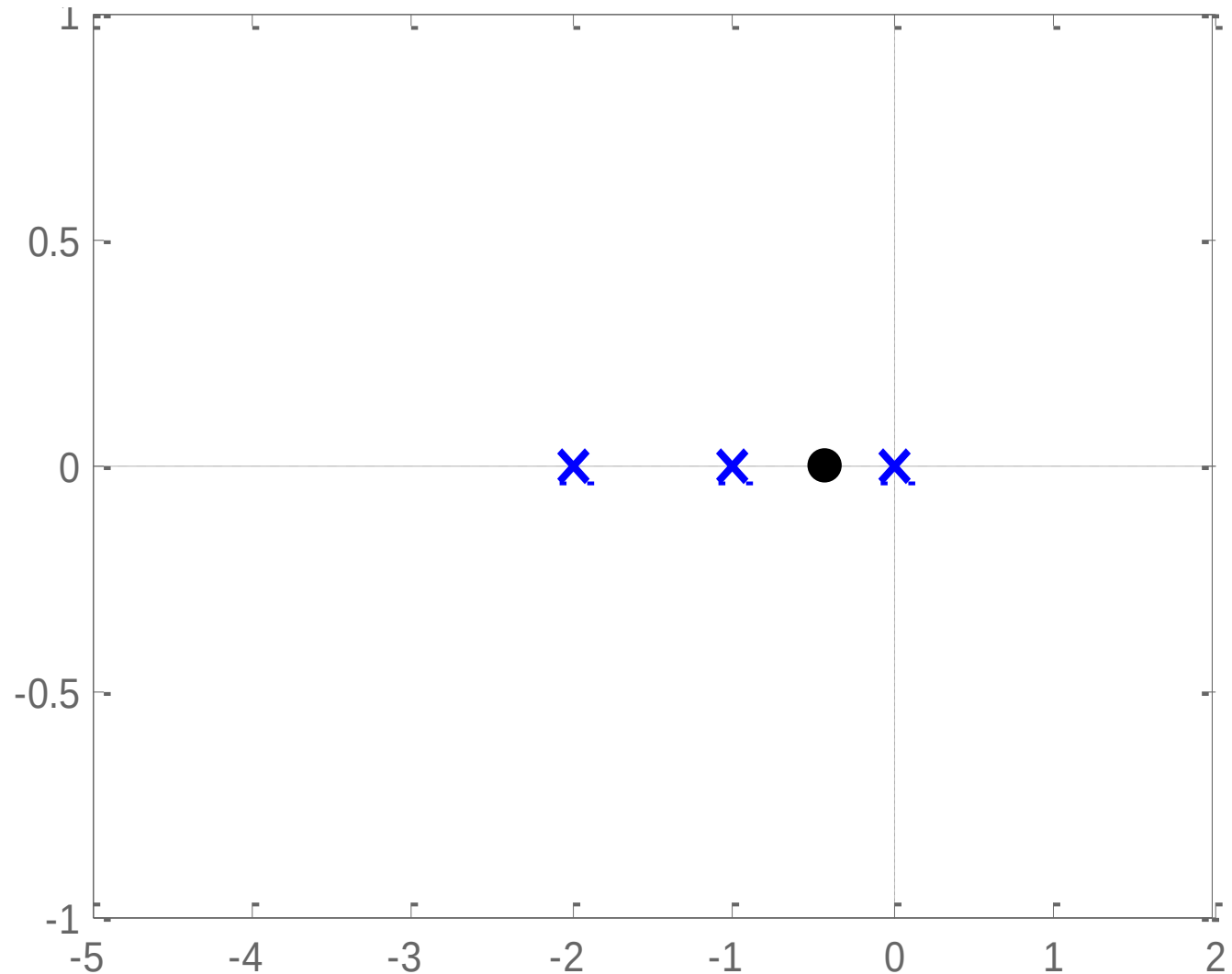
$$s = -0.4226$$



# Construction of root loci

- **Step-4:** Determine the *breakaway point*.

$$s = -0.4226$$



# Home Work

- Determine the Breakaway and break in points

$$KG(s)H(s) = \frac{K(s-3)(s-5)}{(s+1)(s+2)}$$

## Solution

$$KG(s)H(s) = \frac{K(s-3)(s-5)}{(s+1)(s+2)} = \frac{K(s^2 - 8s + 15)}{(s^2 + 3s + 2)}$$

$$\frac{K(s^2 - 8s + 15)}{s^2 + 3s + 2} = -1$$

$$K = -\frac{(s^2 + 3s + 2)}{(s^2 - 8s + 15)}$$

- Differentiating  $K$  with respect to  $s$  and setting the derivative equal to zero yields;

$$\frac{dK}{ds} = -\frac{[(s^2 - 8s + 15)(2s + 3) - (s^2 + 3s + 2)(2s - 8)]}{(s^2 - 8s + 15)^2} = 0$$

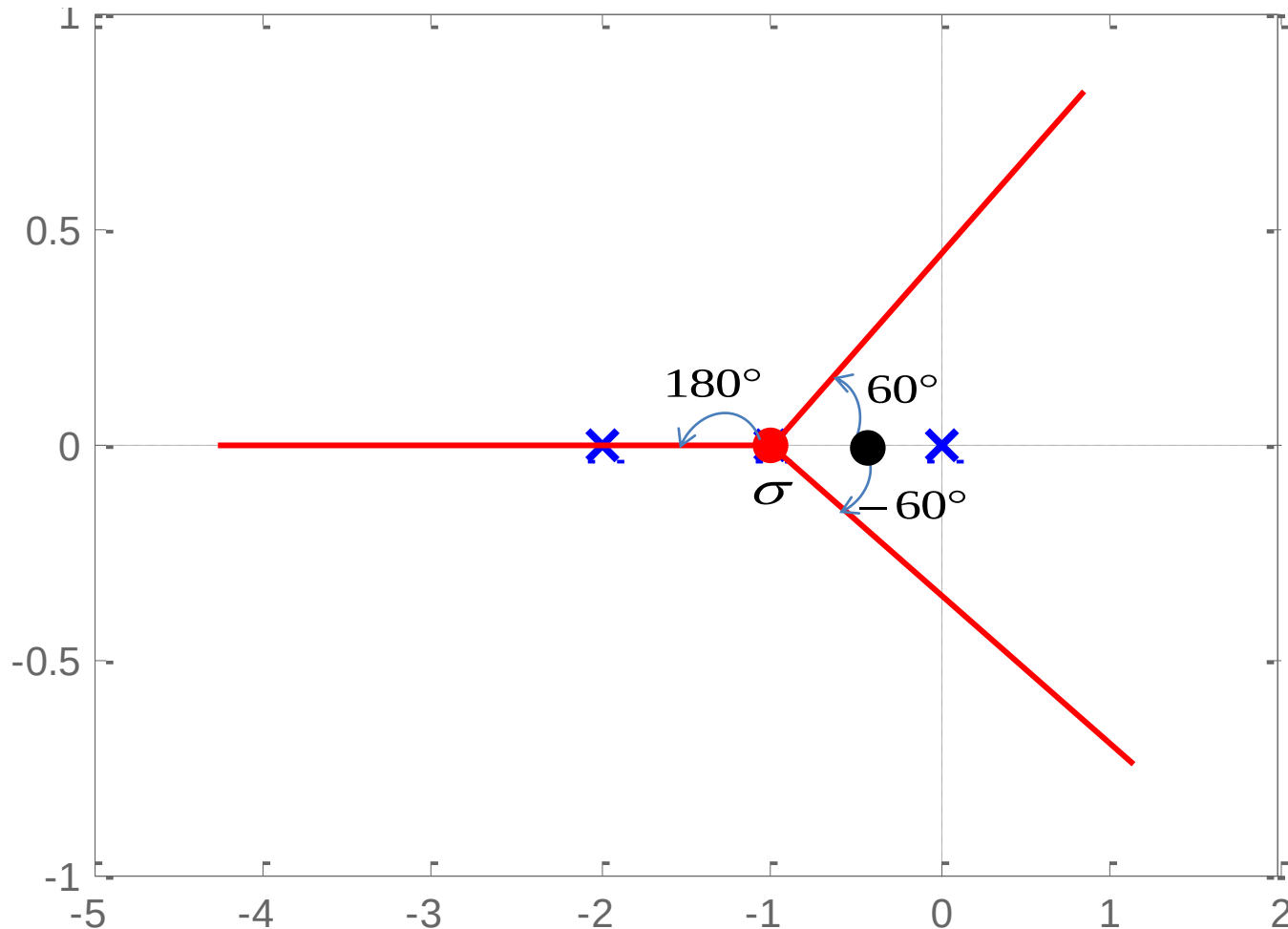
$$11s^2 - 26s - 61 = 0$$

Hence, solving for  $s$ , we find the break-away and break-in points;

$$s = -1.45 \text{ and } 3.82$$

# Construction of root loci

- **Step-5:** Determine the points where root loci cross the imaginary axis.





# Construction of root loci

- **Step-5:** Determine the points where root loci cross the imaginary axis.
  - These points can be found by use of Routh's stability criterion.
  - Since the characteristic equation for the present system is

$$s^3 + 3s^2 + 2s + K = 0$$

- The Routh Array Becomes

$$\begin{array}{ccc} s^3 & 1 & 2 \\ s^2 & 3 & K \\ s^1 & \frac{6 - K}{3} & \\ s^0 & K & \end{array}$$

# Construction of root loci

- **Step-5**: Determine the points where root loci cross the imaginary axis.
- The value(s) of **K** that makes the system marginally stable is **6**.
- The crossing points on the imaginary axis can then be found by solving the auxiliary equation obtained from the  $s^2$  row, that is,

$$3s^2 + K = 3s^2 + 6 = 0$$

- Which yields

$$s = \pm j\sqrt{2}$$

|       |                   |     |
|-------|-------------------|-----|
| $s^3$ | 1                 | 2   |
| $s^2$ | 3                 | $K$ |
| $s^1$ | $\frac{6 - K}{3}$ |     |
| $s^0$ | $K$               |     |

# Construction of root loci

- **Step-5:** Determine the points where root loci cross the imaginary axis.
- An alternative approach is to let  $s=j\omega$  in the characteristic equation, equate both the real part and the imaginary part to zero, and then solve for  $\omega$  and  $K$ .
- For present system the characteristic equation is

$$s^3 + 3s^2 + 2s + K = 0$$

$$(j\omega)^3 + 3(j\omega)^2 + 2j\omega + K = 0$$

$$(K - 3\omega^2) + j(2\omega - \omega^3) = 0$$

# Construction of root loci

- **Step-5:** Determine the points where root loci cross the imaginary axis.

$$(K - 3\omega^2) + j(2\omega - \omega^3) = 0$$

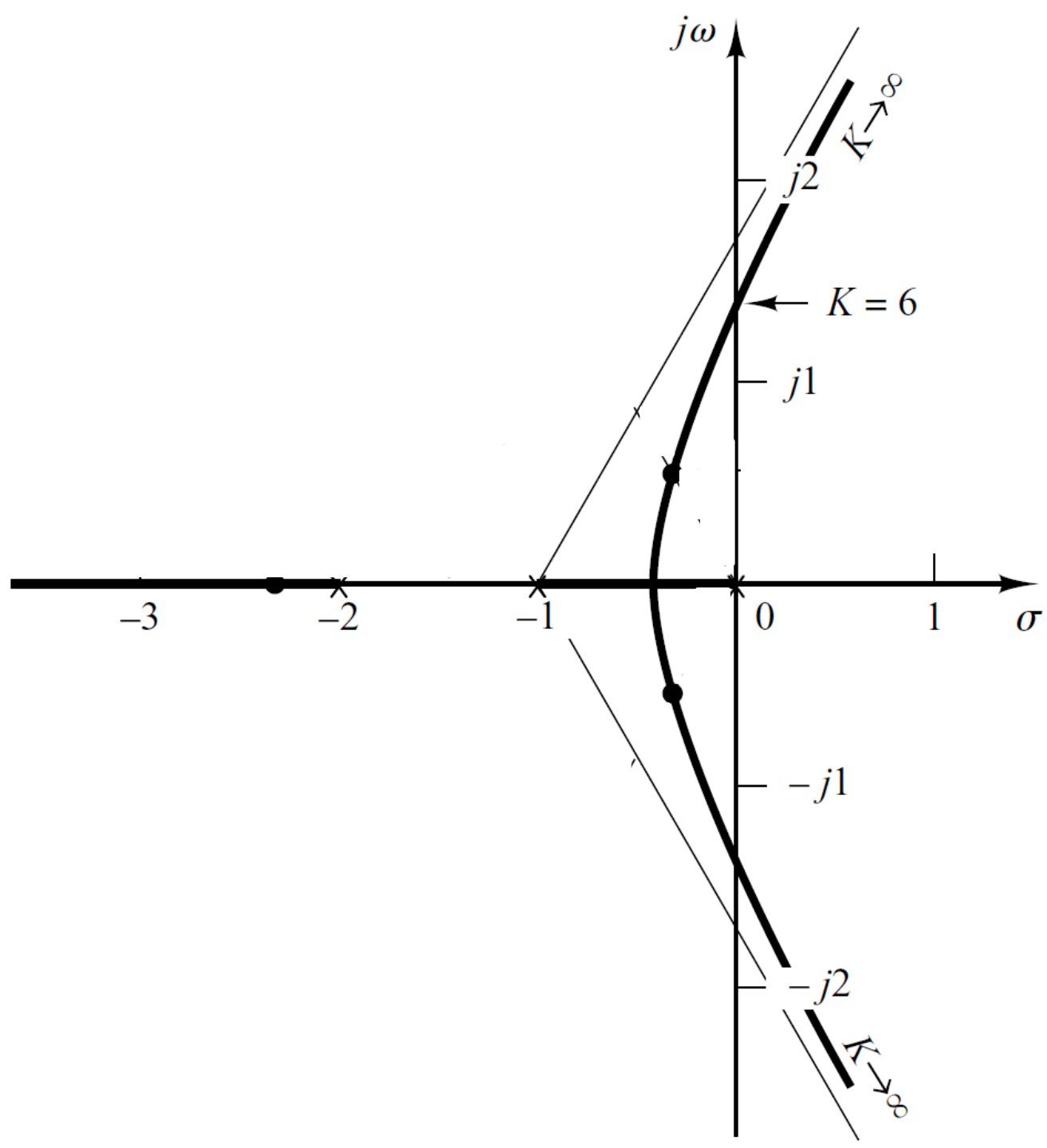
- Equating both real and imaginary parts of this equation to zero

$$(2\omega - \omega^3) = 0$$

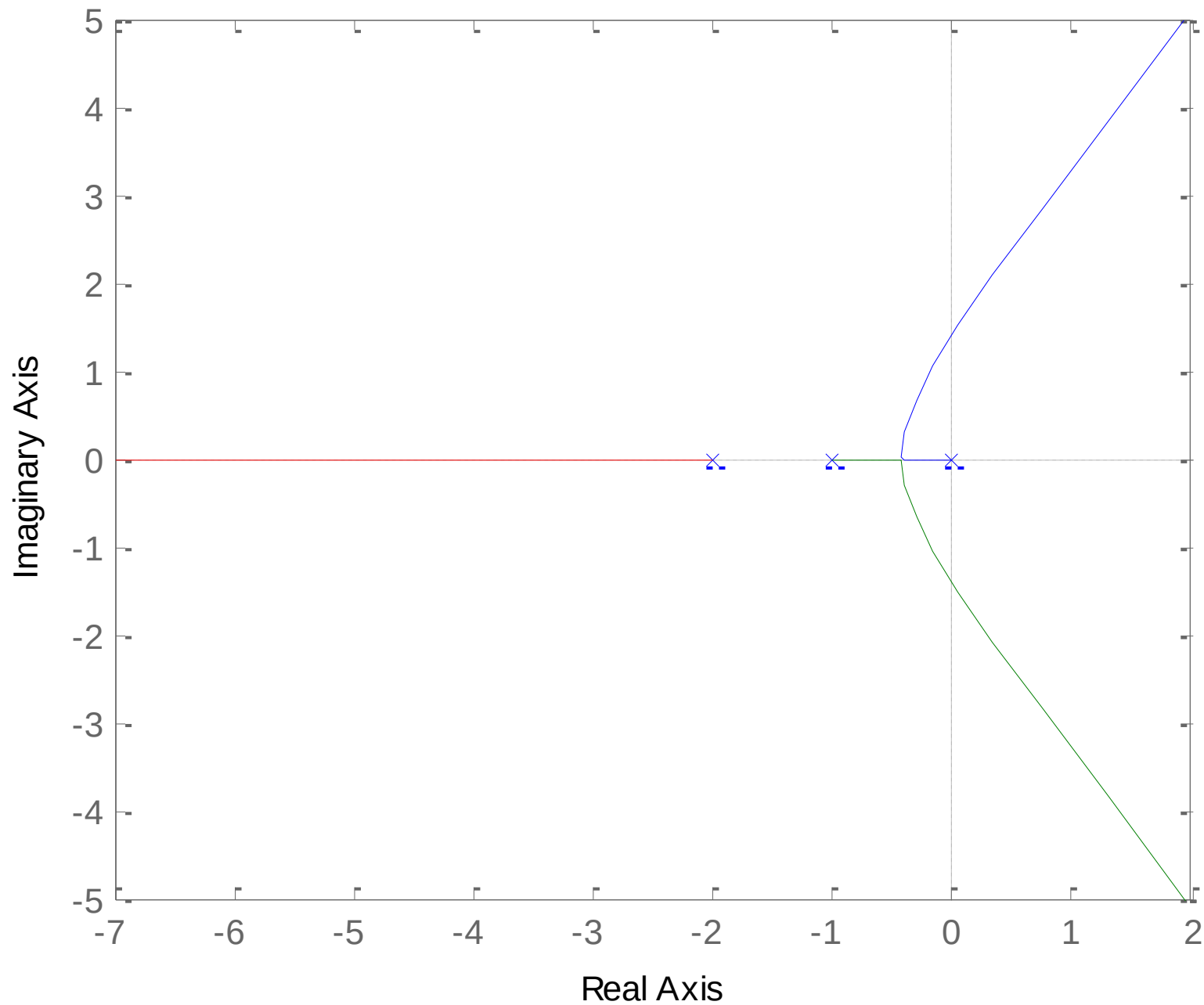
$$(K - 3\omega^2) = 0$$

- Which yields

$$\omega = \pm\sqrt{2}, \quad K = 6 \quad \text{or} \quad \omega = 0, \quad K = 0$$

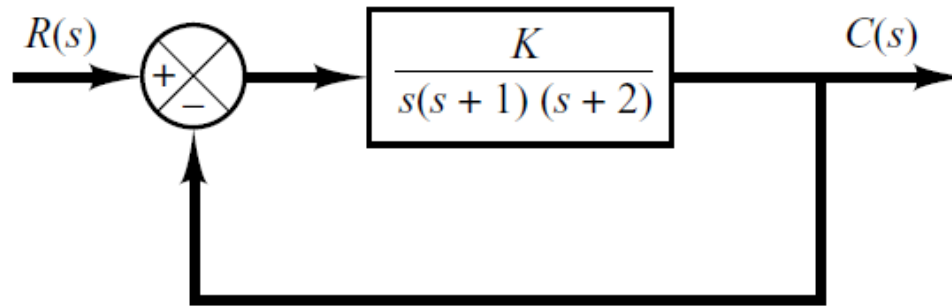


Root Locus



# Example#1

- Consider following unity feedback system.



- Determine the value of  $K$  such that the damping ratio of a pair of dominant complex-conjugate closed-loop poles is **0.5**.

$$G(s)H(s) = \frac{K}{s(s+1)(s+2)}$$

# Example#1

- The damping ratio of **0.5** corresponds to

$$\zeta = \cos \theta$$

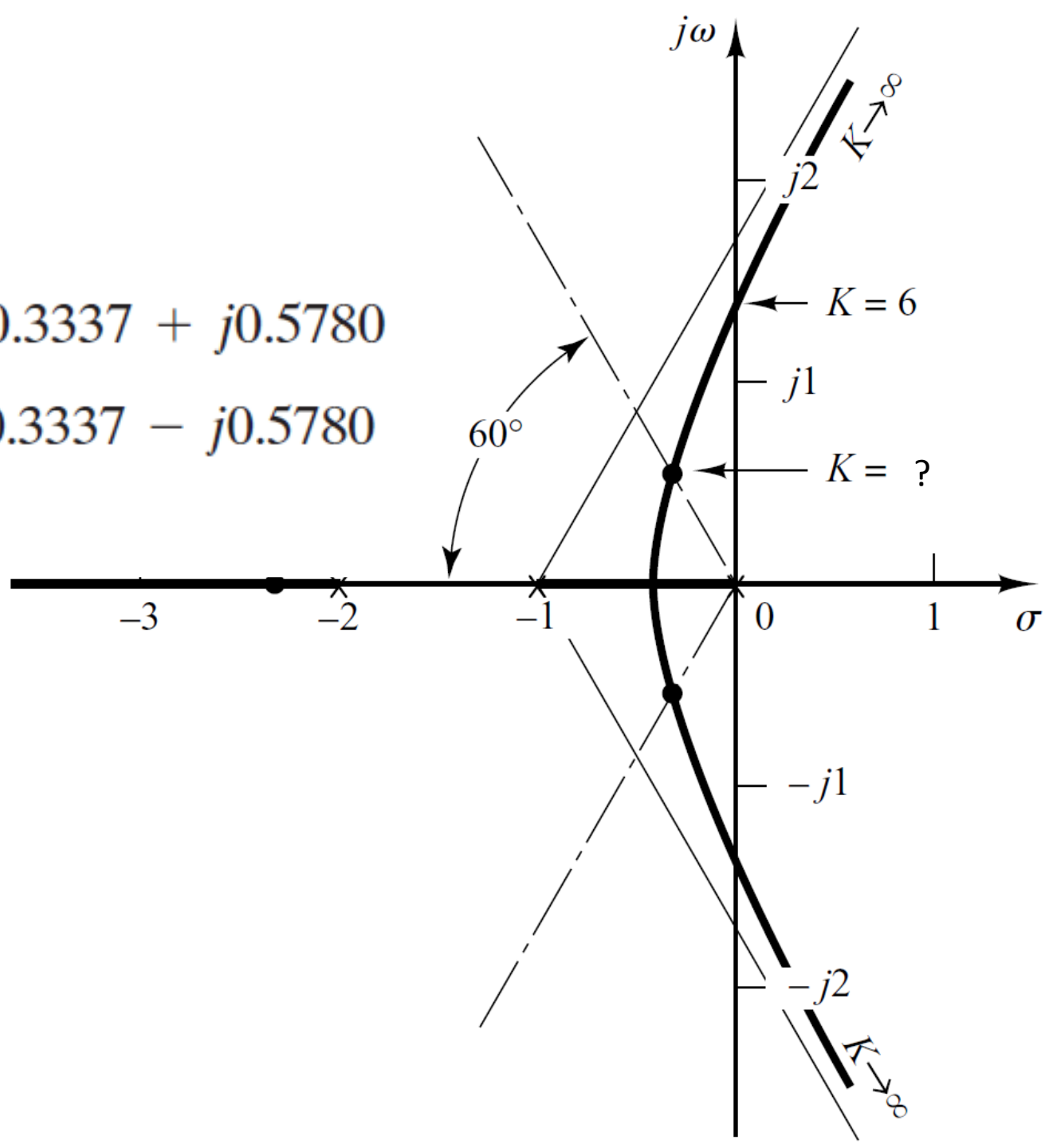
$$\theta = \cos^{-1} \zeta$$

$$\theta = \cos^{-1}(0.5) = 60^\circ$$



$$s_1 = -0.3337 + j0.5780$$

$$s_2 = -0.3337 - j0.5780$$

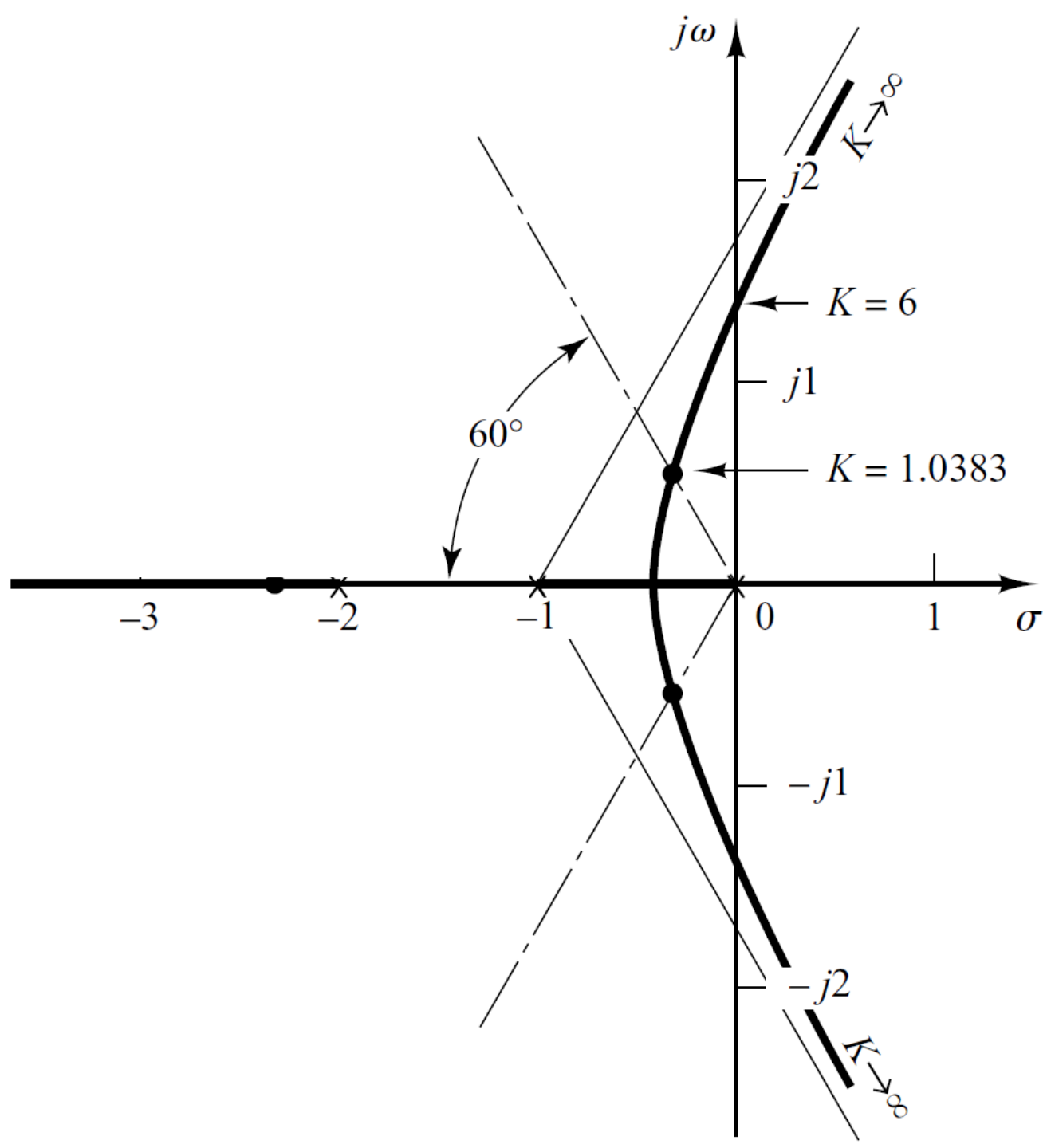


# Example#1

- The value of K that yields such poles is found from the magnitude condition

$$\left| \frac{K}{s(s+1)(s+2)} \right|_{s=-0.3337+j0.5780} = 1$$

$$\begin{aligned} K &= |s(s+1)(s+2)|_{s=-0.3337+j0.5780} \\ &= 1.0383 \end{aligned}$$



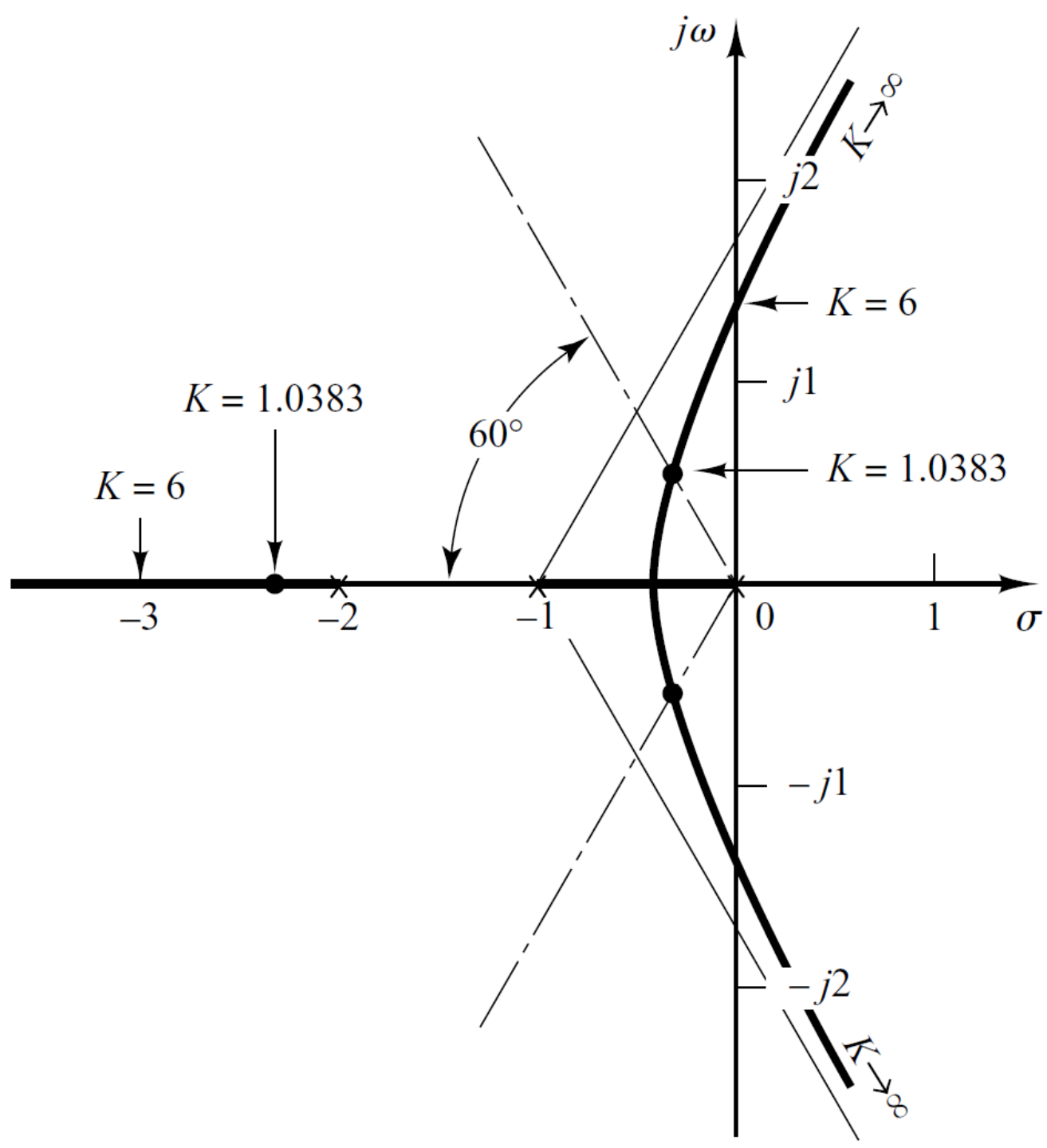
# Example#1

- The third closed loop pole at  $K=1.0383$  can be obtained as

$$1 + G(s)H(s) = 1 + \frac{K}{s(s+1)(s+2)} = 0$$

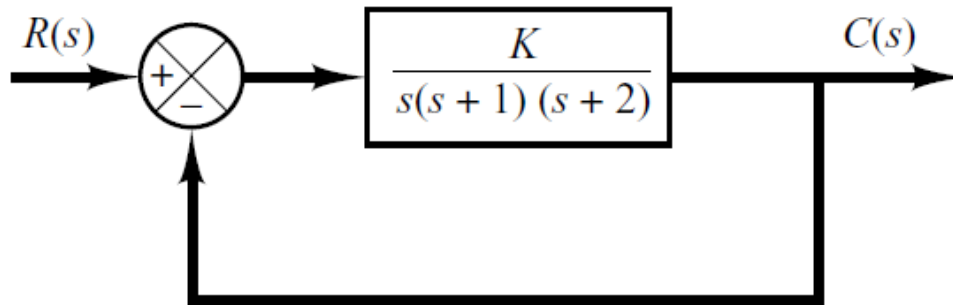
$$1 + \frac{1.0383}{s(s+1)(s+2)} = 0$$

$$s(s+1)(s+2) + 1.0383 = 0$$



# Home Work

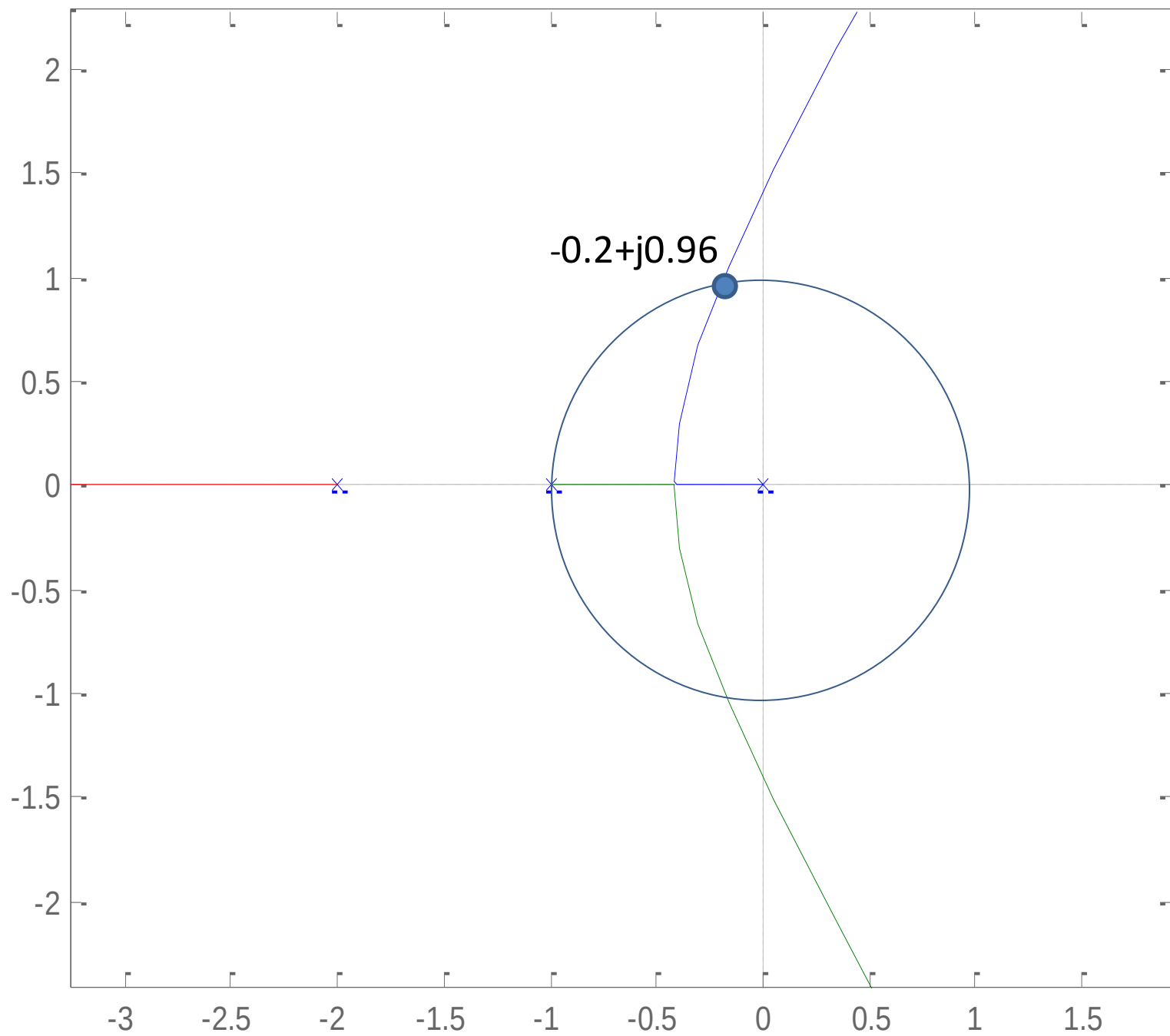
- Consider following unity feedback system.



- Determine the value of  $K$  such that the natural undamped frequency of dominant complex-conjugate closed-loop poles is **1 rad/sec**.

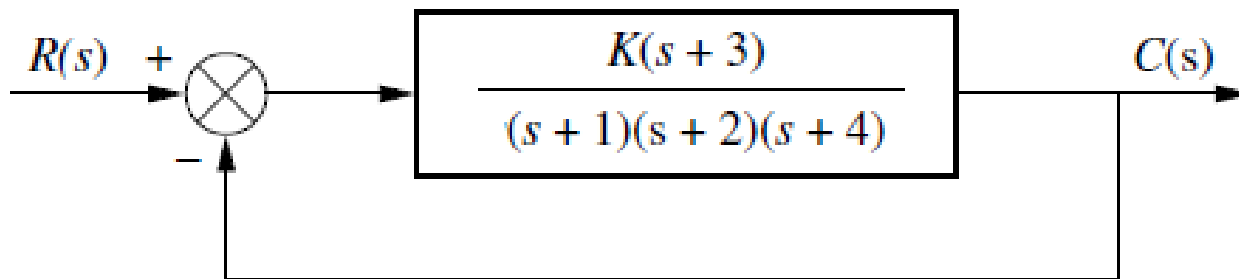
$$G(s)H(s) = \frac{K}{s(s+1)(s+2)}$$

Imaginary Axis



## Example#2

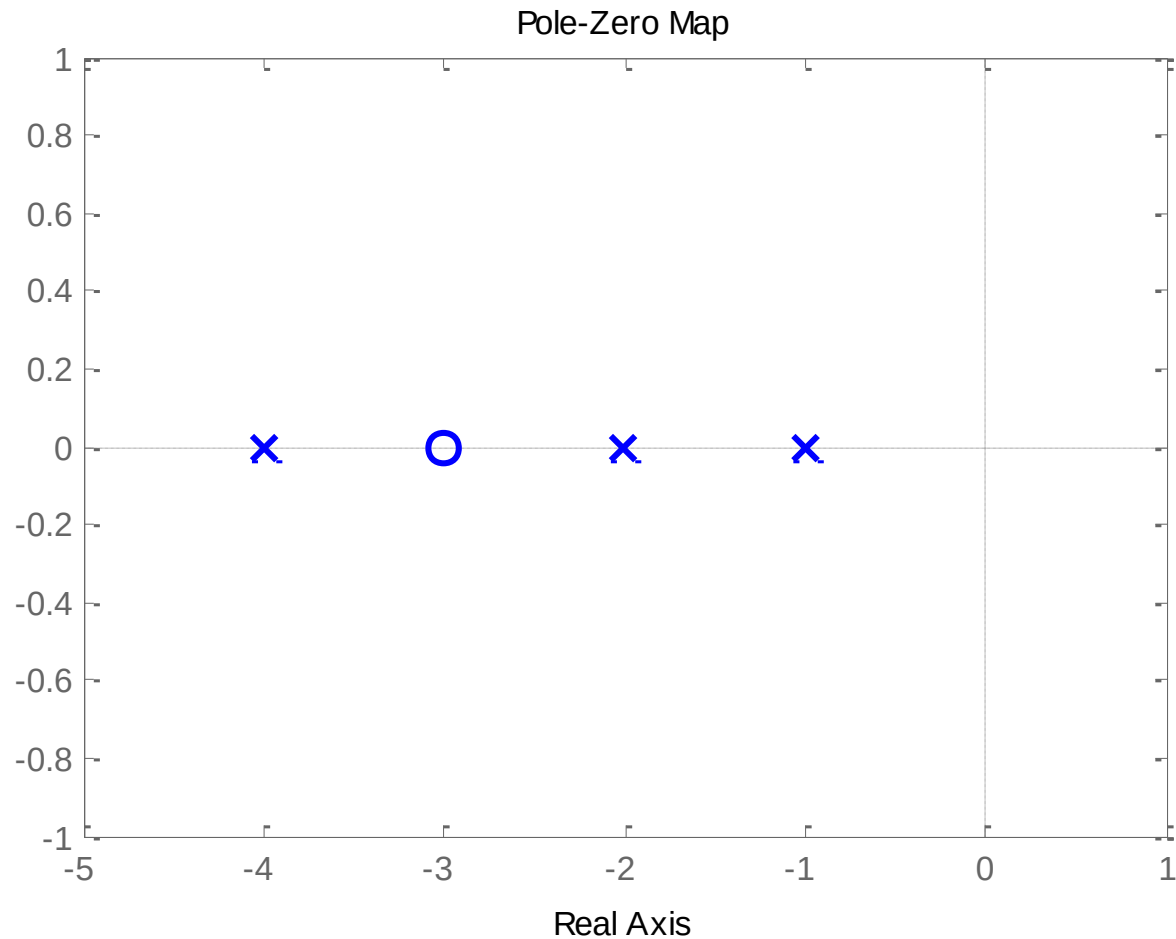
- Sketch the root locus of following system and determine the location of dominant closed loop poles to yield maximum overshoot in the step response less than 30%.





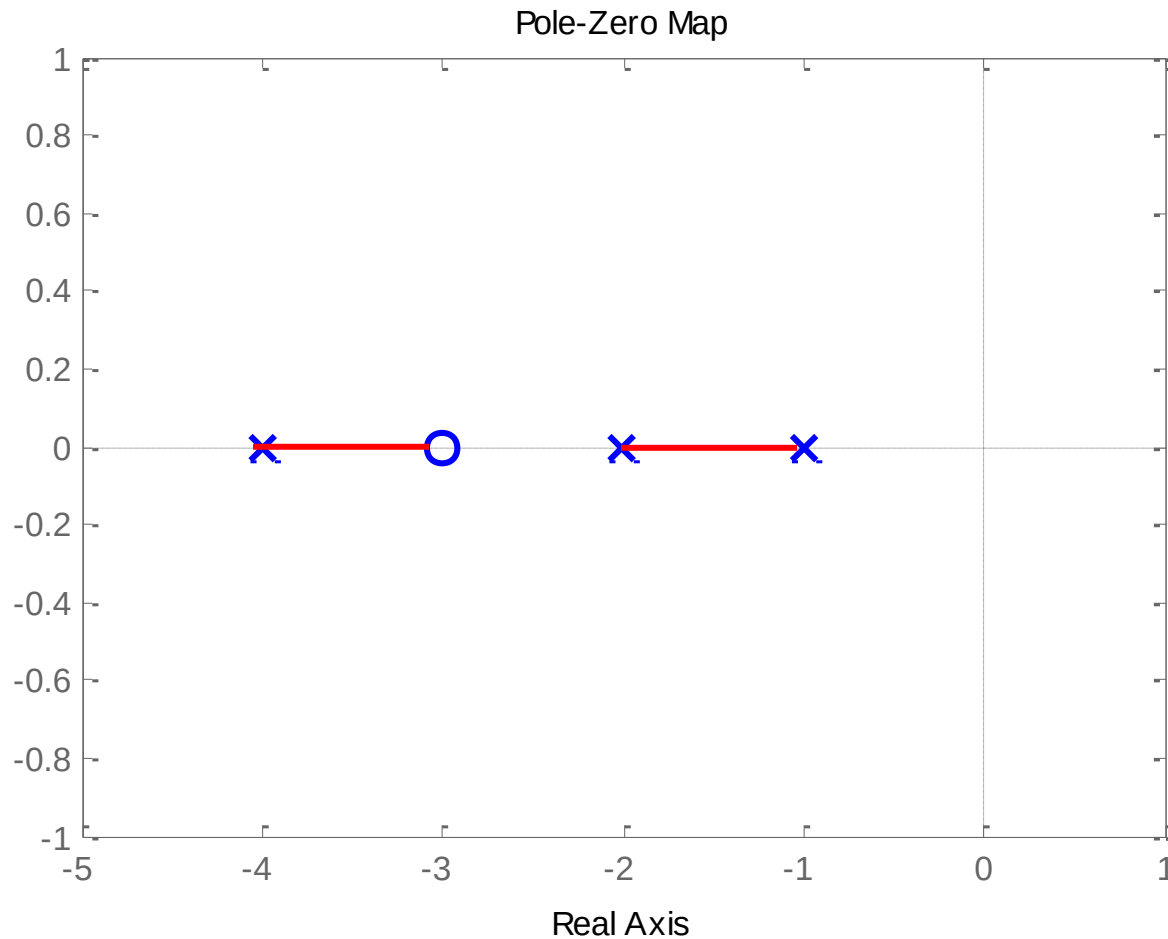
# Example#2

- Step-1: Pole-Zero Map



# Example#2

- Step-2: Root Loci on Real axis

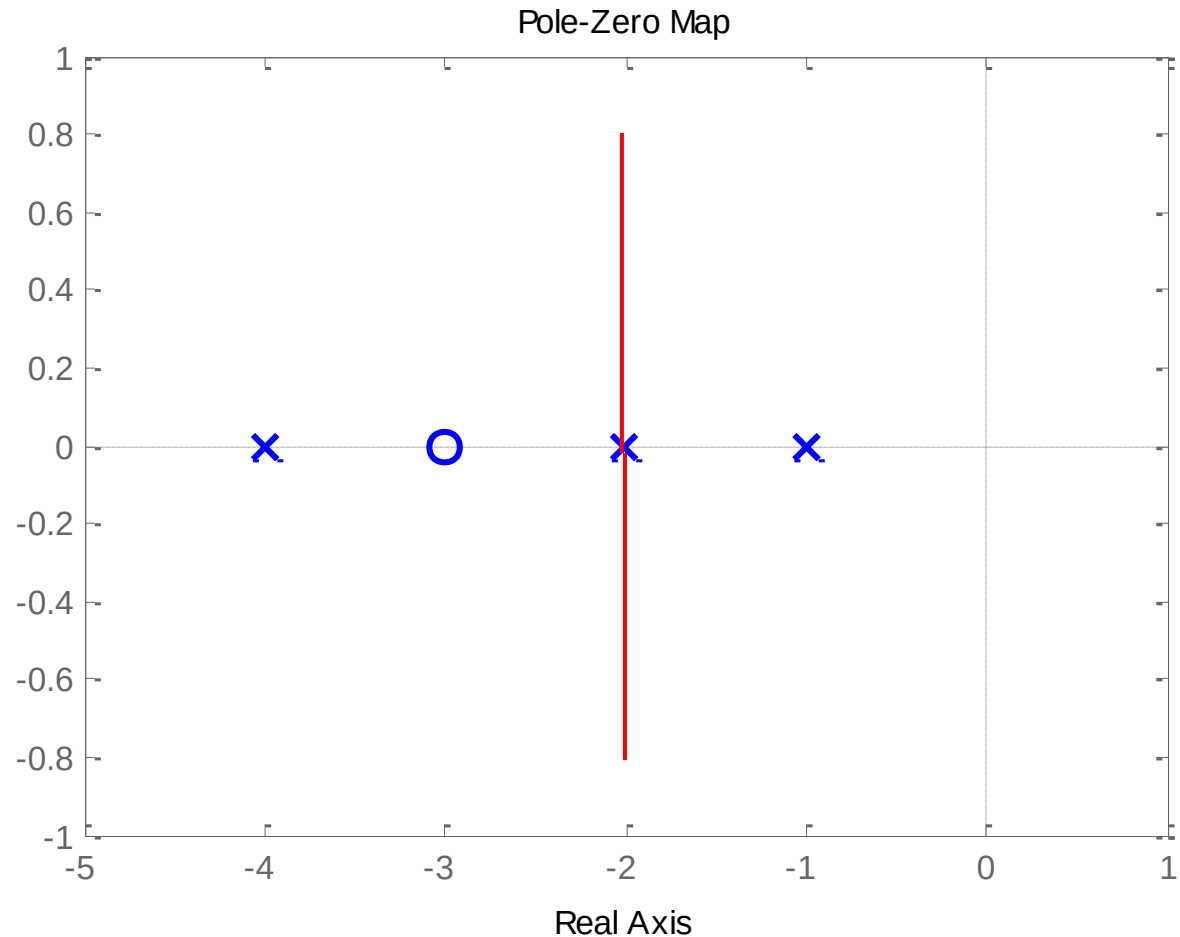


# Example#2

- Step-3: Asymptotes

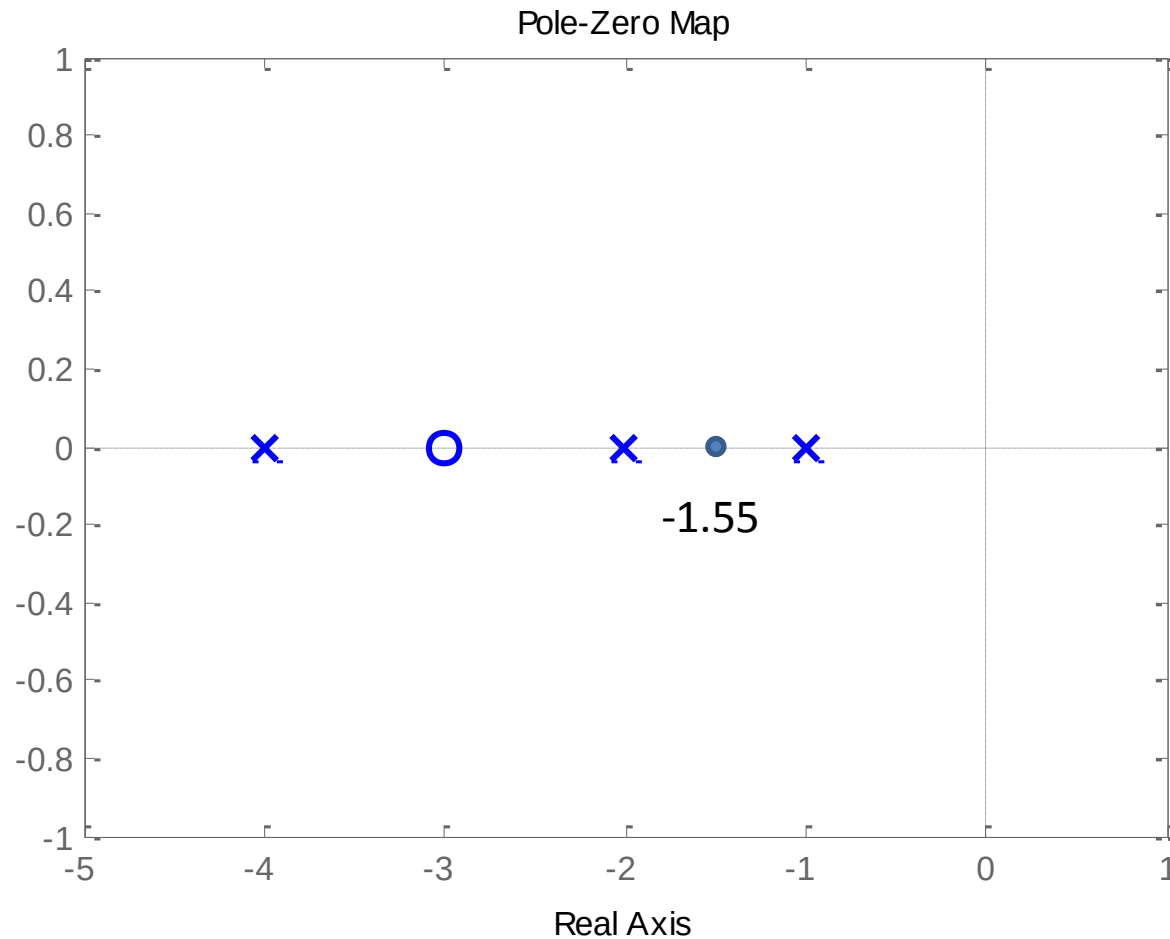
$$\psi = \pm 90^\circ$$

$$\sigma = -2$$

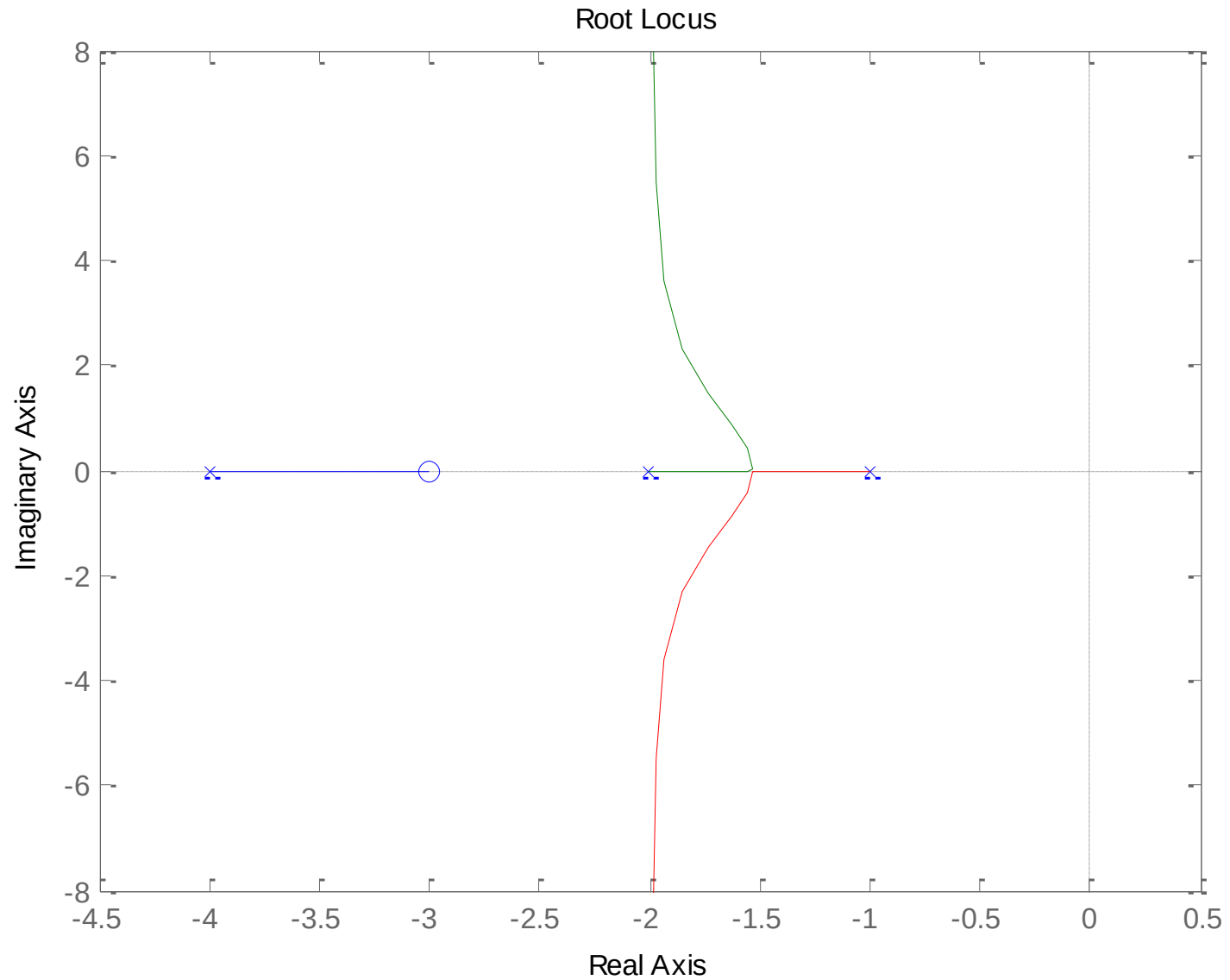


# Example#2

- Step-4: breakaway point



# Example#2



# Example#2

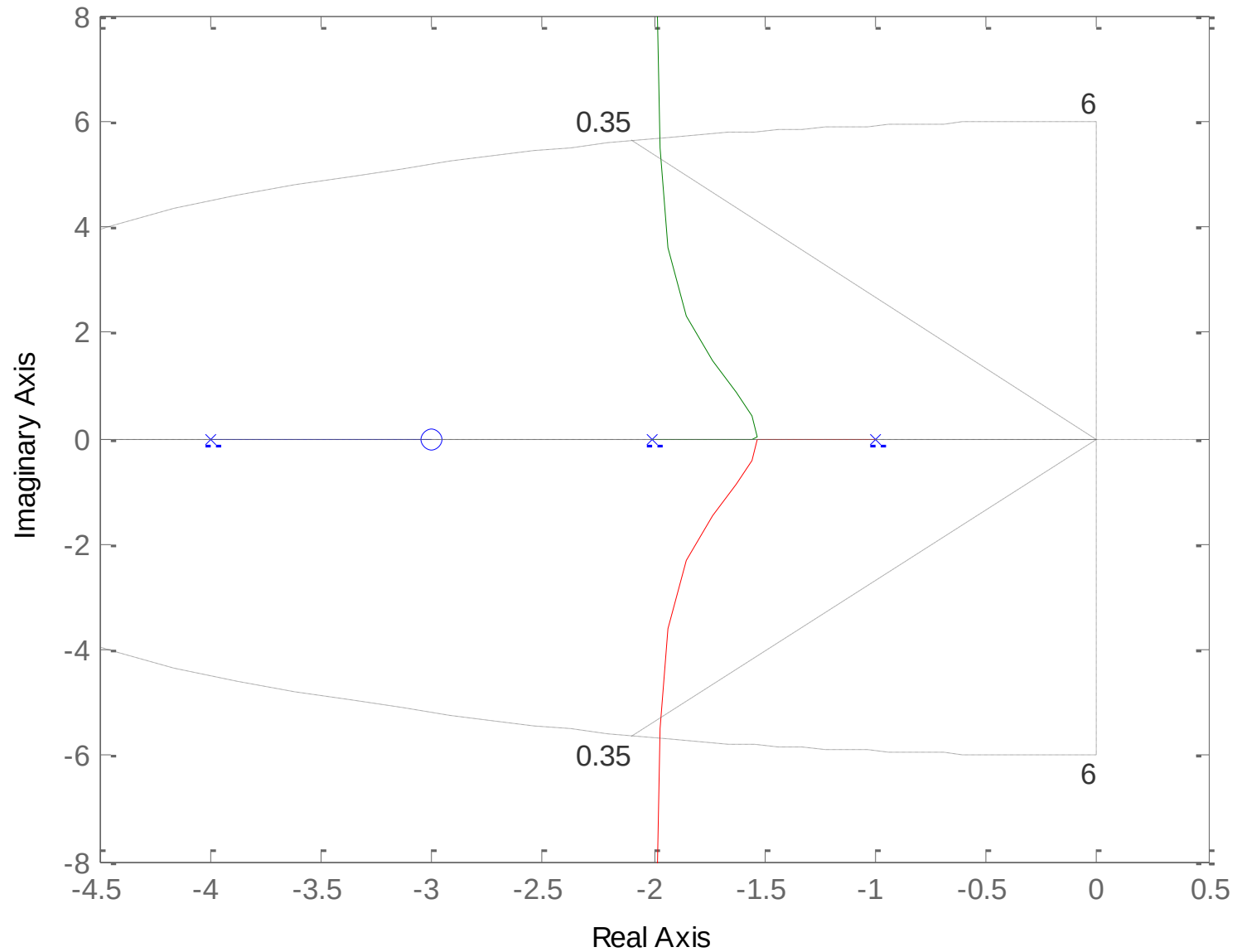
- $M_p < 30\%$  corresponds to

$$M_p = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} \times 100$$

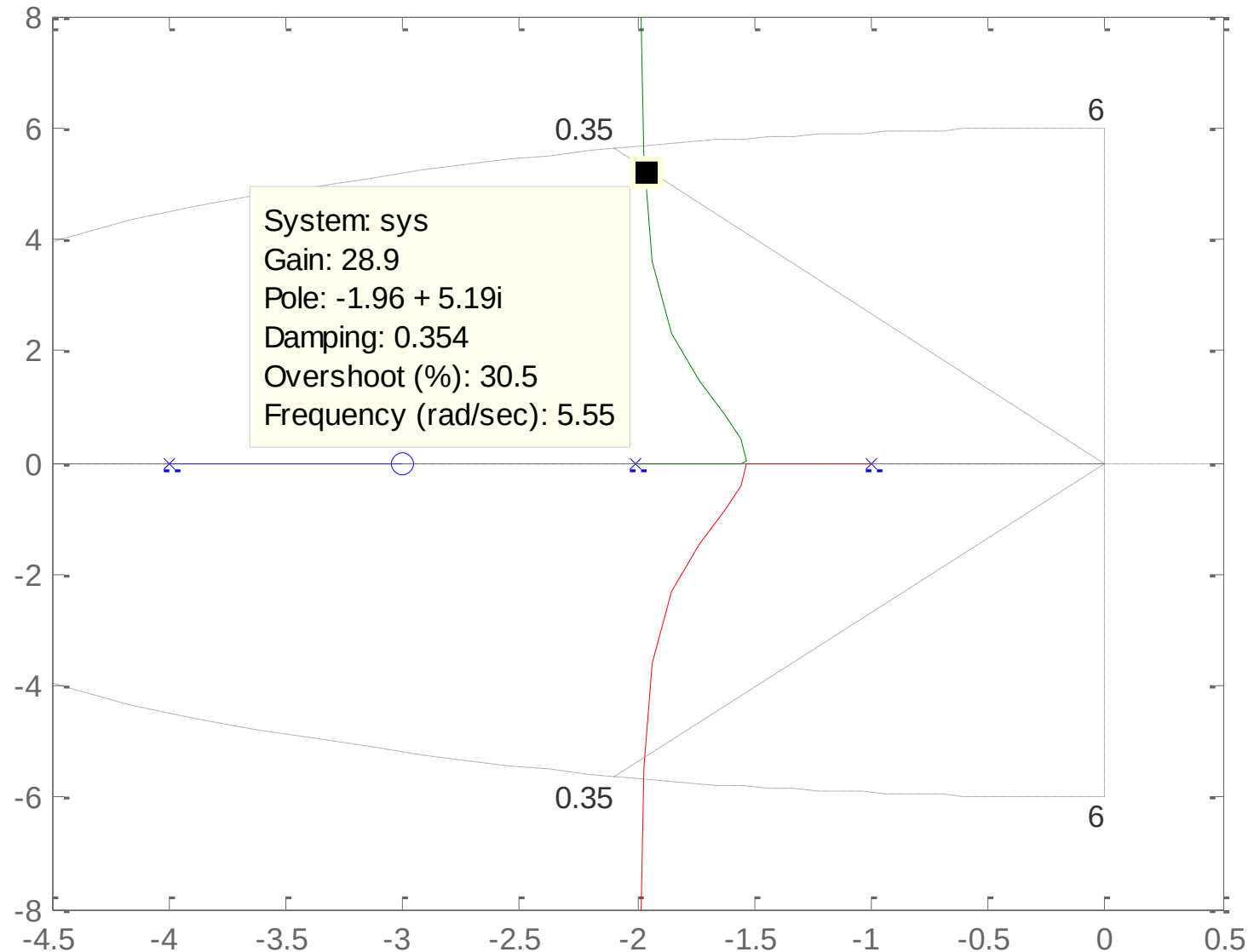
$$30\% = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} \times 100$$

$$\zeta > 0.35$$

# Example#2



# Example#2





# PID CONTROLLER COSTRUCTION AND DESIGEN

المسيطر التناسبي التكاملي  
التفاضلي

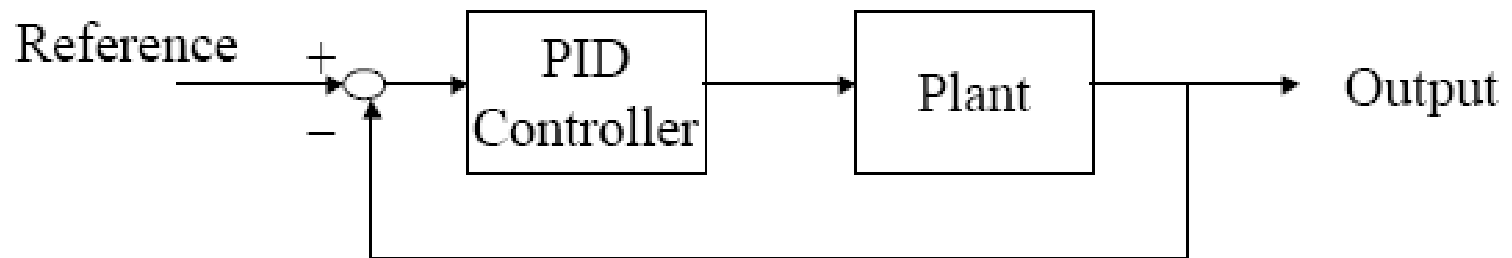
LECTUER 10

# Lecture Outline

- ❑ Introduction to PID
- ❑ Modes of Control
  - ❑ On-Off Control
  - ❑ Proportional Control
  - ❑ Proportional + Integral Control
  - ❑ Proportional + Derivative Control
  - ❑ Proportional + Integral + Derivative Control
- ❑ PID Tuning Rules
  - ❑ Zeigler-Nichol's Tuning Rules
    - ❑ 1<sup>st</sup> Method
    - ❑ 2<sup>nd</sup> Method

# Introduction

- PID Stands for
  - P → تناسبی Proportional
  - I → تكامل Integral
  - D → مشتقة Derivative



# Introduction

- The usefulness of PID controls lies in their general applicability to most control systems.
- In particular, when the mathematical model of the plant is not known and therefore analytical design methods cannot be used, PID controls prove to be most useful.
- In the field of process control systems, it is well known that the basic and modified PID control schemes have proved their usefulness in providing satisfactory control, although in many given situations they may not provide optimal control.

# Introduction

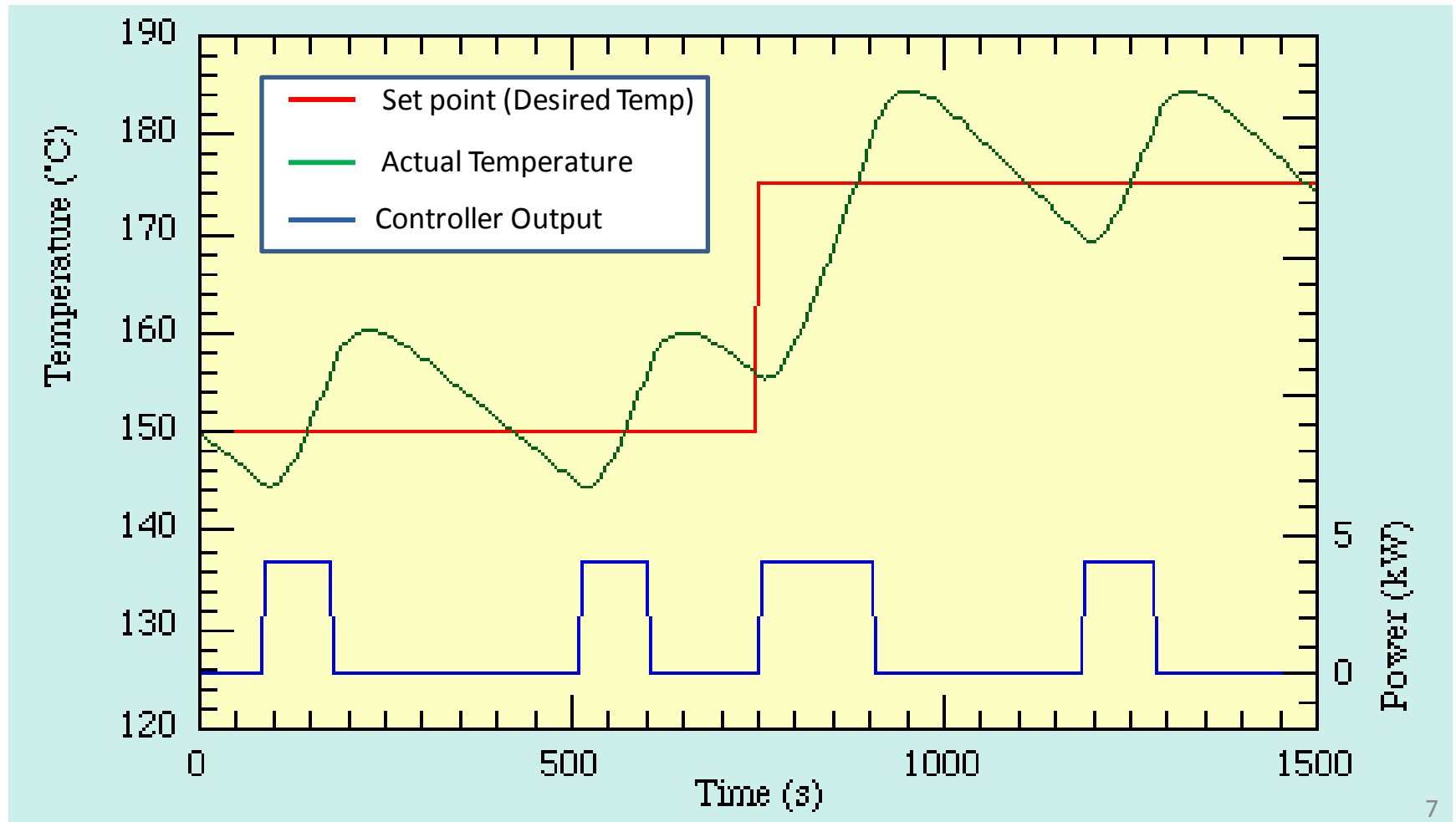
- It is interesting to note that more than half of the industrial controllers in use today are PID controllers or modified PID controllers.
- Because most PID controllers are adjusted on-site, many different types of tuning rules have been proposed in the literature.
- Using these tuning rules, delicate and fine tuning of PID controllers can be made on-site.

# Four Modes of Controllers

- Each mode of control has specific advantages and limitations.
  - On-Off (Bang Bang) Control
  - Proportional (**P**)
  - Proportional plus Integral (**PI**)
  - Proportional plus Derivative (**PD**)
  - Proportional plus Integral plus Derivative (**PID**)

# On-Off Control

- This is the simplest form of control.



# تصميم المسيطرات باستخدام المحل الهندسي

- يستخدم خاصيتي
- Magnitude Condition
- Angle condition
-



## Example 1

The open-loop transfer function of a control system is given by

$$KGH(s) = \frac{K}{s(s+1)(s+4)}$$

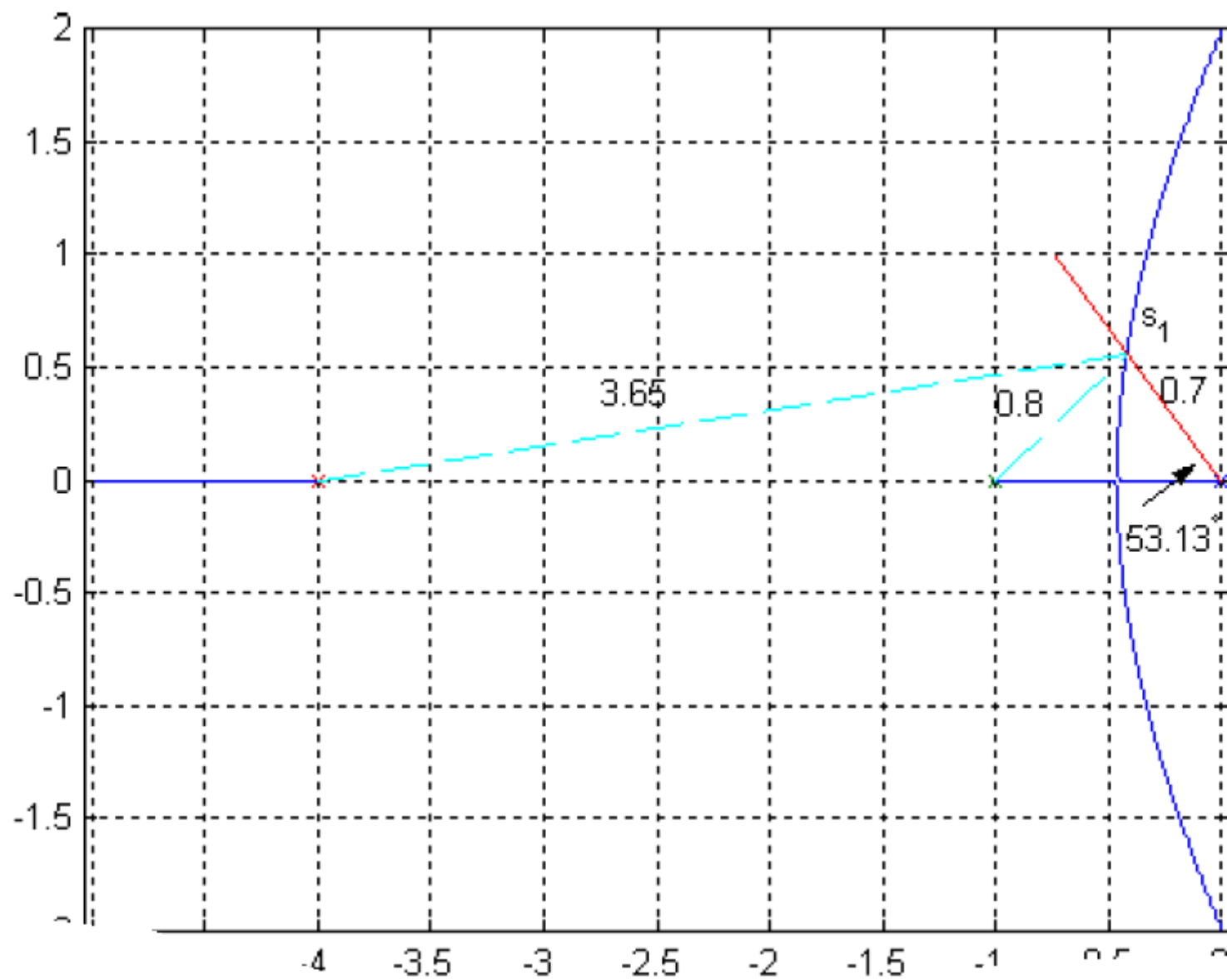
(a) Obtain the gain  $K_0$  of a proportional controller such that the damping ratio of the closed-loop poles will be equal 0.6. Obtain root-locus, step response and the time-domain specifications for the compensated system.

The root-locus plot is shown in Figure 2. For  $\zeta = 0.6$ ,

$$\theta = \cos^{-1} 0.6 = 53.13^\circ$$

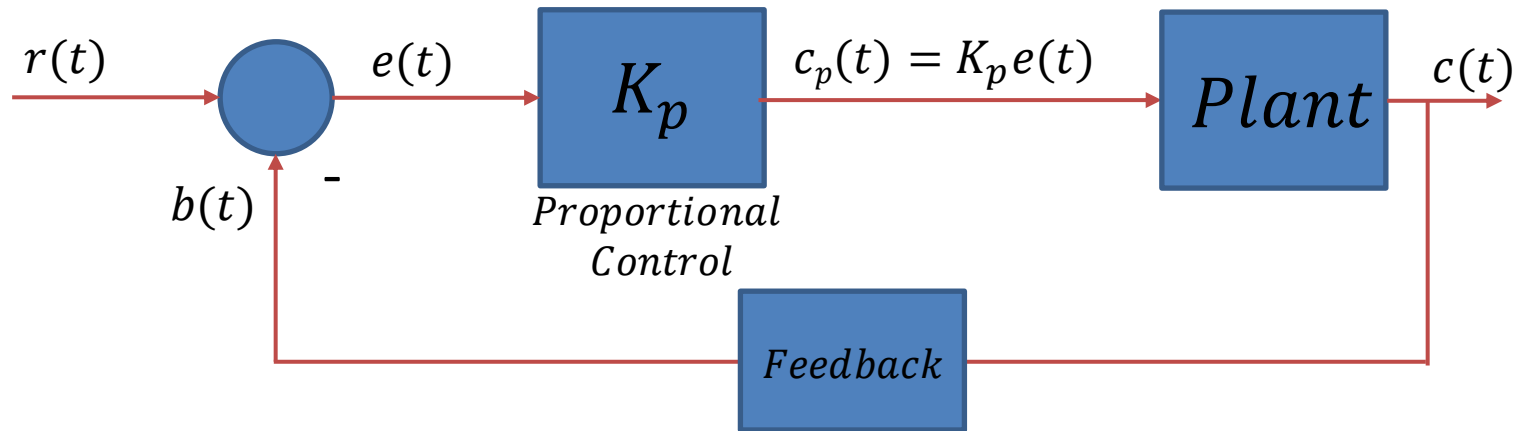
The line drawn at this angle intersects the root-locus at approximately,  $s_1 \simeq -0.41 + j0.5$

The vector lengths from  $s_1$  to the poles are marked on the diagram



# Proportional Control (P)

- In *proportional mode*, there is a continuous linear relation between value of the controlled variable and position of the final control element.



- Output of proportional controller is

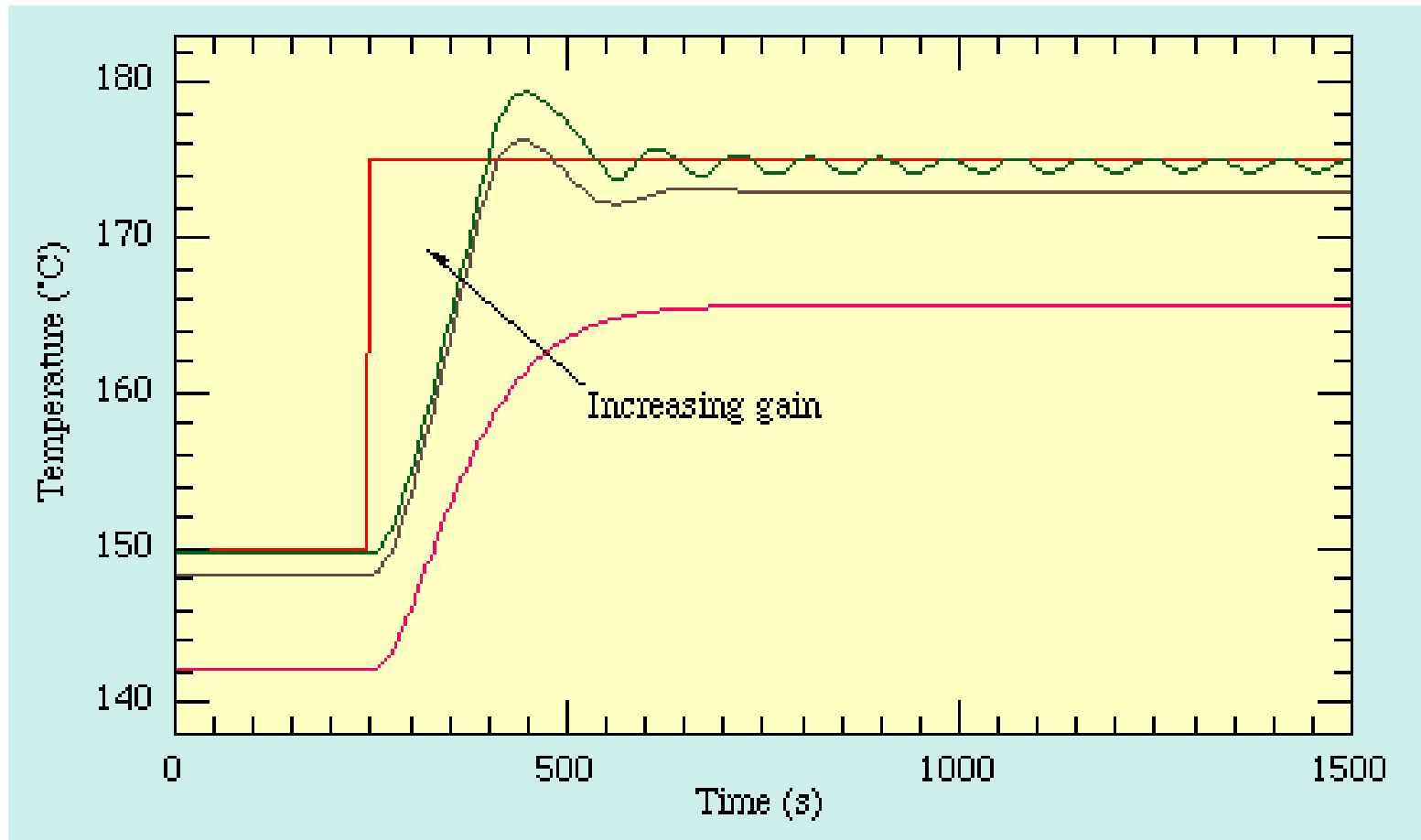
$$c_p(t) = K_p e(t)$$

- The transfer function can be written as

$$\frac{C_p(s)}{E(s)} = K_p$$

# Proportional Controllers (P)

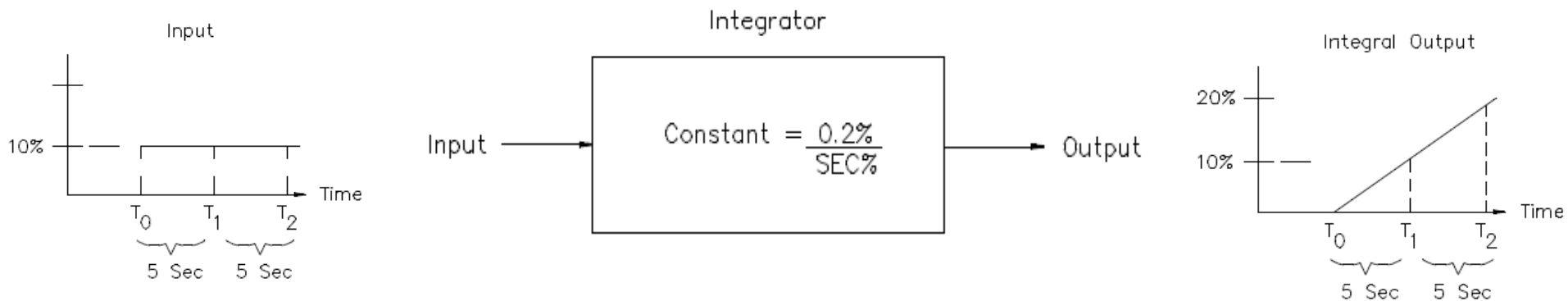
- As the gain is increased the system responds faster to changes in set-point but becomes progressively underdamped and eventually unstable.



# Proportional Plus Integral Controllers تناسبي

## (PI) تكاملي

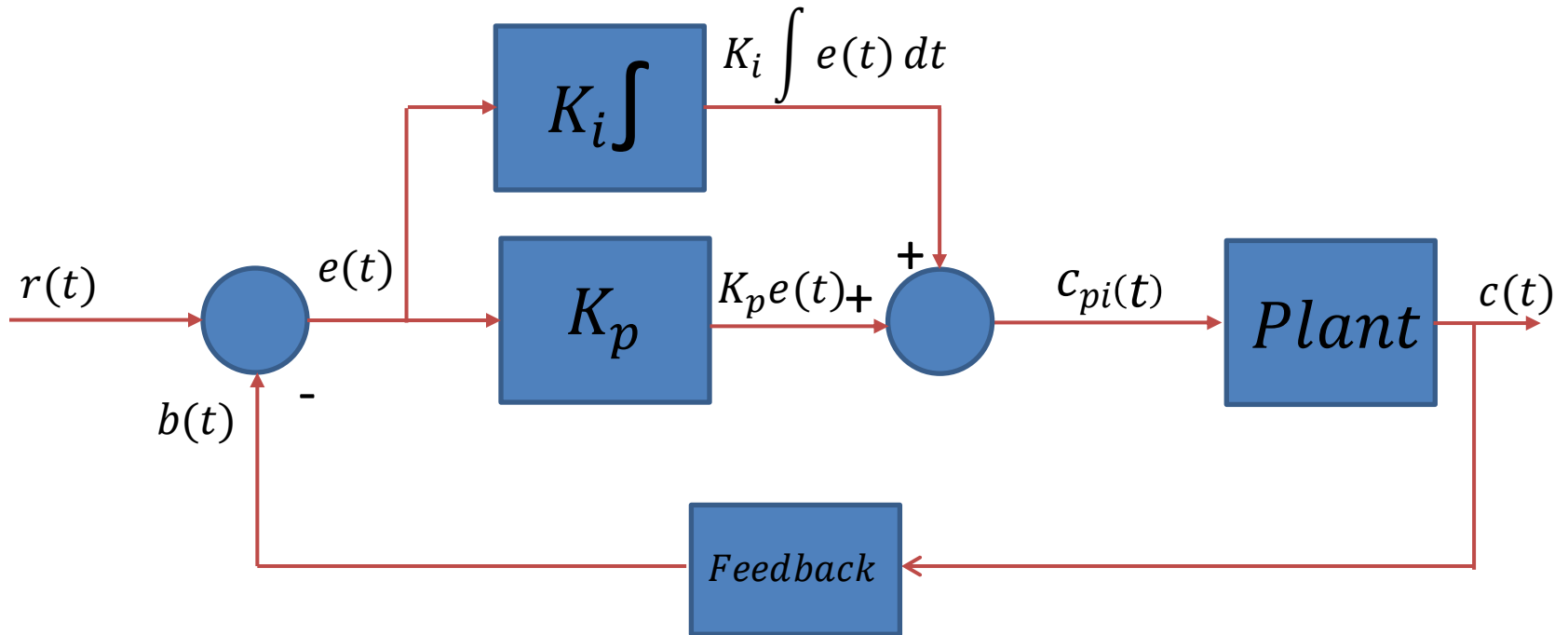
- Integral control describes a controller in which the output rate of change is dependent on the magnitude of the input.
- Specifically, a smaller amplitude input causes a slower rate of change of the output.



# Proportional Plus Integral Controllers (PI)

- The major advantage of integral controllers is that they have the unique ability to return the controlled variable back to the exact set point following a disturbance.
- Disadvantages of the integral control mode are that it responds relatively slowly to an error signal and that it can initially allow a large deviation at the instant the error is produced.
- This can lead to system instability and cyclic operation. For this reason, the integral control mode is not normally used alone, but is combined with another control mode.

# Proportional Plus Integral Control (PI)



$$c_{pi}(t) = K_p e(t) + K_i \int e(t) dt$$

# Proportional Plus Integral Control (**PI**)

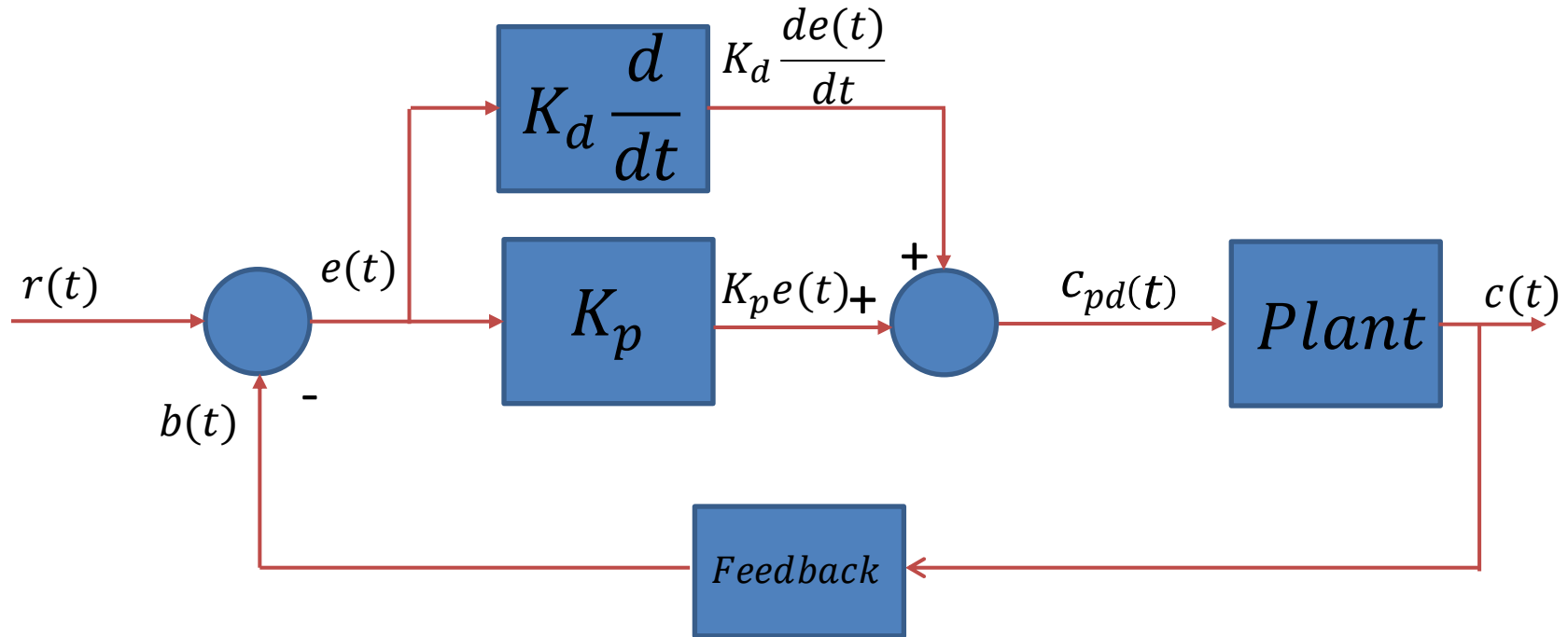
$$c_{pi}(t) = K_p e(t) + K_i \int e(t) dt$$

- The transfer function can be written as

$$\frac{C_{pi}(s)}{E(s)} = K_p + K_i \frac{1}{s}$$



# Proportional Plus derivative Control (PD) تناسبي تفاضلي



$$c_{pd}(t) = K_p e(t) + K_d \frac{de(t)}{dt}$$

# Proportional Plus derivative Control (PD)

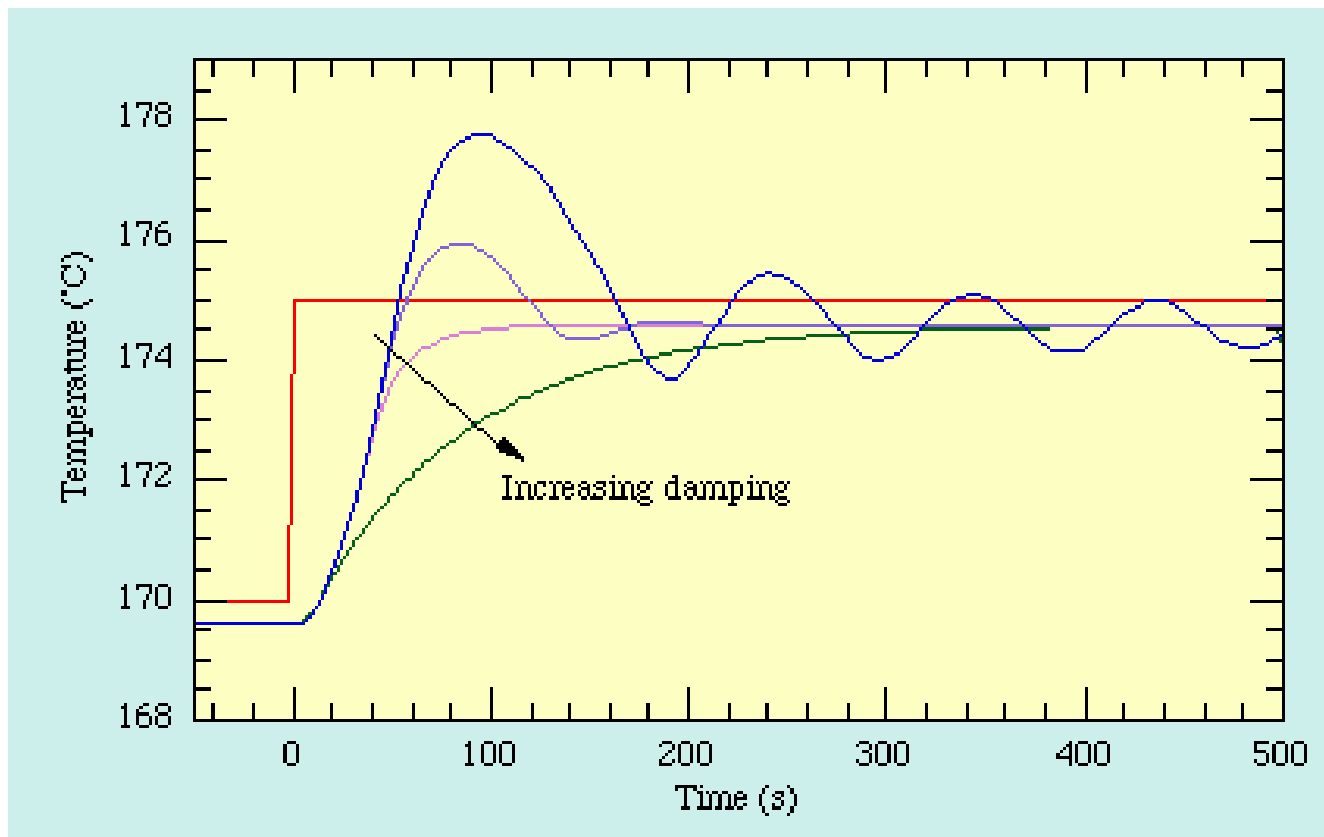
$$c_{pd}(t) = K_p e(t) + K_d \frac{de(t)}{dt}$$

- The transfer function can be written as

$$\frac{C_{pd}(s)}{E(s)} = K_p + K_d s$$

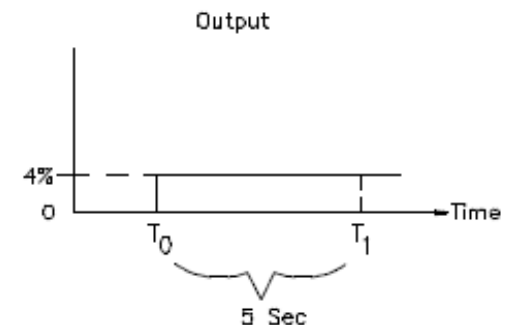
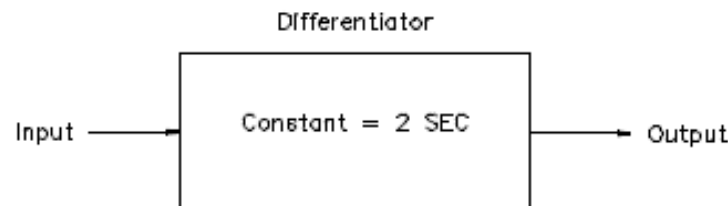
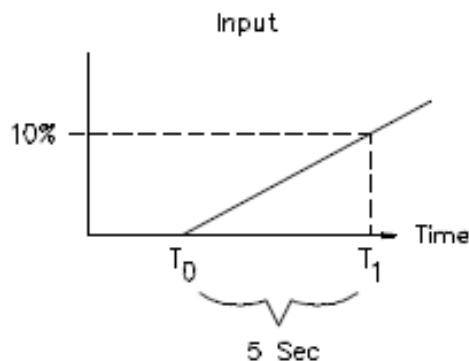
# Proportional Plus derivative Control (PD)

- The stability and overshoot problems that arise when a proportional controller is used at high gain can be mitigated by adding a term proportional to the time-derivative of the error signal. The value of the damping can be adjusted to achieve a critically damped response.

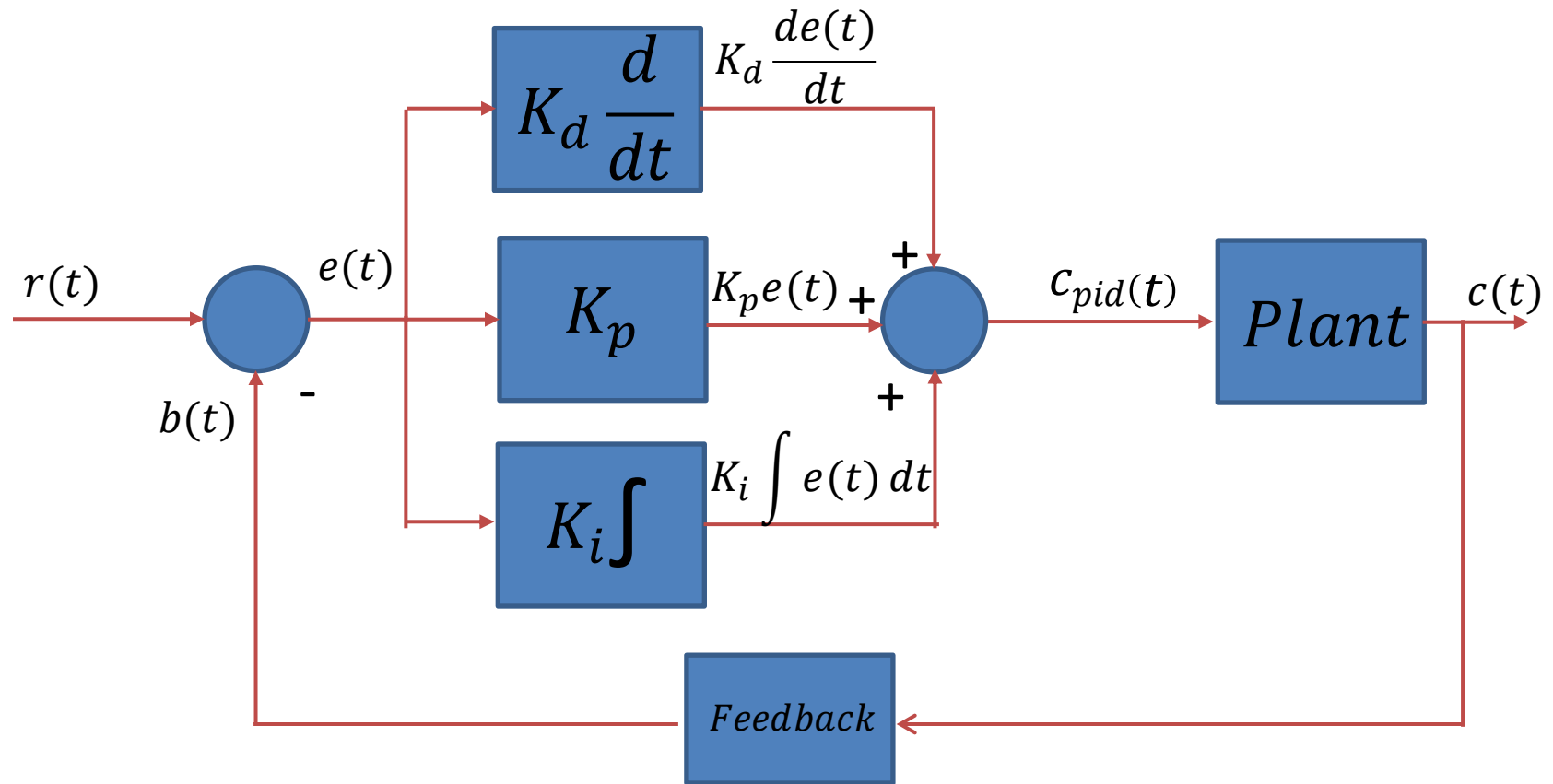


# Proportional Plus derivative Control (PD)

- The higher the error signal rate of change, the sooner the final control element is positioned to the desired value.
- The added derivative action reduces initial overshoot of the measured variable, and therefore aids in stabilizing the process sooner.
- This control mode is called proportional plus derivative (PD) control because the derivative section responds to the rate of change of the error signal



# Proportional Plus Integral Plus Derivative Control (PID) تناسبي تكاملي تفاضلي



$$c_{pid}(t) = K_p e(t) + K_i \int e(t) dt + K_d \frac{de(t)}{dt}$$

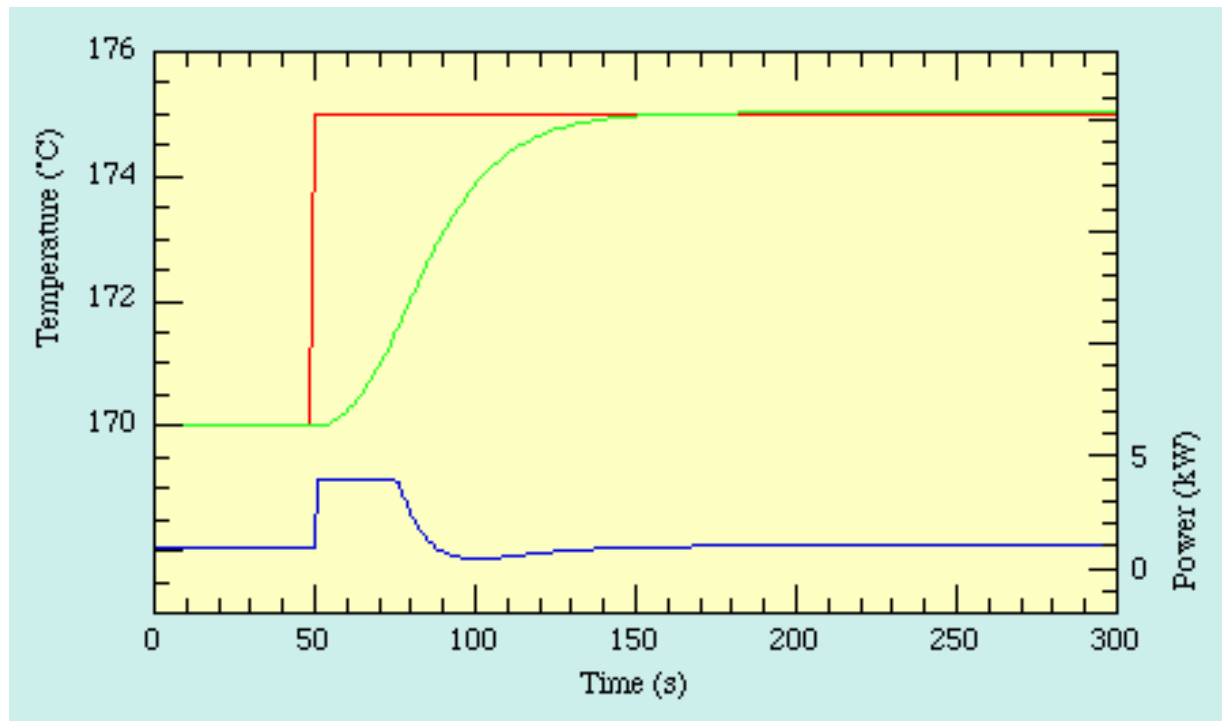
# Proportional Plus Integral Plus Derivative Control (PID)

$$c_{pid}(t) = K_p e(t) + K_i \int e(t) dt + K_d \frac{de(t)}{dt}$$

$$\frac{C_{pid}(s)}{E(s)} = K_p + K_i \frac{1}{s} + K_d s$$

# Proportional Plus Integral Plus Derivative Control (PID)

- Although PD control deals neatly with the overshoot and ringing problems associated with proportional control it does not cure the problem with the steady-state error. Fortunately it is possible to eliminate this while using relatively low gain by adding an integral term to the control function which becomes



# The Characteristics الخواص والفوائد of P, I, and D controllers

| CL RESPONSE | RISE TIME    | OVERSHOOT | SETTLING TIME | S-S ERROR    |
|-------------|--------------|-----------|---------------|--------------|
| $K_p$ ↑     | Decrease     | Increase  | Small Change  | Decrease     |
| $K_i$       | Decrease     | Increase  | Increase      | Eliminate    |
| $K_d$       | Small Change | Decrease  | Decrease      | Small Change |



# Tips for Designing a PID Controller

1. Obtain an open-loop response and determine what needs to be improved
  2. Add a proportional control to improve the rise time
  3. Add a derivative control to improve the overshoot
  4. Add an integral control to eliminate the steady-state error
  5. Adjust each of  $K_p$ ,  $K_i$ , and  $K_d$  until you obtain a desired overall response.
- Lastly, please keep in mind that you do not need to implement all three controllers (proportional, derivative, and integral) into a single system, if not necessary. For example, if a PI controller gives a good enough response (like the above example), then you don't need to implement derivative controller to the system. Keep the controller as simple as possible.

Part-II

# **PID TUNING RULES**

# PID Tuning توليف تنعيم المسيطر

- The transfer function of PID controller is given as

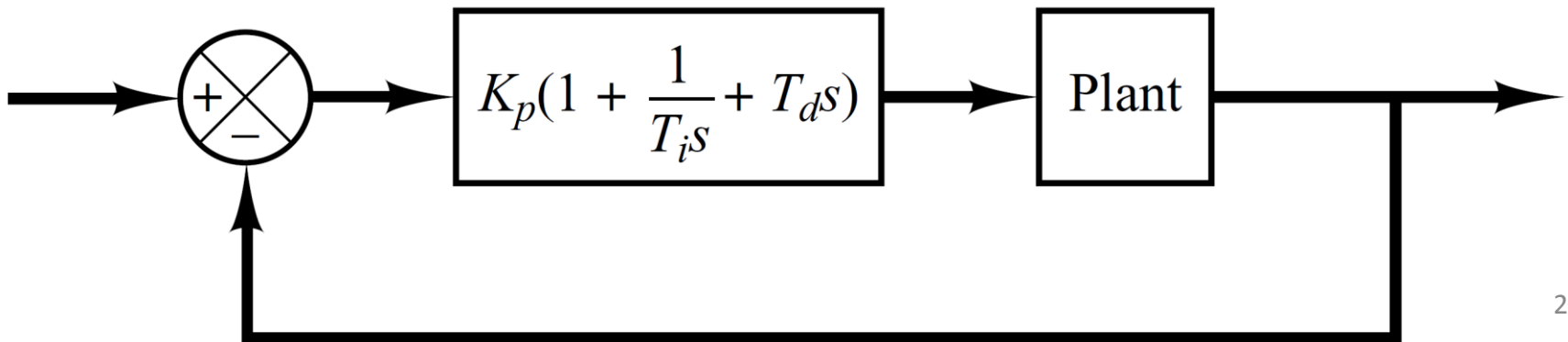
$$\frac{C_{pid}(s)}{E(s)} = K_p + K_i \frac{1}{s} + K_d s$$

- It can be simplified as

$$\frac{C_{pid}(s)}{E(s)} = K_p \left( 1 + \frac{1}{T_i s} + T_d s \right)$$

- Where

$$T_i = \frac{K_p}{K_i} \qquad T_d = \frac{K_d}{K_p}$$



# PID Tuning

- The process of selecting the controller parameters ( $K_p$ ,  $T_i$  and  $T_d$ ) to meet given performance specifications is known as controller tuning.
- Ziegler and Nichols suggested rules for tuning PID controllers experimentally.
- Which are useful when mathematical models of plants are not known.
- These rules can, of course, be applied to the design of systems with known mathematical models.

# PID Tuning

- Such rules suggest a set of values of  $K_p$ ,  $T_i$  and  $T_d$  that will give a stable operation of the system.
- However, the resulting system may exhibit a large maximum overshoot in the step response, which is unacceptable.
- In such a case we need series of fine tunings until an acceptable result is obtained.
- In fact, the Ziegler–Nichols tuning rules give an educated guess for the parameter values and provide a starting point for fine tuning, rather than giving the final settings for  $K_p$ ,  $T_i$  and  $T_d$  in a single shot.

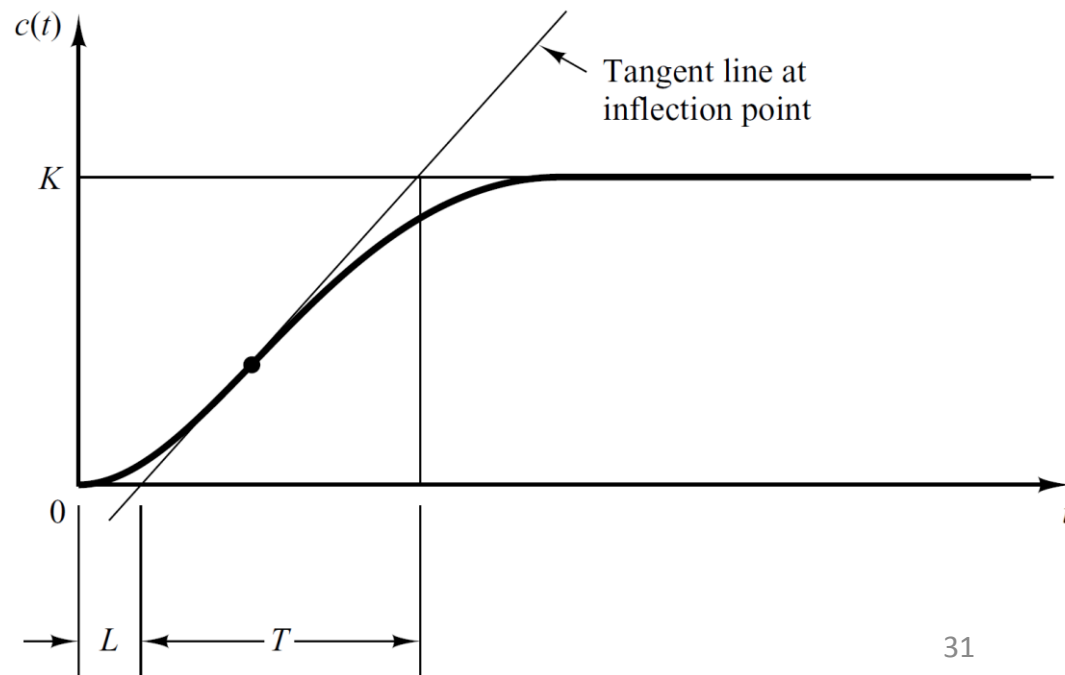
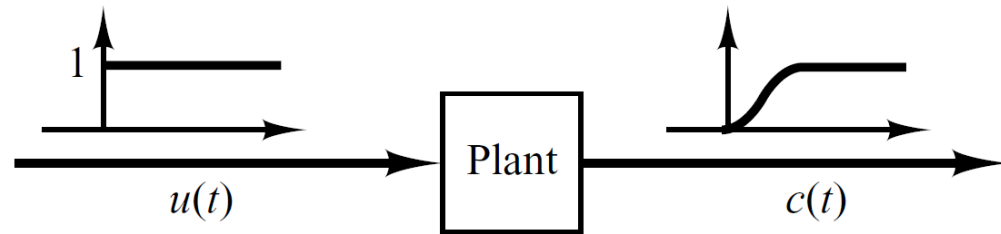
# Zeigler-Nichol's PID Tuning Methods

## طريقة زيكلر-نيكولس

- Ziegler and Nichols proposed rules for determining values of the  $K_p$ ,  $T_i$  and  $T_d$  based on the transient response characteristics of a given plant.
- Such determination of the parameters of PID controllers or tuning of PID controllers can be made by engineers on-site by experiments on the plant.
- There are two methods called Ziegler–Nichols tuning rules:
  - First method (open loop Method)
  - Second method (Closed Loop Method)

# Zeigler-Nichol's First Method الطريقة الاولى

- In the first method, we obtain experimentally the response of the plant to a unit-step input.
- If the plant involves neither integrator(s) nor dominant complex-conjugate poles, then such a unit-step response curve may look S-shaped



# Zeigler-Nichol's First Method

- This method applies if the response to a step input exhibits an S-shaped curve.
- Such step-response curves may be generated experimentally or from a dynamic simulation of the plant.

**Table-1**

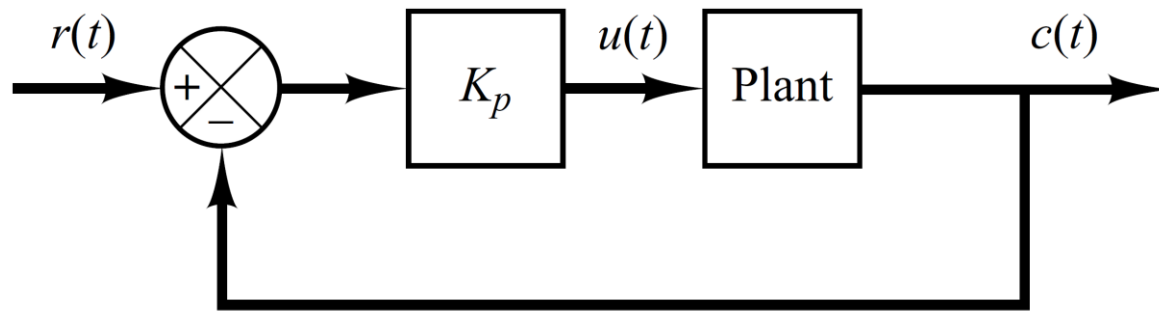
| Type of Controller | $K_p$             | $T_i$           | $T_d$  |
|--------------------|-------------------|-----------------|--------|
| P                  | $\frac{T}{L}$     | $\infty$        | 0      |
| PI                 | $0.9 \frac{T}{L}$ | $\frac{L}{0.3}$ | 0      |
| PID                | $1.2 \frac{T}{L}$ | $2L$            | $0.5L$ |



# Zeigler-Nichols Second Method الطريقة

## الثانية

- In the second method, we first set  $T_i = \infty$  and  $T_d = 0$ .
- Using the proportional control action only (as shown in figure), increase  $K_p$  from 0 to a critical value  $K_{cr}$  at which the output first exhibits sustained oscillations.



- If the output does not exhibit sustained oscillations for whatever value  $K_p$  may take, then this method does not apply.

# Zeigler-Nichol's Second Method

- Thus, the critical gain  $K_{cr}$  and the corresponding period  $P_{cr}$  are determined.

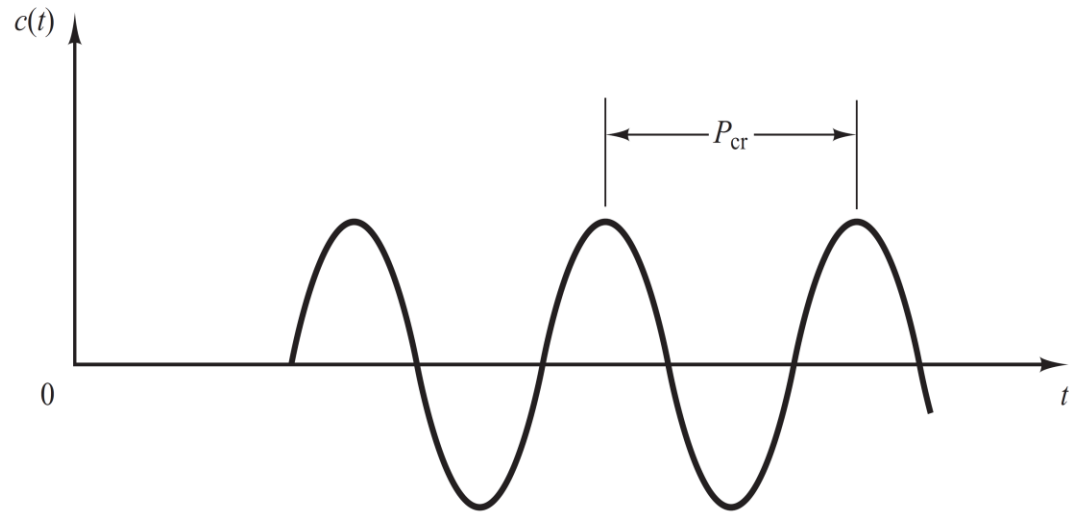
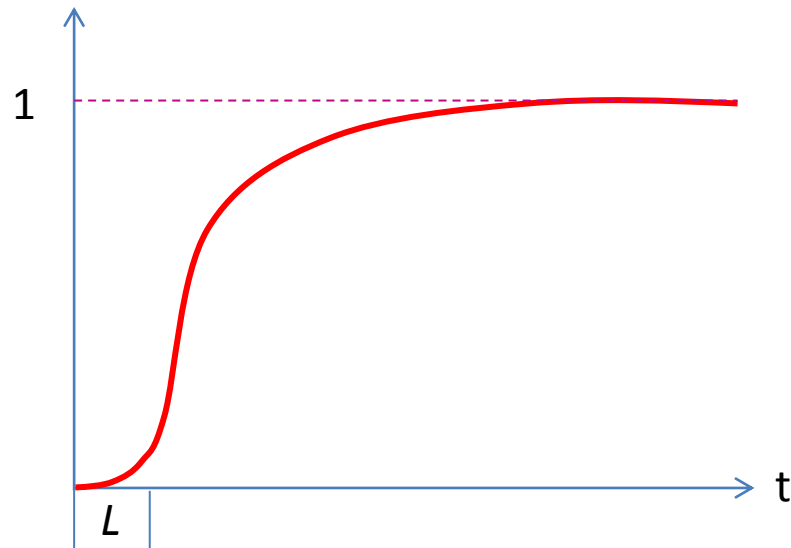


**Table-2**

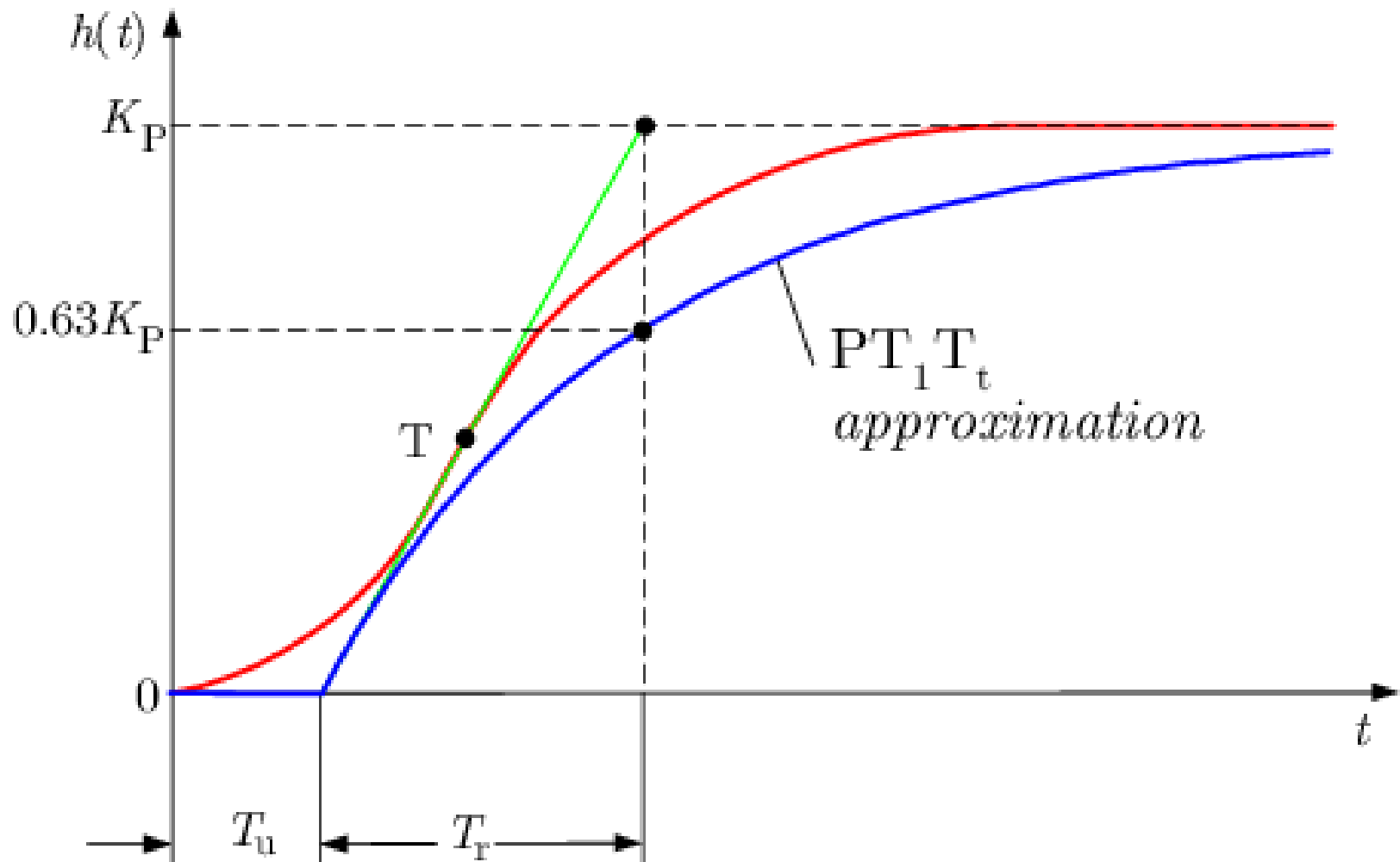
| Type of Controller | $K_p$        | $T_i$                  | $T_d$         |
|--------------------|--------------|------------------------|---------------|
| P                  | $0.5K_{cr}$  | $\infty$               | 0             |
| PI                 | $0.45K_{cr}$ | $\frac{1}{1.2} P_{cr}$ | 0             |
| PID                | $0.6K_{cr}$  | $0.5P_{cr}$            | $0.125P_{cr}$ |

# Example-1

$$\frac{C(s)}{R(s)} = \frac{K}{Ts + 1} e^{-sL}$$

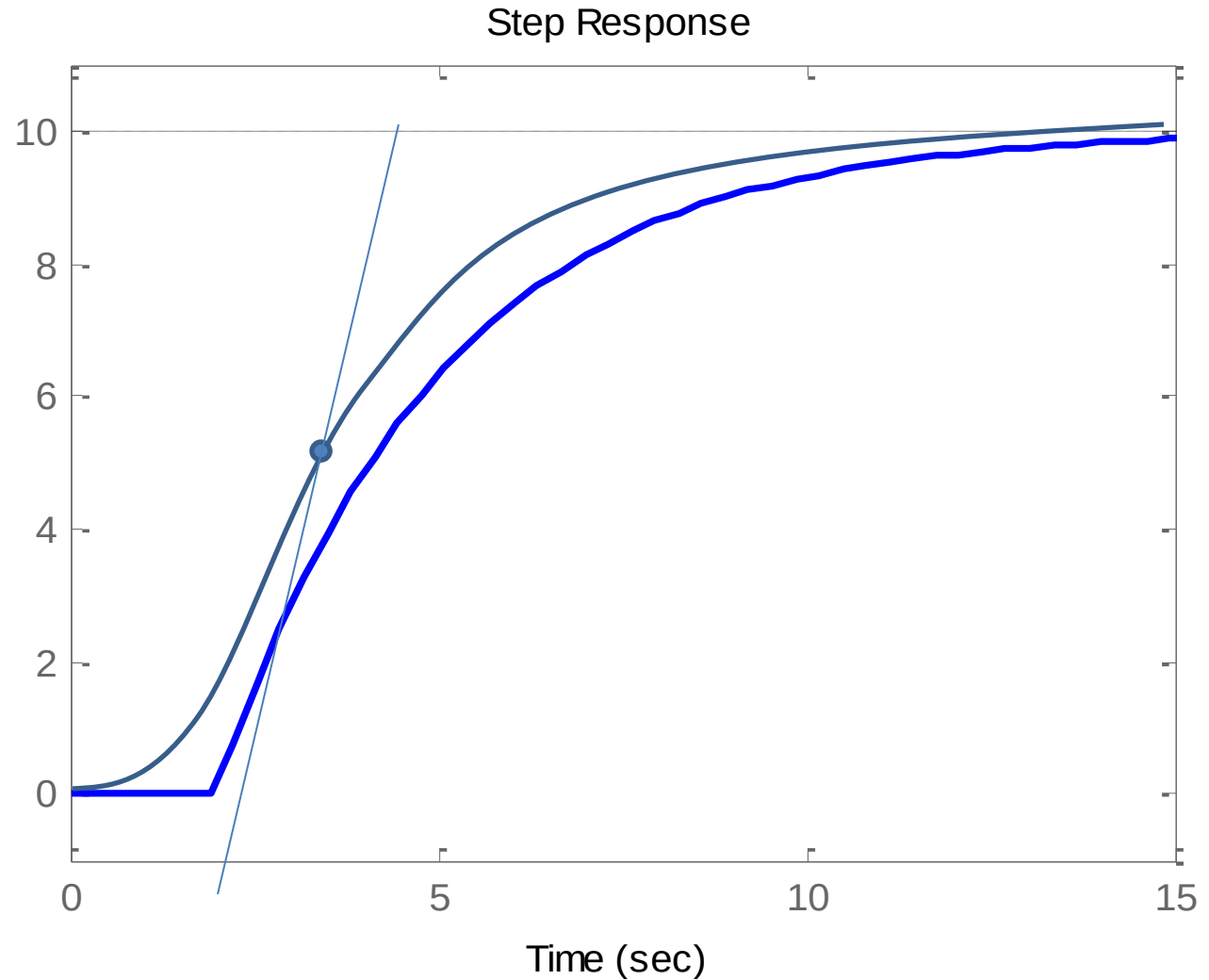


# Example-1



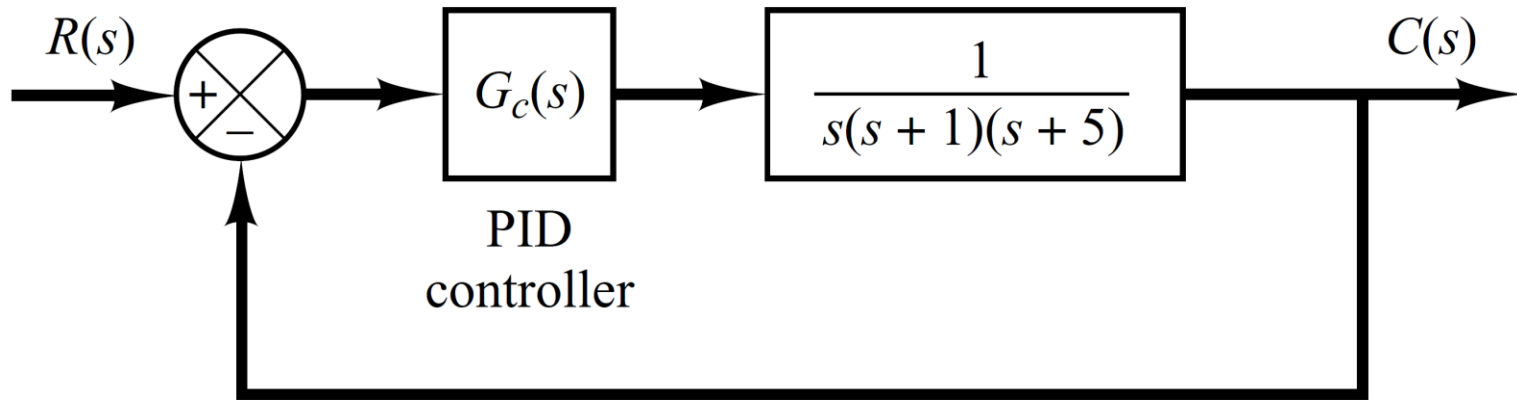
# Example-1

$$\frac{C(s)}{R(s)} = \frac{10}{3s+1} e^{-2s}$$



# Example-2

- Consider the control system shown in following figure.



- Apply a Ziegler–Nichols tuning rule for the determination of the values of parameters  $K_p$ ,  $T_i$  and  $T_d$ .

# Example-2

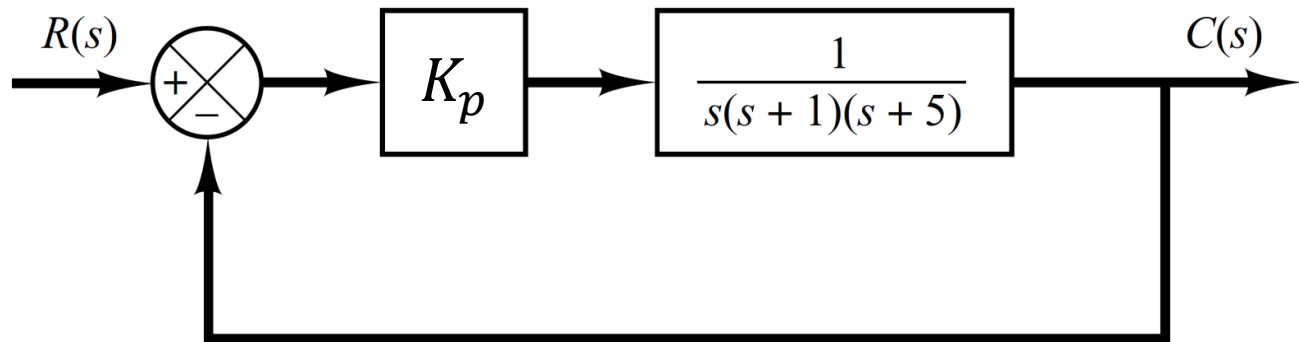
- Transfer function of the plant is

$$G(s) = \frac{1}{s(s+1)(s+5)}$$

- Since plant has an integrator therefore Ziegler-Nichol's first method is not applicable.
- According to second method proportional gain is varied till sustained oscillations are produced.
- That value of  $K_c$  is referred as  $K_{cr}$ .

# Example-2

- Here, since the transfer function of the plant is known we can find  $K_{cr}$  using
  - Root Locus
  - Routh-Herwitz Stability Criterion
- By setting  $T_i = \infty$  and  $T_d = 0$  closed loop transfer function is obtained as follows.



$$\frac{C(s)}{R(s)} = \frac{K_p}{s(s+1)(s+5) + K_p}$$



# Example-2

- The value of  $K_p$  that makes the system marginally unstable so that sustained oscillation occurs can be obtained as

$$s^3 + 6s^2 + 5s + K_p = 0$$

- The Routh array is obtained as
- Examining the coefficients of first column of the Routh array we find that sustained oscillations will occur if  $K_p = 30$ .
- Thus the critical gain  $K_{cr}$  is

|       |                      |       |
|-------|----------------------|-------|
| $s^3$ | 1                    | 5     |
| $s^2$ | 6                    | $K_p$ |
| $s^1$ | $\frac{30 - K_p}{6}$ |       |
| $s^0$ | $K_p$                |       |

$$K_{cr} = 30$$

# Example-2

- With gain  $K_p$  set equal to 30, the characteristic equation becomes

$$s^3 + 6s^2 + 5s + 30 = 0$$

- To find the frequency of sustained oscillations, we substitute  $s = j\omega$  into the characteristic equation.

$$(j\omega)^3 + 6(j\omega)^2 + 5j\omega + 30 = 0$$

- Further simplification leads to

$$6(5 - \omega^2) + j\omega(5 - \omega^2) = 0$$

$$6(5 - \omega^2) = 0$$

$$\omega = \sqrt{5} \text{ rad/sec}$$

# Example-2

$$\omega = \sqrt{5} \text{ rad/sec}$$

- Hence the period of sustained oscillations  $P_{cr}$  is

$$P_{cr} = \frac{2\pi}{\omega}$$

$$P_{cr} = \frac{2\pi}{\sqrt{5}} = 2.8099 \text{ sec}$$

- Referring to **Table-2**

$$K_p = 0.6K_{cr} = 18$$

$$T_i = 0.5P_{cr} = 1.405$$

$$T_d = 0.125P_{cr} = 0.35124$$

# Example-2

$$K_p = 18$$

$$T_i = 1.405$$

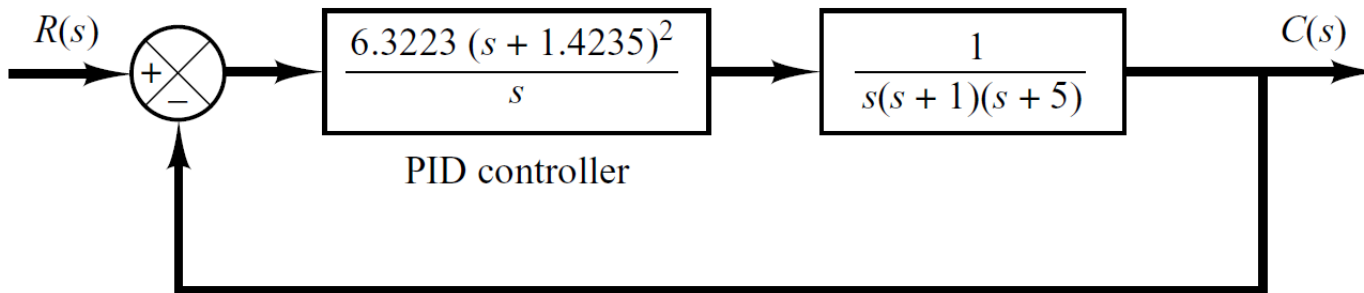
$$T_d = 0.35124$$

- Transfer function of PID controller is thus obtained as

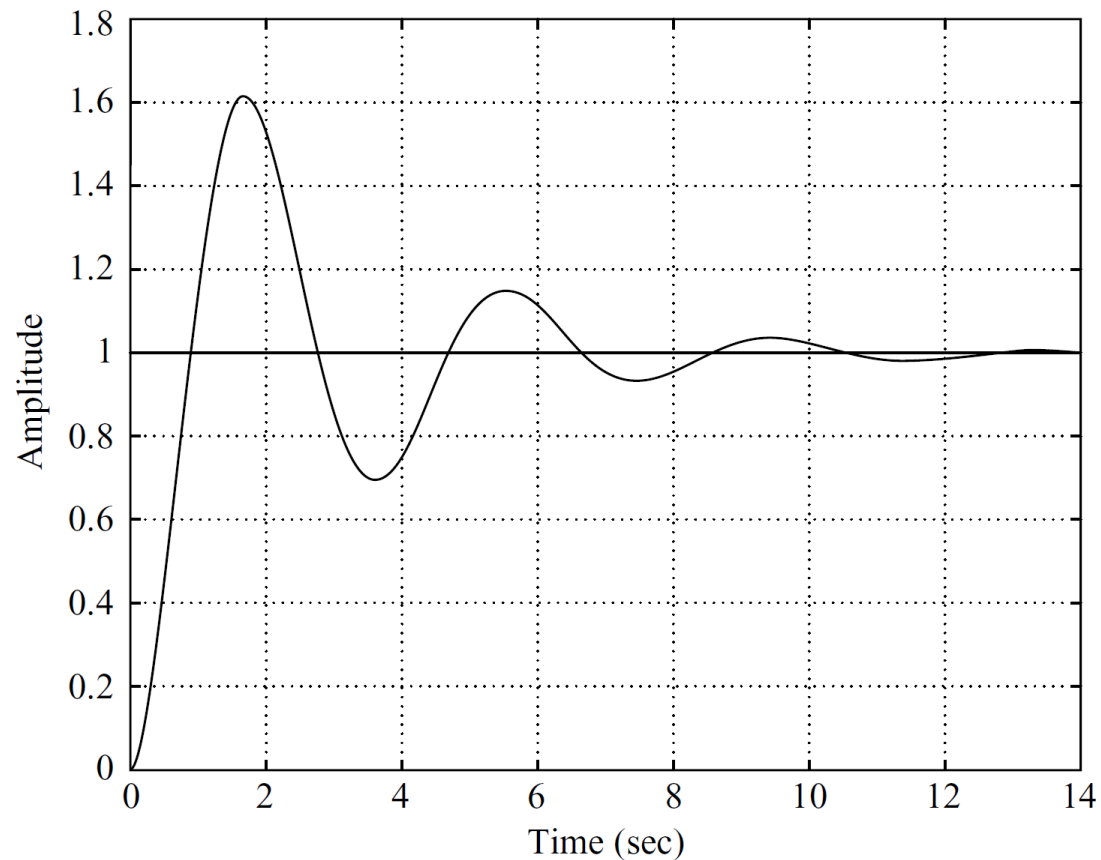
$$G_c(s) = K_p \left( 1 + \frac{1}{T_i s} + T_d s \right)$$

$$G_c(s) = 18 \left( 1 + \frac{1}{1.405s} + 0.35124s \right)$$

# Example-2



Unit-Step Response



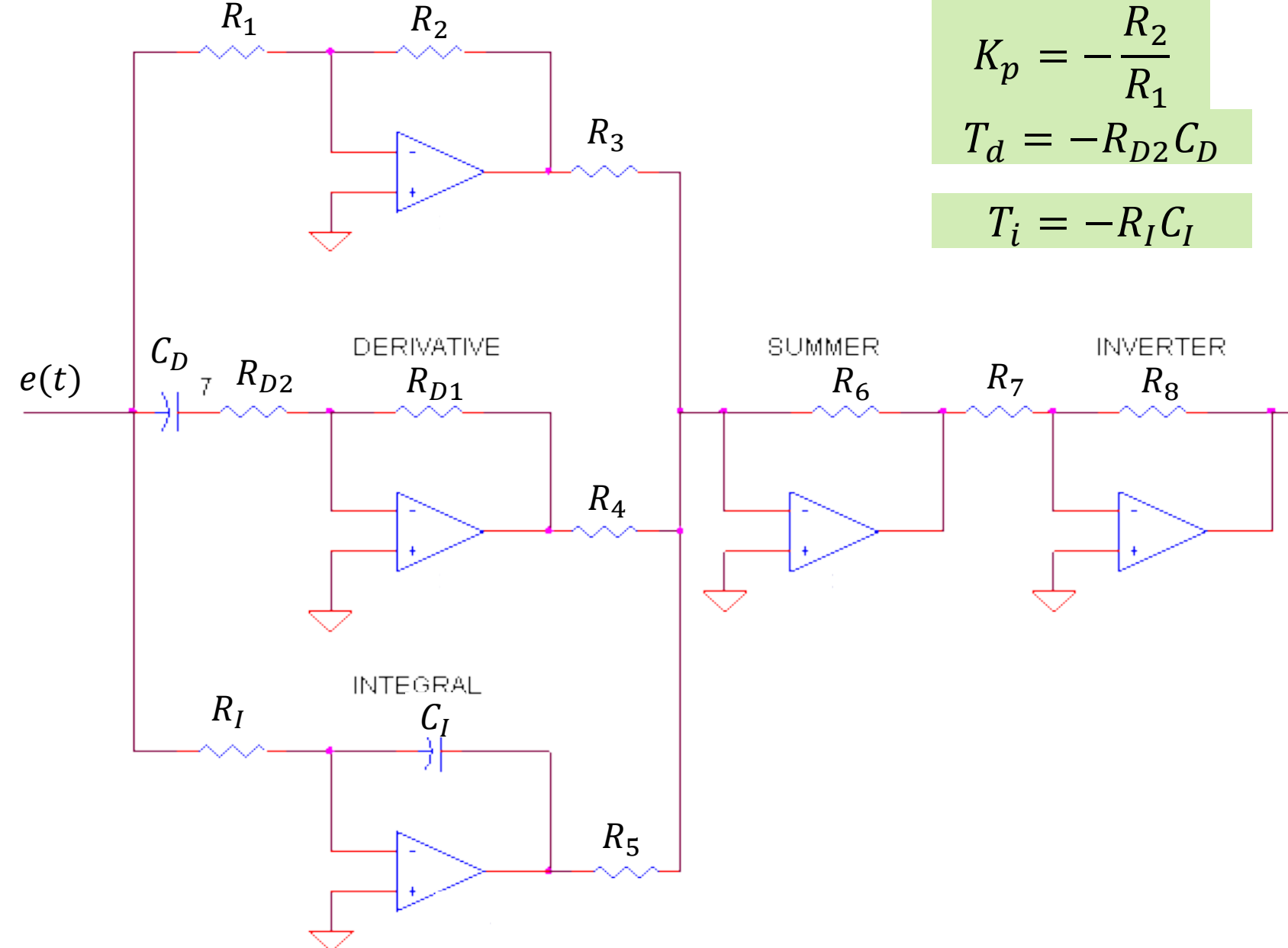
# Electronic PID Controller

PROPORTIONAL

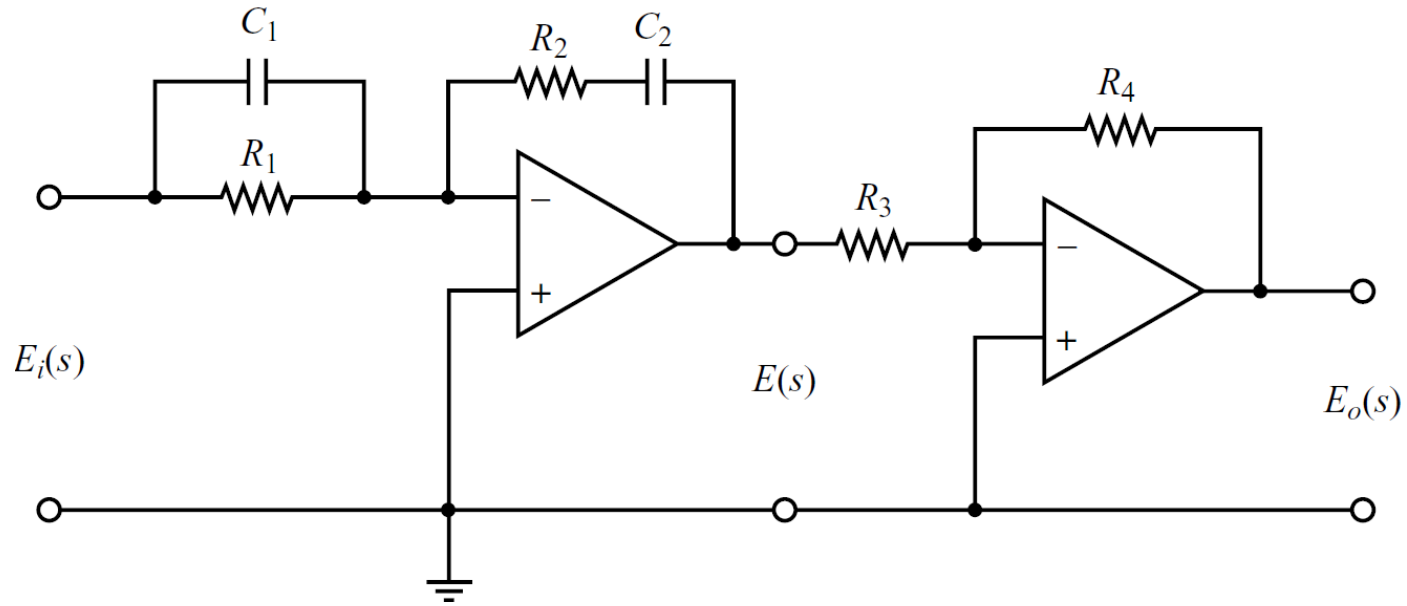
$$K_p = -\frac{R_2}{R_1}$$

$$T_d = -R_{D2}C_D$$

$$T_i = -R_I C_I$$



# Electronic PID Controller



$$\frac{E_o(s)}{E_i(s)} = \frac{R_4}{R_3} \frac{(R_1 C_1 s + 1)(R_2 C_2 s + 1)}{R_2 C_2 s}$$

$$\frac{E_o(s)}{E_i(s)} = \frac{R_4 R_2}{R_3 R_1} \left( \frac{R_1 C_1 + R_2 C_2}{R_2 C_2} + \frac{1}{R_2 C_2 s} + R_1 C_1 s \right)$$

# Electronic PID Controller

$$\frac{E_o(s)}{E_i(s)} = \frac{R_4 R_2}{R_3 R_1} \left( \frac{R_1 C_1 + R_2 C_2}{R_2 C_2} + \frac{1}{R_2 C_2 s} + R_1 C_1 s \right)$$

$$\frac{E_o(s)}{E_i(s)} = \frac{R_4 (R_1 C_1 + R_2 C_2)}{R_3 R_1 C_2} \left[ 1 + \frac{1}{(R_1 C_1 + R_2 C_2) s} + \frac{R_1 C_1 R_2 C_2}{R_1 C_1 + R_2 C_2} s \right]$$

$$K_p = \frac{R_4 (R_1 C_1 + R_2 C_2)}{R_3 R_1 C_2}$$

$$T_i = R_1 C_1 + R_2 C_2$$

$$T_d = \frac{R_1 C_1 R_2 C_2}{R_1 C_1 + R_2 C_2}$$

- In terms of  $K_p$ ,  $K_i$ ,  $K_d$  we have

$$K_p = \frac{R_4 (R_1 C_1 + R_2 C_2)}{R_3 R_1 C_2}$$

$$K_i = \frac{R_4}{R_3 R_1 C_2}$$

$$K_d = \frac{R_4 R_2 C_1}{R_3}$$



معادلات فضاء الحالة State Space

Modeling & Analysis

# Lecture Outline

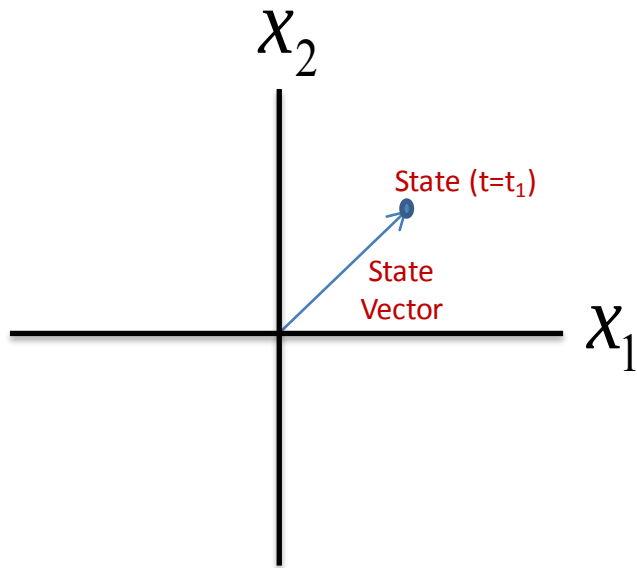
- Basic Definitions
- State Equations
- State Diagram
- Transfer Matrix
- Solution of State Equation
- State Controllability
- State Observability
- Output Controllability

# التعاريف الأساسية Definitions

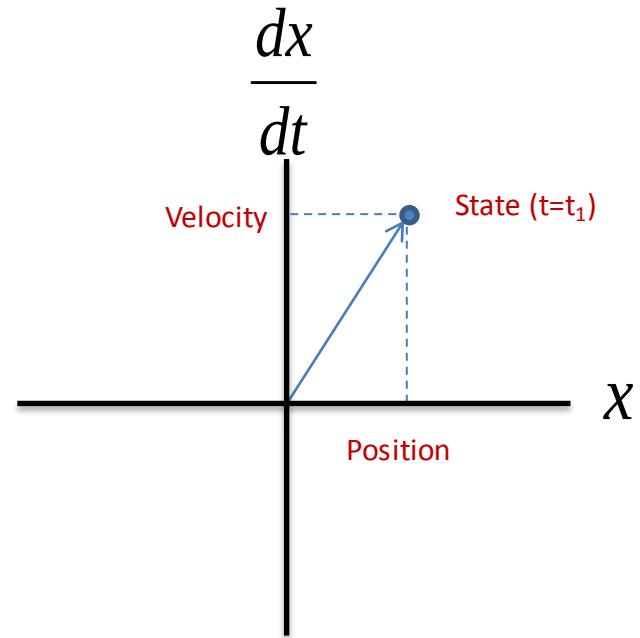
- **State of a system:** We define the state of a system at time  $t_0$  as the amount of information that must be provided at time  $t_0$ , which, together with the input signal  $u(t)$  for  $t \geq t_0$ , uniquely determine the output of the system for all  $t \geq t_0$ .
- **State Variable:** متغير الحالة The state variables of a dynamic system are the smallest set of variables that determine the state of the dynamic system.
- **State Vector:** منجهات الحالة If  $n$  variables are needed to completely describe the behaviour of the dynamic system then  $n$  variables can be considered as  $n$  components of a vector  $x$ , such a vector is called state vector.
- **State Space:** حالة الفضاء The state space is defined as the  $n$ -dimensional space in which the components of the state vector represents its coordinate axes.

# Definitions

- Let  $x_1$  and  $x_2$  are two states variables that define the state of the system completely .



Two Dimensional State space



State space of a Vehicle

# State Space Equations معادلة الحالة

- In state-space analysis we are concerned with three types of variables that are involved in the modeling of dynamic systems: input variables, output variables, and state variables.
- The dynamic system must involve elements that memorize the values of the input for  $t > t_1$ .
- Since integrators in a continuous-time control system serve as memory devices, the outputs of such integrators can be considered as the variables that define the internal state of the dynamic system.
- Thus the outputs of integrators serve as state variables.
- The number of state variables to completely define the dynamics of the system is equal to the number of integrators involved in the system.

# State Space Equations

- Assume that a multiple-input, multiple-output system involves  $n$  integrators.
- Assume also that there are  $r$  inputs  $u_1(t), u_2(t), \dots, u_r(t)$  and  $m$  outputs  $y_1(t), y_2(t), \dots, y_m(t)$ .
- Define  $n$  outputs of the integrators as state variables:  $x_1(t), x_2(t), \dots, x_n(t)$ .
- Then the system may be described by

$$\dot{x}_1(t) = f_1(x_1, x_2, \dots, x_n; u_1, u_2, \dots, u_r; t)$$

$$\dot{x}_2(t) = f_2(x_1, x_2, \dots, x_n; u_1, u_2, \dots, u_r; t)$$

•

•

•

$$\dot{x}_n(t) = f_n(x_1, x_2, \dots, x_n; u_1, u_2, \dots, u_r; t)$$

# State Space Equations

- The outputs  $y_1(t), y_2(t), \dots, y_m(t)$  of the system may be given as.

$$y_1(t) = g_1(x_1, x_2, \dots, x_n; u_1, u_2, \dots, u_r; t)$$

$$y_2(t) = g_2(x_1, x_2, \dots, x_n; u_1, u_2, \dots, u_r; t)$$

⋮

$$y_m(t) = g_m(x_1, x_2, \dots, x_n; u_1, u_2, \dots, u_r; t)$$

- If we define

$$\mathbf{x}(t) = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$\mathbf{f}(\mathbf{x}, \mathbf{u}, t) = \begin{bmatrix} f_1(x_1, x_2, \dots, x_n; u_1, u_2, \dots, u_r; t) \\ f_2(x_1, x_2, \dots, x_n; u_1, u_2, \dots, u_r; t) \\ \vdots \\ f_n(x_1, x_2, \dots, x_n; u_1, u_2, \dots, u_r; t) \end{bmatrix}$$

$$\mathbf{u}(t) = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_r \end{bmatrix}$$

$$\mathbf{y}(t) = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}$$

$$\mathbf{g}(\mathbf{x}, \mathbf{u}, t) = \begin{bmatrix} g_1(x_1, x_2, \dots, x_n; u_1, u_2, \dots, u_r; t) \\ g_2(x_1, x_2, \dots, x_n; u_1, u_2, \dots, u_r; t) \\ \vdots \\ g_m(x_1, x_2, \dots, x_n; u_1, u_2, \dots, u_r; t) \end{bmatrix}$$

# State Space Modeling

- State space equations can then be written as

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}, \mathbf{u}, t) \quad \text{State Equation}$$

$$\mathbf{y}(t) = \mathbf{g}(\mathbf{x}, \mathbf{u}, t) \quad \text{Output Equation}$$

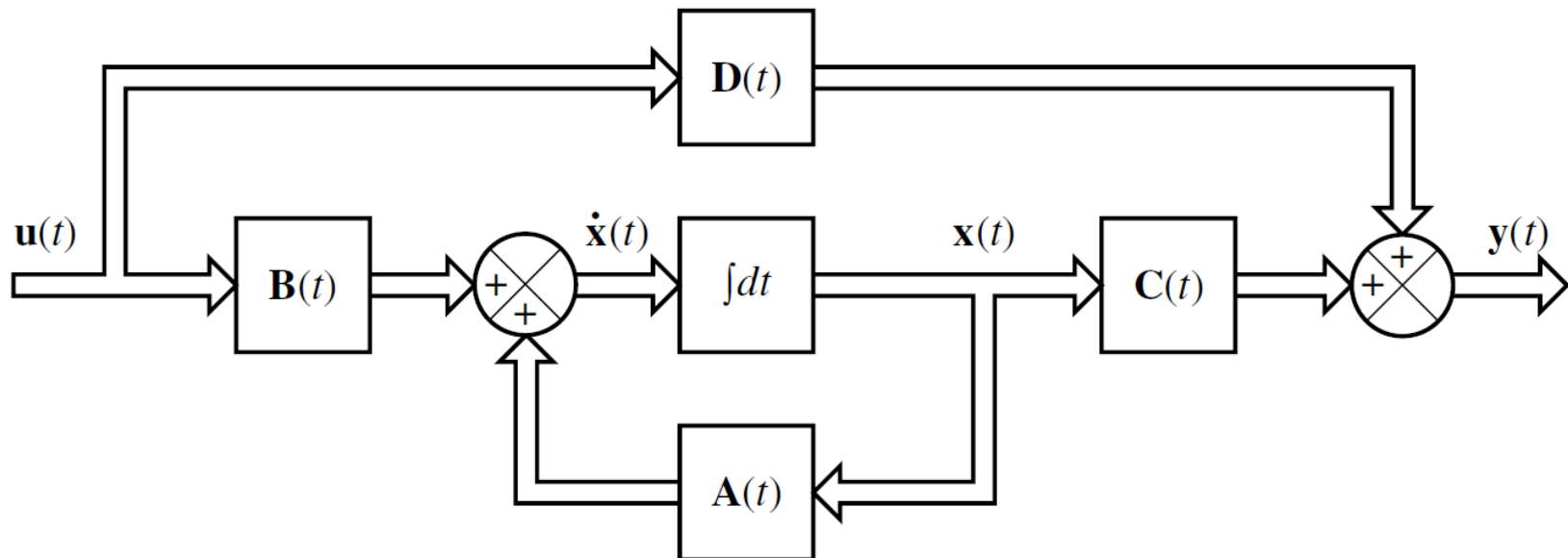
- If vector functions  $\mathbf{f}$  and/or  $\mathbf{g}$  involve time  $t$  explicitly, then the system is called a time varying system.



# State Space Modeling

- If above equations are linearised about the operating state, then we have the following linearised state equation and output equation:

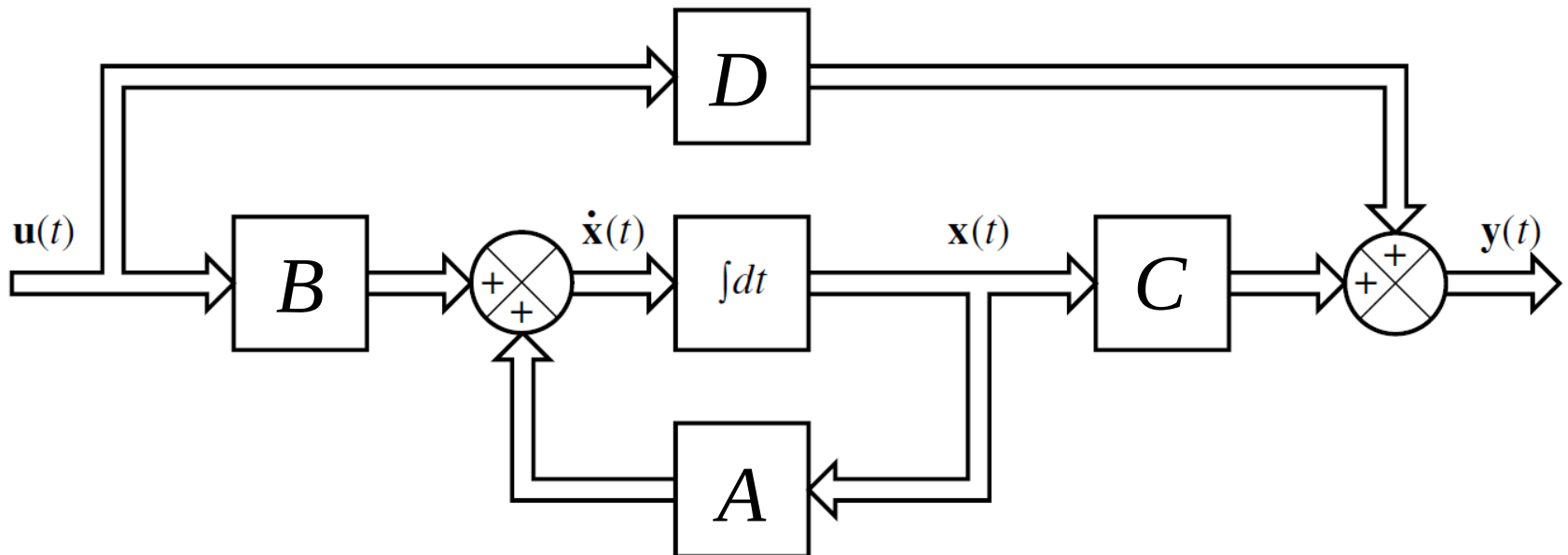
$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t) \quad \mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t) + \mathbf{D}(t)\mathbf{u}(t)$$



# State Space Modeling

- If vector functions  $f$  and  $g$  do not involve time  $t$  explicitly then the system is called a time-invariant system.
- In this case, state and output equations can be simplified to

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \qquad \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$$



# Example-1

- Consider the mechanical system shown in figure. We assume that the system is linear. The external force  $u(t)$  is the input to the system, and the displacement  $y(t)$  of the mass is the output. The displacement  $y(t)$  is measured from the equilibrium position in the absence of the external force. This system is a single-input, single-output system.

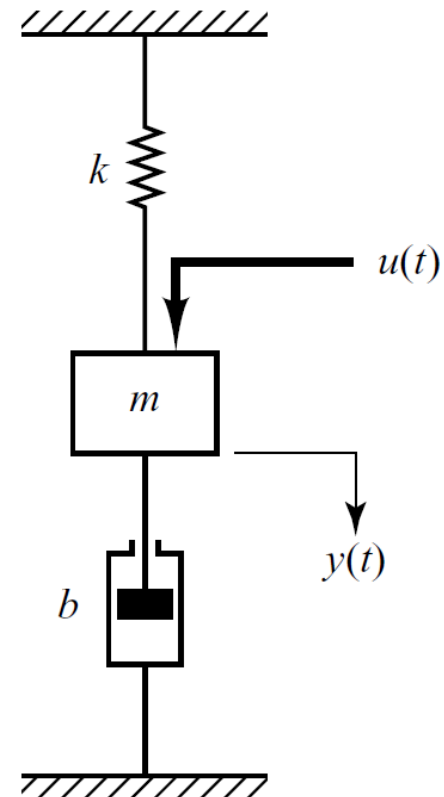
- From the diagram, the system equation is

$$m\ddot{y}(t) + b\dot{y}(t) + ky(t) = u(t)$$

- This system is of second order. This means that the system involves two integrators. Let us define state variables  $x_1(t)$  and  $x_2(t)$  as

$$x_1(t) = y(t)$$

$$x_2(t) = \dot{y}(t)$$



# Example-1

$$x_1(t) = y(t) \quad x_2(t) = \dot{y}(t) \quad m\ddot{y}(t) + b\dot{y}(t) + ky(t) = u(t)$$

- Then we obtain

$$\dot{x}_1(t) = x_2(t)$$

$$\dot{x}_2(t) = -\frac{b}{m}\dot{y}(t) - \frac{k}{m}y(t) + \frac{1}{m}u(t)$$

- Or

$$\dot{x}_1(t) = x_2(t)$$

$$\dot{x}_2(t) = -\frac{b}{m}x_2(t) - \frac{k}{m}x_1(t) + \frac{1}{m}u(t)$$

- The output equation is

$$y(t) = x_1(t)$$

# Example-1

$$\dot{x}_1(t) = x_2(t) \quad \dot{x}_2(t) = -\frac{b}{m}x_2(t) - \frac{k}{m}x_1(t) + \frac{1}{m}u(t) \quad y(t) = x_1(t)$$

- In a vector-matrix form,

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

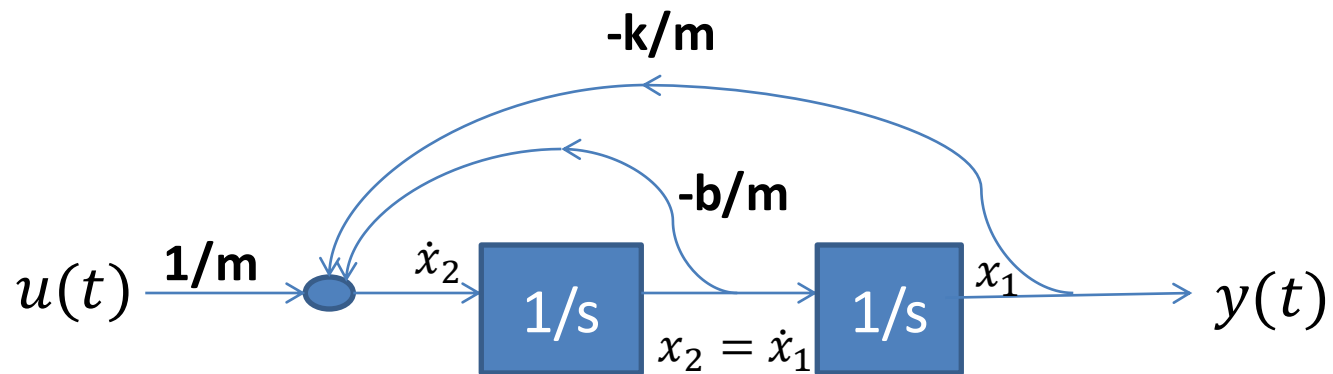
# Example-1

- State diagram of the system is

$$\dot{x}_1(t) = x_2(t)$$

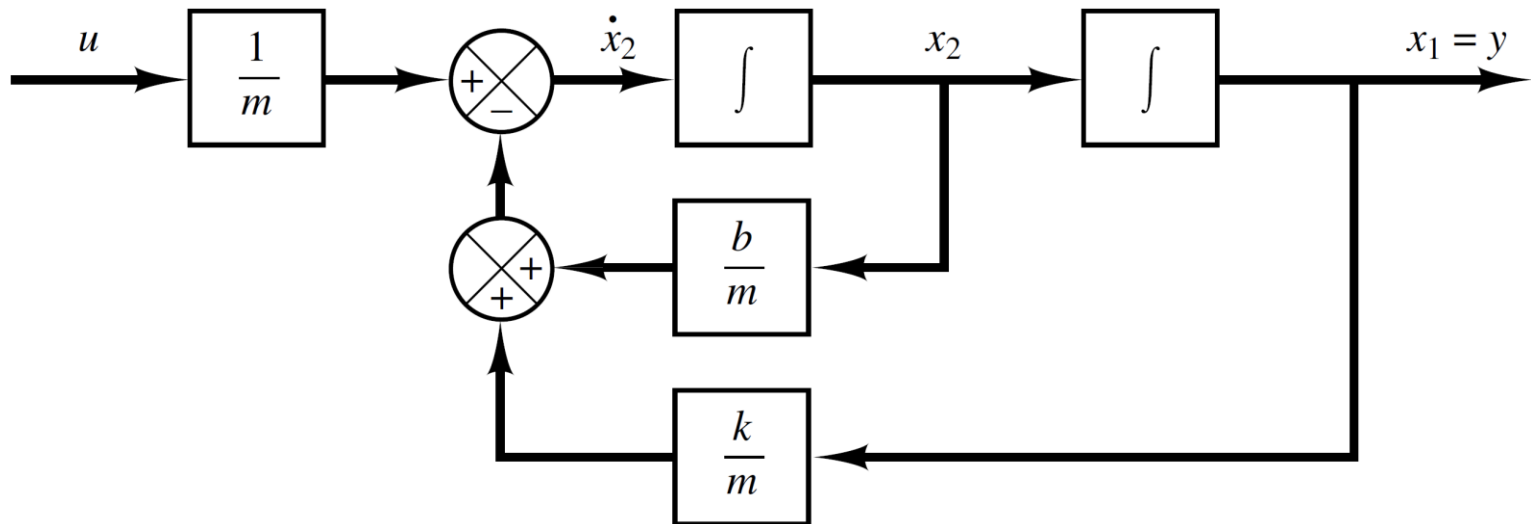
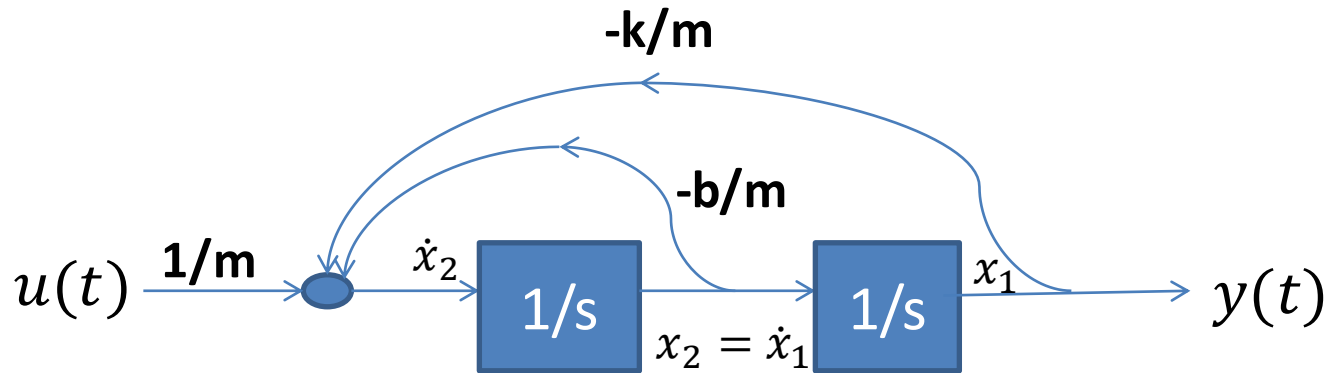
$$\dot{x}_2(t) = -\frac{b}{m}x_2(t) - \frac{k}{m}x_1(t) + \frac{1}{m}u(t)$$

$$y(t) = x_1(t)$$



# Example-1

- State diagram in signal flow and block diagram format



# Transfer Matrix (State Space to T.F)

- Now Let us convert a space model to a transfer function model.

$$\dot{x}(t) = Ax(t) + Bu(t) \longrightarrow (1)$$

$$y(t) = Cx(t) + Du(t) \longrightarrow (2)$$

- Taking Laplace transform of equation (1) and (2) considering initial conditions to zero.

$$sX(s) = AX(s) + BU(s) \longrightarrow (3)$$

$$Y(s) = CX(s) + DU(s) \longrightarrow (4)$$

- From equation (3)



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$$(sI - A)X(s) = BU(s)$$

$$X(s) = (sI - A)^{-1} BU(s) \longrightarrow (5)$$



# Transfer Matrix (State Space to T.F)

- Substituting equation (5) into equation (4) yields

$$Y(s) = C(sI - A)^{-1}BU(s) + DU(s)$$

$$Y(s) = [C(sI - A)^{-1}B + D]U(s)$$

$$\frac{Y(s)}{U(s)} = C(sI - A)^{-1}B + D$$

# Example#3

- Convert the following State Space Model to Transfer Function Model if  $K=3$ ,  $B=1$  and  $M=10$ ;

$$\begin{bmatrix} \dot{x} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{K}{M} & -\frac{B}{M} \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix} f(t)$$

$$y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix}$$

## Example#3

- Substitute the given values and obtain A, B, C and D matrices.

$$\begin{bmatrix} \dot{x} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{3}{10} & -\frac{1}{10} \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{10} \end{bmatrix} f(t)$$

$$y(t) = [0 \quad 1] \begin{bmatrix} x \\ v \end{bmatrix}$$

## Example#3

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{3}{10} & -\frac{1}{10} \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ \frac{1}{10} \end{bmatrix}$$

$$C = [0 \quad 1]$$

$$D = 0$$

$$\frac{Y(s)}{U(s)} = C(sI - A)^{-1}B + D$$

## Example#3

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{3}{10} & -\frac{1}{10} \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ \frac{1}{10} \end{bmatrix}$$

$$C = [0 \quad 1]$$

$$D = 0$$

$$\frac{Y(s)}{U(s)} = [0 \quad 1] \left( \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -\frac{3}{10} & -\frac{1}{10} \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 \\ \frac{1}{10} \end{bmatrix}$$

# Example#3

$$\frac{Y(s)}{U(s)} = \begin{bmatrix} 0 & 1 \end{bmatrix} \left( \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -\frac{3}{10} & -\frac{1}{10} \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 \\ \frac{1}{10} \end{bmatrix}$$

$$\frac{Y(s)}{U(s)} = \begin{bmatrix} 0 & 1 \end{bmatrix} \left( \begin{bmatrix} s & -1 \\ \frac{3}{10} & s + \frac{1}{10} \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 \\ \frac{1}{10} \end{bmatrix}$$

$$\frac{Y(s)}{U(s)} = \begin{bmatrix} 0 & 1 \end{bmatrix} \frac{1}{s(s + \frac{1}{10}) + \frac{3}{10}} \begin{bmatrix} s + \frac{1}{10} & 1 \\ -\frac{3}{10} & s \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{10} \end{bmatrix}$$

# Example#3

$$\frac{Y(s)}{U(s)} = \begin{bmatrix} 0 & 1 \end{bmatrix} \frac{1}{s(s + \frac{1}{10}) + \frac{3}{10}} \begin{bmatrix} s + \frac{1}{10} & 1 \\ -\frac{3}{10} & s \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{10} \end{bmatrix}$$

$$\frac{Y(s)}{U(s)} = \frac{1}{s(s + \frac{1}{10}) + \frac{3}{10}} \begin{bmatrix} -\frac{3}{10} & s \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{10} \end{bmatrix}$$

$$\frac{Y(s)}{U(s)} = \frac{1}{s(s + \frac{1}{10}) + \frac{3}{10}} \frac{s}{10}$$

# Example#3

$$\frac{Y(s)}{U(s)} = \frac{1}{s(s + \frac{1}{10}) + \frac{3}{10}} \frac{s}{10}$$

$$\frac{Y(s)}{U(s)} = \frac{s}{s(10s + 1) + 3}$$



# Home Work

- Obtain the transfer function  $T(s)$  from following state space representation.

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix} u$$
$$y = [1 \ 0 \ 0] \mathbf{x}$$

Answer

$$T(s) = \frac{10(s^2 + 3s + 2)}{s^3 + 3s^2 + 2s + 1}$$

# Forced and Unforced Response منظومات ذات الادخال وعديمة الادخال

- Forced Response, with  $u(t)$  as forcing function

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u(t)$$

- Unforced Response (response due to initial conditions)

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$$

## Equations

- Consider the state equation given below

$$\dot{x}(t) = Ax(t) \longrightarrow (1)$$

- Taking Laplace transform of the equation (1)

$$sX(s) - x(0) = AX(s)$$

$$sX(s) - AX(s) = x(0)$$

$$(sI - A)X(s) = x(0)$$

$$X(s) = (sI - A)^{-1} x(0)$$

$$X(s) = \frac{1}{sI - A} x(0)$$

# Solution of State Equations

$$X(s) = \frac{1}{sI - A} x(0)$$

- Taking inverse Laplace

$$x(t) = e^{At} x(0)$$

$$\phi(t) = e^{At}$$

State Transition Matrix

# Solution of State Equations

- Consider the state equation with  $u(t)$  as forcing function

$$\dot{x}(t) = Ax(t) + Bu(t) \longrightarrow (1)$$

- Taking Laplace transform of the equation (1)

$$sX(s) - x(0) = AX(s) + BU(s)$$

$$sX(s) - AX(s) = x(0) + BU(s)$$

$$(sI - A)X(s) = x(0) + BU(s)$$

$$X(s) = \frac{x(0) + BU(s)}{sI - A}$$

# Solution of State Equations

$$X(s) = \frac{x(0) + BU(s)}{sI - A}$$

$$X(s) = \frac{x(0)}{sI - A} + \frac{BU(s)}{sI - A}$$

- Taking the inverse Laplace transform of above equation.

$$x(t) = \phi(t)x(0) + \int_0^t \phi(t-\tau)u(\tau)dt$$

Natural Response

Forced Response

# State Space Trajectories

- The unforced response of a system released from any initial point  $\mathbf{x}(t_0)$  traces a curve or trajectory in state space, with time  $t$  as an implicit function along the trajectory.
- Unforced system's response depend upon initial conditions.

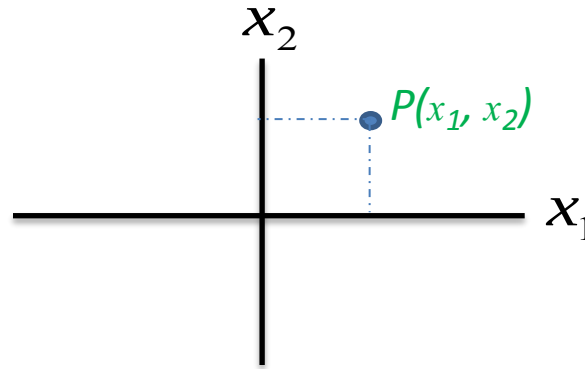
$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)$$

- Response due to initial conditions can be obtained as

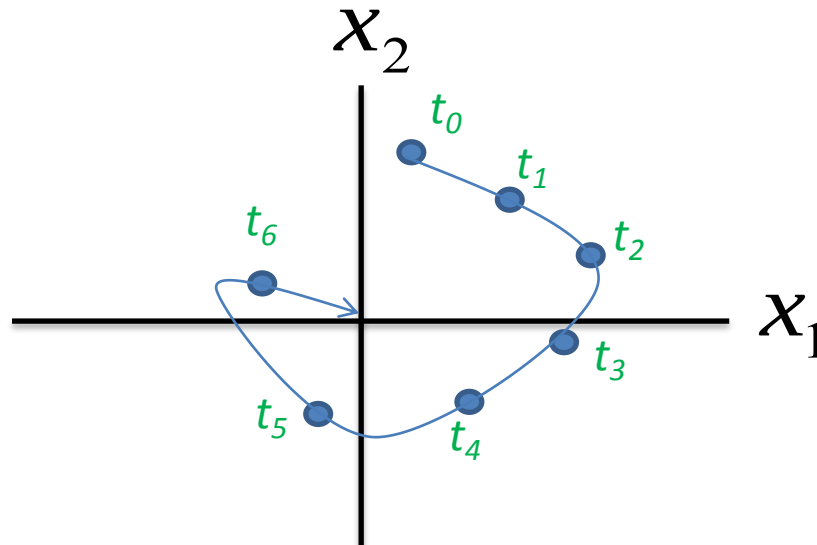
$$\mathbf{x}(t) = \phi(t)\mathbf{x}(0)$$

# State Transition

- Any point  $P$  in state space represents the state of the system at a specific time  $t$ .



- State transitions provide complete picture of the system

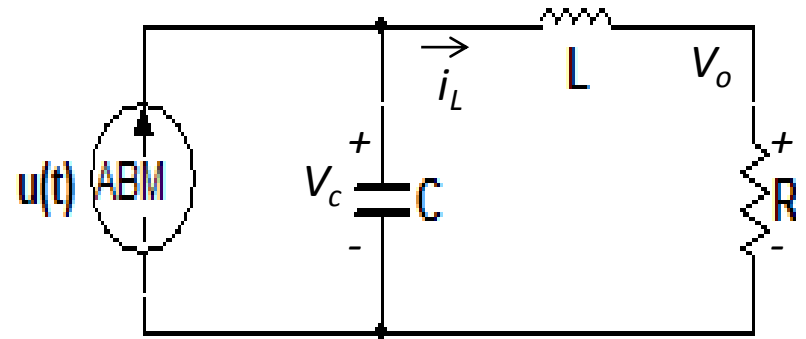




# Example-4

- Consider RLC Circuit obtain the state transition matrix  $\phi(t)$ .

$$\begin{bmatrix} \dot{v}_c \\ \dot{i}_L \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{C} \\ \frac{1}{L} & -\frac{R}{L} \end{bmatrix} \begin{bmatrix} v_c \\ i_L \end{bmatrix} + \begin{bmatrix} \frac{1}{C} \\ 0 \end{bmatrix} u(t)$$



$$R = 3, \quad L = 1 \quad \text{and} \quad C = 0.5$$

$$\begin{bmatrix} \dot{v}_c \\ \dot{i}_L \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} v_c \\ i_L \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} u(t)$$

## Example-4 (cont...)

$$\begin{bmatrix} \dot{v}_c \\ \dot{i}_L \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} v_c \\ i_L \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} u(t)$$

- State transition matrix can be obtained as

$$\phi(t) = \ell^{-1}[(SI - A)^{-1}] = \ell^{-1} \left\{ \left( \begin{bmatrix} S & 0 \\ 0 & S \end{bmatrix} - \begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix} \right)^{-1} \right\}$$

- Which is further simplified as

$$\phi(t) = \ell^{-1} \begin{bmatrix} \frac{(S+3)}{(S+1)(S+2)} & -\frac{2}{(S+1)(S+2)} \\ \frac{1}{(S+1)(S+2)} & \frac{S}{(S+1)(S+2)} \end{bmatrix}$$

## Example-4 (cont...)

$$\phi(t) = \ell^{-1} \begin{bmatrix} \frac{(S+3)}{(S+1)(S+2)} & -\frac{2}{(S+1)(S+2)} \\ \frac{1}{(S+1)(S+2)} & \frac{S}{(S+1)(S+2)} \end{bmatrix}$$

- Taking the inverse Laplace transform of each element

$$\phi(t) = \begin{bmatrix} (2e^{-t} - e^{-2t}) & (-2e^{-t} + 2e^{-2t}) \\ (e^{-t} - e^{-2t}) & (-e^{-t} + 2e^{-2t}) \end{bmatrix}$$

# Example-5

- For the RLC circuit of example-4 draw the state space trajectory with following initial conditions.

$$\begin{bmatrix} v_c(0) \\ i_L(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

- Solution

$$x(t) = \phi(t)x(0)$$

$$\begin{bmatrix} v_c \\ i_L \end{bmatrix} = \begin{bmatrix} (2e^{-t} - e^{-2t}) & (-2e^{-t} + 2e^{-2t}) \\ (e^{-t} - e^{-2t}) & (-e^{-t} + 2e^{-2t}) \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} v_c \\ i_L \end{bmatrix} = \begin{bmatrix} -3e^{-t} + 3e^{-2t} \\ -e^{-t} + 3e^{-2t} \end{bmatrix}$$

$$v_c = -3e^{-t} + 3e^{-2t}$$

$$i_L = -e^{-t} + 3e^{-2t}$$

► **EXAMPLE 10-5-1** Consider the state equation

$$\begin{bmatrix} \frac{dx_1(t)}{dt} \\ \frac{dx_2(t)}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \quad (10-81)$$

The problem is to determine the state-transition matrix  $\phi(t)$  and the state vector  $\mathbf{x}(t)$  for  $t \geq 0$  when the input is  $u(t) = 1$  for  $t \geq 0$ . The coefficient matrices are identified to be

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \mathbf{E} = 0 \quad (10-82)$$

Therefore,

$$s\mathbf{I} - \mathbf{A} = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix} \quad (10-83)$$

The inverse matrix of  $(s\mathbf{I} - \mathbf{A})$  is

$$(s\mathbf{I} - \mathbf{A})^{-1} = \frac{1}{s^2 + 3s + 2} \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix} \quad (10-84)$$

The state-transition matrix of  $\mathbf{A}$  is found by taking the inverse Laplace transform of Eq. (10-84). Thus,

$$\phi(t) = \mathcal{L}^{-1}[(s\mathbf{I} - \mathbf{A})^{-1}] = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix} \quad (10-85)$$

$$\begin{aligned} \mathbf{x}(t) = & \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix} \mathbf{x}(0) \\ & + \int_0^t \begin{bmatrix} 2e^{-(t-\tau)} - e^{-2(t-\tau)} & e^{-(t-\tau)} - e^{-2(t-\tau)} \\ -2e^{-(t-\tau)} + e^{-2(t-\tau)} & -e^{-(t-\tau)} + 2e^{-2(t-\tau)} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} d\tau \end{aligned} \quad (10-86)$$

or

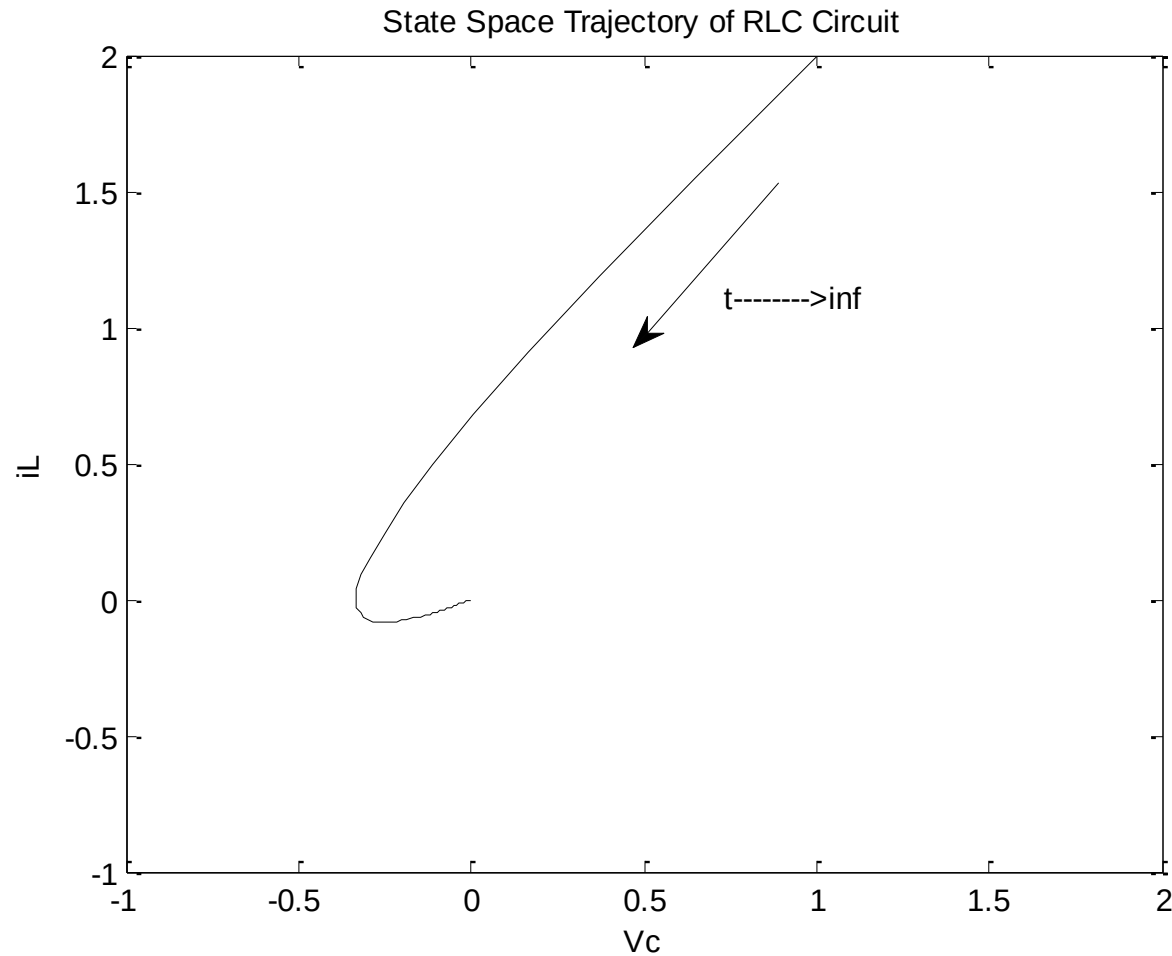
$$\mathbf{x}(t) = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix} \mathbf{x}(0) + \begin{bmatrix} 0.5 - e^{-t} + 0.5e^{-2t} \\ e^{-t} - e^{-2t} \end{bmatrix} \quad t \geq 0 \quad (10-87)$$

As an alternative, the second term of the state-transition equation can be obtained by taking the inverse Laplace transform of  $(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s)$ . Thus, we have

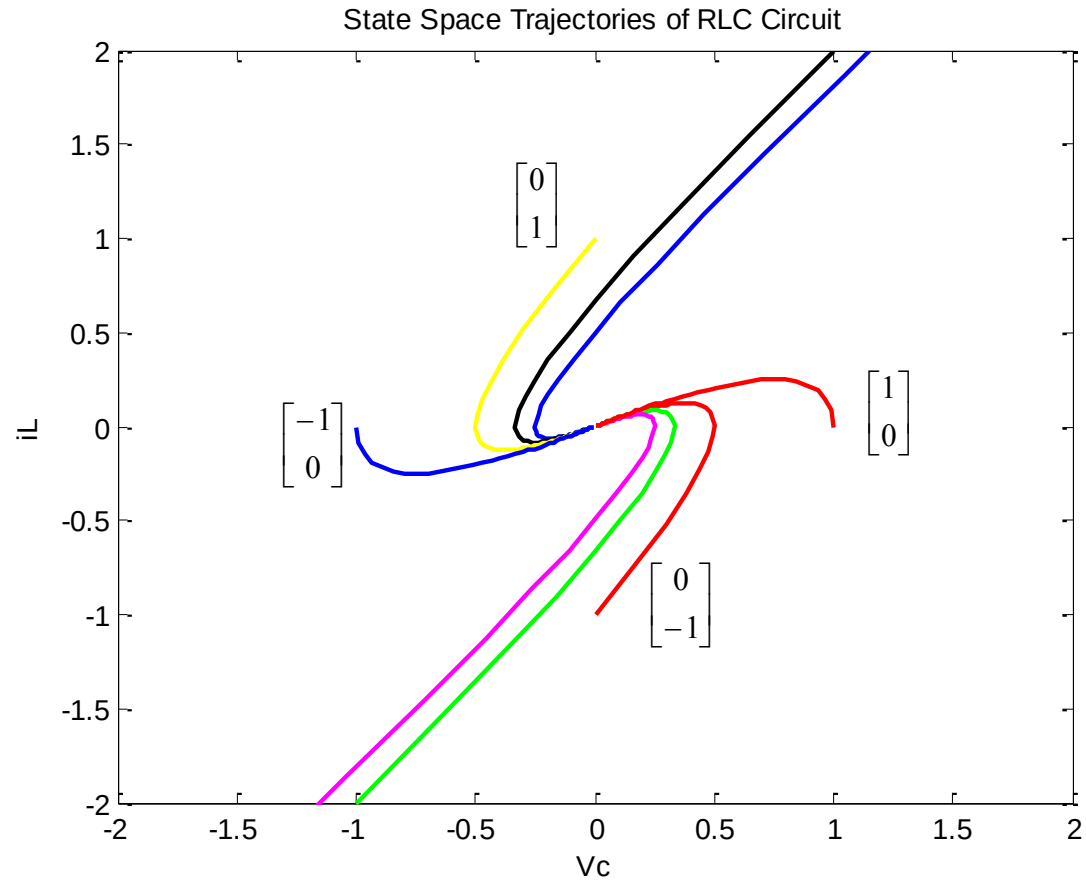
$$\begin{aligned} \mathcal{L}^{-1}[(s\mathbf{I} - \mathbf{A})^{-1}]\mathbf{B}\mathbf{U}(s) &= \mathcal{L}^{-1}\left(\frac{1}{s^2 + 3s + 2} \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \frac{1}{s}\right) \\ &= \mathcal{L}^{-1}\left(\frac{1}{s^2 + 3s + 2} \begin{bmatrix} 1 \\ \frac{1}{s} \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0.5 - e^{-t} + 0.5e^{-2t} \\ e^{-t} - e^{-2t} \end{bmatrix} \quad t \geq 0 \end{aligned} \quad (10-88)$$

# Example-5 (cont...)

- Following trajectory is obtained



# Example-5 (cont...)

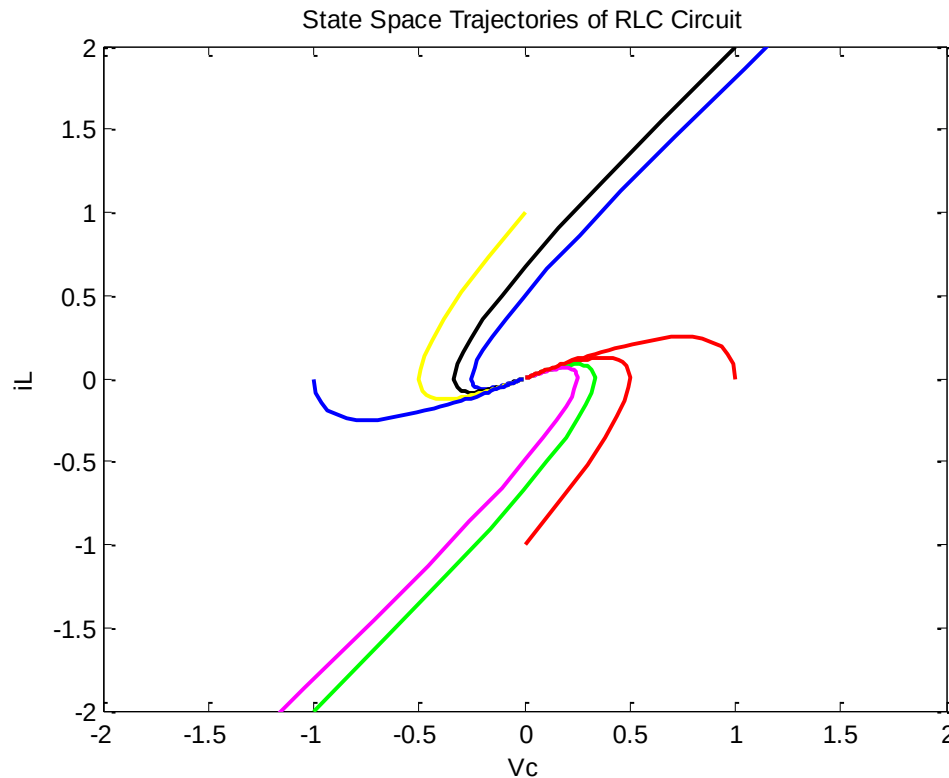




# Equilibrium Point

- The equilibrium or stationary state of the system is when

$$\dot{x}(t) = 0$$



## 10-5 STATE-TRANSITION EQUATION

The **state-transition equation** is defined as the solution of the linear homogeneous state equation. The linear time-invariant state equation

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{E}\mathbf{w}(t) \quad (10-73)$$

can be solved using either the classical method of solving linear differential equations or the Laplace transform method. The Laplace transform solution is presented in the following equations.

Taking the Laplace transform on both sides of Eq. (10-73), we have

$$s\mathbf{X}(s) - \mathbf{x}(0) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{U}(s) + \mathbf{E}\mathbf{W}(s) \quad (10-74)$$

where  $\mathbf{x}(0)$  denotes the initial-state vector evaluated at  $t = 0$ . Solving for  $\mathbf{X}(s)$  in Eq. (10-74) yields

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{x}(0) + (s\mathbf{I} - \mathbf{A})^{-1}[\mathbf{B}\mathbf{U}(s) + \mathbf{E}\mathbf{W}(s)] \quad (10-75)$$

The state-transition equation of Eq. (10-73) is obtained by taking the inverse Laplace transform on both sides of Eq. (10-75):

$$\begin{aligned} \mathbf{x}(t) &= \mathcal{L}^{-1}[(s\mathbf{I} - \mathbf{A})^{-1}]\mathbf{x}(0) + \mathcal{L}^{-1}\{(s\mathbf{I} - \mathbf{A})^{-1}[\mathbf{B}\mathbf{U}(s) + \mathbf{E}\mathbf{W}(s)]\} \\ &= \boldsymbol{\phi}(t)\mathbf{x}(0) + \int_0^t \boldsymbol{\phi}(t - \tau)[\mathbf{B}\mathbf{u}(\tau) + \mathbf{E}\mathbf{w}(\tau)]d\tau \quad t \geq 0 \end{aligned} \quad (10-76)$$

# Example#6

- Obtain the time response of the following system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

- Where  $u(t)$  is unit step function occurring at  $t=0$ .  
consider  $x(0)=0$ .

## Solution

- Calculate the state transition matrix

$$\phi(t) = \ell^{-1}[(SI - A)^{-1}]$$

$$\Phi(t) = e^{At} = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$$

# Example#6

- Obtain the state transition equation of the system

$$x(t) = \phi(t)x(0) + \int_0^t \phi(t-\tau)u(\tau)d\tau$$

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t \begin{bmatrix} 2e^{-(t-\tau)} - e^{-2(t-\tau)} & e^{-(t-\tau)} - e^{-2(t-\tau)} \\ -2e^{-(t-\tau)} + 2e^{-2(t-\tau)} & -e^{-(t-\tau)} + 2e^{-2(t-\tau)} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} [1] d\tau$$

$$\mathbf{x}(t) = \int_0^t \begin{bmatrix} 2e^{-(t-\tau)} - e^{-2(t-\tau)} & e^{-(t-\tau)} - e^{-2(t-\tau)} \\ -2e^{-(t-\tau)} + 2e^{-2(t-\tau)} & -e^{-(t-\tau)} + 2e^{-2(t-\tau)} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} [1] d\tau$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} - e^{-t} + \frac{1}{2} e^{-2t} \\ e^{-t} - e^{-2t} \end{bmatrix}$$

# Home Work

- Compute the state transition matrix if

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

**Solution**

$$\phi(t) = \ell^{-1}[(SI - A)^{-1}]$$

# STATE DIAGRAM

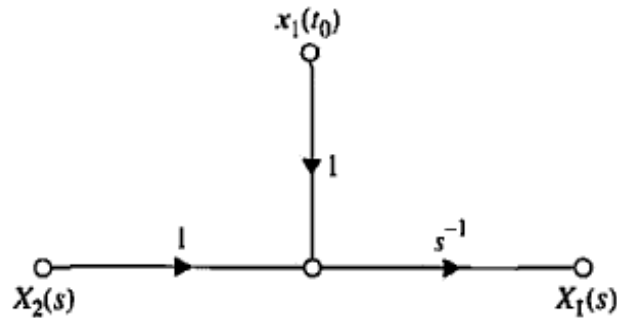


**Figure 10-3** Signal-flow graph representation of  $X_1(s) = [X_2(s)/s] + [x_1(t_0)/s]$ .

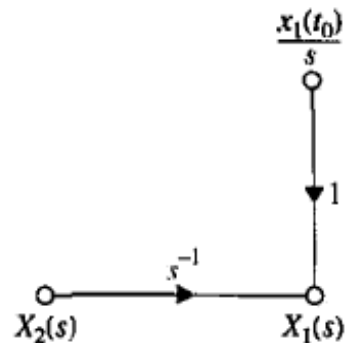
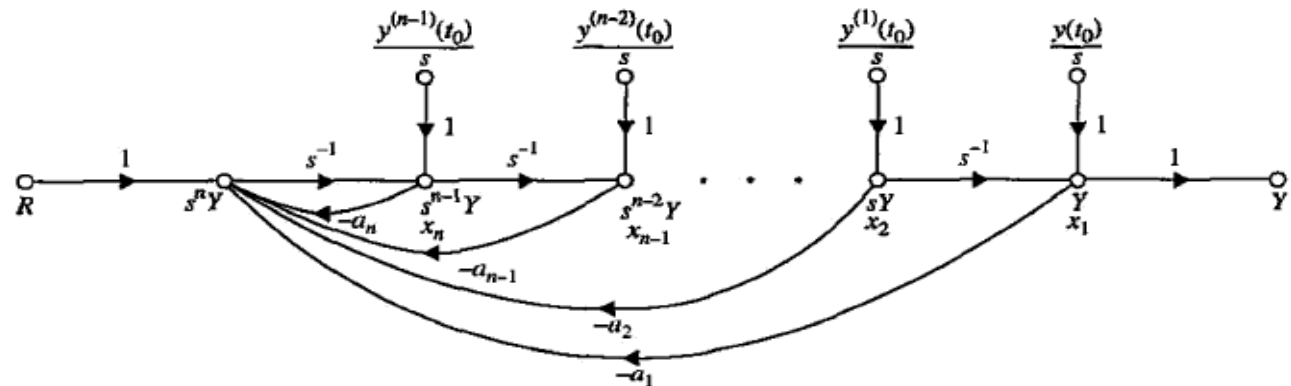
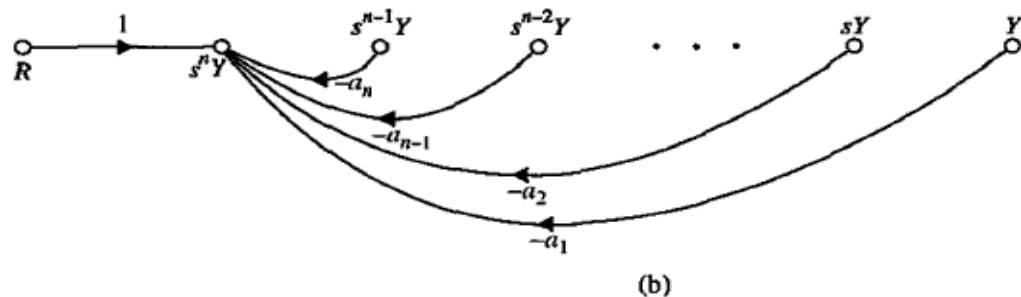
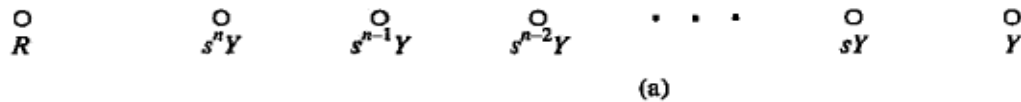


**Figure 10-4** Signal-flow graph representation of  $X_1(s) = [X_2(s)/s] + [x_1(t_0)/s]$ .

# From Differential Equations to State Diagrams

$$\frac{d^n y(t)}{dt^n} + a_n \frac{d^{n-1} y(t)}{dt^{n-1}} + \cdots + a_2 \frac{dy(t)}{dt} + a_1 y(t) = r(t)$$



➤ **EXAMPLE 10-2-2** Consider the differential equation

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = r(t) \quad (10-26)$$

Equating the highest-ordered term of the last equation to the rest of the terms, we have

$$\frac{d^2y(t)}{dt^2} = -3\frac{dy(t)}{dt} - 2y(t) + r(t) \quad (10-27)$$

Following the procedure just outlined, the state diagram of the system is drawn as shown in Fig. 10-6. The state variables  $x_1$  and  $x_2$  are assigned as shown.

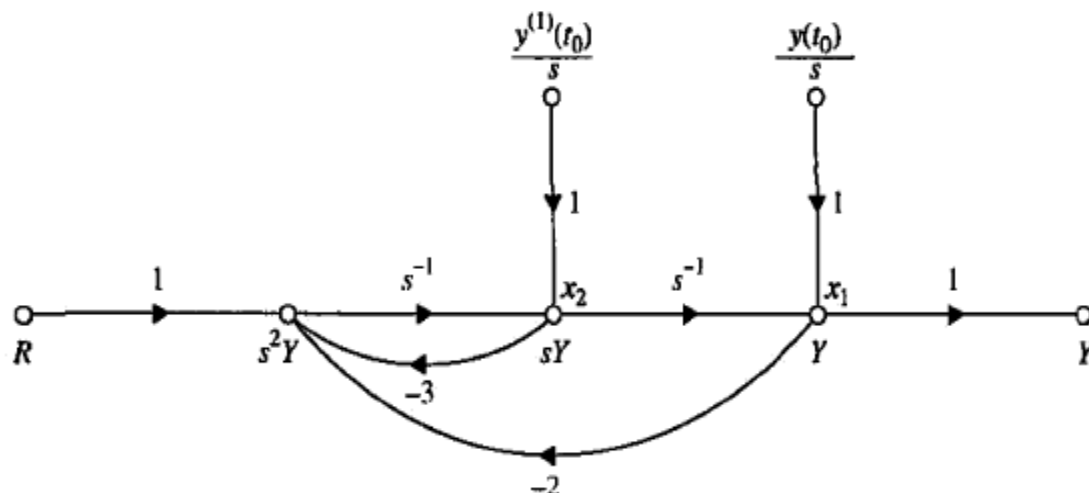


**Figure 10-6** State diagram for Eq. (10-26).



# From State Diagrams to State and Output Equations

1. Delete the initial states and the integrator branches with gains  $s^{-1}$  from the state diagram, since the state and output equations do not contain the Laplace operator  $s$  or the initial states.
2. For the state equations, regard the nodes that represent the derivatives of the state variables as output nodes, since these variables appear on the left-hand side of the state equations. The output  $y(t)$  in the output equation is naturally an output node variable.
3. Regard the state variables and the inputs as input variables on the state diagram, since these variables are found on the right-hand side of the state and output equations.
4. Apply the SFG gain formula to the state diagram.

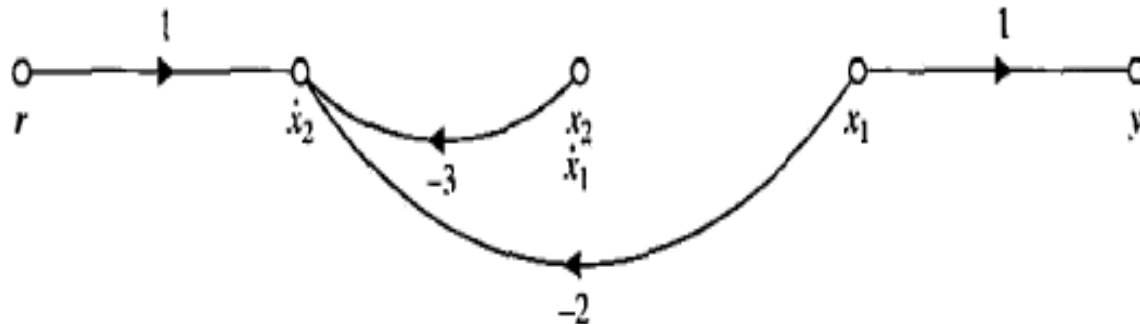
# EXAMPLE

## EXAMPLE

shows the state diagram of Fig. 10-6 with the integrator branches and the initial states eliminated. Using  $dx_1(t)/dt$  and  $dx_2(t)/dt$  as the output nodes and  $x_1(t)$ ,  $x_2(t)$ , and  $r(t)$  as input nodes, and applying the gain formula between these nodes, the state equations are obtained as

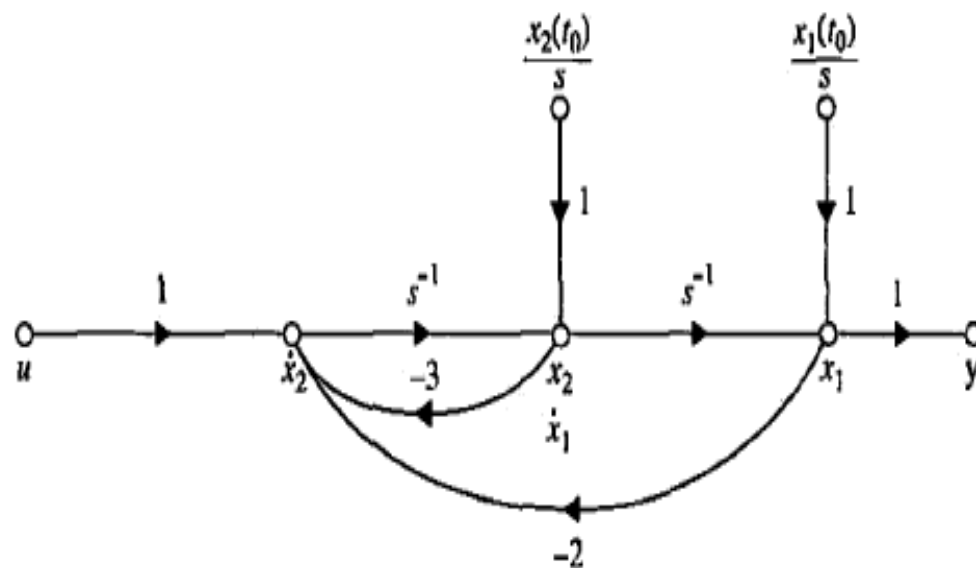
$$\frac{dx_1(t)}{dt} = x_2(t)$$

$$\frac{dx_2(t)}{dt} = -2x_1(t) - 3x_2(t) + r(t)$$



## State-Transition Equation Determined from the State Diagram

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1} \mathbf{x}(t_0) + (s\mathbf{I} - \mathbf{A})^{-1} [\mathbf{B}\mathbf{U}(s) + \mathbf{E}\mathbf{W}(s)] \quad t \geq t_0$$



$$\begin{aligned}
 X_1(s) &= \frac{s^{-1}(1+3s^{-1})}{\Delta} x_1(t_0) + \frac{s^{-2}}{\Delta} x_2(t_0) + \frac{s^{-2}}{\Delta} U(s) \\
 &\quad - \frac{2s^{-2}}{\Delta} x_1(t_0) + \frac{s^{-1}}{\Delta} x_2(t_0) + \frac{s^{-1}}{\Delta} U(s)
 \end{aligned}$$

$$\begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix} = \frac{1}{(s+1)(s+2)} \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix} \begin{bmatrix} x_1(t_0) \\ x_2(t_0) \end{bmatrix} + \frac{1}{(s+1)(s+2)} \begin{bmatrix} 1 \\ s \end{bmatrix} U(s)$$

# CHARACTERISTIC EQUATIONS, EIGENVALUES, AND EIGENVECTORS

## Characteristic Equation from State Equations

From the state-variable approach, we can write Eq. (10-126) as

$$\begin{aligned} \mathbf{G}_u(s) &= \mathbf{C} \frac{\text{adj}(s\mathbf{I} - \mathbf{A})}{(s\mathbf{I} - \mathbf{A})} \mathbf{B} + \mathbf{D} \\ &= \frac{\mathbf{C}[\text{adj}(s\mathbf{I} - \mathbf{A})]\mathbf{B} + |s\mathbf{I} - \mathbf{A}|\mathbf{D}}{|s\mathbf{I} - \mathbf{A}|} \end{aligned}$$

Setting the denominator of the transfer-function matrix  $\mathbf{G}_u(s)$  to zero, we get the characteristic equation

$$|s\mathbf{I} - \mathbf{A}| = 0$$

which is an alternative form of the characteristic equation but should lead to the same equation as in Eq. (10-146). *An important property of the characteristic equation is that, if the coefficients of  $\mathbf{A}$  are real, then the coefficients of  $|s\mathbf{I} - \mathbf{A}|$  are also real.*

# EXAMPLE ON MULTIVARIABLE SYSTEM

$$\frac{d^2 y_1(t)}{dt^2} + 4 \frac{dy_1(t)}{dt} - 3y_2(t) = u_1(t) + 2w(t)$$

$$\frac{dy_1(t)}{dt} + \frac{dy_2(t)}{dt} + y_1(t) + 2y_2(t) = u_2(t)$$

The state variables of the system are assigned as:

$$x_1(t) = y_1(t)$$

$$x_2(t) = \frac{dy_1(t)}{dt}$$

$$x_3(t) = y_2(t)$$

$$\begin{bmatrix} \frac{dx_1(t)}{dt} \\ \frac{dx_2(t)}{dt} \\ \frac{dx_3(t)}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -4 & 3 \\ -1 & -1 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} w(t)$$

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \mathbf{C}\mathbf{x}(t)$$

# Eigenvalues & Eigenvectors

## **Eigenvalues**

The roots of the characteristic equation are often referred to as the eigenvalues of the matrix  $A$ .



## Eigenvectors

Eigenvectors are useful in modern control methods, one of which is the similarity transformation, which will be discussed in a later section.

*Any nonzero vector  $\mathbf{p}_i$  that satisfies the matrix equation*

$$(\lambda_i \mathbf{I} - \mathbf{A})\mathbf{p}_i = \mathbf{0} \quad (10-155)$$

*where  $\lambda_i$ ,  $i = 1, 2, \dots, n$ , denotes the  $i$ th eigenvalue of  $\mathbf{A}$ , called the eigenvector of  $\mathbf{A}$  associated with the eigenvalue  $\lambda_i$ . If  $\mathbf{A}$  has distinct eigenvalues, the eigenvectors can be solved directly from Eq. (10-155).*

➤ **EXAMPLE 10-8-5** Consider that the state equation of Eq. (10-44) has the coefficient matrices

$$\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \mathbf{E} = \mathbf{0}$$

The characteristic equation of  $\mathbf{A}$  is

$$|s\mathbf{I} - \mathbf{A}| = s^2 - 1$$

The eigenvalues are  $\lambda_1 = 1$  and  $\lambda_2 = -1$ . Let the eigenvectors be written as

$$\mathbf{p}_1 = \begin{bmatrix} p_{11} \\ p_{21} \end{bmatrix} \quad \mathbf{p}_2 = \begin{bmatrix} p_{12} \\ p_{22} \end{bmatrix}$$

Substituting  $\lambda_1 = 1$  and  $\mathbf{p}_1$  into Eq. (10-155), we get

$$\begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} p_{11} \\ p_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

which leads to

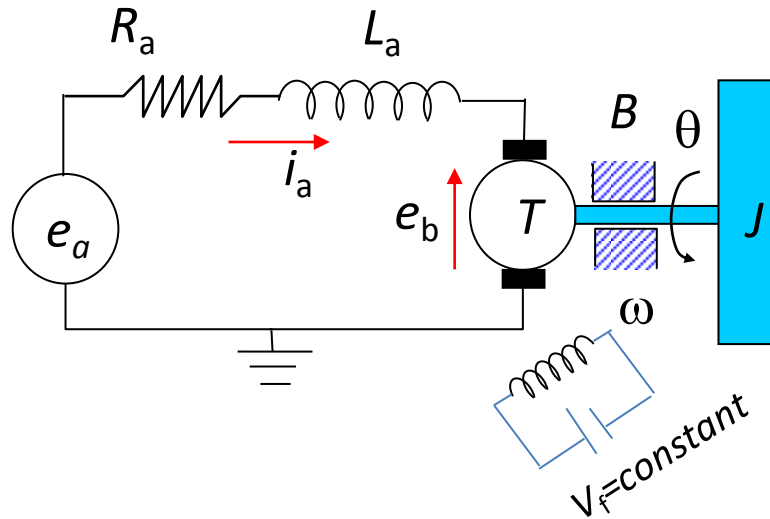
$$-2p_{12} + p_{22} = 0$$

The last equation has two unknowns, which means that one can be set arbitrarily. Let  $p_{12} = 1$ , then  $p_{22} = 2$ . The eigenvectors are

$$\mathbf{p}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \mathbf{p}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

## Example-2

- State space representation of armature Controlled D.C Motor.



- $e_a$  is armature voltage (i.e. input) and  $\theta$  is output.

$$e_a = R_a i_a + L_a \frac{di_a}{dt} + e_b$$

$$T = J\ddot{\theta} + B\dot{\theta}$$

## Example-2

$$T = K_t i_a$$

$$e_b = K_b \dot{\theta}$$

$$J\ddot{\theta} + B\dot{\theta} - K_t i_a = 0$$

$$L_a \frac{di_a}{dt} + R_a i_a + K_b \dot{\theta} = e_a$$

- Choosing  $\theta$ ,  $\dot{\theta}$  and  $i_a$  as state variables

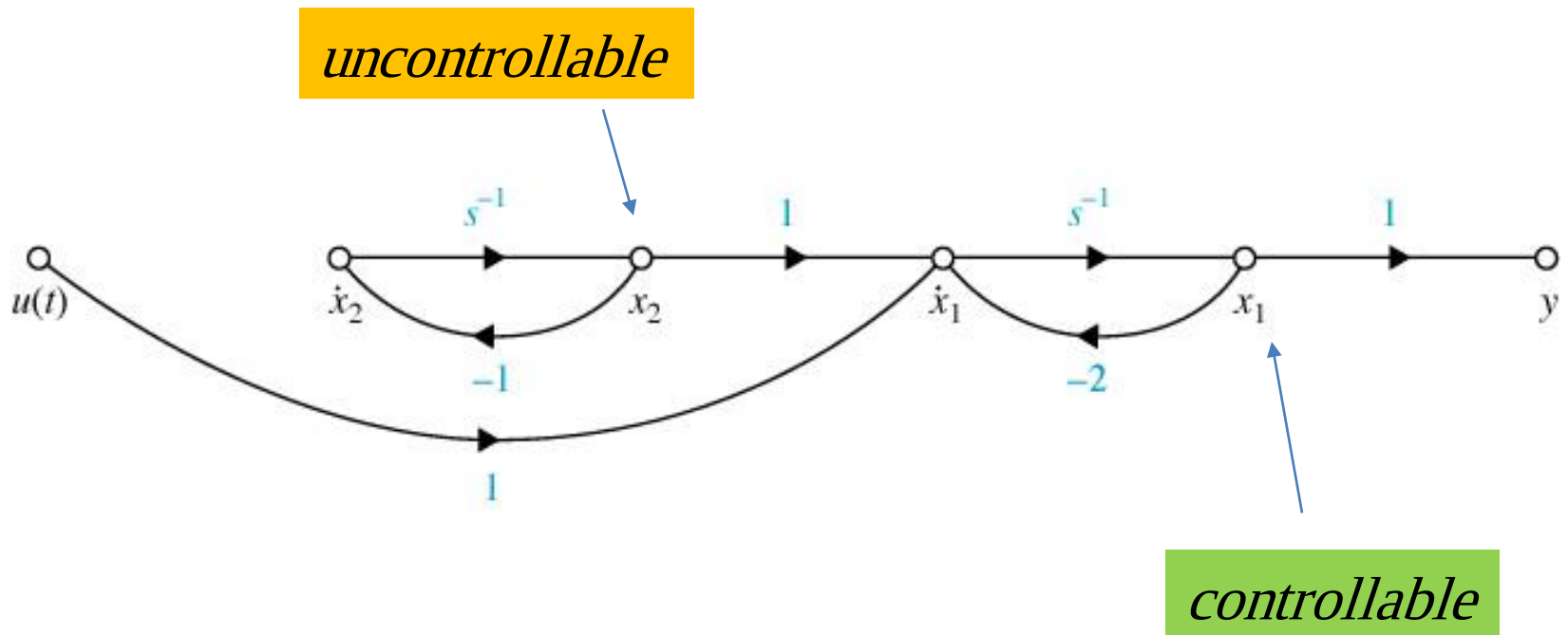
$$\frac{d}{dt} \begin{bmatrix} \theta \\ \dot{\theta} \\ i_a \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -\frac{B}{J} & \frac{K_t}{J} \\ 0 & -\frac{K_b}{L_a} & -\frac{R_a}{L_a} \end{bmatrix} \begin{bmatrix} \theta \\ \dot{\theta} \\ i_a \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{L_a} \end{bmatrix} e_a$$

- Since  $\theta$  is output of the system therefore output equation is given as

$$y(t) = [1 \quad 0 \quad 0] \begin{bmatrix} \theta \\ \dot{\theta} \\ i_a \end{bmatrix}$$

# State Controllability

- A system is completely controllable if there exists an unconstrained control  $u(t)$  that can transfer any initial state  $x(t_0)$  to any other desired location  $x(t)$  in a finite time,  $t_0 \leq t \leq T$ .



# State Controllability

- Controllability Matrix **CM**

$$CM = [B \quad AB \quad A^2B \quad \cdots \quad A^{n-1}B]$$

- System is said to be state controllable if

$$\text{rank}(CM) = n$$

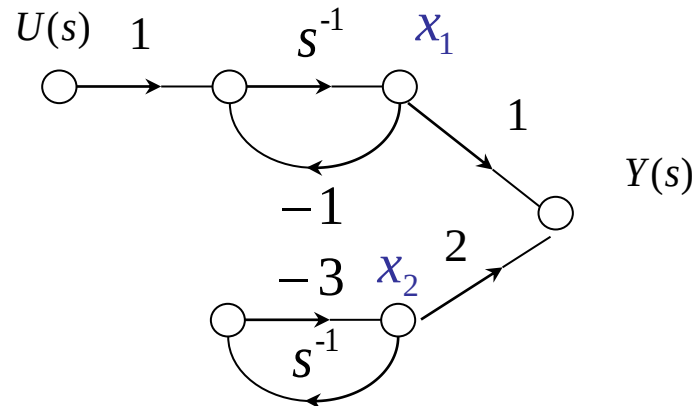
# State Controllability (Example)

- Consider the system given below

$$\dot{x} = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 2 \end{bmatrix} x$$

- State diagram of the system is





# State Controllability (Example)

- Controllability matrix  $CM$  is obtained as

$$CM = [B \quad AB]$$

$$B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad AB = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

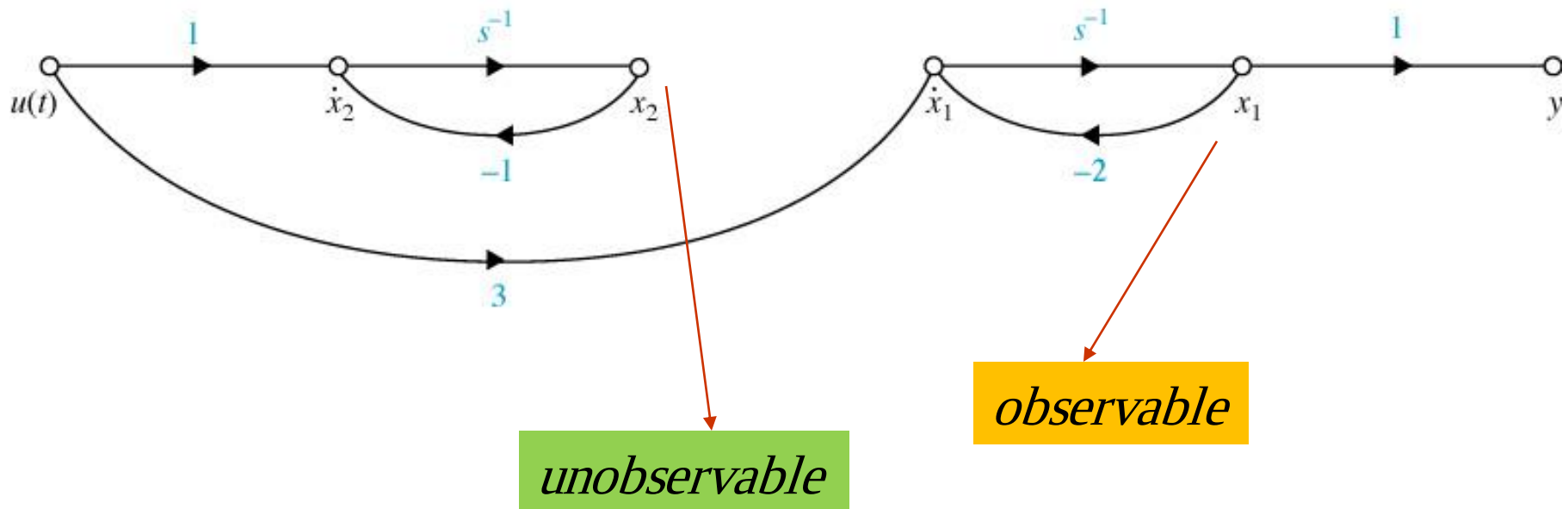
- Thus

$$CM = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

- Since  $\text{rank}(CM) \neq n$  therefore system is not completely state controllable.

# State Observability

- A system is completely observable if and only if there exists a finite time  $T$  such that the initial state  $x(0)$  can be determined from the observation history  $y(t)$  given the control  $u(t)$ ,  $0 \leq t \leq T$ .



# State Observability

- Observable Matrix (**OM**)

$$\text{Observability Matrix } OM = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

- The system is said to be completely state observable if

$$\text{rank}(OM) = n$$

# State Observability (Example)

- Consider the system given below

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 0 & 4 \end{bmatrix} x$$

- OM is obtained as

$$OM = \begin{bmatrix} C \\ CA \end{bmatrix}$$

- Where  $C = \begin{bmatrix} 0 & 4 \end{bmatrix}$

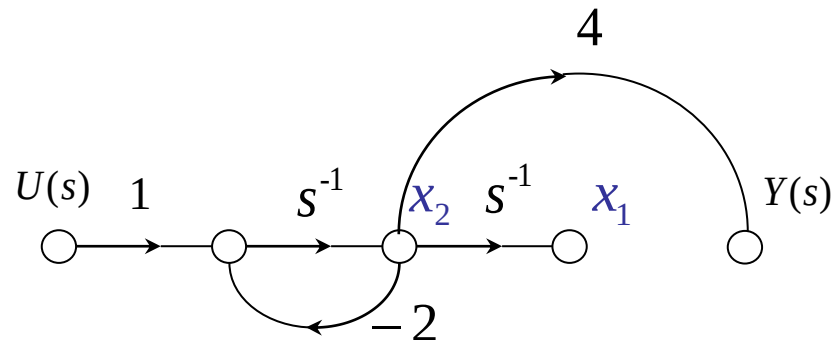
$$CA = \begin{bmatrix} 0 & 4 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 0 & -12 \end{bmatrix}$$

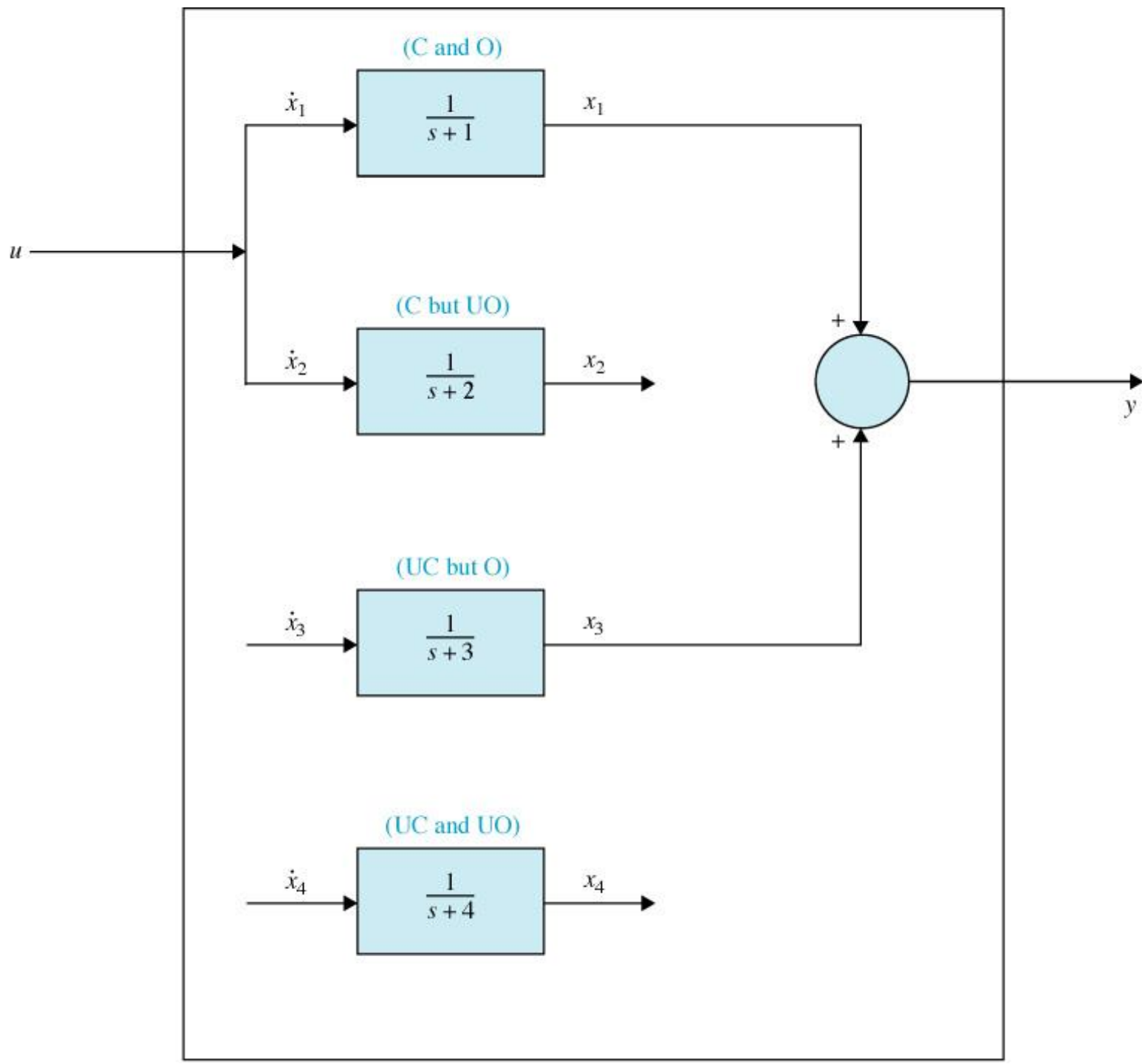
# State Observability (Example)

- Therefore OM is given as

$$OM = \begin{bmatrix} 0 & 4 \\ 0 & -12 \end{bmatrix}$$

- Since  $\text{rank}(OM) \neq n$  therefore system is not completely state observable.





# Output Controllability

- Output controllability describes the ability of an external input to move the output from any initial condition to any final condition in a finite time interval.
- Output controllability matrix (OCM) is given as

$$OCM = \begin{bmatrix} CB & CAB & CA^2B & \dots & CA^{n-1}B \end{bmatrix}$$

# THE DESIGN OF STATE VARIABLE FEEDBACK SYSTEMS

The time-domain method, expressed in terms of state variables, can also be utilized to design a suitable compensation scheme for a control system. Typically, we are interested in controlling the system with a control signal,  $u(t)$ , which is a function of several measurable state variables. Then we develop a state variable controller that operates on the information available in measured form.

State variable design is typically comprised of three steps. In the first step, we assume that all the state variables are measurable and utilize them in a **full-state feedback control law**. Full-state feedback is not usually practical because it is not possible (in general) to measure all the states. In practice, only certain states (or linear combinations thereof) are measured and provided as system outputs. The second step in state variable design is to construct an **observer** to estimate the states that are not directly sensed and available as outputs. Observers can either be full-state observers or reduced-order observers. Reduced-order observers account for the fact that certain states are already available as system outputs; hence they do not need to be estimated. The final step in the design process is to appropriately connect the observer to the full-state feedback control law. It is common to refer to the state-variable controller as a compensator. Additionally, it is possible to consider reference inputs to the state variable compensator to complete the design.



## CONTROLLABILITY:

Full-state feedback design commonly relies on **pole-placement techniques**. It is important to note that a system must be completely controllable and completely observable to allow the flexibility to place all the closed-loop system poles arbitrarily. The concepts of controllability and observability were introduced by Kalman in the 1960s.

A system is completely controllable if there exists an unconstrained control  $u(t)$  that can transfer any initial state  $x(t_0)$  to any other desired location  $x(t)$  in a finite time,  $t_0 \leq t \leq T$ .

For the system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

we can determine whether the system is controllable by examining the algebraic condition

$$\text{rank}[\mathbf{B} \quad \mathbf{AB} \quad \mathbf{A}^2\mathbf{B} \cdots \mathbf{A}^{n-1}\mathbf{B}] = n$$

The matrix  $\mathbf{A}$  is an  $n \times n$  matrix and  $\mathbf{B}$  is an  $n \times 1$  matrix. For multi input systems,  $\mathbf{B}$  can be  $n \times m$ , where  $m$  is the number of inputs.

For a single-input, single-output system, the controllability matrix  $\mathbf{P}_c$  is described in terms of  $\mathbf{A}$  and  $\mathbf{B}$  as

$$\mathbf{P}_c = [\mathbf{B} \quad \mathbf{AB} \quad \mathbf{A}^2\mathbf{B} \cdots \mathbf{A}^{n-1}\mathbf{B}]$$

which is  $n \times n$  matrix. Therefore, if the determinant of  $\mathbf{P}_c$  is nonzero, the system is controllable.

Example:

Consider the system

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u, \quad y = [1 \ 0 \ 0] \mathbf{x} + [0] u$$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad AB = \begin{bmatrix} 0 \\ 1 \\ -a_2 \end{bmatrix}, \quad A^2B = \begin{bmatrix} 1 \\ -a_2 \\ (a_2^2 - a_1) \end{bmatrix}$$

$$P_c = [B \quad AB \quad A^2B] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -a_2 \\ 1 & -a_2 & (a_2^2 - a_1) \end{bmatrix}$$

The determinant of  $P_c = 1$  and  $\neq 0$ , hence this system is controllable.

Example.

Consider a system represented by the two state equations

$$\dot{x}_1 = -2x_1 + u, \quad \dot{x}_2 = -3x_2 + d x_1$$

The output of the system is  $y=x_2$ . Determine the condition of controllability.

$$\dot{\mathbf{x}} = \begin{bmatrix} -2 & 0 \\ d & -3 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u, \quad y = \begin{bmatrix} 0 & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \end{bmatrix} u$$

$$\mathbf{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } \mathbf{AB} = \begin{bmatrix} -2 & 0 \\ d & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ d \end{bmatrix}$$

$$\mathbf{P}_c = \begin{bmatrix} 1 & -2 \\ 0 & d \end{bmatrix}$$

The determinant of  $\mathbf{P}_c$  is equal to  $d$ , which is nonzero only when  $d$  is nonzero.

The controllability matrix  $P_c$  can be constructed in Matlab by using **ctrb** command.

From two-mass system,

$$B = \begin{bmatrix} 0 \\ 0 \\ 50 \\ 0 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -500 & 500 & -20.5 & 0 \\ 20000 & -20000 & 0 & -8.2 \end{bmatrix}$$

```
clc
clear
A=[0 0 1 0;0 0 0 1;-500 500 -20.5
0;20000 -20000 0 -8.2];
B=[0;0;50;0];
Pc=ctrb(A,B)
rank_Pc=rank(Pc)
det_Pc=det(Pc)
```

Pc =

1.0e+007 \*

```
    0    0.0000   -0.0001   -0.0004
    0     0         0    0.1000
0.0000  -0.0001   -0.0004    0.0594
    0     0    0.1000   -2.8700
```

rank\_Pc =

4

det\_Pc =

-2.5000e+015

The system is  
controllable.

## OBSERVABILITY:

All the poles of the closed-loop system can be placed arbitrarily in the complex plane if and only if the system is **observable** and **controllable**. Observability refers to the ability to estimate a state variable.

A system is completely observable if and only if there exists a finite time  $T$  such that the initial state  $x(0)$  can be determined from the observation history  $y(t)$  given the control  $u(t)$ .

Consider the single-input, single-output system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u \quad \text{and} \quad y = \mathbf{C}\mathbf{x}$$

where  $\mathbf{C}$  is a  $1 \times n$  row vector, and  $\mathbf{x}$  is an  $n \times 1$  column vector. This system is completely observable when the determinant of the **observability matrix**  $\mathbf{P}_0$  is nonzero.

The observability matrix, which is an nxn matrix, is written as

$$P_O = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

Example:

Consider the previously given system

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix}, \quad C = [1 \quad 0 \quad 0]$$

$$CA = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \quad , \quad CA^2 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

Thus, we obtain

$$P_O = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The  $\det P_O = 1$ , and the system is completely observable. Note that determination of observability does not utilize the B and C matrices.

Example: Consider the system given by

$$\dot{\mathbf{x}} = \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} \mathbf{u} \quad \text{and} \quad \mathbf{y} = \begin{bmatrix} 1 & 1 \end{bmatrix} \mathbf{x}$$



We can check the system controllability and observability using the  $P_c$  and  $P_o$  matrices.

From the system definition, we obtain

$$B = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \text{and} \quad AB = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

Therefore, the controllability matrix is determined to be

$$P_c = [B \quad AB] = \begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}$$

$\det P_c = 0$  and  $\text{rank}(P_c) = 1$ . Thus, the system is not controllable.

From the system definition, we obtain

$$C = \begin{bmatrix} 1 & 1 \end{bmatrix} \quad \text{and} \quad CA = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

Therefore, the observability matrix is determined to be

$$P_o = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$\det P_o = 0$  and  $\text{rank}(P_o) = 1$ . Thus, the system is not observable.

If we look again at the state model, we note that

$$y = x_1 + x_2$$

However,

$$\dot{x}_1 + \dot{x}_2 = 2x_1 + (x_2 - x_1) + u - u = x_1 + x_2$$

Thus, the system state variables do not depend on  $u$ , and the system is not controllable. Similarly, the output  $(x_1+x_2)$  depends on  $x_1(0)$  plus  $x_2(0)$  and does not allow us to determine  $x_1(0)$  and  $x_2(0)$  independently. Consequently, the system is not observable.

The observability matrix  $P_O$  can be constructed in Matlab by using **obsv** command.

From two-mass system,

```
clc
clear
A=[2 0;-1 1];
C=[1 1];
Po=obsv(A,C)
rank_Po=rank(Po)
det_Po=det(Po)
```

Po =

```
1  1
1  1
```

rank\_Po =

1

det\_Po =

0

The system is not observable.

# Home Work

- Check the *state controllability*, *state observability* and *output controllability* of the following system

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C = [0 \quad 1]$$

# POLE-PLACEMENT DESIGN THROUGH STATE FEEDBACK

To investigate the condition required for arbitrary pole placement in an  $n$ th-order system, let us consider that the process is described by the following state equation:

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t) \quad (10-317)$$

where  $\mathbf{x}(t)$  is an  $n \times 1$  state vector, and  $u(t)$  is the scalar control. The state-feedback control is

$$u(t) = -\mathbf{K}\mathbf{x}(t) + r(t) \quad (10-318)$$

where  $\mathbf{K}$  is the  $1 \times n$  feedback matrix with constant-gain elements. By substituting Eq. (10-318) into Eq. (10-317), the closed-loop system is represented by the state equation

$$\frac{d\mathbf{x}(t)}{dt} = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x}(t) + \mathbf{B}r(t) \quad (10-319)$$

It will be shown in the following that if the pair  $[\mathbf{A}, \mathbf{B}]$  is completely controllable, then a matrix  $\mathbf{K}$  exists that can give an arbitrary set of eigenvalues of  $(\mathbf{A} - \mathbf{B}\mathbf{K})$ ; that is, the  $n$  roots of the characteristic equation

$$|s\mathbf{I} - \mathbf{A} + \mathbf{B}\mathbf{K}| = 0 \quad (10-320)$$

can be arbitrarily placed. To show that this is true, that if a system is completely controllable, it can always be represented in the controllable canonical form (CCF); that is, in Eq. (10-317),

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \quad (10-321)$$

The feedback gain matrix  $\mathbf{K}$  is expressed as

$$\mathbf{K} = [k_1 \quad k_2 \quad \cdots \quad k_n] \quad (10-322)$$

where  $k_1, k_2, \dots, k_n$  are real constants. Then,

$$\mathbf{A} - \mathbf{BK} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 - k_1 & -a_1 - k_2 & -a_2 - k_3 & \cdots & -a_{n-1} - k_n \end{bmatrix} \quad (10-323)$$

The eigenvalues of  $\mathbf{A} - \mathbf{BK}$  are then found from the characteristic equation

$$|s\mathbf{I} - (\mathbf{A} - \mathbf{BK})| = s^n + (a_{n-1} + k_n)s^{n-1} + (a_{n-2} + k_{n-1})s^{n-2} + \cdots + (a_0 + k_1) = 0 \quad (10-324)$$

Clearly, the eigenvalues can be arbitrarily assigned, because the feedback gains  $k_1, k_2, \dots, k_n$  are isolated in each coefficient of the characteristic equation. Intuitively, it makes sense that a system must be controllable for the poles to be placed arbitrarily. If one or more state variables are uncontrollable, then the poles associated with these state variables are also uncontrollable and cannot be moved as desired. The following example illustrates the design of a control system with state feedback.

# Pole Placement

ازاحة الاقطاب  
التغذية العكسية من كل حالة



# Pole Placement

A majority of the design techniques in modern control theory is based on the state-feedback configuration. That is, instead of using controllers with fixed configurations in the forward or feedback path, control is achieved by feeding back the state variables through real constant gains.

يتم استخدام التغذية العكسية من كل حالة  
وبكسب ثابت اي

$$U = -KX$$

# Definition of Pole Placement

- **(Pole Assignment, Pole Allocation)**

Placing the poles or eigenvalues of the closed-loop system at specified locations. وضع الاقطاب  
او القيم الذاتية للمنظومة المغلقة في ماق مختارة

- Poles can be arbitrarily placed if and only if the system is controllable. يجب ان تكون المنظومة  
مسيطر عليها

- Pole placement is easier if the system is given in controllable form.

# Pole Placement

In this section, we consider full-state variable feedback to achieve the desired pole locations of the closed-loop system.

The first step in the state variable design process requires us to assume that all the states are available for feedback—that is, we have access to the complete state  $\mathbf{x}(t)$  for all  $t$ . The system input  $u(t)$  is given by

$$u = -\mathbf{K}\mathbf{x}.$$

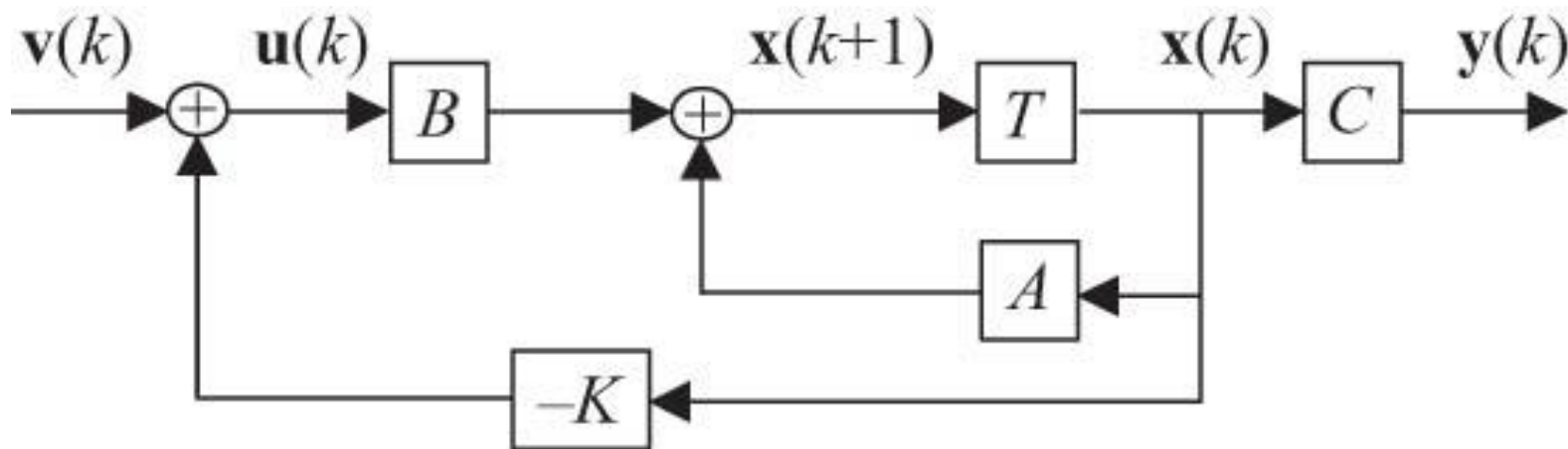
Determining the gain matrix  $\mathbf{K}$  is the objective of the full-state feedback design procedure. The beauty of the state variable design process is that the problem naturally separates into a full-state feedback component and an observer design component.

# Pole Placement

These two design procedures can occur independently, and in fact, the **separation principle** provides the proof that this approach is optimal. We will show later that the stability of the closed-loop system is guaranteed if the full-state feedback control law stabilizes the system (under the assumption of access to the complete state) and the observer is stable (the tracking error is asymptotically stable).

# State Feedback

State Feedback: involves the use of the state vector to compute the control action for specified system dynamics.



# State Feedback

$$X(k+1)=Ax(k)+Bu(k)$$

$$Y(k)=C x(k)$$

$$U(k)=-K x(k)+v(k)$$

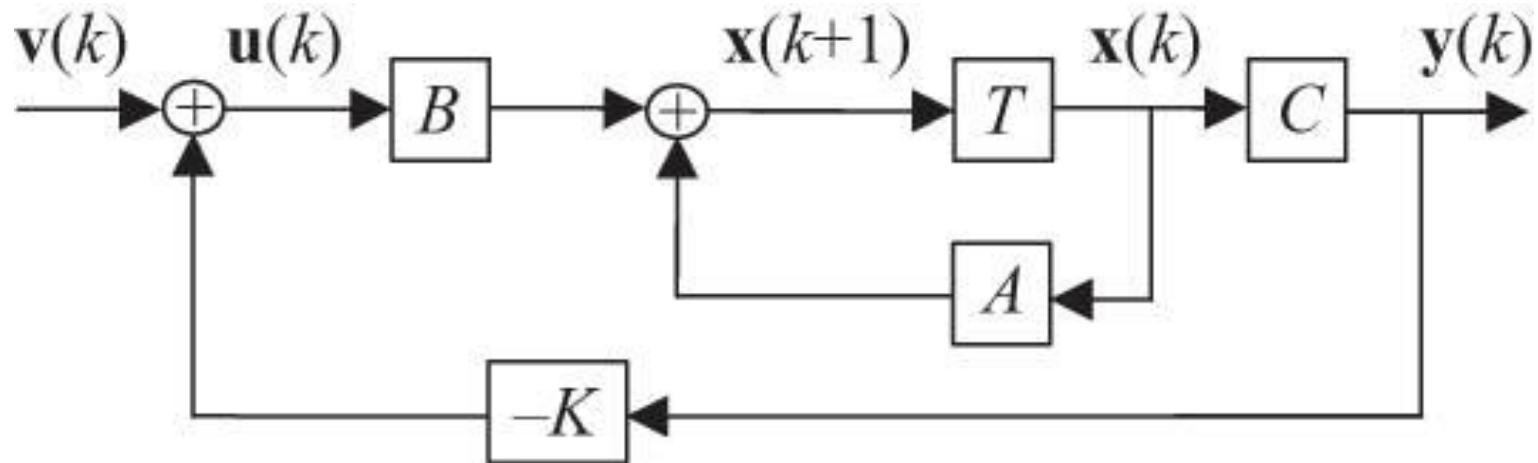
$$X(k+1)=[A-BK]x(k)+B v(k)$$

Let  $A_d = A-BK$  The closed-loop system state space:  $X(k+1)= A_d x(k)+B v(k)$

$$Y(k)=C x(k)$$

# Output Feedback

Output Feedback



# Output Feedback

$$U(k) = -K y(k) + v(k) = -K C x(k) + v(k)$$

$$X(k+1) = [A - BKC]x(k) + B v(k)$$



# Pole Placement

**Theorem 9.1: State Feedback.** If the pair  $(A,B)$  is controllable, then there exists a feedback gain matrix  $K$  that arbitrarily assigns the system poles to any set  $[\lambda_i \ i=1,\dots,n]$ . Furthermore, if the pair  $(A,B)$  is stabilizable, then the controllable modes can all be arbitrarily assigned.

# Procedure 9.1: Pole Placement by Equating Coefficients خطوات العمل للتصميم

1. Evaluate the desired characteristic polynomial from the specified eigenvalues using the expression

$$\Delta_c^d(\lambda) = \prod_{i=1}^n (\lambda - \lambda_i) \quad (9.10)$$

2. Evaluate the closed-loop characteristic polynomial using the expression

$$\det\{\lambda I_n - (A - BK)\} \quad (9.11)$$

3. Equate the coefficients of the two polynomials to obtain  $n$  equations to be solved for the entries of the matrix  $K$ .

# Example 9.1

Assign the eigenvalues  $[0.3 \pm j0.21]$  to the pair

$$A = \begin{bmatrix} 0 & 1 \\ 3 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$(\lambda - 0.3 - j0.2)(\lambda - 0.3 + j0.2) = \lambda^2 - 0.6\lambda + 0.13$$

The closed loop matrix is

$$A - bk^T = \begin{bmatrix} 0 & 1 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} [k_1 \quad k_2] = \begin{bmatrix} 0 & 1 \\ 3 - k_1 & 4 - k_2 \end{bmatrix}$$

The closed loop characteristic polynomial is

$$\text{Det}\{\lambda I - (A - bk^T)\} = \lambda^2 - (4 - k_2)\lambda - (3 - k_1)$$

$$4 - k_2 = 0.6 \rightarrow k_2 = 3.4 \quad 3 - k_1 = 0.13 \rightarrow k_1 = 3.13$$

# MATLAB

```
>> A = [0, 1; 3, 4];
```

```
>> B = [0; 1];
```

```
>> poles = [0.3 + j*.2, 0.3 - j*0.2];
```

```
>> K = place(A, B, poles)
```

```
K = 3.1300 3.4000
```

# Pole Placement by Transformation to Controllable Form

Any controllable single-input-single-output (SISO) system can be transformed into controllable form using the transformation

$$T_c = \mathcal{C}\mathcal{C}^{-1} = [\mathbf{b} \mid A\mathbf{b} \mid \dots \mid A^{n-1}\mathbf{b}] \begin{bmatrix} a_1 & a_2 & \dots & a_{n-1} & 1 \\ a_2 & a_3 & \dots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1} & 1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{bmatrix} \quad (9.12)$$

$$T_c^{-1} = \mathcal{C}\mathcal{C}^{-1} = \begin{bmatrix} 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & t_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 1 & \dots & t_{n-2,n} \\ 0 & 1 & t_{2n} & \dots & t_{n-1,n} \\ 1 & t_{2n} & t_{3n} & \dots & t_{nn} \end{bmatrix} [\mathbf{b} \mid A\mathbf{b} \mid \dots \mid A^{n-1}\mathbf{b}]^{-1} \quad (9.13)$$

where  $\mathcal{C}$  is the controllability matrix, the subscript  $c$  denotes the controllable form, and the terms  $t_{jn}$ ,  $j = 2, \dots, n$ , are given by

$$t_{2n} = -a_{n-1}$$

$$t_{j+1,n} = -\sum_{t=0}^{j-1} a_{n-t-1} t_{j-t,n}, \quad j = 2, \dots, n-1$$

The state feedback for a system in controllable form is

$$u = -\mathbf{k}_c^T \mathbf{x}_c = -\mathbf{k}_c^T (T_c^{-1} \mathbf{x}) \quad (9.14)$$

$$\mathbf{k}^T = [a_0^d - a_0 \quad a_1^d - a_1 \quad \dots \quad a_{n-1}^d - a_{n-1}] T_c^{-1} \quad (9.15)$$

## Procedure 9.2

1. Obtain the characteristic polynomial of the pair  $(A, B)$  using the Leverrier algorithm described in Section 7.4.1.
2. Obtain the transformation matrix  $T^{-1}$  using the coefficients of the polynomial from step 1.
3. Obtain the desired characteristic polynomial coefficients from the given eigenvalues using (9.10).
4. Compute the state feedback matrix using (9.15).

## Example 9.2

Design a feedback controller for the pair

$$A = \begin{bmatrix} 0.1 & 0 & 0.1 \\ 0 & 0.5 & 0.2 \\ 0.2 & 0 & 0.4 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 0.01 \\ 0 \\ 0.005 \end{bmatrix}$$

to obtain the eigenvalues  $\{0.1, 0.4 \pm j0.4\}$ .

# Solution

The characteristic polynomial of the state matrix is

$$\lambda^3 - \lambda^2 + 0.27\lambda - 0.01 \quad \text{i.e.,} \quad a_2 = -1, a_1 = 0.27, a_0 = -0.01$$

The transformation matrix  $T_c^{-1}$  is

$$T_c^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0.73 \end{bmatrix} \times 10^3 \begin{bmatrix} 10 & 1.5 & 0.6 \\ 0 & 1 & 1.3 \\ 5 & 4 & 1.9 \end{bmatrix}^{-1} = 10^3 \begin{bmatrix} 0.1923 & 1.25 & -0.3846 \\ -0.0577 & 0.625 & 0.1154 \\ 0.0173 & 0.3125 & 0.1654 \end{bmatrix}$$

The desired characteristic polynomial is

$$\lambda^3 - 0.9\lambda^2 + 0.4\lambda - 0.032 \quad \text{i.e.,} \quad a_2^d = -0.9, a_1^d = 0.4, a_0^d = -0.032$$

Hence, we have the feedback gain vector

$$\begin{aligned} \mathbf{k}^T &= [a_0^d - a_0 \quad a_1^d - a_1 \quad a_2^d - a_2] T_c^{-1} \\ &= [-0.032 + 0.01 \quad 0.4 - 0.27 \quad -0.9 + 1] \times 10^3 \begin{bmatrix} 0.1923 & 1.25 & -0.3846 \\ -0.0577 & 0.625 & 0.1154 \\ 0.0173 & 0.3125 & 0.1654 \end{bmatrix} \\ &= [-10 \quad 85 \quad 40] \end{aligned}$$



# Servo Problem

The schemes shown in [Figures 9.1 and 9.2](#) are regulators that drive the system state to zero starting from any initial condition capable of rejecting impulse disturbances. In practice, it is often necessary to track a constant reference input  $r$  with zero steady-state error.

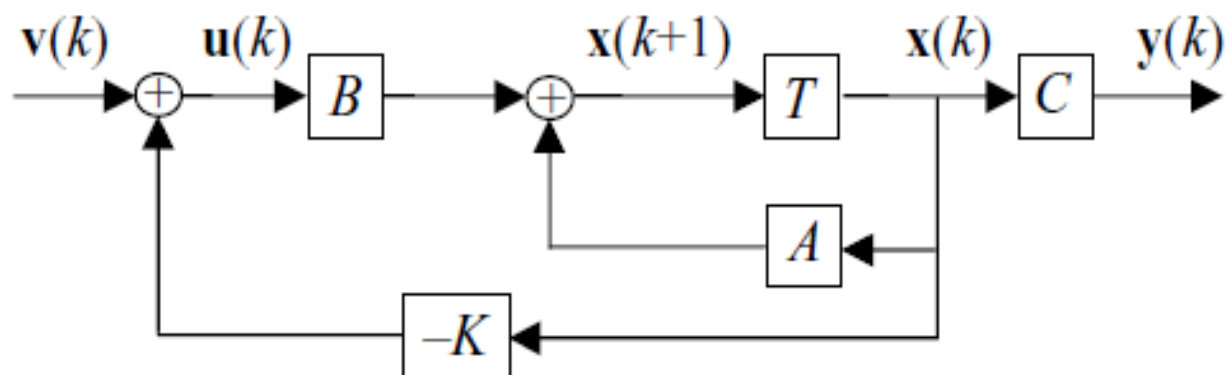


**FIGURE 9.1**

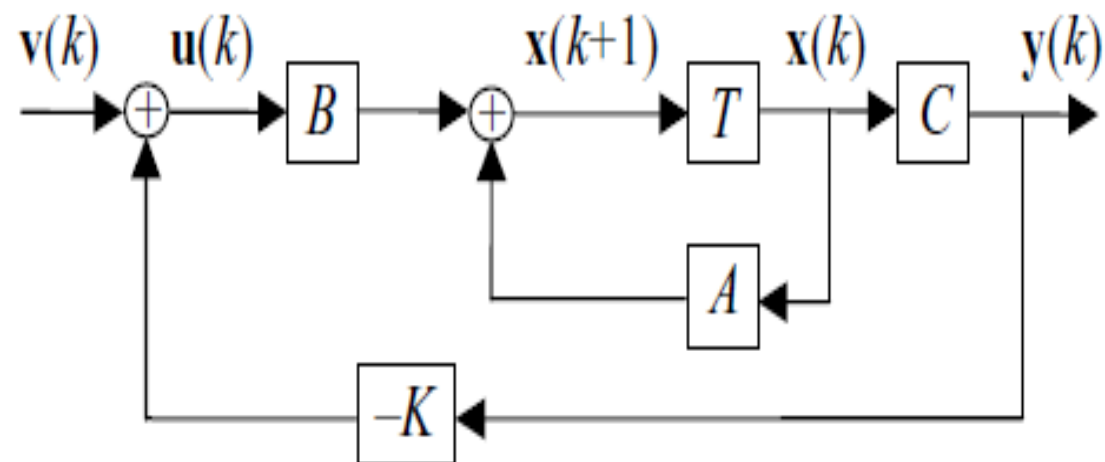
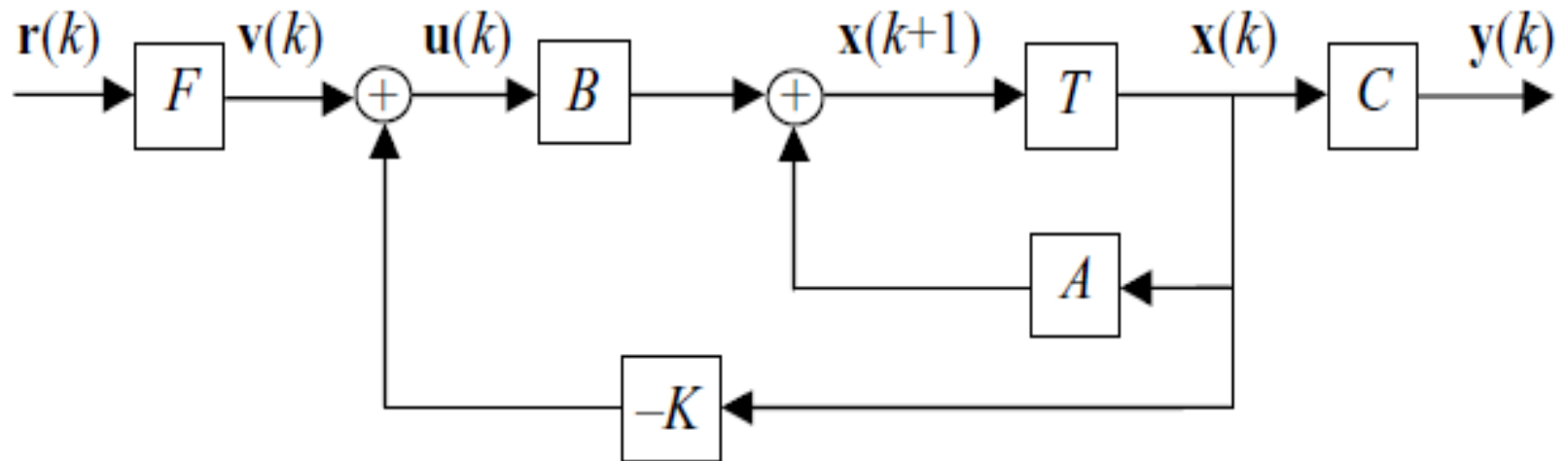


**FIGURE 9.2**

# Servo Problem



# Servo Problem

The reference input of (9.2) becomes  $\mathbf{v}(k) = F \mathbf{r}(k)$ , and the control law is chosen as

$$\mathbf{u}(k) = -K \mathbf{x}(k) + F \mathbf{r}(k)$$

with  $\mathbf{r}(k)$  the reference input to be tracked. The corresponding closed-loop system equations are

$$\mathbf{x}(k+1) = A_{cl} \mathbf{x}(k) + B F \mathbf{r}(k)$$

$$y(k) = C \mathbf{x}(k)$$

where the closed-loop state matrix is

$$A_{cl} = A - BK$$

The  $z$ -transform of the corresponding output is given by

$$\mathbf{Y}(z) = C [zI_n - A_{cl}]^{-1} BF \mathbf{R}(z)$$

The steady-state tracking error for a unit step input is given by

$$\begin{aligned} \lim_{z \rightarrow 1} (z-1) \{ \mathbf{Y}(z) - \mathbf{R}(z) \} &= \lim_{z \rightarrow 1} \{ C [zI_n - A_{cl}]^{-1} BF - I \} \\ &= C [I_n - A_{cl}]^{-1} BF - I \end{aligned}$$

# Servo Problem

For zero steady-state error, we require the condition

$$C[I_n - A_{cl}]^{-1}BF = I_n$$

If the system is square ( $m = l$ ) and  $A_{cl}$  is stable (no unity eigenvalues), we solve for the reference gain

$$F = [C(I_n - A_{cl})^{-1}B]^{-1}$$

## Example 9.5

Design a state–space controller for the discretized state–space model of the DC motor speed control system described in Example 6.8 (with  $T = 0.02$ ) to obtain

- (1) zero steady-state error due to a unit step,
- (2) a damping ratio of 0.7, and
- (3) a settling time of about 1 s.

$$G(s) = \frac{1}{(s+1)(s+10)}$$

# Solution

The discretized transfer function of the system with digital-to-analog converter (DAC) and analog-to-digital converter (ADC) is

$$G_{ZAS}(z) = (1 - z^{-1}) \mathcal{Z} \left\{ \frac{G(s)}{s} \right\} = 1.8604 \times 10^{-4} \frac{z + 0.9293}{(z - 0.8187)(z - 0.9802)}$$

The corresponding state–space model, computed with MATLAB, is

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 1.799 & -0.8025 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0.01563 \\ 0 \end{bmatrix} u(k)$$

$$y(k) = [0.01191 \quad 0.01107] \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$



# Solution

The desired eigenvalues of the closed-loop system are selected as  $\{0.9 \pm j0.09\}$  (see Example 6.8). This yields the feedback gain vector

$$K = [-0.068517 \quad 0.997197]$$

and the closed-loop state matrix

$$A_d = \begin{bmatrix} 1.8 & -0.8181 \\ 1 & 0 \end{bmatrix}$$

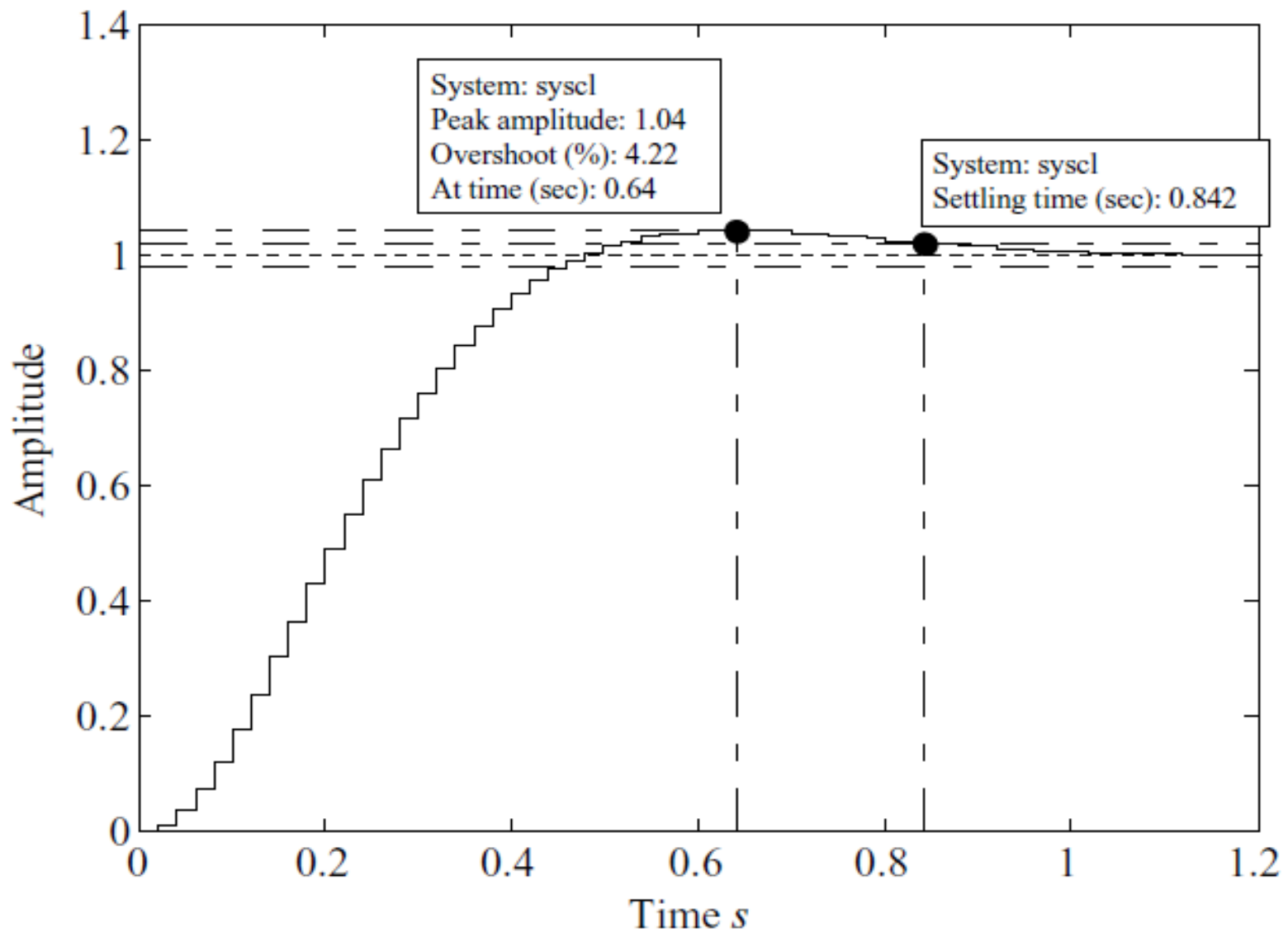
# Solution

The feedforward gain is

$$F = \left[ C(I_n - A_{cl})^{-1} B \right]^{-1}$$
$$= \left[ \begin{bmatrix} 0.01191 & 0.01107 \end{bmatrix} \left( \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1.8 & -0.8181 \\ 1 & 0 \end{bmatrix} \right)^{-1} \begin{bmatrix} 0.01563 \\ 0 \end{bmatrix} \right]^{-1} = 50.42666$$

The response of the system to a step reference input  $r$  is shown in [Figure 9.8](#). The system has a settling time of about 0.84 s and percentage overshoot of about 4% with a peak time of about 1 s. All design specifications are met.

# Solution



# Integral Control

The control law (9.21) is equivalent to a **feedforward action** determined by  $F$  to yield zero steady-state error for a constant reference input  $r$ . Because the forward action does not include any form of feedback, this approach is not robust to modeling uncertainties. Thus, modeling errors (which always occur in practice) will result in nonzero steady-state error. To eliminate such errors, we introduce the **integral control** shown in Figure 9.9, with a new state added for each control error integrated.

# Integral Control

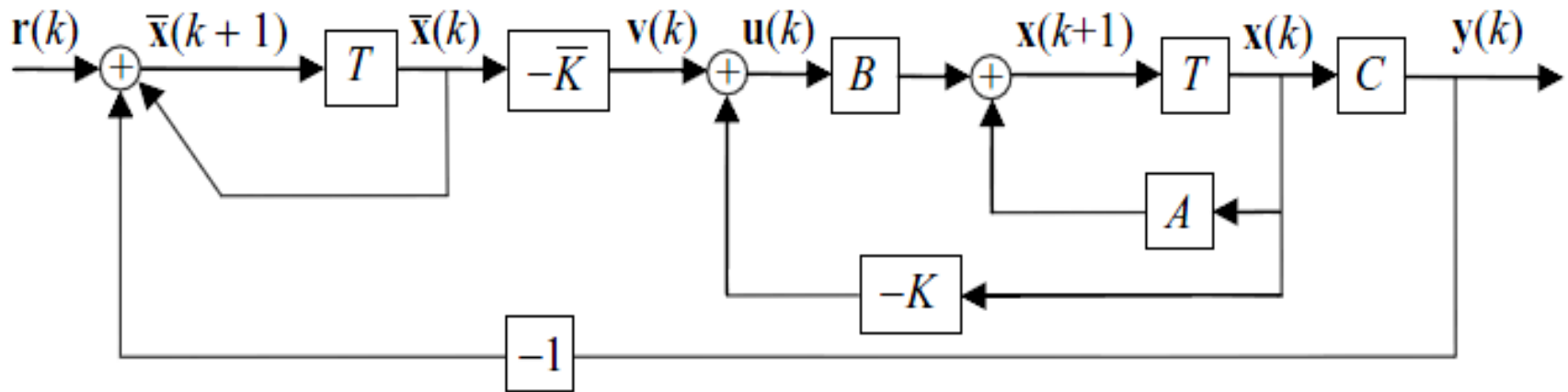


**FIGURE 9.9**

Control scheme with integral control.

The resulting state–space equations are

$$\begin{aligned}\mathbf{x}(k+1) &= A\mathbf{x}(k) + B\mathbf{u}(k) \\ \bar{\mathbf{x}}(k+1) &= \bar{\mathbf{x}}(k) + \mathbf{r}(k) - \mathbf{y}(k) \\ \mathbf{y}(k) &= C\mathbf{x}(k) \\ \mathbf{u}(k) &= -K\mathbf{x}(k) - \bar{K}\bar{\mathbf{x}}(k)\end{aligned}\tag{9.25}$$

where  $\bar{\mathbf{x}}$  is  $l \times 1$ . The state-space equations can be combined and rewritten in terms of an augmented state vector  $\tilde{\mathbf{x}}(k) = [\mathbf{x}(k) \quad \bar{\mathbf{x}}(k)]^T$  as

$$\begin{aligned}\begin{bmatrix} \mathbf{x}(k+1) \\ \bar{\mathbf{x}}(k+1) \end{bmatrix} &= \begin{bmatrix} A & \mathbf{0} \\ -C & I_l \end{bmatrix} \begin{bmatrix} \mathbf{x}(k) \\ \bar{\mathbf{x}}(k) \end{bmatrix} - \begin{bmatrix} B \\ \mathbf{0} \end{bmatrix} \begin{bmatrix} K & \bar{K} \end{bmatrix} \begin{bmatrix} \mathbf{x}(k) \\ \bar{\mathbf{x}}(k) \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ I_l \end{bmatrix} \mathbf{r}(k) \\ \mathbf{y}(k) &= \begin{bmatrix} C & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{x}(k+1) \\ \bar{\mathbf{x}}(k+1) \end{bmatrix}\end{aligned}$$

# Integral Control

That is,

$$\tilde{\mathbf{x}}(k+1) = (\tilde{A} - \tilde{B}\tilde{K})\tilde{\mathbf{x}}(k) + \begin{bmatrix} \mathbf{0} \\ I_l \end{bmatrix} \mathbf{r}(k) \quad (9.26)$$

$$\mathbf{y}(k) = [C \quad \mathbf{0}]\tilde{\mathbf{x}}(k)$$

where

$$\tilde{A} = \begin{bmatrix} A & \mathbf{0} \\ -C & I_l \end{bmatrix} \quad \tilde{B} = \begin{bmatrix} B \\ \mathbf{0} \end{bmatrix} \quad \tilde{K} = [K \quad \bar{K}] \quad (9.27)$$

The eigenvalues of the closed-loop system state matrix  $A_{cl} = (\tilde{A} - \tilde{B}\tilde{K})$  can be arbitrarily assigned by computing the gain matrix  $K$  using any of the procedures for the regulator problem as described in [Sections 9.1](#) and [9.2](#).

## Example 9.6

Solve the design problem presented in **Example 9.5** using integral control.



# Solution

The state–space matrices of the system are

$$A = \begin{bmatrix} 1.799 & -0.8025 \\ 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0.01563 \\ 0 \end{bmatrix}$$
$$C = [0.01191 \quad 0.01107]$$

Adding integral control, we obtain

$$\tilde{A} = \begin{bmatrix} A & \mathbf{0} \\ -C & 1 \end{bmatrix} = \begin{bmatrix} 1.799 & -0.8025 & 0 \\ 1 & 0 & 0 \\ -0.01191 & -0.01107 & 1 \end{bmatrix} \quad \tilde{B} = \begin{bmatrix} B \\ 0 \end{bmatrix} = \begin{bmatrix} 0.01563 \\ 0 \\ 0 \end{bmatrix}$$

In [Example 9.5](#), the eigenvalues were selected as  $\{0.9 \pm j0.09\}$ . Using integral control increases the order of the system by one, and an additional eigenvalue must be selected. The desired eigenvalues are selected as  $\{0.9 \pm j0.09, 0.2\}$ , and the additional eigenvalue at 0.2 is chosen for its negligible effect on the overall dynamics. This yields the feedback gain vector

# Solution

$$\tilde{K} = [51.1315 \quad -40.4431 \quad -40.3413]$$

The closed-loop system state matrix is

$$A_{cl} = \begin{bmatrix} 1 & -0.1706 & 0.6303 \\ 1 & 0 & 0 \\ -0.0119 & -0.0111 & 1 \end{bmatrix}$$

The response of the system to a unit step reference signal  $r$  is shown in [Figure 9.10](#). The figure shows that the control specifications are satisfied. The settling time of 0.87 is well below the specified value of 1 s, and the percentage overshoot is about 4.2%, which is less than the value corresponding to  $\zeta = 0.7$  for the dominant pair.

# Solution

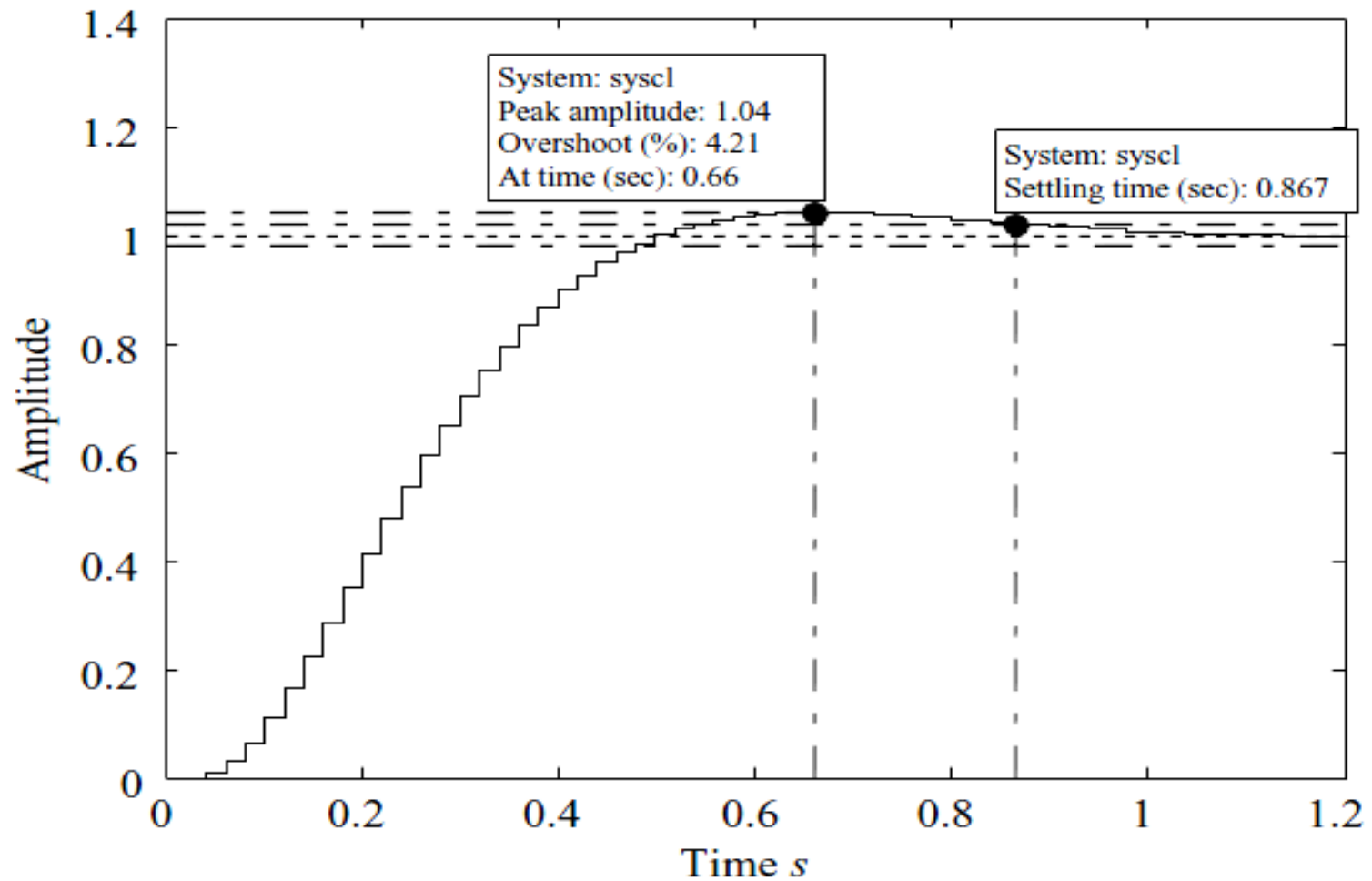


FIGURE 9.10

# State Estimation

In most applications, measuring the entire state vector is impossible or prohibitively expensive. To implement state feedback control, an estimate  $\hat{\mathbf{x}}(k)$  of the state vector can be used. The state vector can be estimated from the input and output measurements by using a **state estimator** or **observer**.

# Estimators

In the previous section, we have designed a state feedback controller, assuming that all states are measured. We also know, however, that not all states can be measured, for a detailed discussion on this topic. Does this mean that the state feedback controllers cannot be implemented? The answer is that we can obtain a reasonably equivalent state feedback controller through *estimated* state measurements. A *state estimator*, or simply an estimator, is a mathematical construct that helps estimate the states of a system.

# Full-Order Observer

To estimate all the states of the system, one could in theory use a system with the same state equation as the plant to be observed. In other words, one could use the open-loop system

$$\hat{\mathbf{x}}(k+1) = A\hat{\mathbf{x}}(k) + B\mathbf{u}(k)$$

However, this open-loop estimator assumes perfect knowledge of the system dynamics and lacks the feedback needed to correct the errors that are inevitable in any implementation. The limitations of this observer become obvious on examining its error dynamics. We obtain the error dynamics by subtracting the open-loop observer dynamics from the system dynamics (9.1).

$$\tilde{\mathbf{x}}(k+1) = A\tilde{\mathbf{x}}(k) \quad \text{Where} \quad \tilde{\mathbf{x}} = \mathbf{x} - \hat{\mathbf{x}}$$

# Full-Order Observer

The error dynamics are determined by the state matrix of the system and cannot be chosen arbitrarily. For an unstable system, the observer will be unstable and cannot track the state of the system.

A practical alternative is to feed back the difference between the measured and the estimated output of the system, as shown in [Figure 9.12](#). This yields to the following observer:

# Full-Order Observer

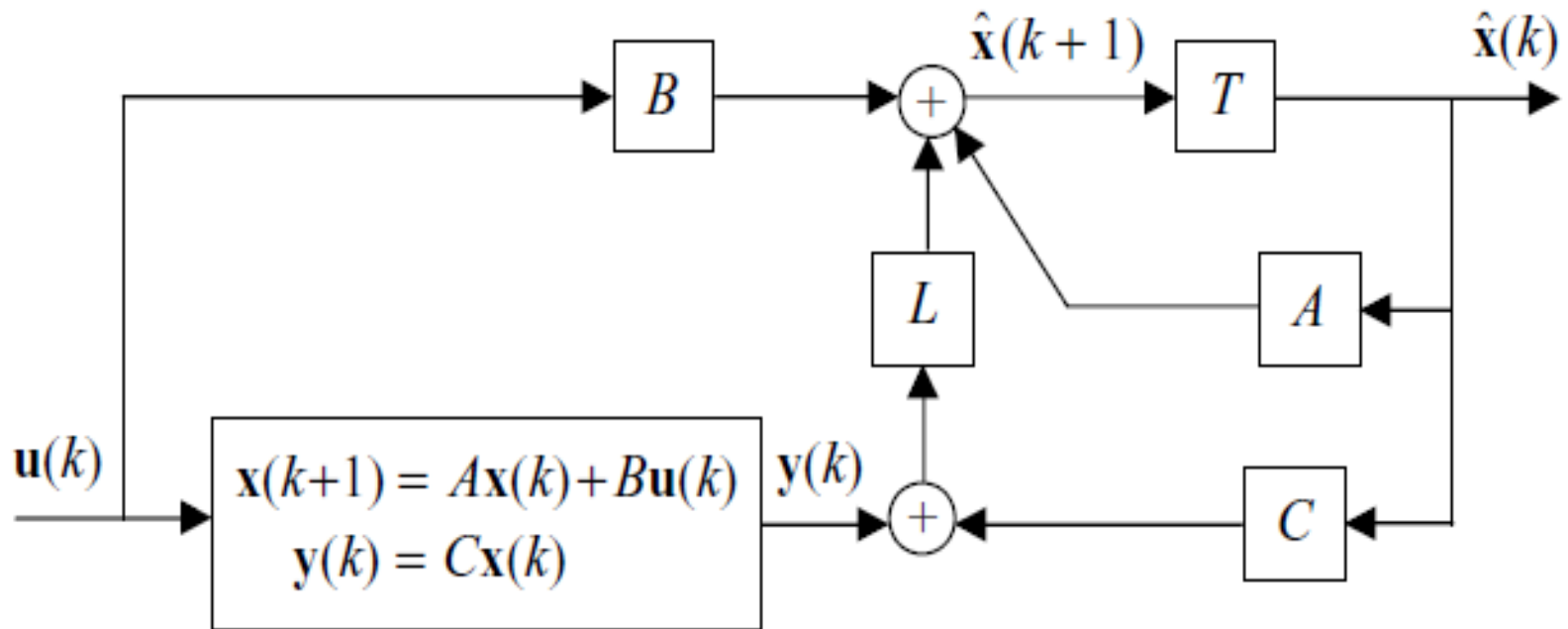


FIGURE 9.12



# Full-Order Observer

$$\hat{\mathbf{x}}(k+1) = A\hat{\mathbf{x}}(k) + B\mathbf{u}(k) + L[\mathbf{y}(k) - C\hat{\mathbf{x}}(k)]$$

Subtracting the observer state equation from the system dynamics yields the estimation error dynamics

$$\tilde{\mathbf{x}}(k+1) = (A - LC)\tilde{\mathbf{x}}(k)$$

The error dynamics are governed by the eigenvalues of the observer matrix

•  $A_0 = A - LC$ . We transpose the matrix to obtain

$$A_0^T = A^T - C^T L^T$$

# Full-Order Observer

**Theorem 9.2: State Estimation.** If the pair  $(A, C)$  is observable, then there exists a feedback gain matrix  $L$  that arbitrarily assigns the observer poles to any set  $\{l_i, i = 1, \dots, n\}$ . Furthermore, if the pair  $(A, C)$  is detectable, then the observable modes can all be arbitrarily assigned.

## Example 9.8

Determine the observer gain matrix  $L$  for the discretized state–space model of the armature controlled DC motor described in Example 7.15 with the observer eigenvalues selected as  $\{0.1, 0.2 \pm j0.2\}$ .

# Solution

Recall that the system matrices are

$$A = \begin{bmatrix} 1.0 & 0.1 & 0.0 \\ 0.0 & 0.9995 & 0.0095 \\ 0.0 & -0.0947 & 0.8954 \end{bmatrix} \quad B = \begin{bmatrix} 1.622 \times 10^{-6} \\ 4.821 \times 10^{-4} \\ 9.468 \times 10^{-2} \end{bmatrix} \quad C = [1 \quad 0 \quad 0]$$

The MATLAB command **place** gives the observer gain

$$L = \begin{bmatrix} 2.3949 \\ 18.6734 \\ 4.3621 \end{bmatrix}$$

# HW 7

9.7

9.11

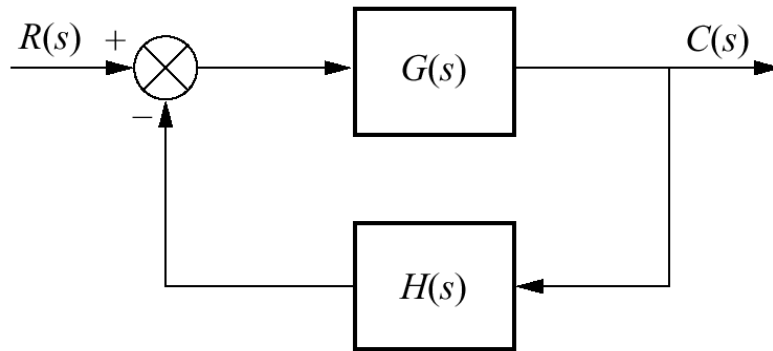
9.15

# قاعدة نايكوست للاستقرارية والاستجابة الترددية

## Frequency Response Method

### Lecture 13

# Knowledge Before Studying Nyquist Criterion



$$T(s) = \frac{G(s)}{1 + G(s)H(s)}$$

**unstable if there is any pole on RHP (right half plane)**

$$G(s) = \frac{N_G(s)}{D_G(s)}$$

$$H(s) = \frac{N_H(s)}{D_H(s)}$$

**Open-loop system:**

$$G(s)H(s) = \frac{N_G(s) N_H(s)}{D_G(s) D_H(s)}$$

**Characteristic equation:**

$$1 + G(s)H(s) = 1 + \frac{N_G N_H}{D_G D_H} = \frac{D_G D_H + N_G N_H}{D_G D_H}$$

**poles of  $G(s)H(s)$  and  $1+G(s)H(s)$  are the same**

**Closed-loop system:**

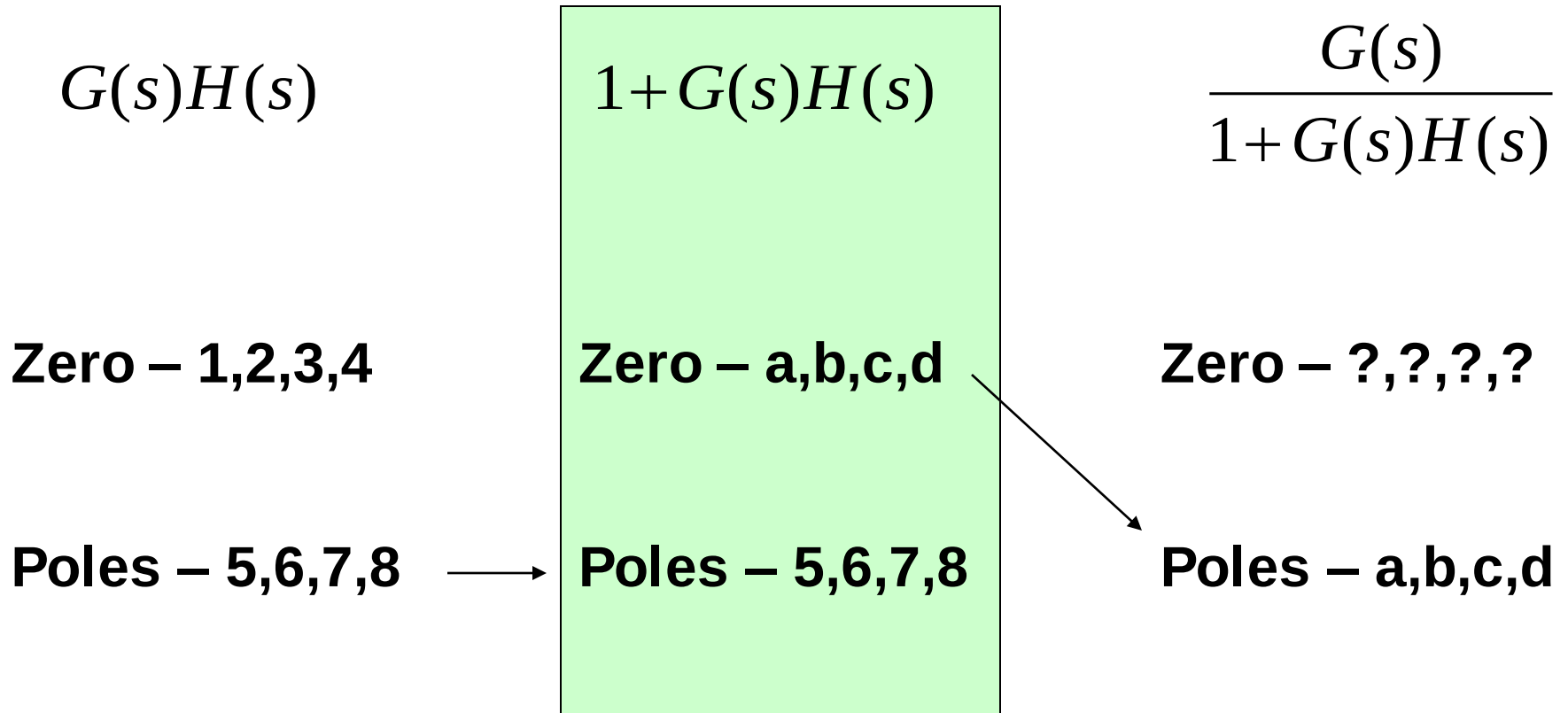
$$T(s) = \frac{G(s)}{1 + G(s)H(s)} = \frac{N_G(s) D_H(s)}{D_G(s) D_H(s) + N_G(s) N_H(s)}$$

**zero of  $1+G(s)H(s)$  is pole of  $T(s)$**



$$G(s)H(s) = \frac{(s-1)(s-2)(s-3)(s-4)}{(s-5)(s-6)(s-7)(s-8)}$$


---



To know stability, we have to know a,b,c,d

# Stability from Nyquist plot

## قاعدة نايكوست للاستقرارية

**From a Nyquist plot, we can tell a number of closed-loop poles on the right half plane.**

- If there is any closed-loop pole on the right half plane, the system goes unstable.**
- If there is no closed-loop pole on the right half plane, the system is stable.**

# Nyquist Criterion

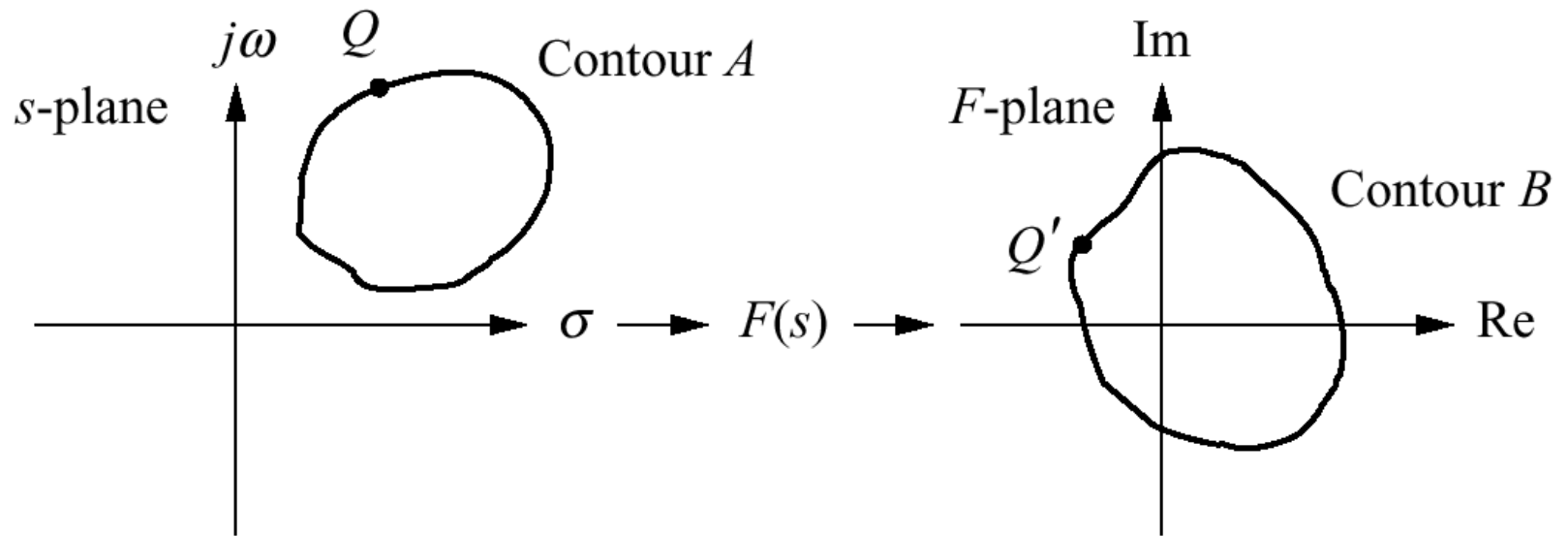
**Nyquist plot is a plot used to verify stability of the system.**

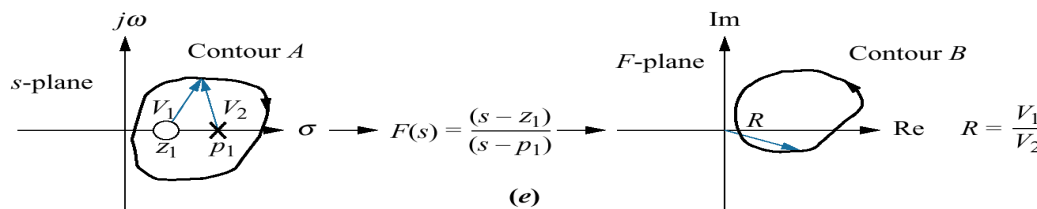
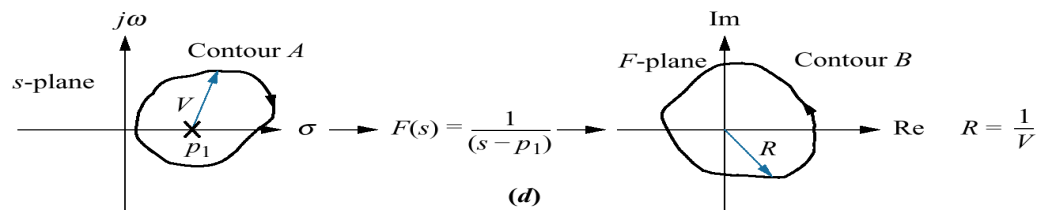
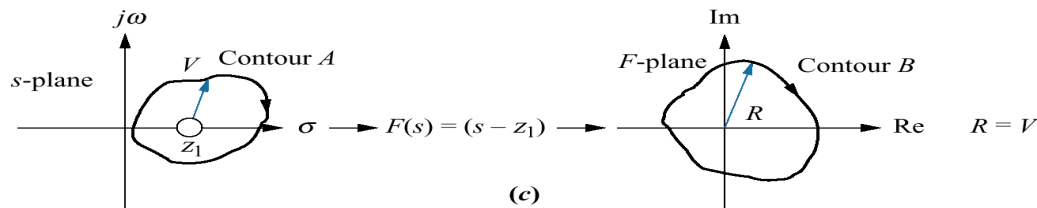
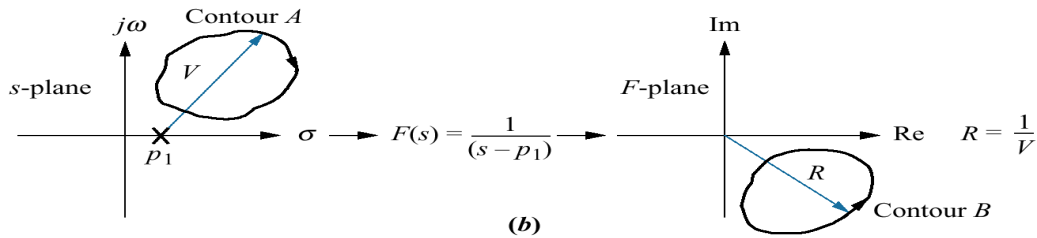
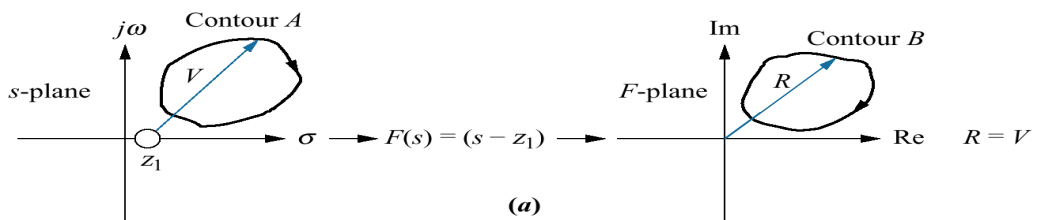
**mapping contour**

**function** 
$$F(s) = \frac{(s - z_1)(s - z_2)}{(s - p_1)(s - p_2)}$$

**mapping all points (contour) from one plane to another by function  $F(s)$ .**

$$F(s) = \frac{(s - z_1)(s - z_2)}{(s - p_1)(s - p_2)}$$

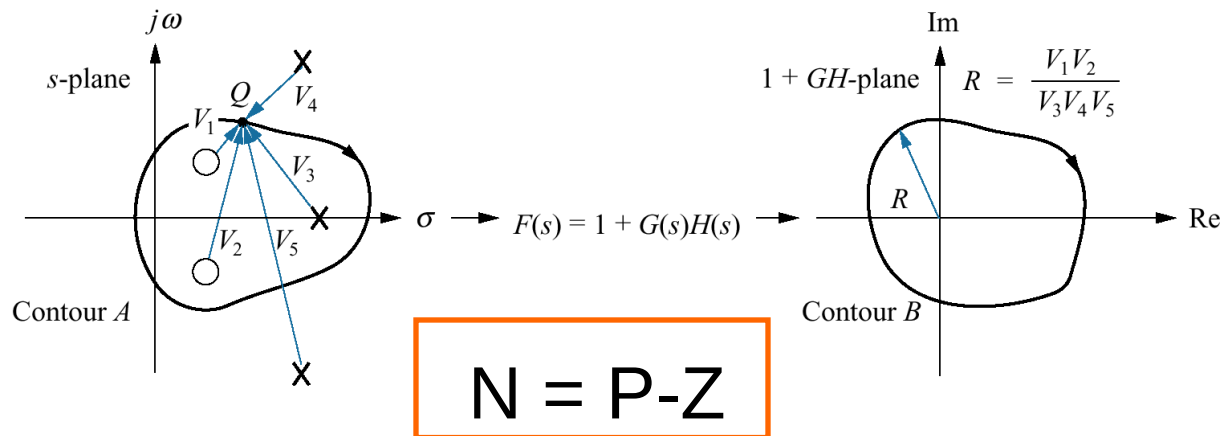




- Pole/zero inside the contour has 360 deg. angular change.
- Pole/zero outside contour has 0 deg. angular change.
- Move clockwise around contour, zero inside yields rotation in clockwise, pole inside yields rotation in counterclockwise

# Characteristic equation

$$F(s) = 1 + G(s)H(s)$$



$N$  = # of counterclockwise direction about the origin

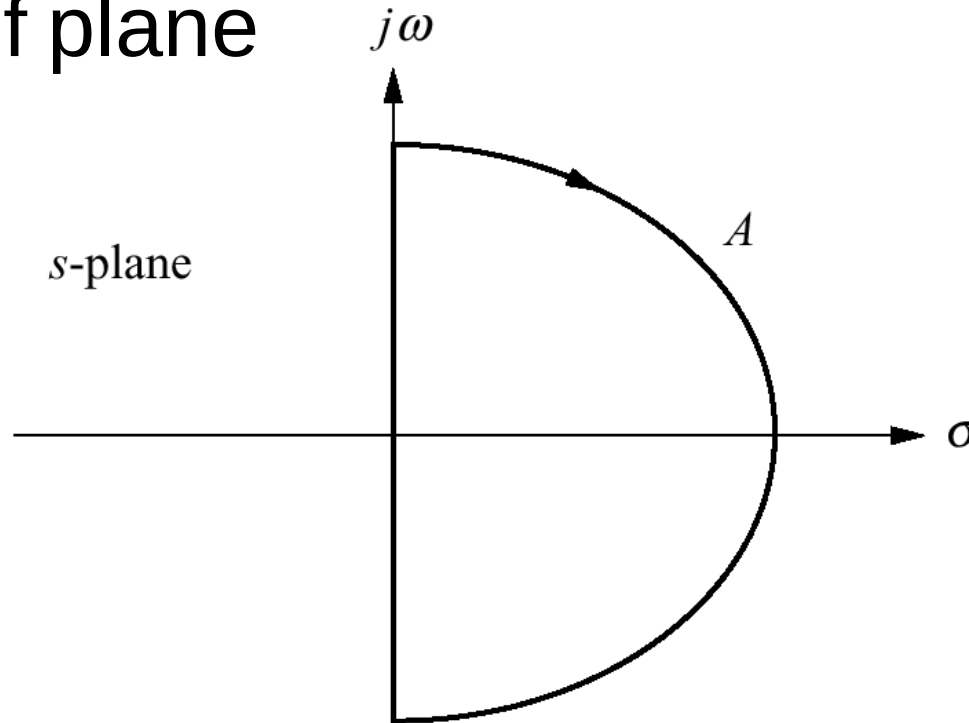
$P$  = # of poles of characteristic equation inside contour  
= # of poles of open-loop system

$z$  = # of zeros of characteristic equation inside contour  
= # of poles of closed-loop system

$$Z = P - N$$

# Characteristic equation

- Increase size of the contour to cover the right half plane



- More convenient to consider the open-loop system (with known pole/zero)

# Nyquist diagram of $G(s)H(s)$

‘Open-loop system’

Mapping from characteristic equ. to open-loop system by shifting to the left one step

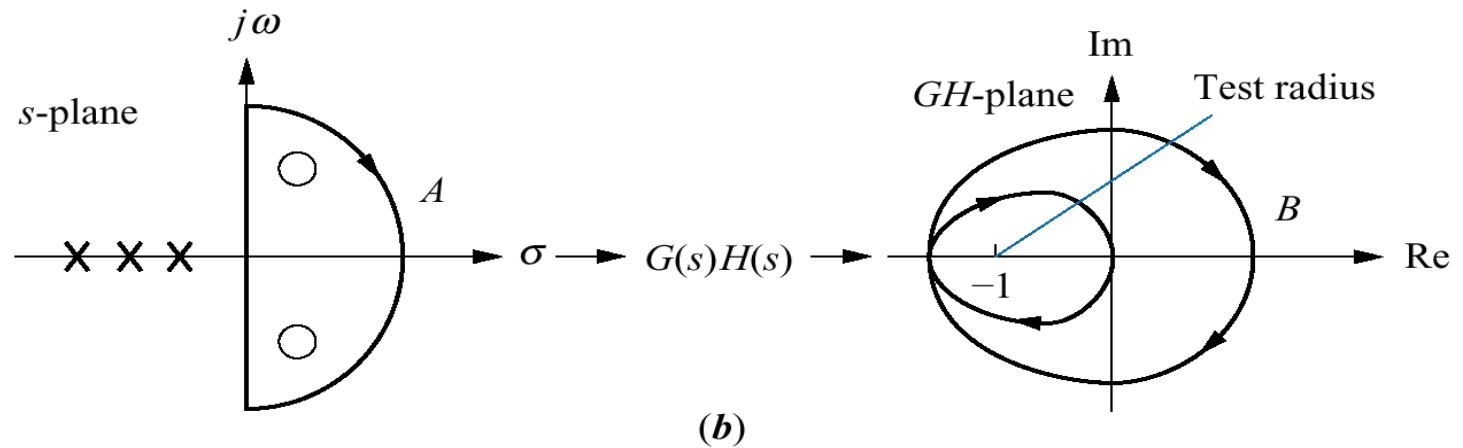
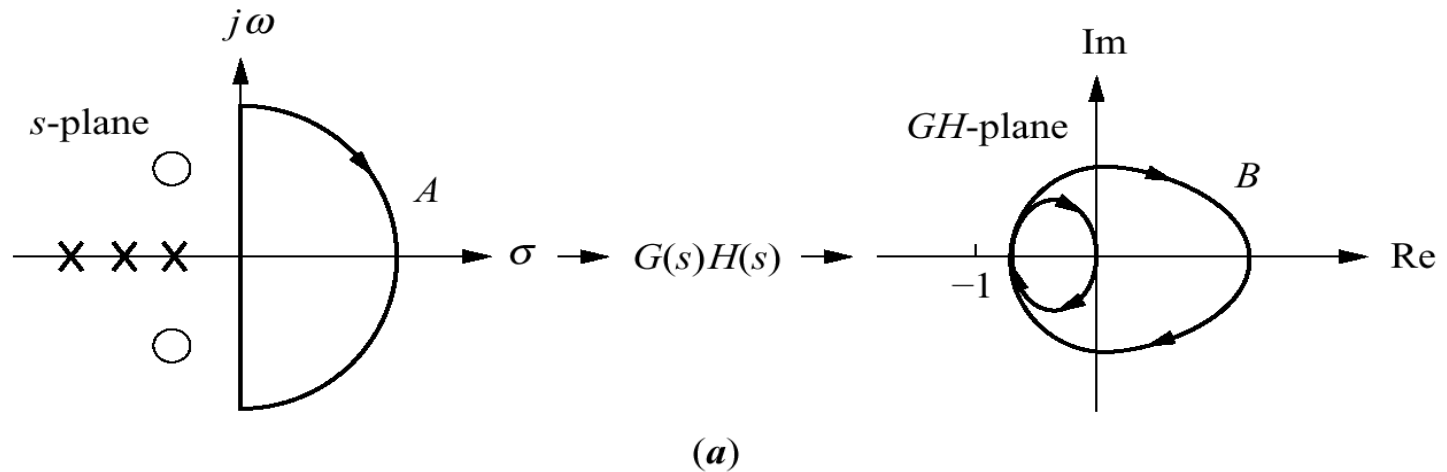
$$Z = P - N$$

$Z$  = # of closed-loop poles inside the right half plane

$P$  = # of open-loop poles inside the right half plane

$N$  = # of counterclockwise revolutions around -1



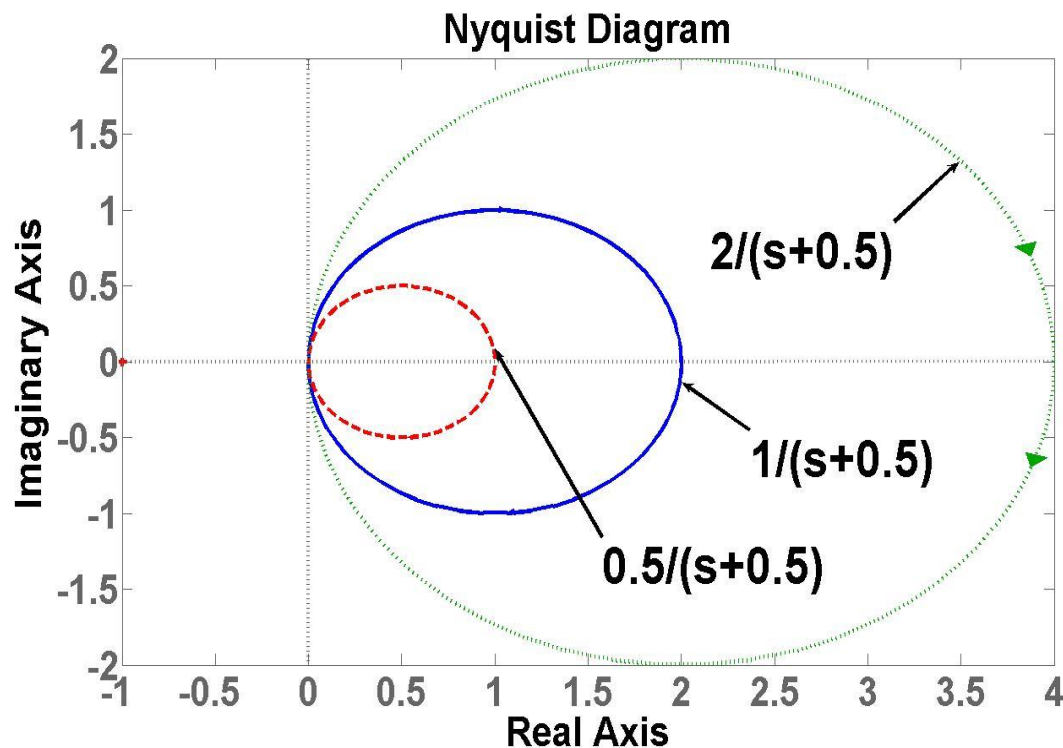


$\circ$  = zeros of  $1 + G(s)H(s)$   
 = poles of closed-loop system  
 Location not known

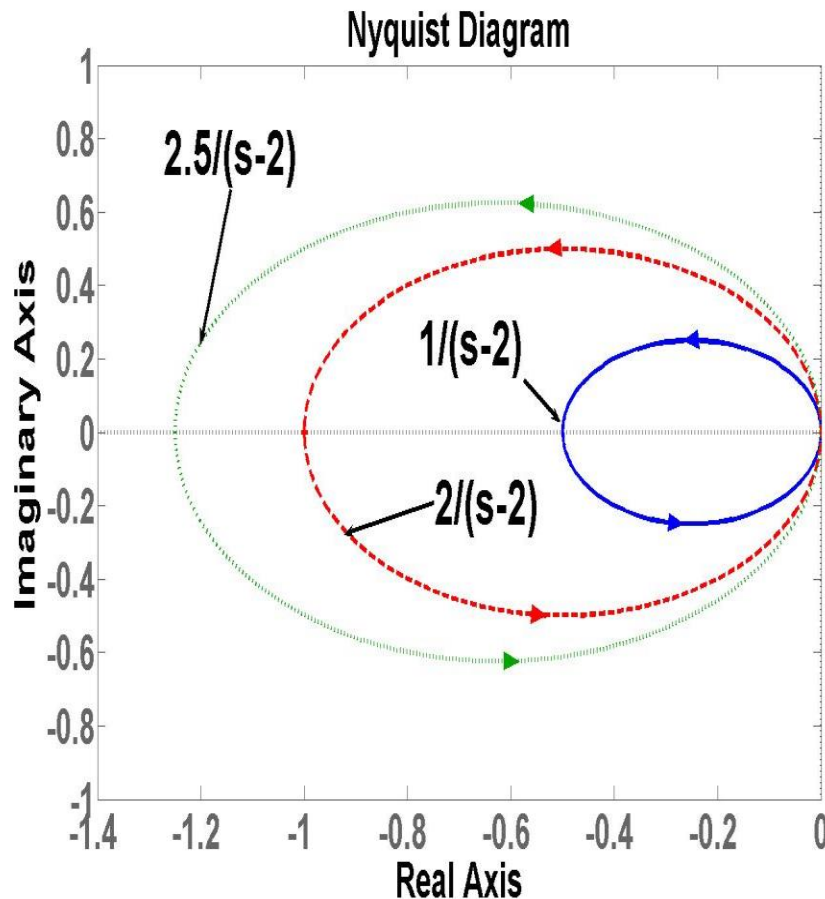
$\times$  = poles of  $1 + G(s)H(s)$   
 = poles of  $G(s)H(s)$   
 Location is known

# Properties of Nyquist plot

If there is a gain,  $K$ , in front of open-loop transfer function, the Nyquist plot will expand by a factor of  $K$ .



# Nyquist plot example



- Open loop system has pole at 2

$$G(s) = \frac{1}{s-2}$$

- Closed-loop system has pole at 1

$$\frac{G(s)}{1+G(s)} = \frac{1}{(s-1)}$$

- If we multiply the open-loop with a gain,  $K$ , then we can move the closed-loop pole's position to the left-half plane

# Nyquist plot example (cont.)

- New look of open-loop system:

$$G(s) = \frac{K}{s-2}$$

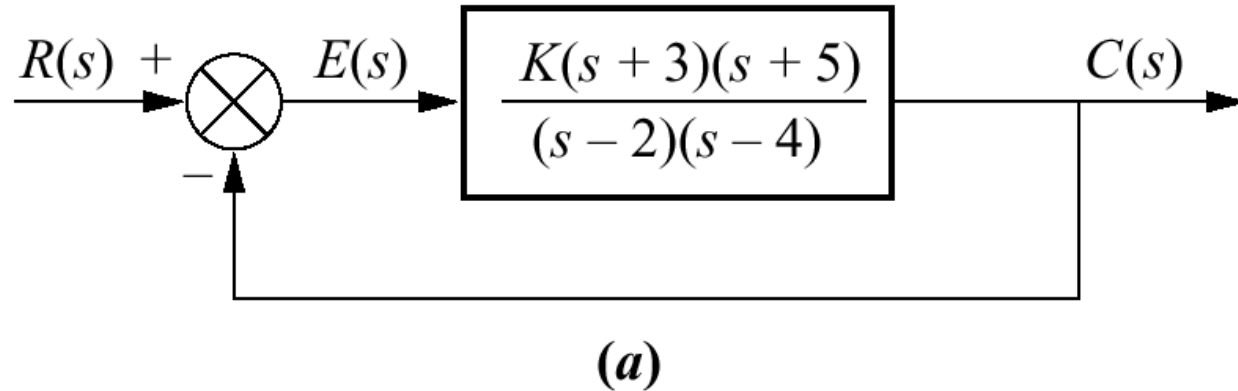
- Corresponding closed-loop system:

$$\frac{G(s)}{1+G(s)} = \frac{K}{s+(K-2)}$$

- Evaluate value of K for stability

$$K \geq 2$$

# Adjusting an open-loop gain to guarantee stability



**Step I:** sketch a Nyquist Diagram

**Step II:** find a range of  $K$  that makes the system stable!

# How to make a Nyquist plot?

Easy way by Matlab

- Nyquist: 'nyquist'
- Bode: 'bode'

# Step I: make a Nyquist plot

- Starts from an open-loop transfer function (set  $K=1$ )
- Set  $s = j\omega$  and find frequency response
  - At dc,  $\omega = 0 \rightarrow s = 0$
  - **Find  $\omega$  at which the imaginary part equals zero**

$$G(s)H(s) = \frac{(s+3)(s+5)}{(s-2)(s-4)} = \frac{s^2 + 8s + 15}{s^2 - 6s + 8}$$

$$G(j\omega)H(j\omega) = \frac{-\omega^2 + 8j\omega + 15}{-\omega^2 - 6j\omega + 8} = \frac{(15 - \omega^2) + 8j\omega}{(8 - \omega^2) - 6j\omega}$$

$$= \frac{(15 - \omega^2) + 8j\omega}{(8 - \omega^2) - 6j\omega} \times \frac{(8 - \omega^2) + 6j\omega}{(8 - \omega^2) + 6j\omega}$$

$$= \frac{(15 - \omega^2)(8 - \omega^2) - 48\omega^2 + j(154\omega - 14\omega^3)}{(8 - \omega^2)^2 + 6^2\omega^2}$$

$$\omega = 0, \sqrt{11}$$

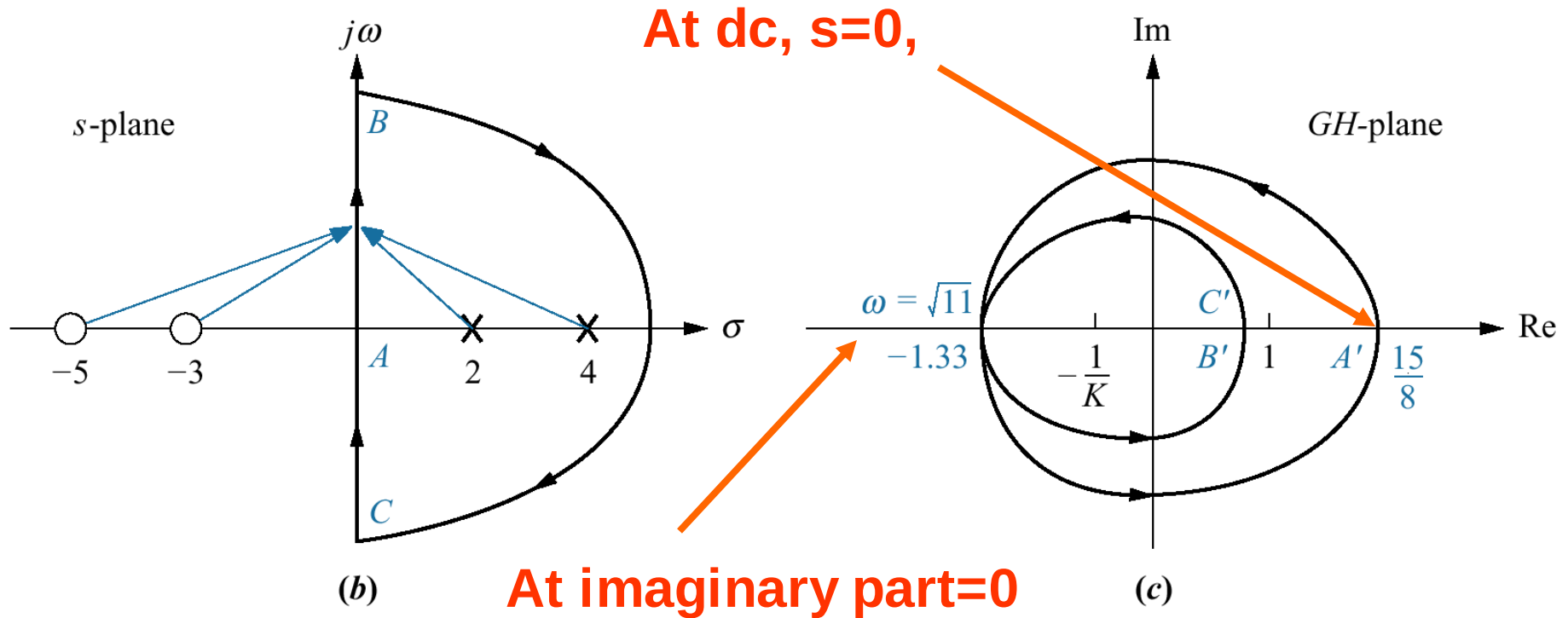
**Need the imaginary term = 0,**

**Substitute  $\omega = \sqrt{11}$  back in to the transfer function**

**And get  $G(s) = -1.33$**

$$\frac{(15 - 11)(8 - 11) - 48(11)}{(8 - 11)^2 + 6^2(11)} = \frac{-540}{412} = -1.31$$





## Step II: satisfying stability condition

- **P = 2, N has to be 2 to guarantee stability**
- **Marginally stable if the plot intersects -1**
- **For stability, 1.33K has to be greater than 1**

$$K > 1/1.33$$

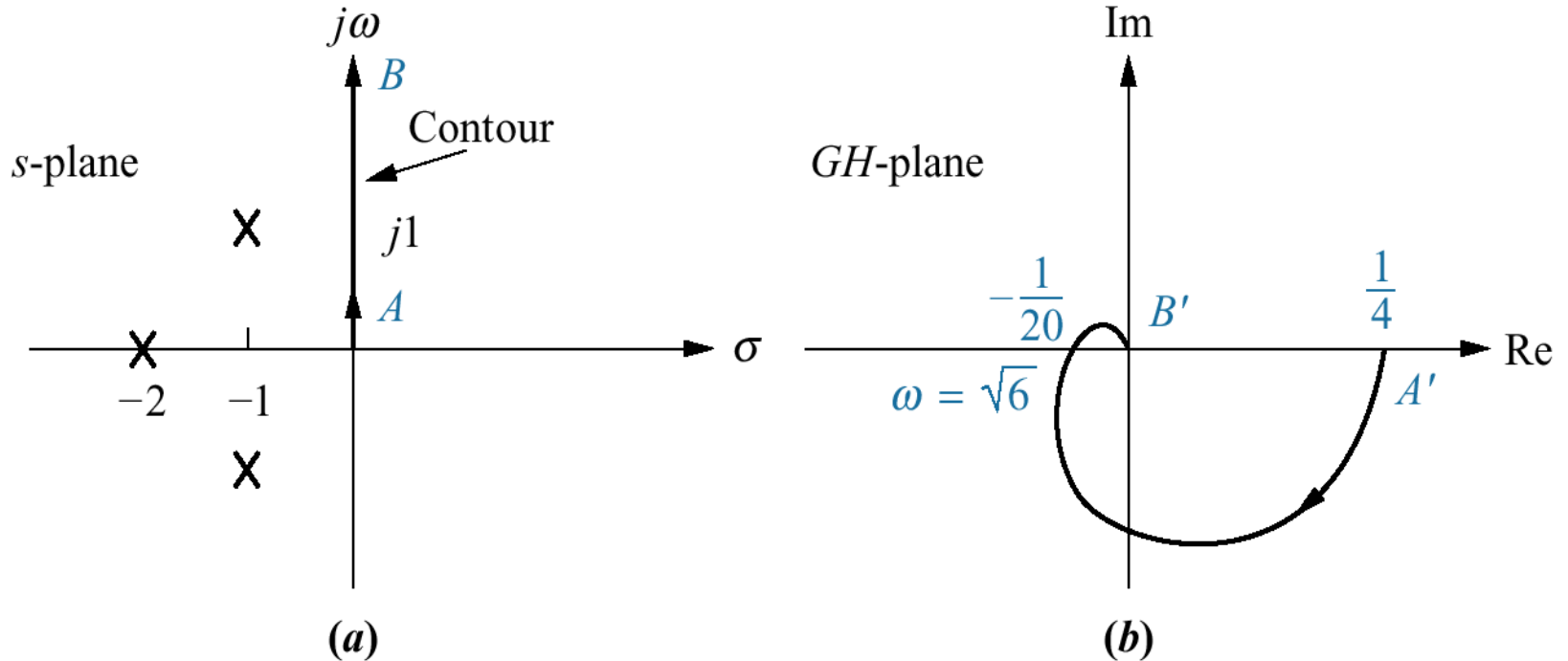
or

$$K > 0.75$$

# Example

Evaluate a range of  $K$  that makes the system stable

$$G(s) = \frac{K}{(s^2 + 2s + 2)(s + 2)}$$



Step I: find frequency at which imaginary part = 0

**Set  $S = j\omega$**

$$G(j\omega) = \frac{K}{((j\omega)^2 + 2j\omega + 2)(j\omega + 2)}$$
$$= \frac{4(1 - \omega^2) - j\omega(6 - \omega^2)}{16(1 - \omega^2)^2 + \omega^2(6 - \omega^2)^2}$$

**At  $\omega = 0, \sqrt{6}$  the imaginary part = 0**

**Plug  $\omega = \sqrt{6}$  back in the transfer function  
and get  $G = -0.05$**

# Step II: consider stability condition

- **P = 0, N has to be 0 to guarantee stability**
- **Marginally stable if the plot intersects -1**
- **For stability, 0.05K has to be less than 1**

$$K < 1/0.05$$

or

$$K < 20$$

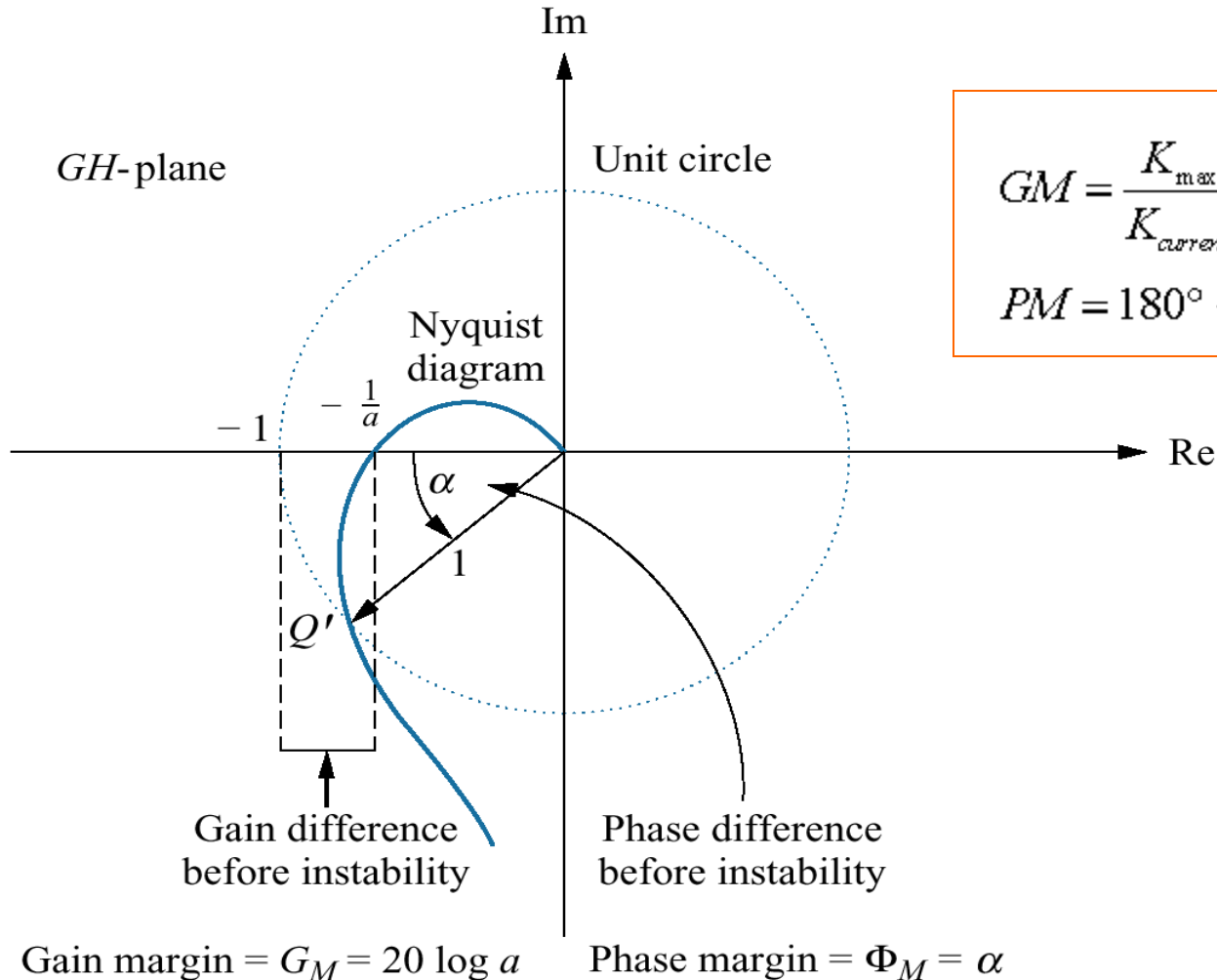
# Gain Margin and Phase Margin

**Gain margin is the change in open-loop gain (in dB), required at 180 of phase shift to make the closed-loop system unstable.**

**Phase margin is the change in open-loop phase shift, required at unity gain to make the closed-loop system unstable.**

**GM/PM tells how much system can tolerate before going unstable!!!**

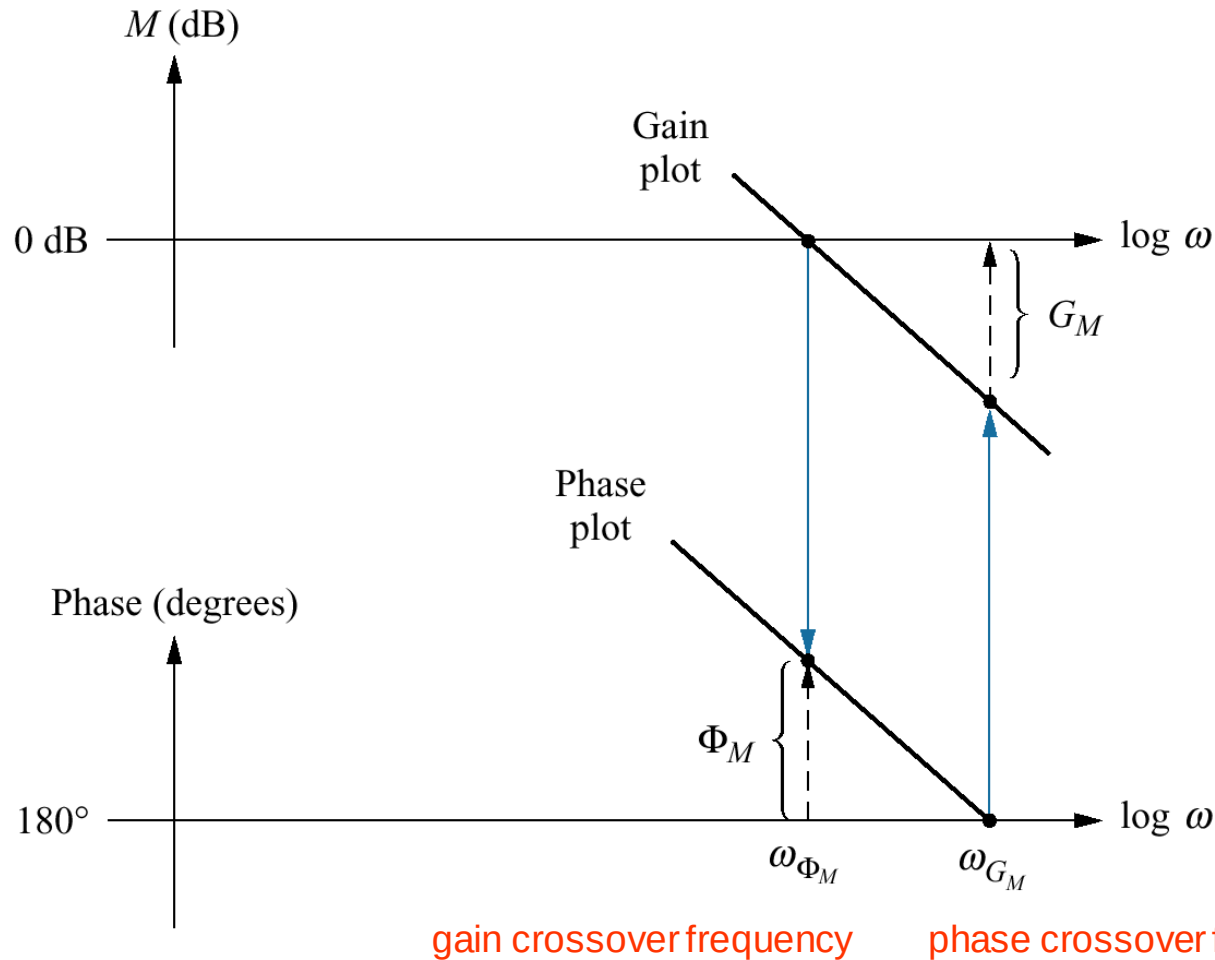
# GM and PM via Nyquist plot



$$GM = \frac{K_{\max}}{K_{\text{current}}} = \frac{1}{|G(j\omega_\phi)H(j\omega_\phi)|}$$

$$PM = 180^\circ + \angle G(j\omega_x)H(j\omega_x)$$

# GM and PM via Bode Plot



- The frequency at which the phase equals  $180^\circ$  is called the phase crossover frequency  $\omega_{G_M}$

- The frequency at which the magnitude equals  $0$  is called the gain crossover frequency  $\omega_{\Phi_M}$



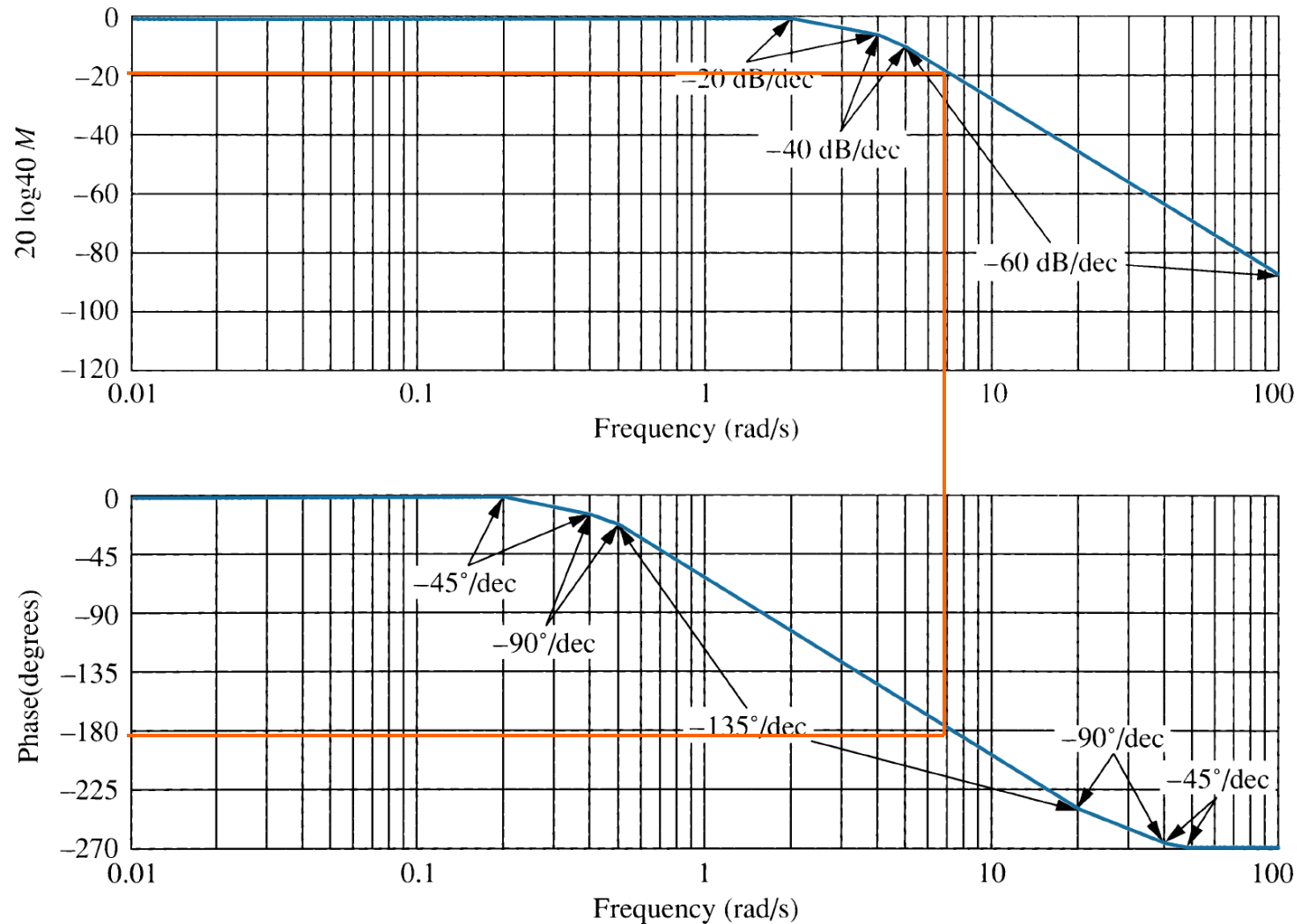
# Example

**Find Bode Plot and evaluate a value of K that makes the system stable**  
**The system has a unity feedback with an open-loop transfer function**

$$G(s) = \frac{K}{(s + 2)(s + 4)(s + 5)}$$

**First, let's find Bode Plot of G(s) by assuming that K=40 (the value at which magnitude plot starts from 0 dB)**

---

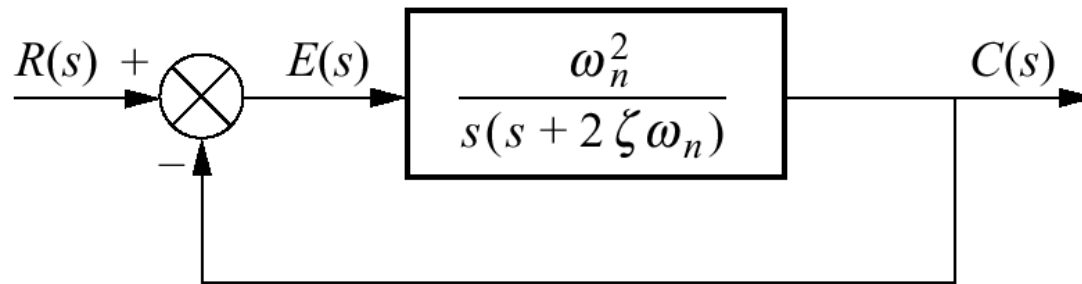


**At phase = -180,  $\omega = 7$  rad/sec, magnitude = -20 dB**

- **GM>0, system is stable!!!**
- **Can increase gain up 20 dB without causing instability (20dB = 10)**
- **Start from K = 40**
- **with K < 400, system is stable**

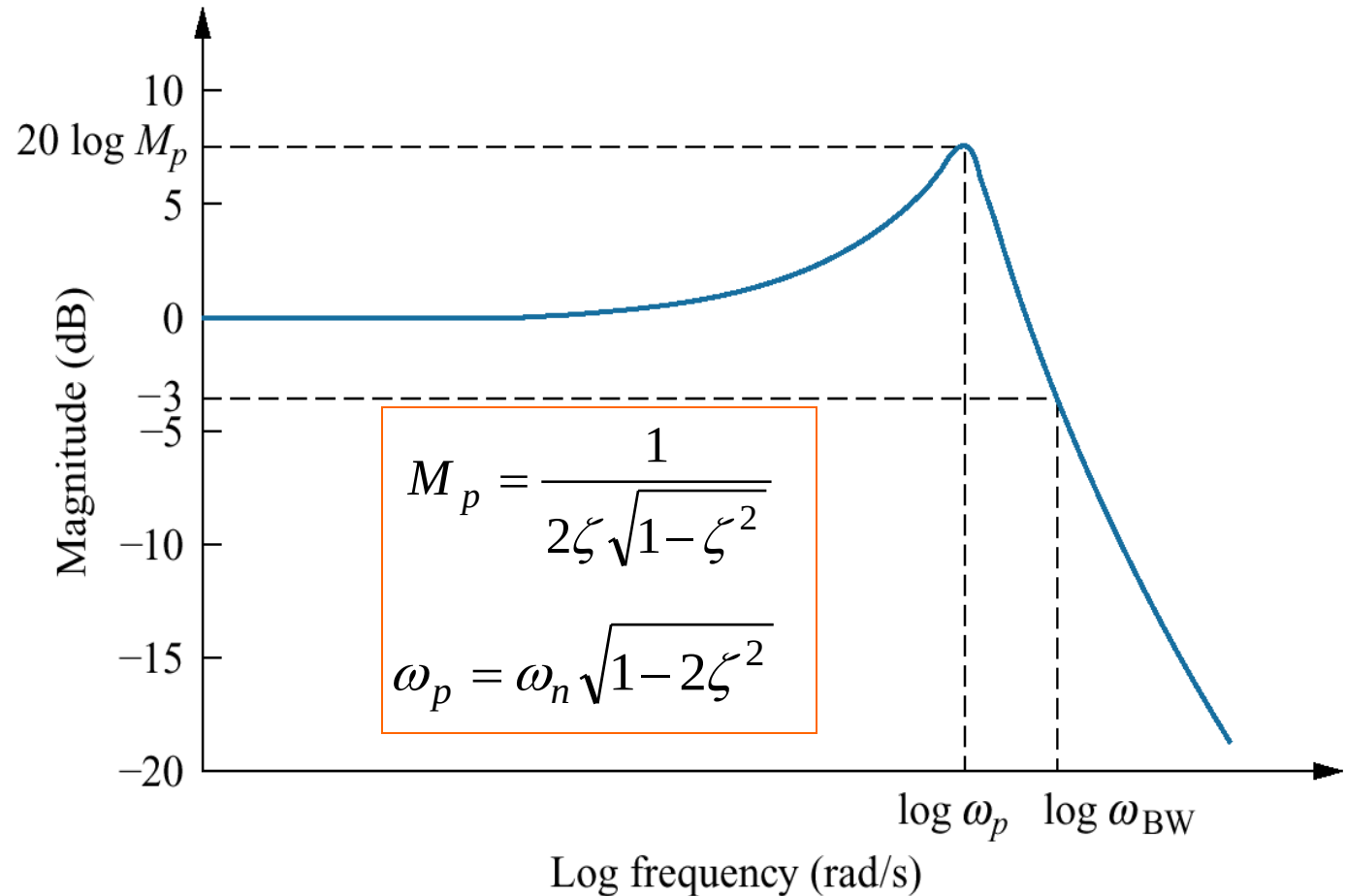
# Closed-loop transient and closed-loop frequency responses

## '2<sup>nd</sup> system'



$$\frac{C(s)}{R(s)} = T(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

# Damping ratio and closed-loop frequency response



## Magnitude Plot of closed-loop system

## Response speed and closed-loop frequency response

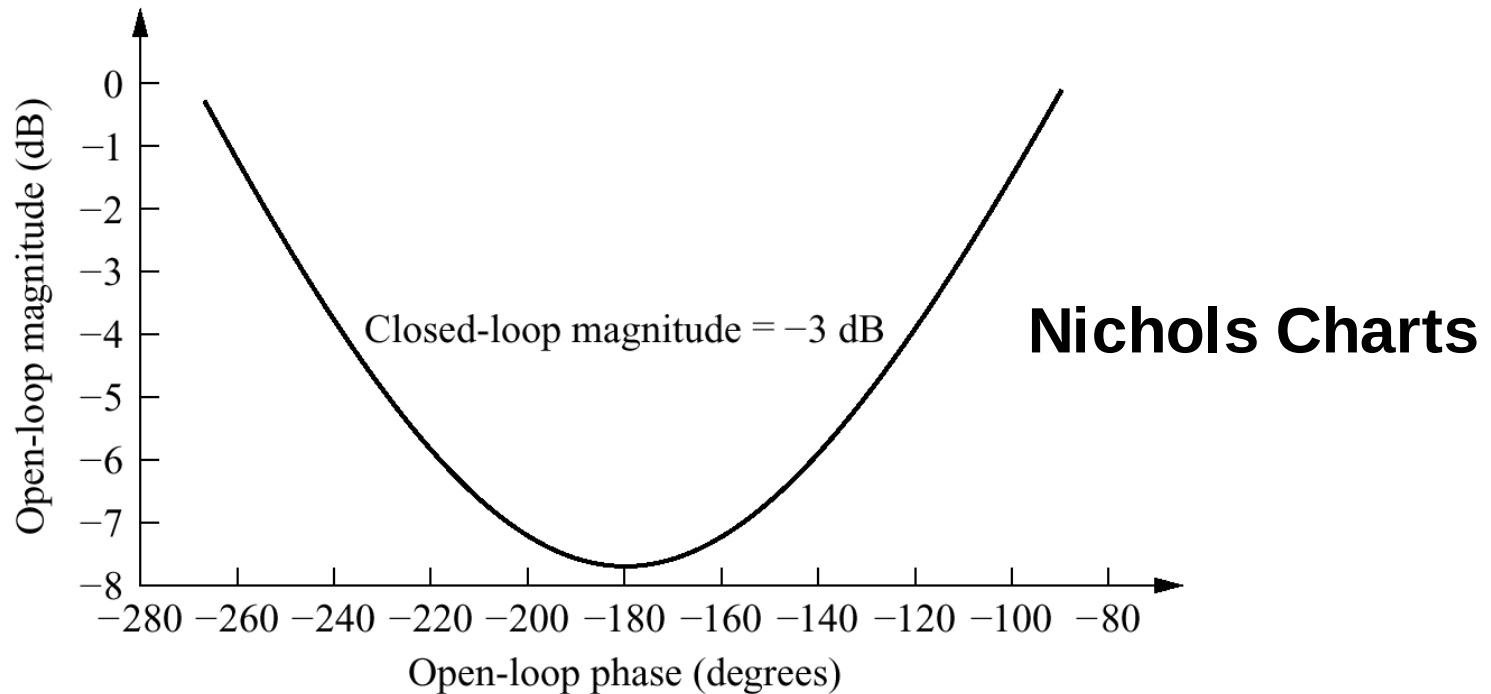
$$\omega_{BW} = \omega_n \sqrt{(1-2\zeta^2) + \sqrt{4\zeta^4 - 4\zeta^2 + 2}}$$

$$\omega_{BW} = \frac{4}{T_s \zeta} \sqrt{(1-2\zeta^2) + \sqrt{4\zeta^4 - 4\zeta^2 + 2}}$$

$$\omega_{BW} = \frac{\pi}{T_p \sqrt{1-\zeta^2}} \sqrt{(1-2\zeta^2) + \sqrt{4\zeta^4 - 4\zeta^2 + 2}}$$

**$\omega_{BW}$  = frequency at which magnitude is 3dB down  
from value at dc (0 rad/sec), or  $M = \frac{1}{\sqrt{2}}$  .**

# Find $\omega_{BW}$ from Open-loop Frequency Response



**From open-loop frequency response, we can find  $\omega_{BW}$  at the open-loop frequency that the magnitude lies between -6dB to -7.5dB (phase between -135 to -225)**

# Relationship between damping ratio and phase margin of open-loop frequency response

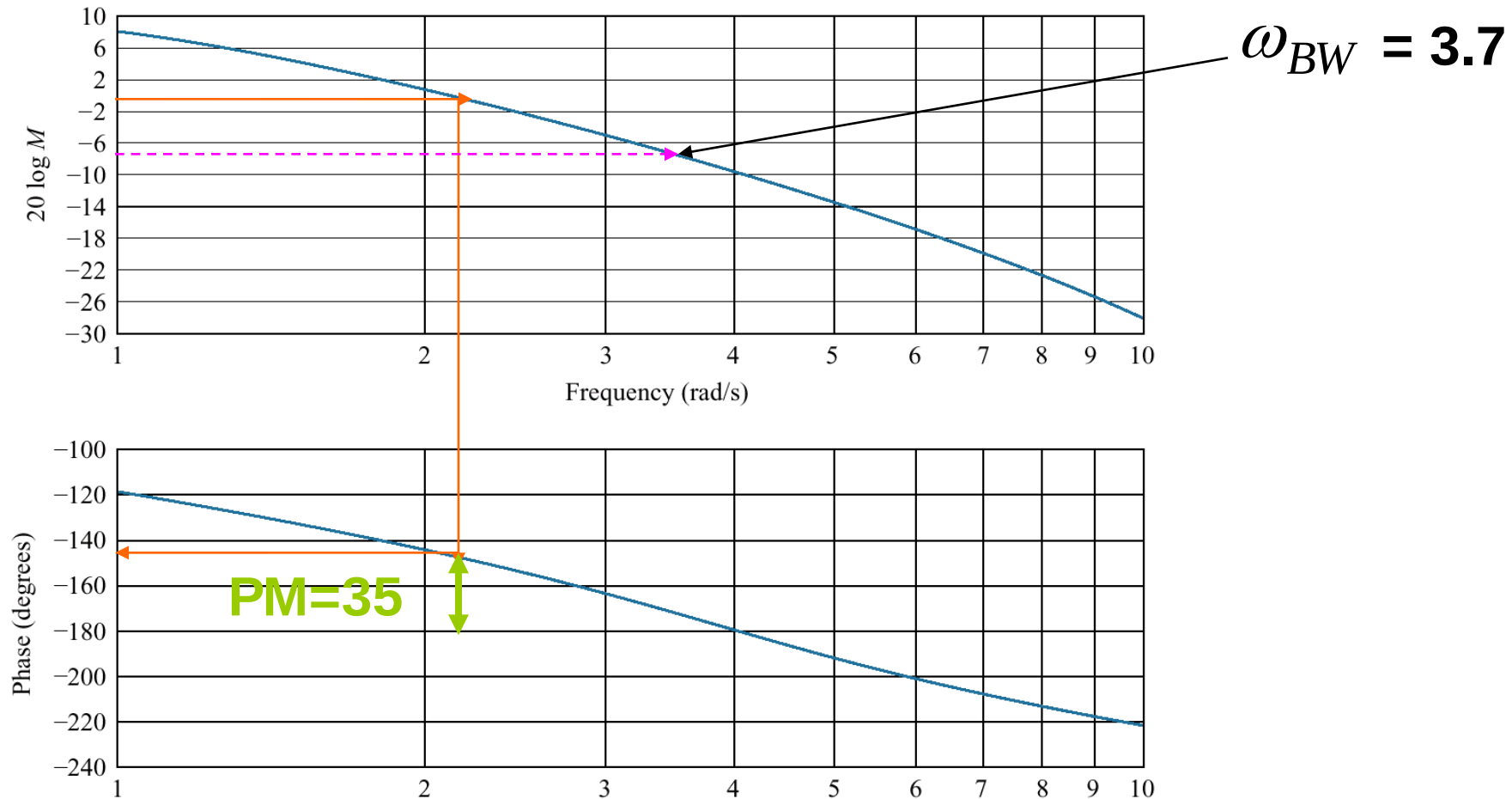
**Phase margin of open-loop frequency response  
Can be written in terms of damping ratio as following**

$$\phi_M = \tan^{-1} \frac{2\zeta}{\sqrt{-2\zeta^2 + \sqrt{1 + 4\zeta^4}}}$$



# Example

Open-loop system with a unity feedback has a bode plot below, approximate settling time and peak time



$$\phi_M = \tan^{-1} \frac{2\zeta}{\sqrt{-2\zeta^2 + \sqrt{1+4\zeta^4}}}$$

**Solve for PM = 35**     $\zeta = 0.32$

$$T_s = \frac{4}{\omega_{BW} \zeta} \sqrt{(1-2\zeta^2) + \sqrt{4\zeta^4 - 4\zeta^2 + 2}}$$

$$= 5.5$$

$$T_p = \frac{\pi}{\omega_{BW} \sqrt{1-\zeta^2}} \sqrt{(1-2\zeta^2) + \sqrt{4\zeta^4 - 4\zeta^2 + 2}}$$

$$= 1.43$$

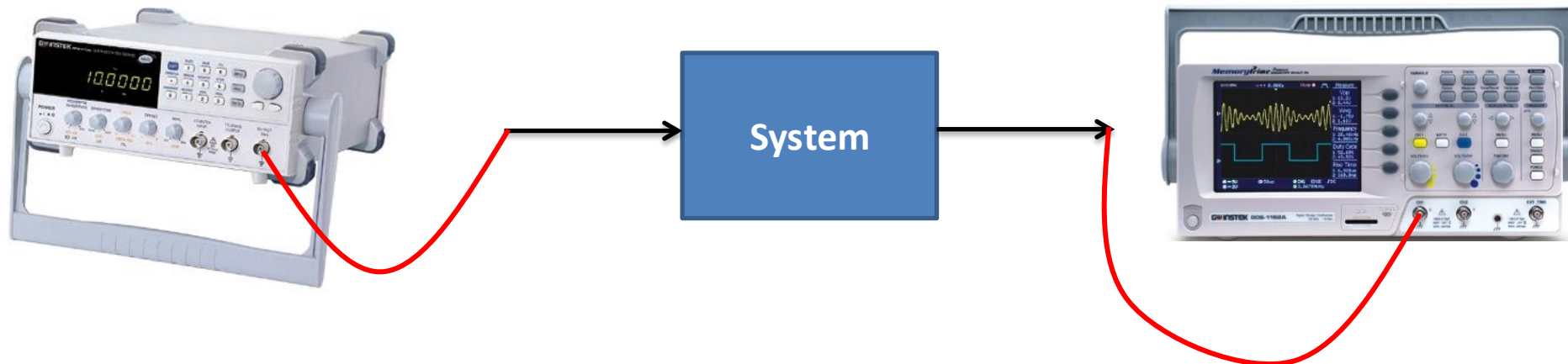
# **Control Systems (CS)**

## **FREQUENCY RESPONSE**

### **الاستجابة الترددية لمنظومة السيطرة**

# Introduction

- Frequency response is the steady-state response of a system to a sinusoidal input.
- In frequency-response methods, the frequency of the input signal is varied over a certain range and the resulting response is studied.



# The Concept of Frequency Response

## مفهوم الاستجابة الترددية Response

- In the steady state, sinusoidal inputs to a linear system generate sinusoidal responses of the same frequency.
- Even though these responses are of the same frequency as the input, they differ in amplitude and phase angle from the input.
- These differences are functions of frequency.

# The Concept of Frequency Response

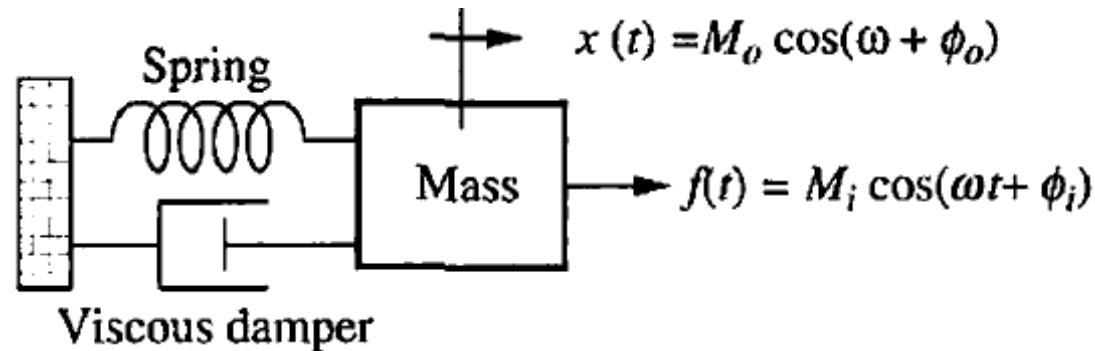
- Sinusoids can be represented as complex numbers called phasors.
- The magnitude of the complex number is the amplitude of the sinusoid, and the angle of the complex number is the phase angle of the sinusoid.
- Thus  $M_1 \cos(\omega t + \phi)$  can be represented as  $M_1 \angle \phi_1$  where the frequency,  $\omega$ , is implicit.

# The Concept of Frequency Response

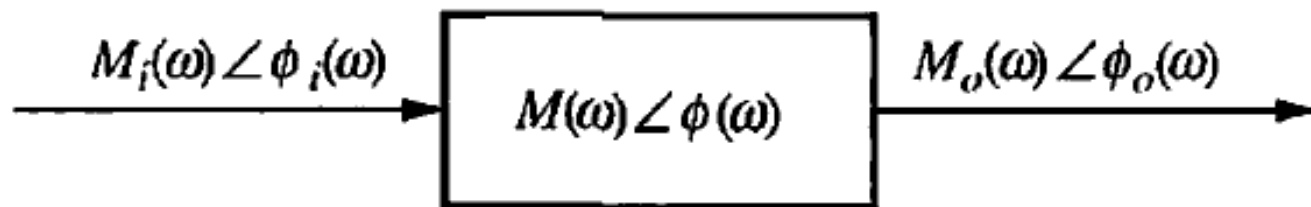
- A system causes both the amplitude and phase angle of the input to be changed.
- Therefore, the system itself can be represented by a complex number.
- Thus, the product of the input phasor and the system function yields the phasor representation of the output.

# The Concept of Frequency Response

- Consider the mechanical system.

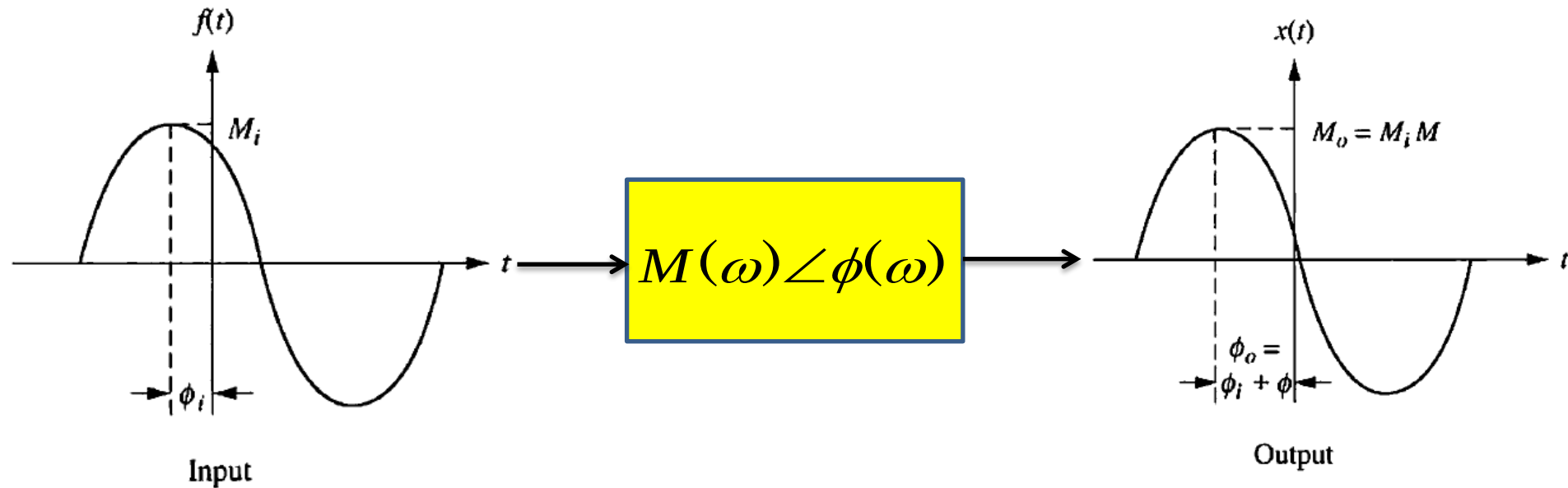


- If the input force,  $f(t)$ , is sinusoidal, the steady-state output response,  $x(t)$ , of the system is also sinusoidal and at the same frequency as the input.





# The Concept of Frequency Response

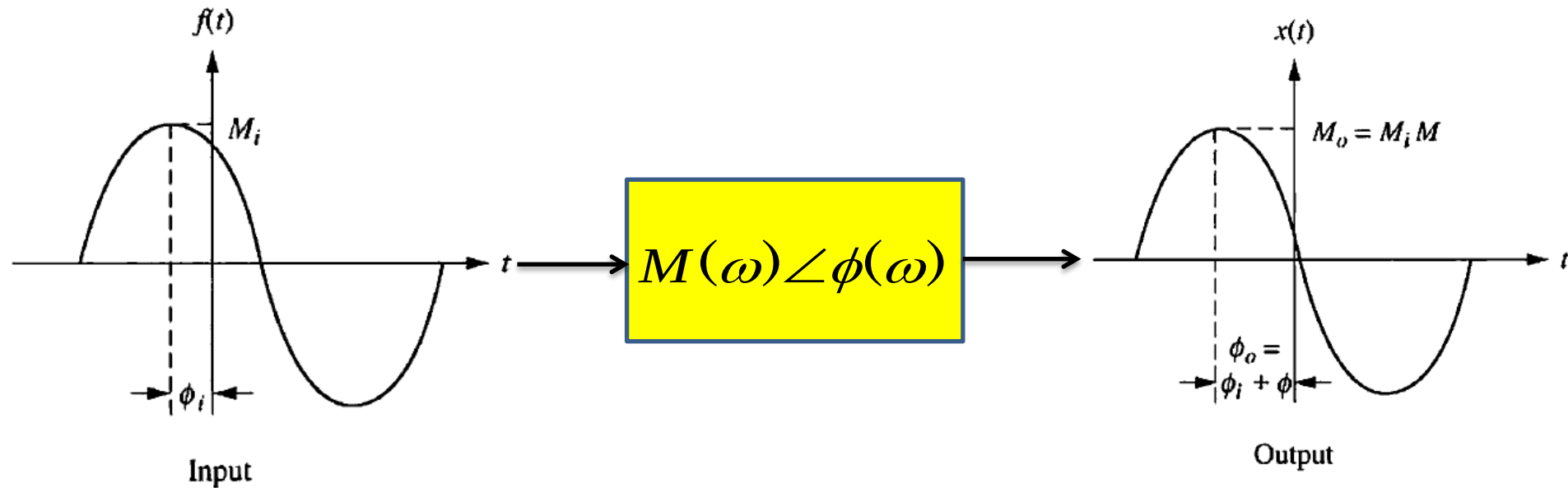


- Assume that the system is represented by the complex number

$$M(\omega) \angle \phi(\omega)$$

- The output is found by multiplying the complex number representation of the input by the complex number representation of the system.

# The Concept of Frequency Response



- Thus, the steady-state output sinusoid is

$$M_o(\omega) \angle \phi_o(\omega) = M(\omega) M_i(\omega) \angle [\phi(\omega) + \phi_i(\omega)]$$

- $M_o(\omega)$  is the magnitude response and  $\Phi(\omega)$  is the phase response. The combination of the magnitude and phase frequency responses is called the frequency response.

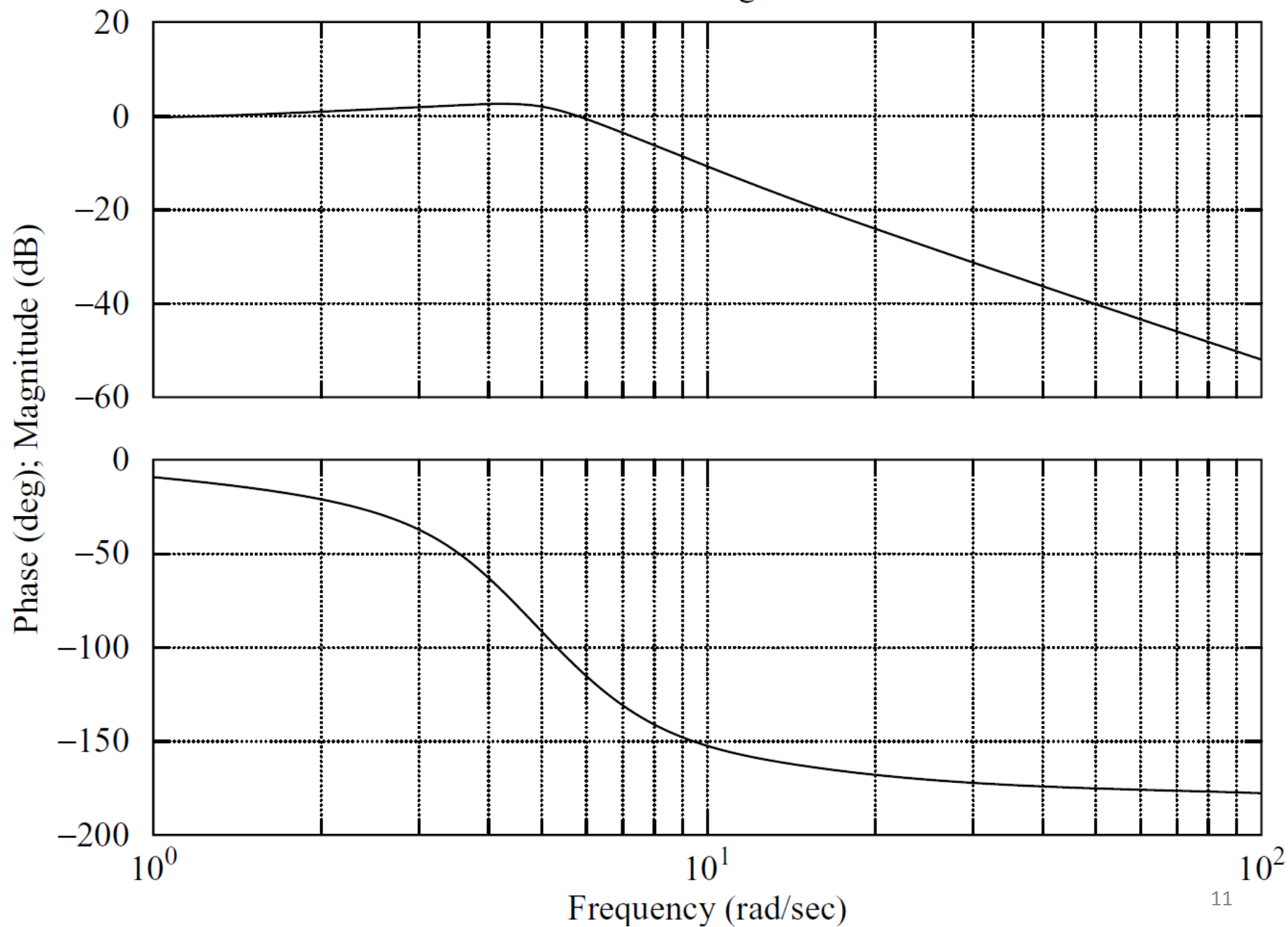
# Frequency Domain Plots

- Bode Plot رسم بود
- Nyquist Plot رسم نايكوست
- Nichol's Chart خارطة نيكلز

# Bode Plot

- A Bode diagram consists of two graphs:
  - One is a plot of the logarithm of the magnitude of a sinusoidal transfer function.
  - The other is a plot of the phase angle.
  - Both are plotted against the frequency on a logarithmic scale.

Bode Diagram



NAME OF STUDENT: \_\_\_\_\_

STUDENT NUMBER: \_\_\_\_\_

# Decade



# Basic Factors of a Transfer Function

- The basic factors that very frequently occur in an arbitrary transfer function are
  1. Gain  $K$
  2. Integral and Derivative Factors  $(j\omega)^{\pm 1}$
  3. First Order Factors  $(j\omega T + 1)^{\pm 1}$
  4. Quadratic Factors  $[1 + 2\zeta(j\omega/\omega_n) + (j\omega/\omega_n)^2]^{\mp 1}$

$$G(s) = \frac{20(3s + 1)}{s(s + 1)(s^2 + 5s + 2)}$$

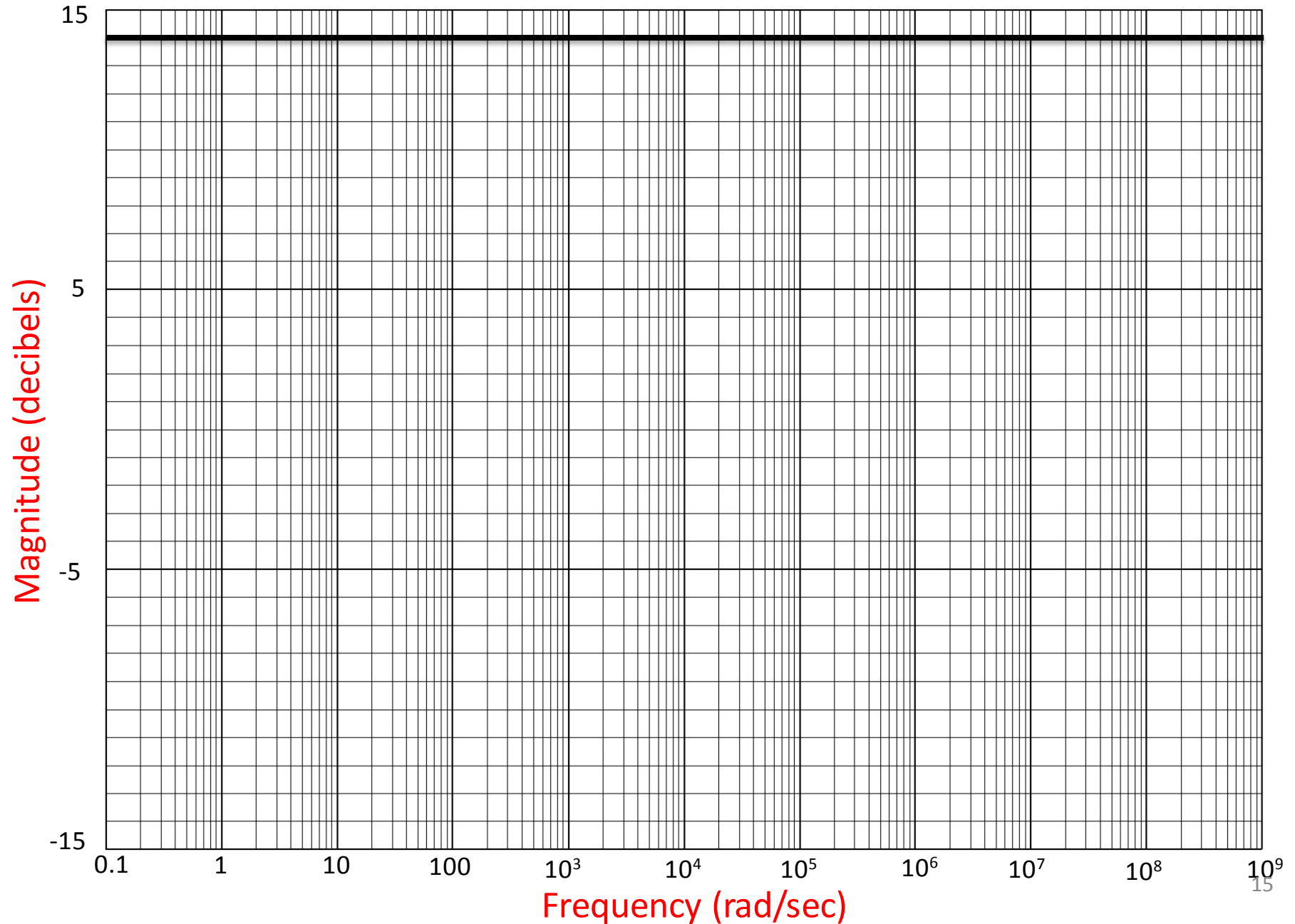
# Basic Factors of a Transfer Function

## 1. Gain $K$

- The log-magnitude curve for a constant gain  $K$  is a horizontal straight line at the magnitude of  $20 \log(K)$  decibels.
- The phase angle of the gain  $K$  is zero.
- The effect of varying the gain  $K$  in the transfer function is that it raises or lowers the log-magnitude curve of the transfer function by the corresponding constant amount, but it has no effect on the phase curve.

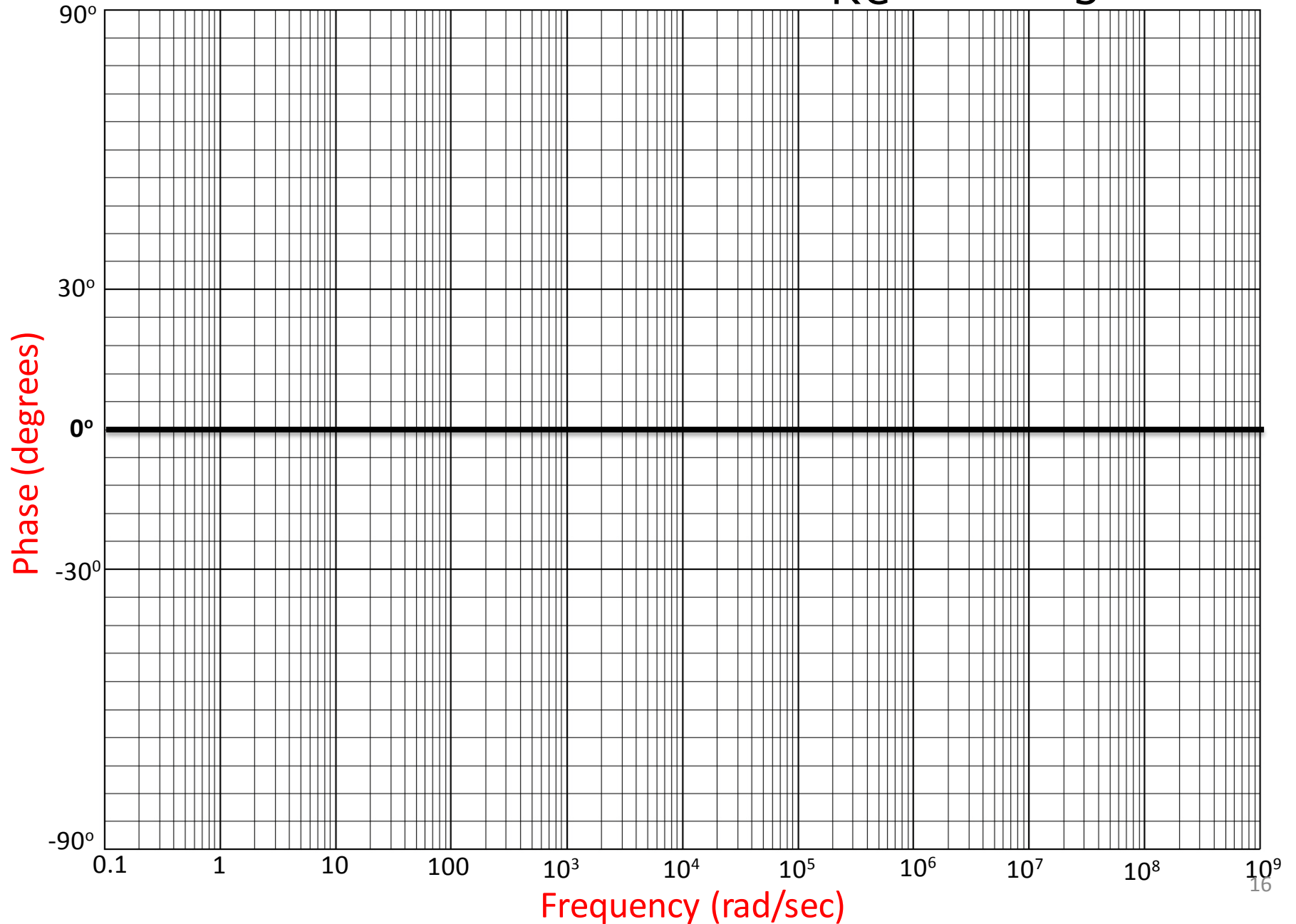


If  $K = 5$  Then  $20\log(K) = 20\log(5) = 14\text{db}$



If  $K = 5$

$$\text{Then } \phi = \tan^{-1}\left(\frac{\text{Im}}{\text{Re}}\right) = \tan^{-1}\left(\frac{0}{5}\right) = 0$$



# Basic Factors of a Transfer Function

## 2. Integral and Derivative Factors $(j\omega)^{\pm 1}$

**Derivative Factor**  $G(s) = s$ , where  $s = j\omega$

**Magnitude**  $|G(j\omega)| = 20\log(\omega)$

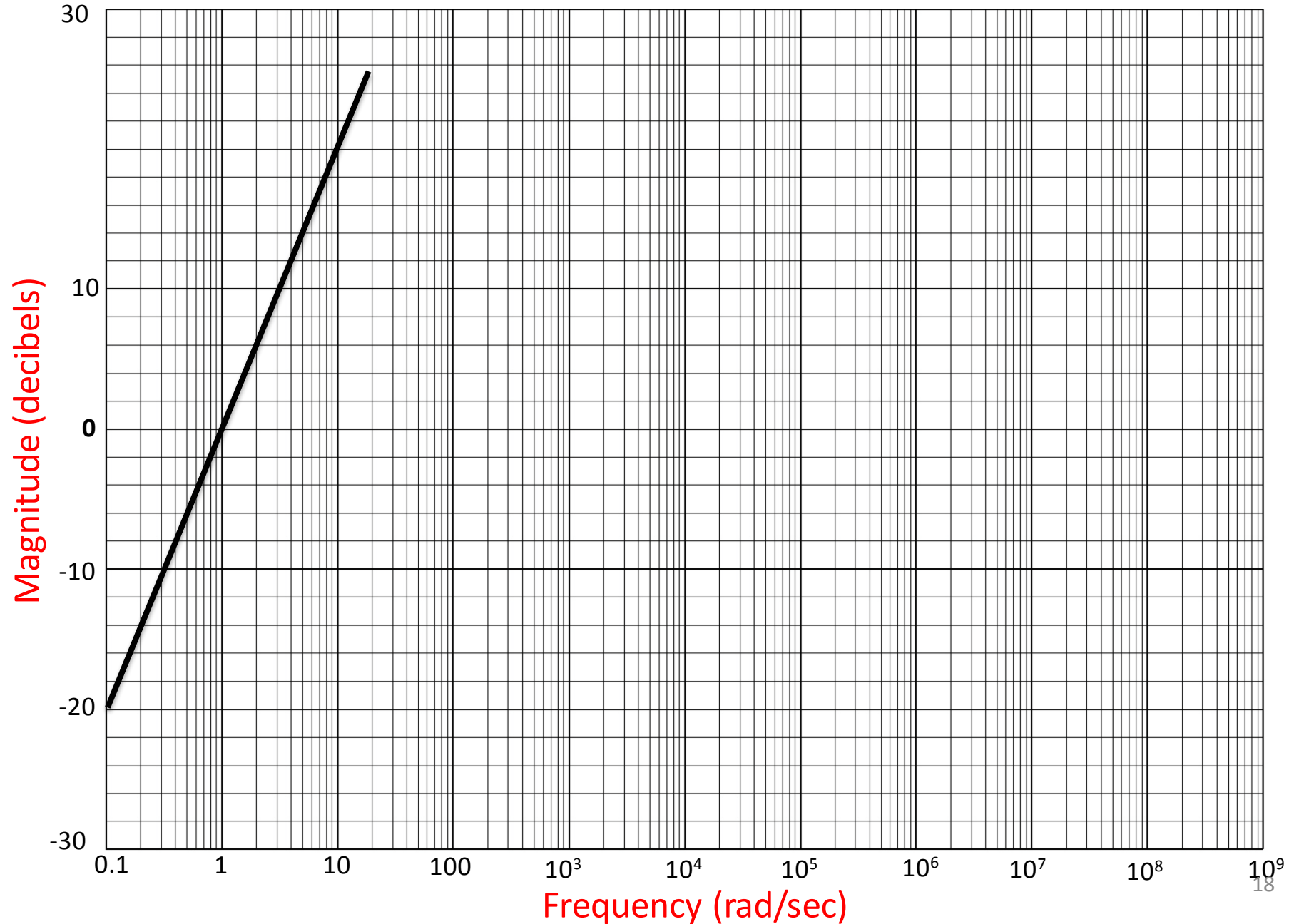
| $\omega$ | 0.1 | 0.2 | 0.4 | 0.5 | 0.7 | 0.8 | 0.9 | 1 |
|----------|-----|-----|-----|-----|-----|-----|-----|---|
| db       | -20 | -14 | -8  | -6  | -3  | -2  | -1  | 0 |

**Slope=6b/octave**

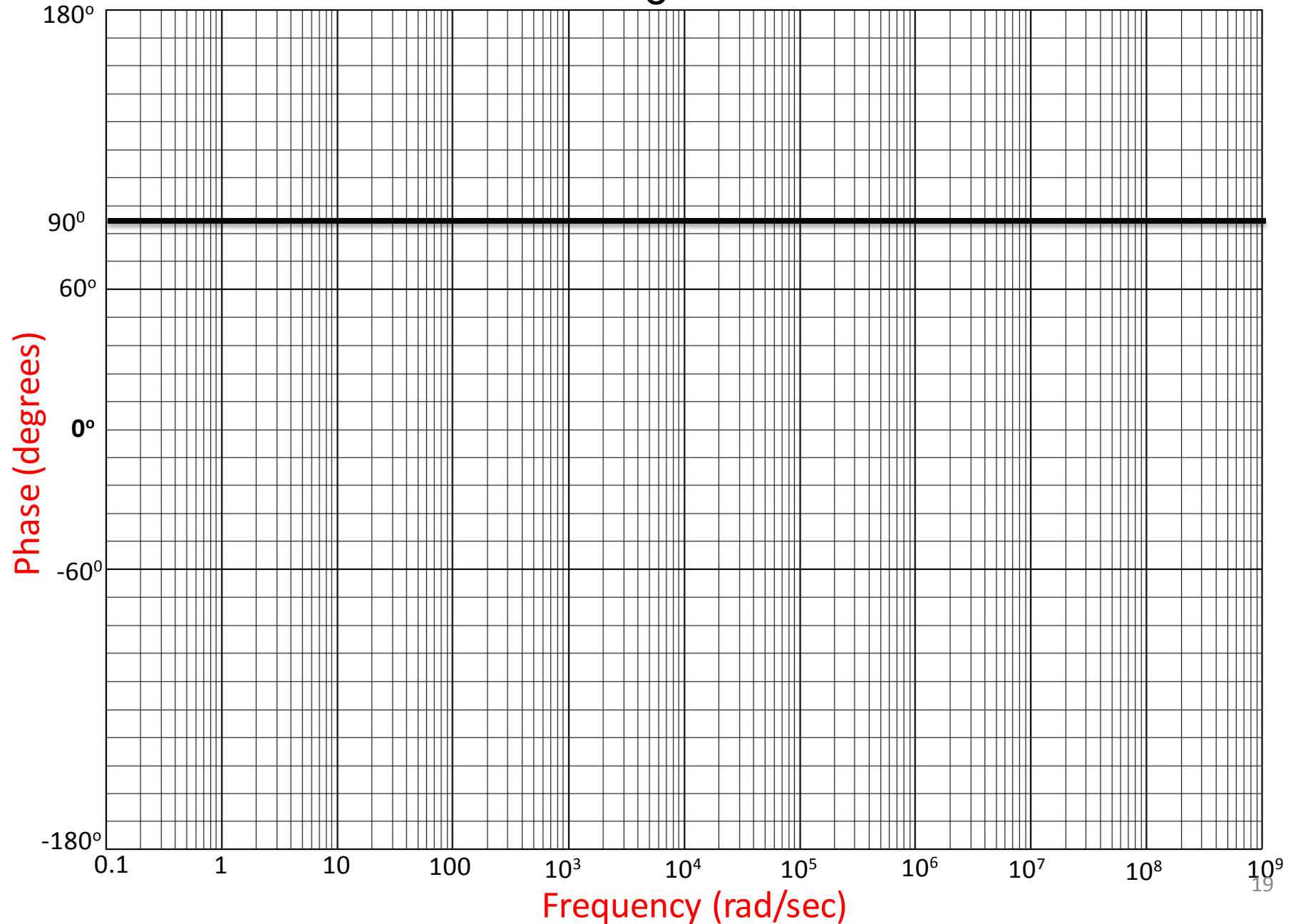
**Slope=20db/decade**

**Phase**  $\angle G(j\omega) = \tan^{-1}\left(\frac{\omega}{0}\right) = 90^\circ$

$20\text{db/decade}$



$$\phi = \tan^{-1}\left(\frac{\omega}{0}\right) = 90^\circ$$



# Basic Factors of a Transfer Function

## 2. Integral and Derivative Factors $(j\omega)^{\pm 1}$

- When expressed in decibels, the reciprocal of a number differs from its value only in sign; that is, for the number  $N$ ,

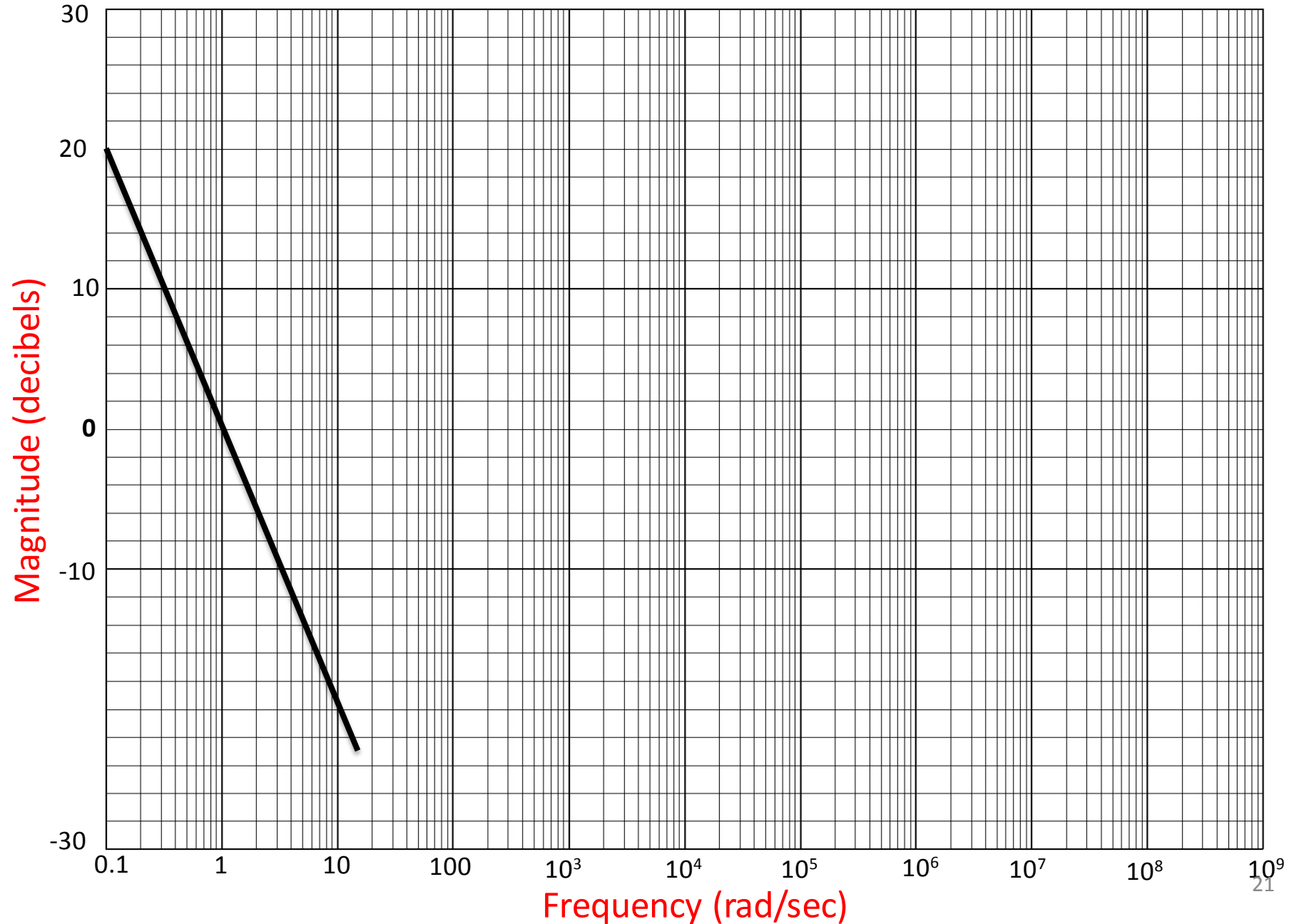
$$20 \log(N) = -20 \log\left(\frac{1}{N}\right)$$

- Therefore, for Integral Factor the slope of the magnitude line would be same but with opposite sign (i.e -6db/octave or -20db/decade).

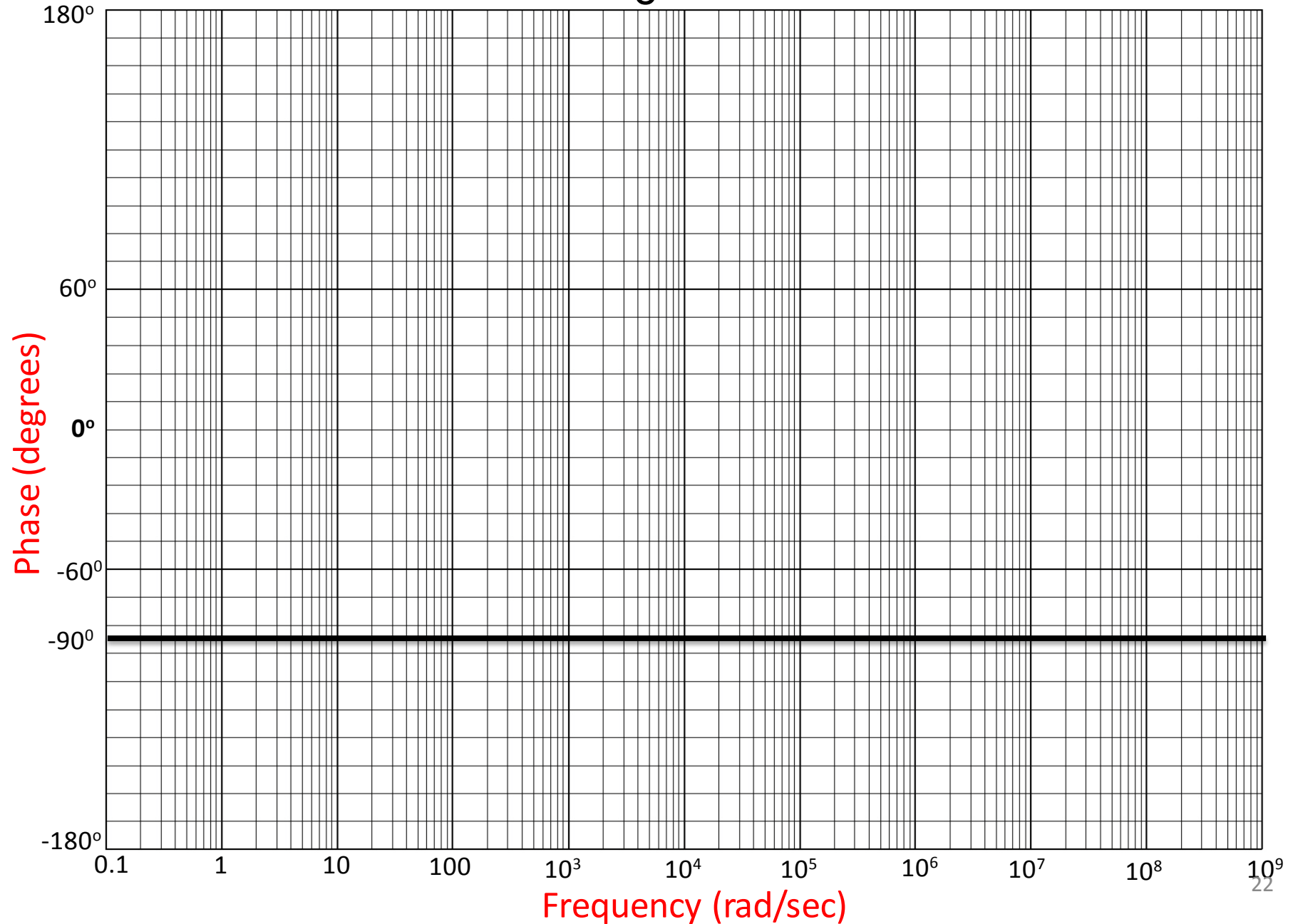
**Magnitude**  $|G(j\omega)| = \left| \frac{1}{j\omega} \right| = -20 \log(\omega)$

**Phase**  $\angle G(j\omega) = -\tan^{-1}\left(\frac{\omega}{0}\right) = -90^\circ$

$-20\text{db/decade}$



$$\phi = \tan^{-1}\left(\frac{\omega}{0}\right) = -90^\circ$$





# Basic Factors of a Transfer Function

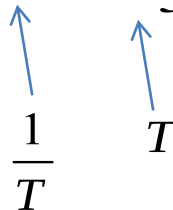
## 3. First Order Factors ( $j\omega T + 1$ )

$$M(\omega) = 20 \log(|1 + j\omega T|)$$

$$M(\omega) = 20 \log(\sqrt{1 + \omega^2 T^2})$$

- For Low frequencies  $\omega \ll 1/T$

$$M(\omega) = 20 \log(1) = 0$$

$$G(s) = (s + 3) = \left(\frac{1}{3}s + 1\right)$$


$\frac{1}{T}$        $T$

- For high frequencies  $\omega \gg 1/T$

$$M(\omega) = 20 \log(\omega T)$$

# Basic Factors of a Transfer Function

## 3. First Order Factors ( $j\omega T + 1$ )

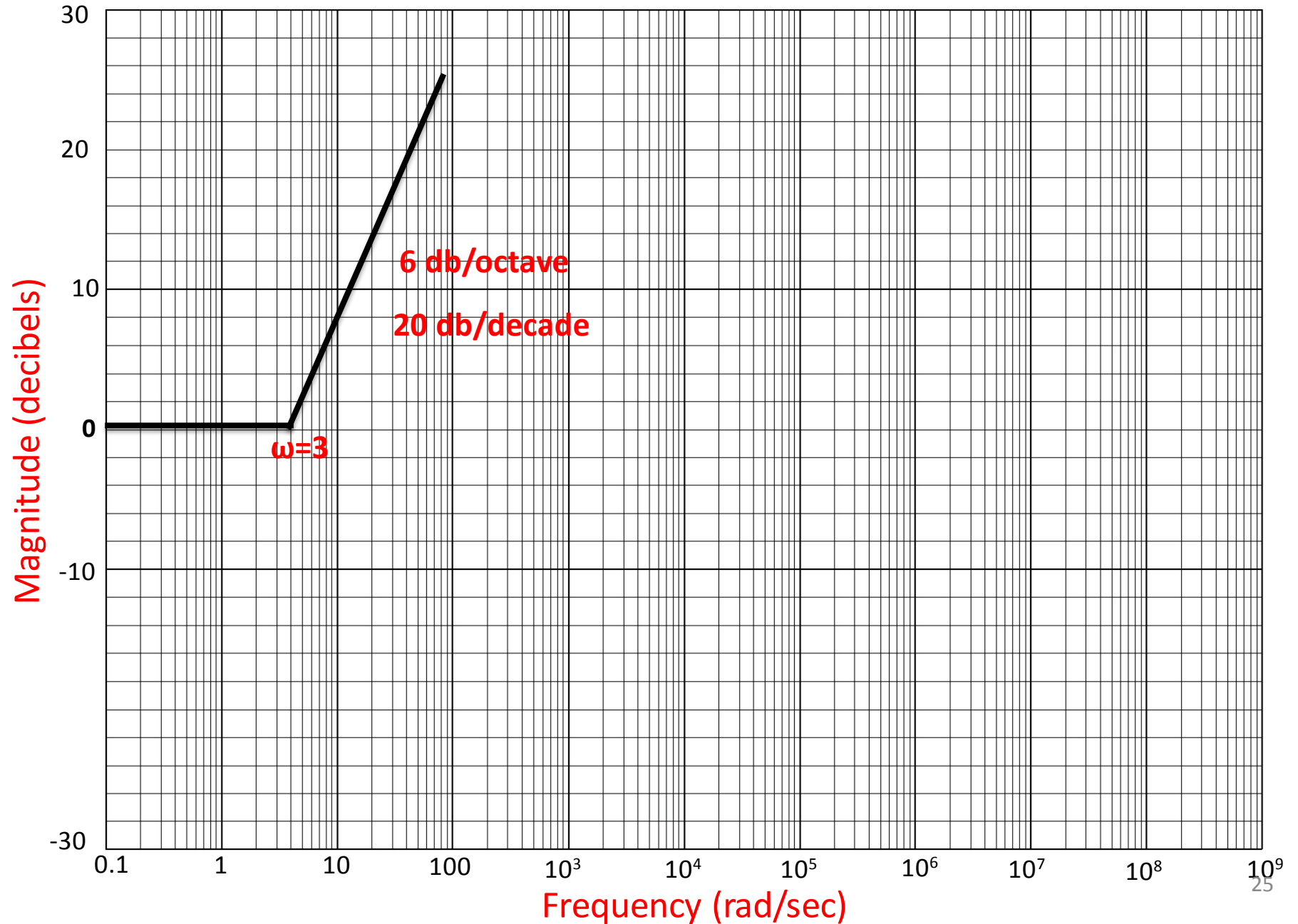
$$\phi(\omega) = \tan^{-1}(\omega T)$$

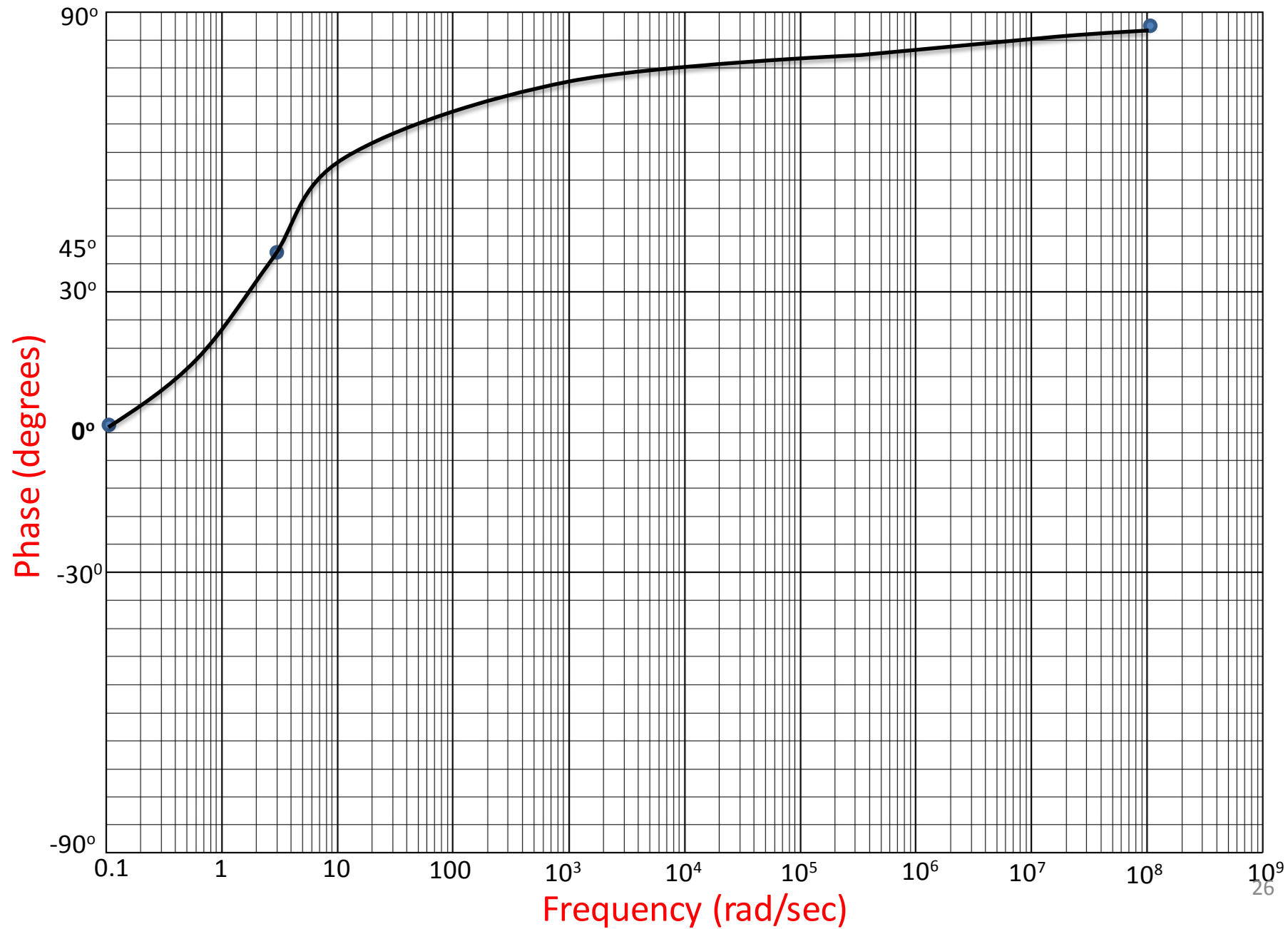
when  $\omega = 0$ ,  $\phi(\omega) = \tan^{-1}(0) = 0^\circ$

when  $\omega = \frac{1}{T}$ ,  $\phi(\omega) = \tan^{-1}(1) = 45^\circ$

when  $\omega = \infty$ ,  $\phi(\omega) = \tan^{-1}(\infty) = 90^\circ$

$$G(s) = (s + 3) = \left(\frac{1}{3}s + 1\right)$$





# Basic Factors of a Transfer Function

## 3. First Order Factors $(j\omega T + 1)^{-1}$

$$M(\omega) = -20 \log(|1 + j\omega T|)$$

$$M(\omega) = -20 \log(\sqrt{1 + \omega^2 T^2})$$

– For Low frequencies  $\omega \ll 1/T$

$$G(s) = \frac{1}{(s + 3)}$$

$$M(\omega) = -20 \log(1) = 0$$

– For high frequencies  $\omega \gg 1/T$

$$M(\omega) = -20 \log(\omega T)$$

# Basic Factors of a Transfer Function

## 3. First Order Factors $(j\omega T + 1)^{-1}$

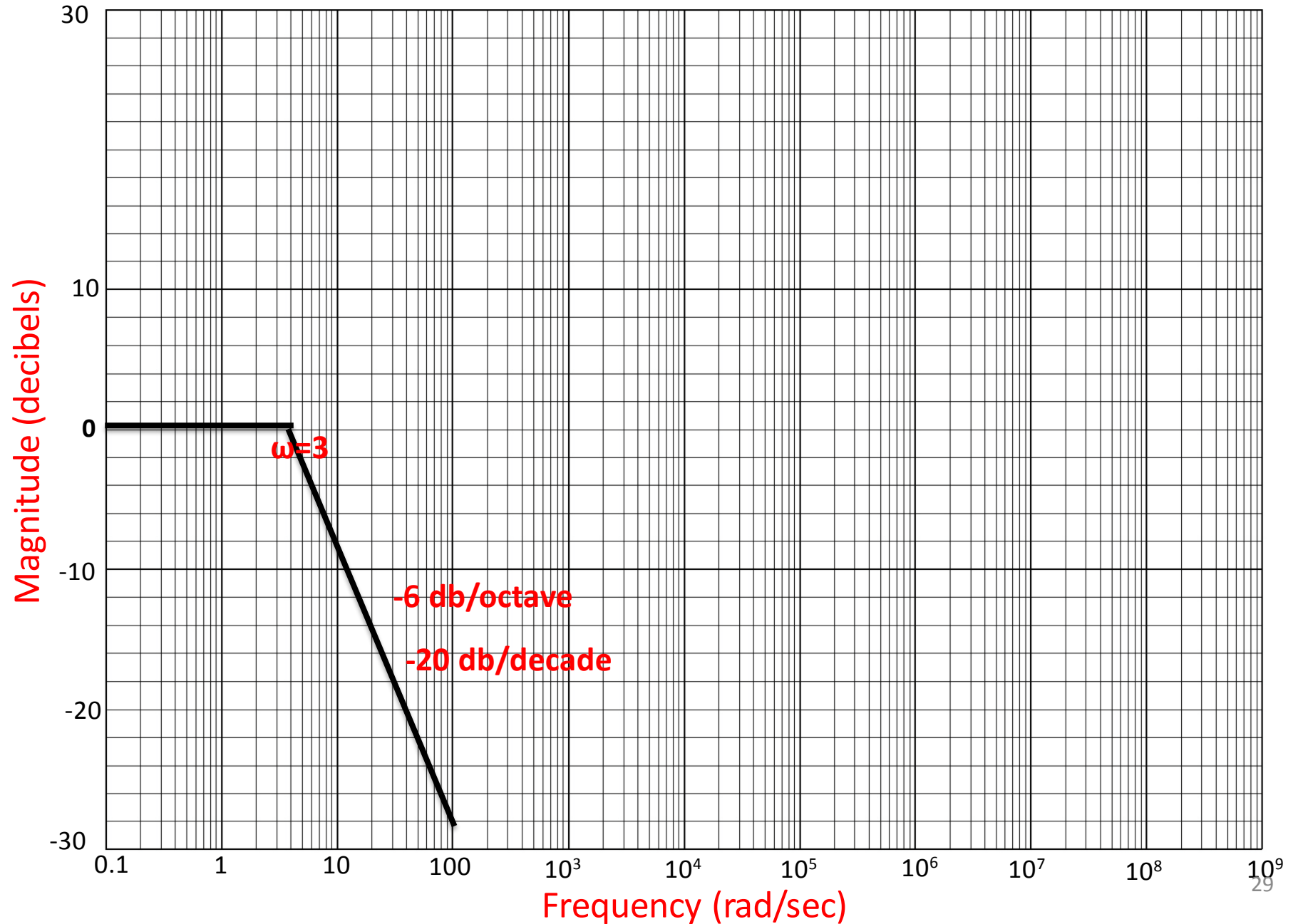
$$\phi(\omega) = -\tan^{-1}(\omega T)$$

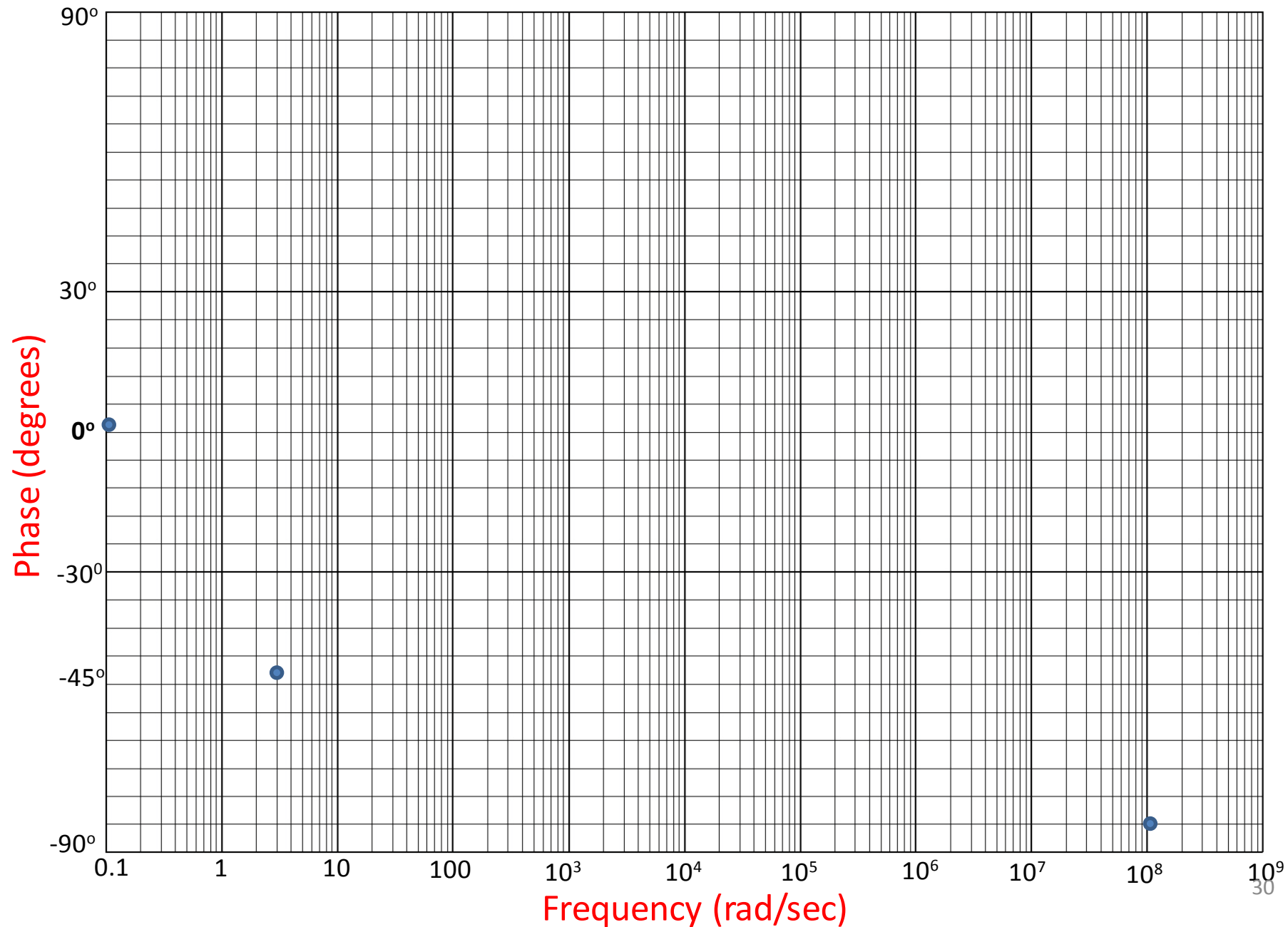
when  $\omega = 0$ ,  $\phi(\omega) = \tan^{-1}(0) = 0^\circ$

when  $\omega = \frac{1}{T}$ ,  $\phi(\omega) = -\tan^{-1}(1) = -45^\circ$

when  $\omega = \infty$ ,  $\phi(\omega) = -\tan^{-1}(\infty) = -90^\circ$

$$G(s) = \frac{1}{(s + 3)}$$







# Example#1

- Draw the Bode Plot of following Transfer function.

$$G(s) = \frac{20s}{(s + 10)}$$

**Solution:**

$$G(s) = \frac{2s}{(0.1s + 1)}$$

- The transfer function contains
  1. Gain Factor (**K=2**)
  2. Derivative Factor (**s**)
  3. 1<sup>st</sup> Order Factor in denominator (**0.1s+1**)<sup>-1</sup>

# Example#1

$$G(s) = \frac{2s}{(0.1s + 1)}$$

1. Gain Factor (**K=2**)

$$|K|_{db} = 20 \log(2) = 6 \text{ db}$$

2. Derivative Factor (**s**)

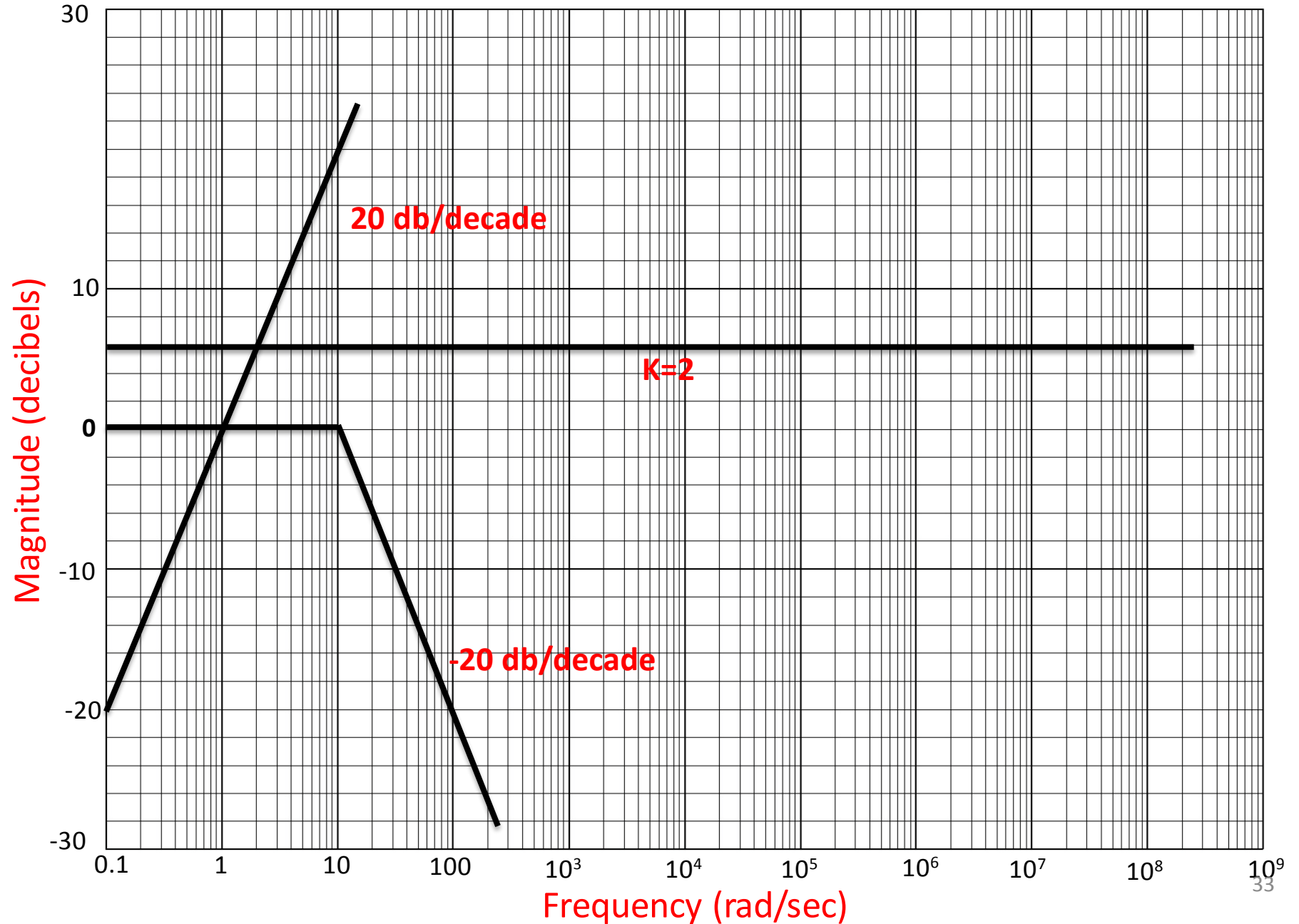
$$|s|_{db} = 20 \log(\omega) = 20 \text{ db/decade}$$

3. 1<sup>st</sup> Order Factor in denominator (**0.1s+1**)

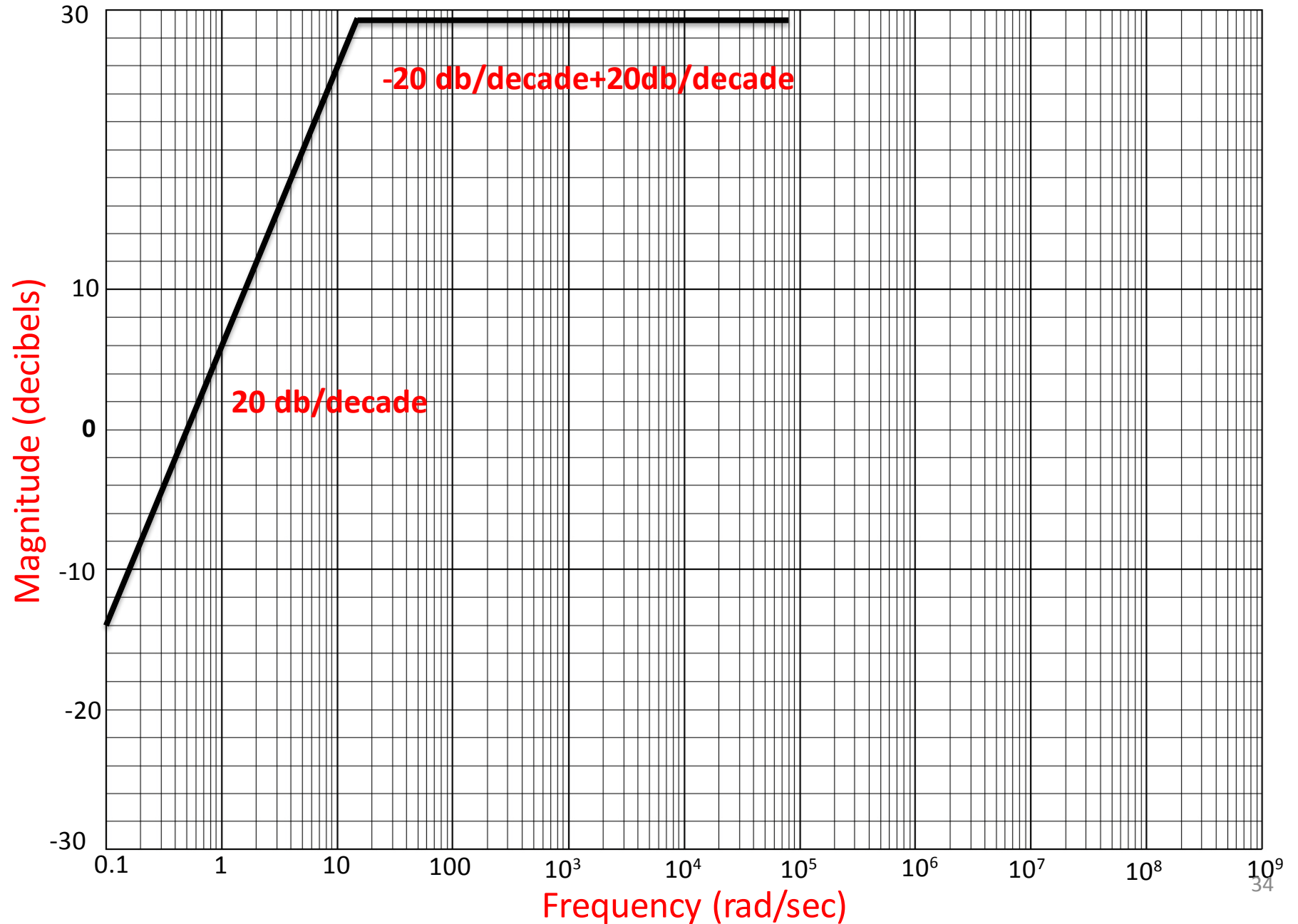
$$\text{when } \omega \ll 10, \left| \frac{1}{0.1j\omega + 1} \right|_{db} = -20 \log(1) = 0$$

$$\text{when } \omega \gg 10, \left| \frac{1}{0.1j\omega + 1} \right|_{db} = -20 \log(0.1\omega) = -20 \text{ db/decade}$$

$$G(s) = \frac{20s}{(s+10)}$$



$$G(s) = \frac{20s}{(s + 10)}$$



# Example#1

$$G(j\omega) = \frac{2j\omega}{(0.1j\omega + 1)}$$

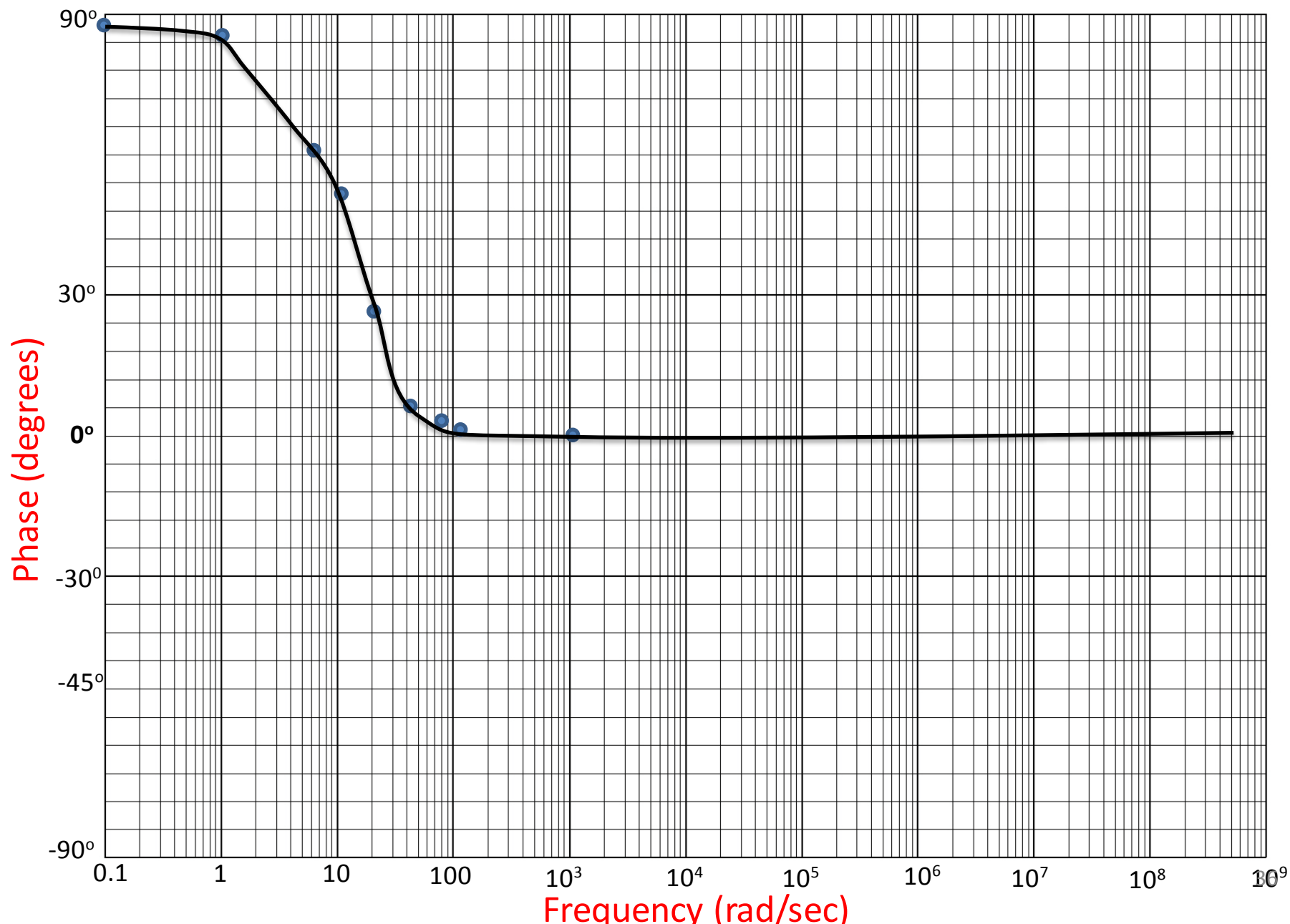
$$\angle G(j\omega) = \angle 2 + \angle j\omega - \angle(0.1j\omega + 1)$$

$$\angle G(j\omega) = \tan^{-1}\left(\frac{0}{2}\right) + \tan^{-1}\left(\frac{\omega}{0}\right) - \tan^{-1}(0.1\omega)$$

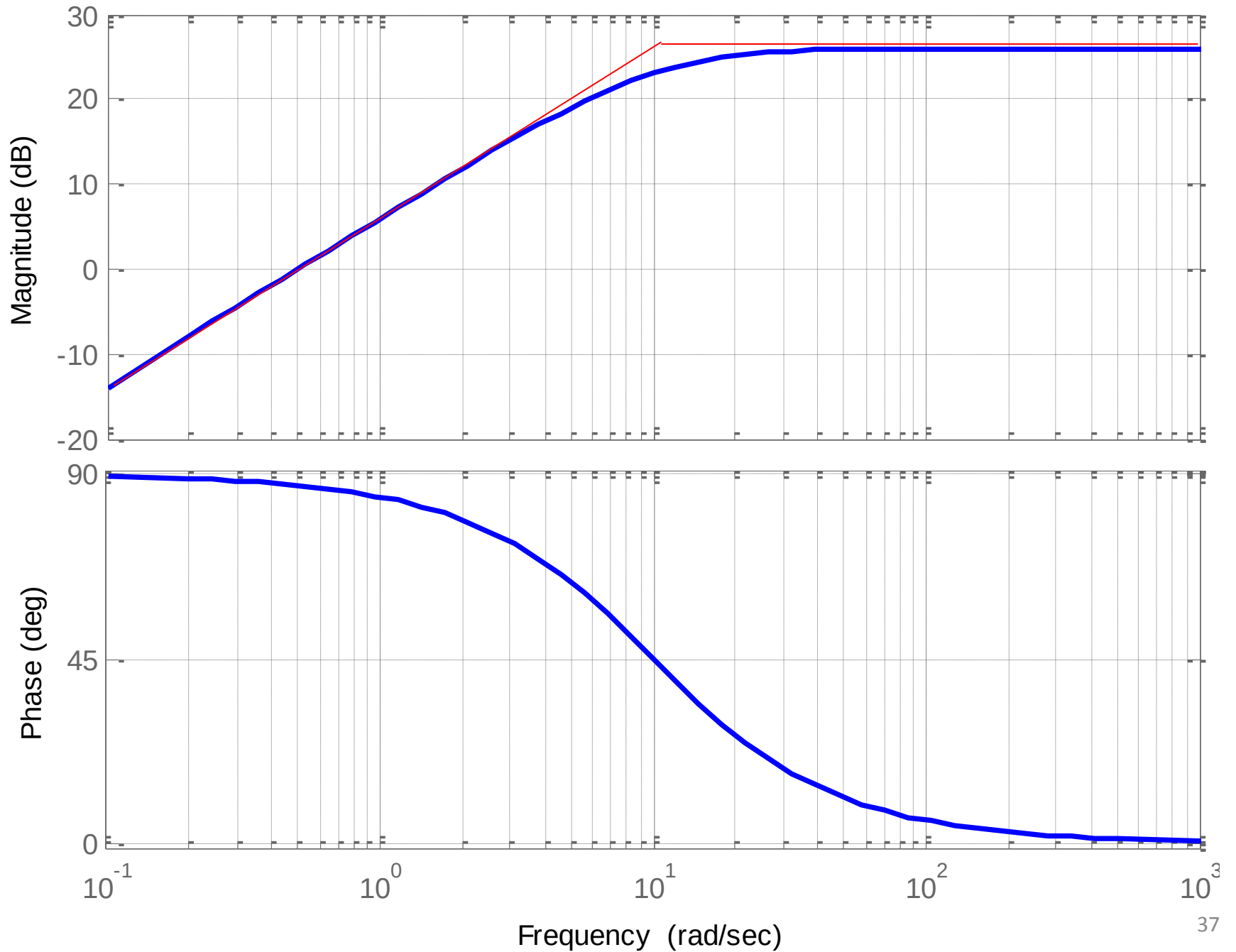
$$\angle G(j\omega) = 90^\circ - \tan^{-1}(0.1\omega)$$

| $\omega$       | 0.1  | 1    | 5    | 10 | 20   | 40 | 70 | 100 | 1000 | $\infty$ |
|----------------|------|------|------|----|------|----|----|-----|------|----------|
| $\Phi(\omega)$ | 89.4 | 84.2 | 63.4 | 45 | 26.5 | 14 | 8  | 5.7 | 0.5  | 0        |

| $\omega$       | 0.1  | 1    | 5    | 10 | 20   | 40 | 70 | 100 | 1000 | $\infty$ |
|----------------|------|------|------|----|------|----|----|-----|------|----------|
| $\Phi(\omega)$ | 89.4 | 84.2 | 63.4 | 45 | 26.5 | 14 | 8  | 5.7 | 0.5  | 0        |



Bode Diagram



## Example#2

$$G(s) = \frac{20(s + 3)}{s(s + 20)(s + 100)}$$

**Solution:**

$$G(s) = \frac{0.03(0.33s + 1)}{s(0.05s + 1)(0.01s + 1)}$$



# Basic Factors of a Transfer Function

## 4. Quadratic Factors $[1 + 2\zeta(j\omega/\omega_n) + (j\omega/\omega_n)^2]$

$$M(\omega) = 20 \log \sqrt{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\zeta \frac{\omega}{\omega_n}\right)^2}$$

- For Low frequencies  $\omega \ll \omega_n$

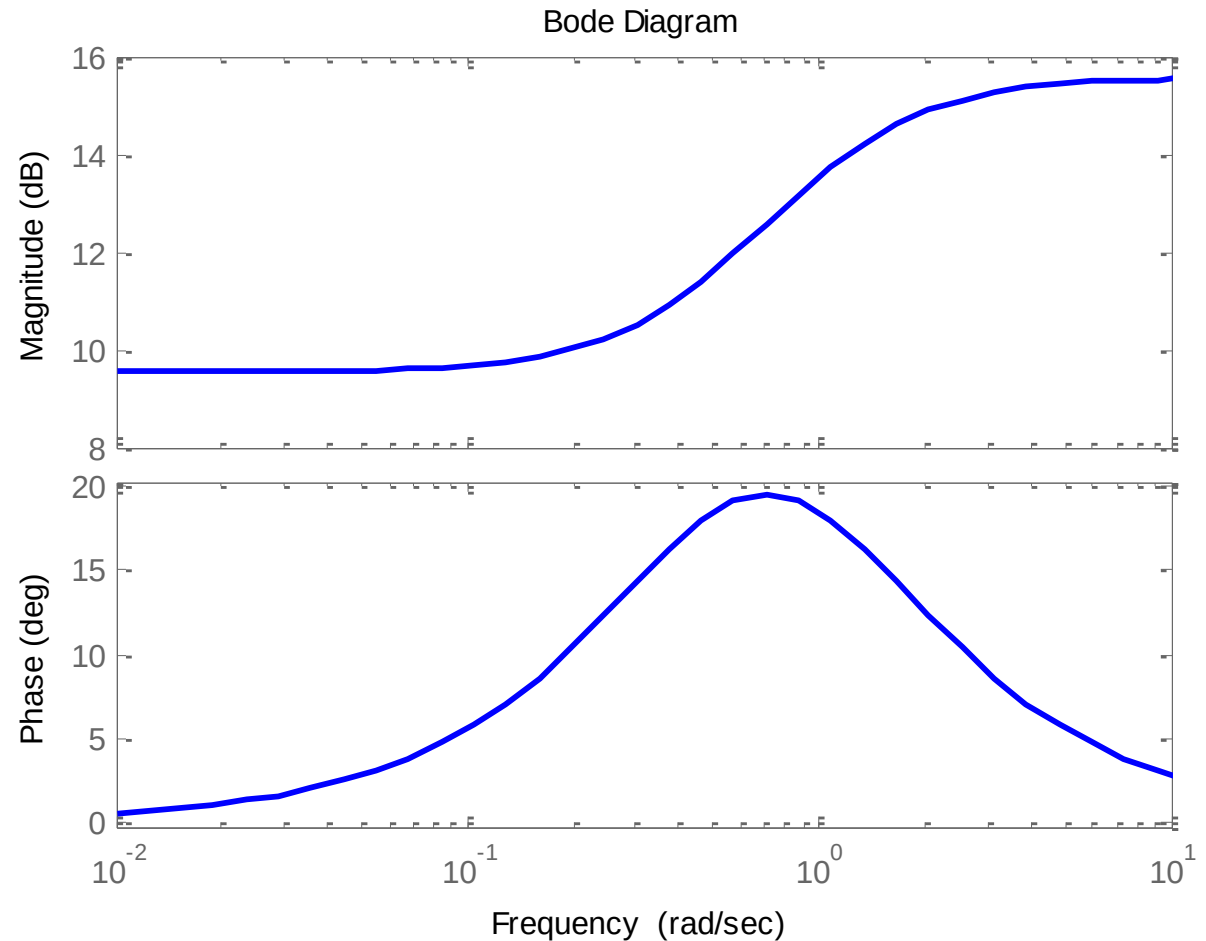
$$M(\omega) = 20 \log(1) = 0$$

- For high frequencies  $\omega \gg \omega_n$

$$M(\omega) = 40 \log\left(\frac{\omega}{\omega_n}\right) = 40 \text{db} / \text{dec}$$

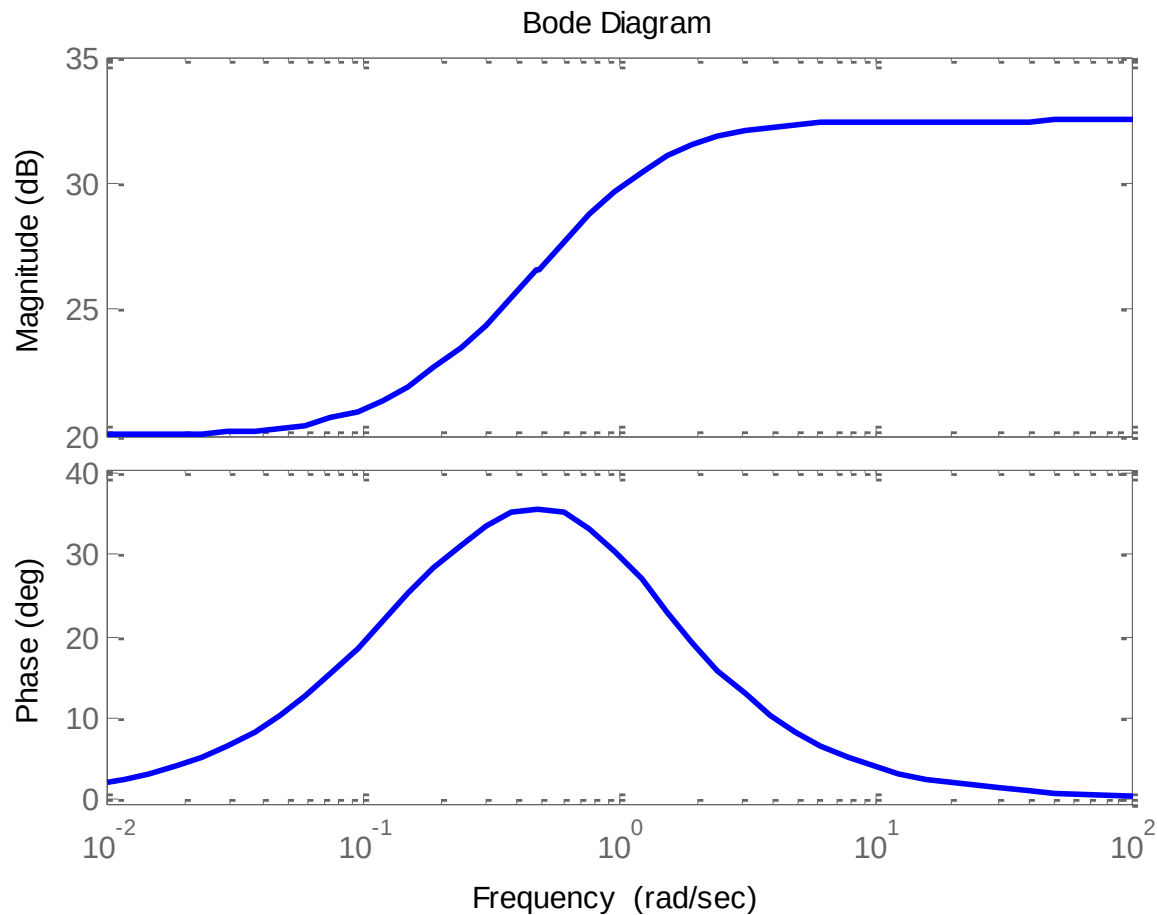
# Minimum-Phase & Non-minimum Phase Systems

$$G(s) = \frac{3(2s + 1)}{(s + 1)}$$



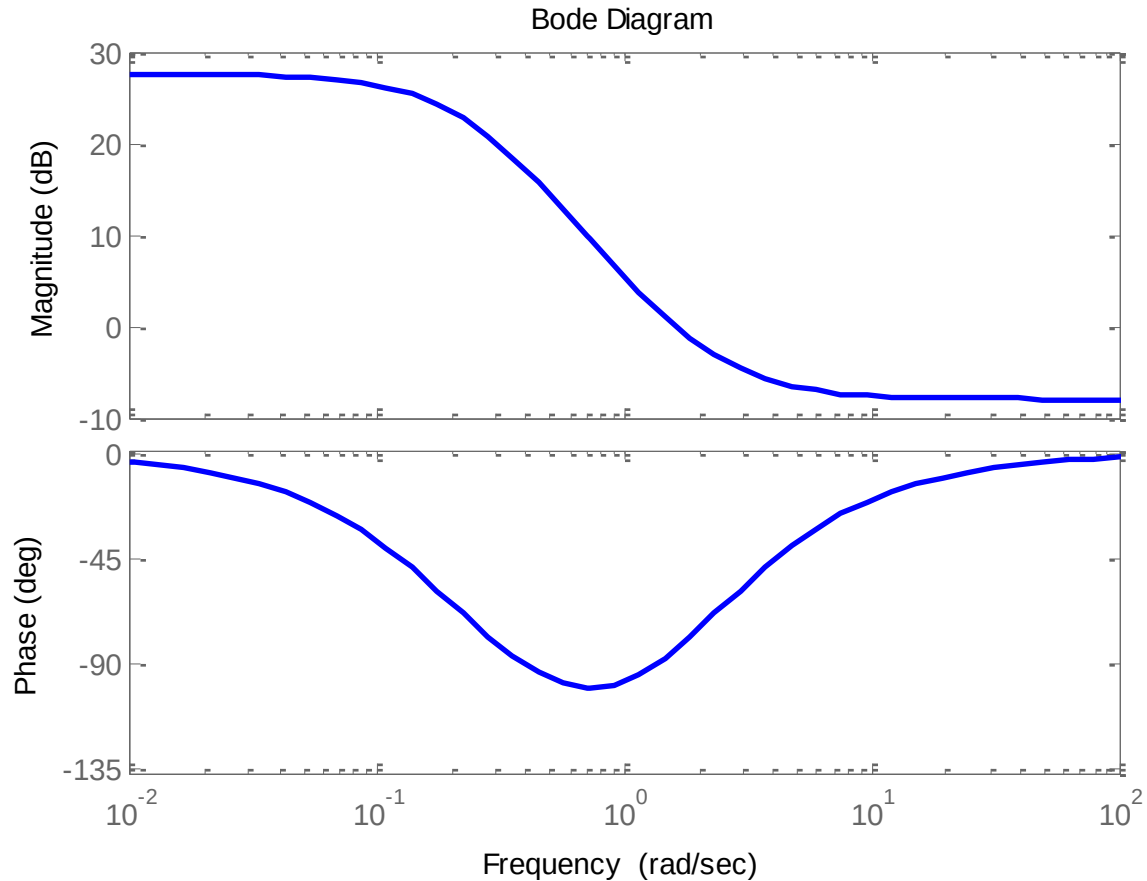
# Minimum-Phase & Non-minimum Phase Systems

$$G(s) = \frac{10(3s + 1)(7s + 1)}{(s + 1)(5s + 1)}$$



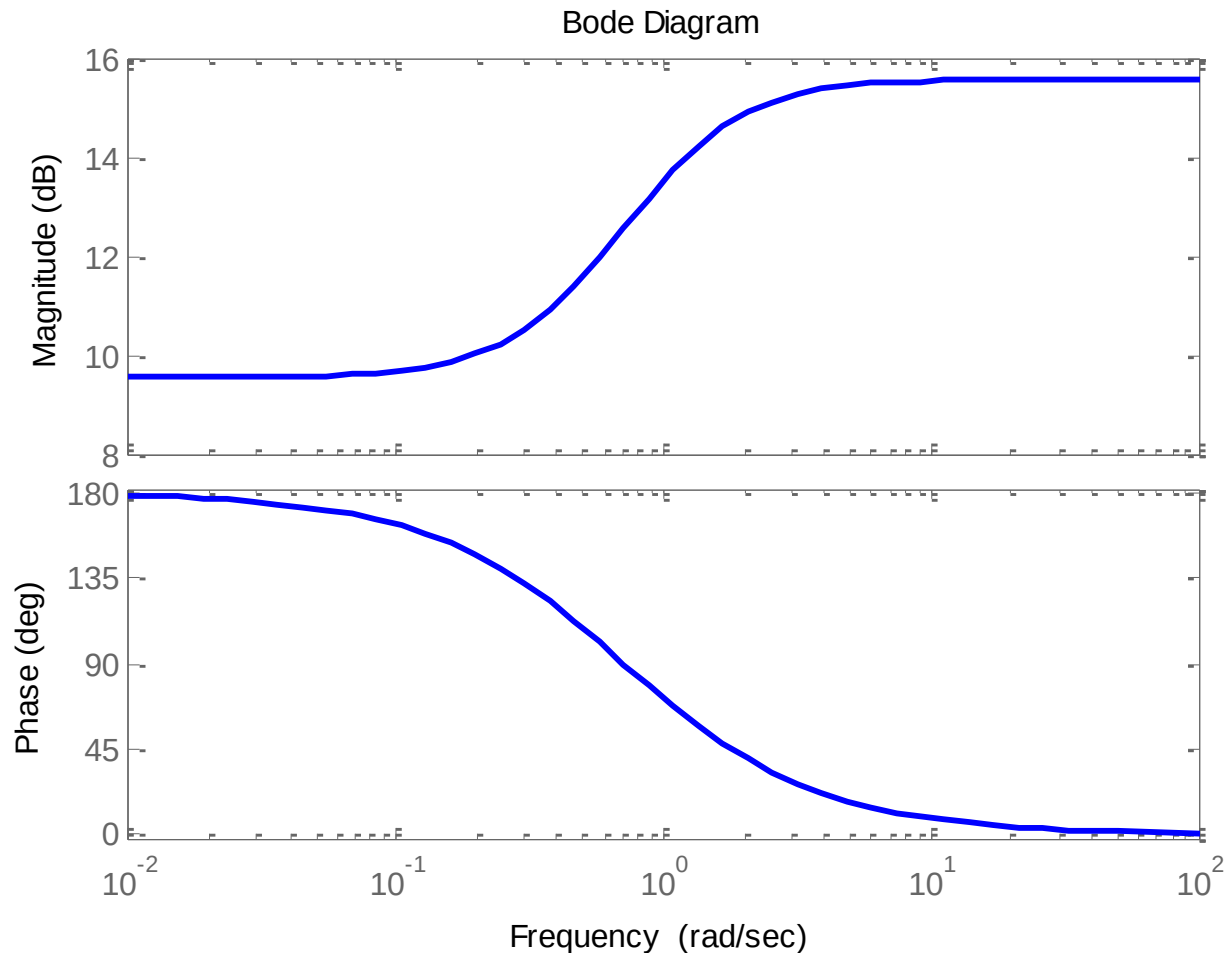
# Minimum-Phase & Non-minimum Phase Systems

$$G(s) = \frac{6(s^2 + 4s + 4)}{(3s + 1)(5s + 1)}$$



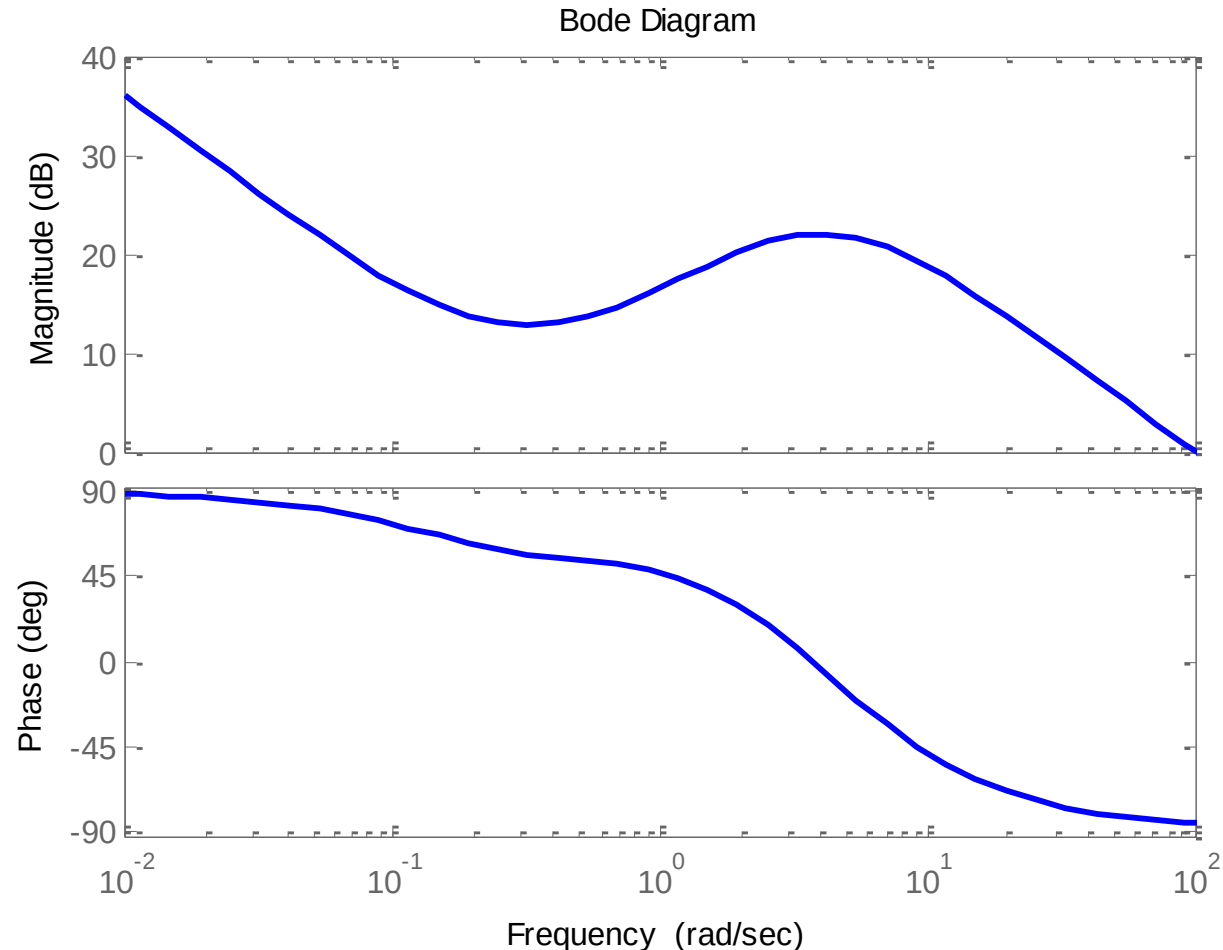
# Minimum-Phase & Non-minimum Phase Systems

$$G(s) = \frac{3(2s - 1)}{(s + 1)}$$



# Minimum-Phase & Non-minimum Phase Systems

$$G(s) = \frac{10(5s - 1)(2s + 1)}{s(s^2 + 8s + 16)}$$



# Minimum-Phase & Non-minimum Phase Systems

- Transfer functions having neither poles nor zeros in the right-half  $s$  plane are minimum-phase transfer functions.
- Whereas, those having poles and/or zeros in the right-half  $s$  plane are non-minimum-phase transfer functions.

# Relative Stability

- ❑ **Phase crossover frequency ( $\omega_p$ )** is the frequency at which the phase angle of the open-loop transfer function equals  $-180^\circ$ .
- ❑ **The gain crossover frequency ( $\omega_g$ )** is the frequency at which the magnitude of the open loop transfer function, is unity.
- ❑ The **gain margin ( $K_g$ )** is the reciprocal of the magnitude of  $G(j\omega)$  at the phase cross over frequency.

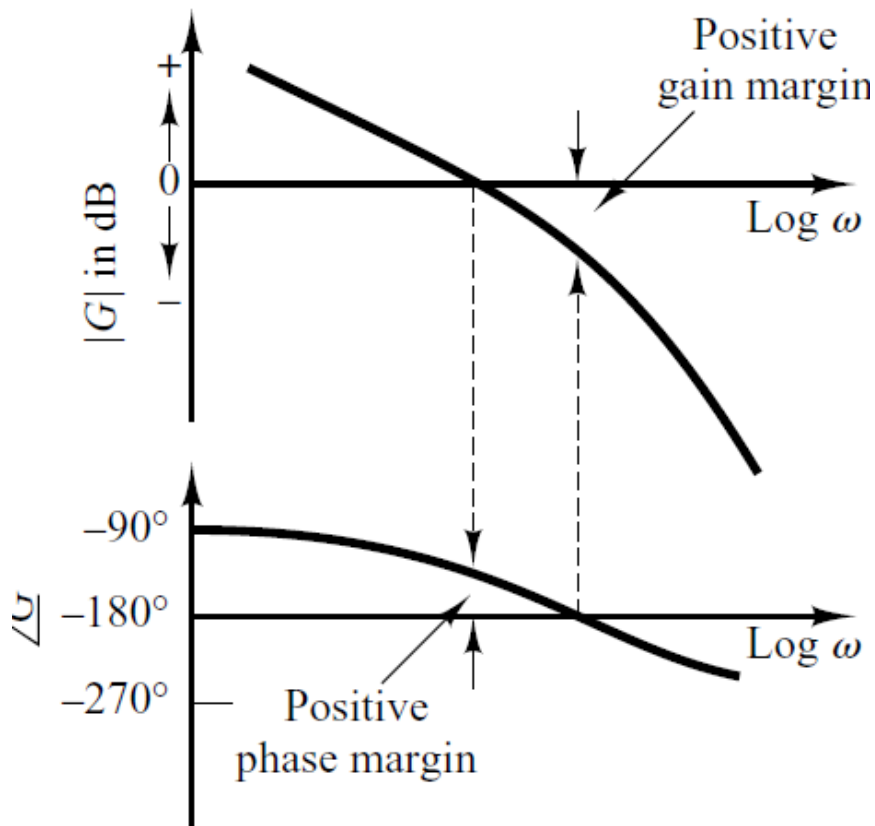
$$K_g = \frac{1}{|G(j\omega_1)|} \quad K_g \text{ dB} = 20 \log K_g = -20 \log |G(j\omega_1)|$$

- ❑ The **phase margin ( $\gamma$ )** is that amount of additional phase lag at the gain crossover frequency required to bring the system to the verge of instability.

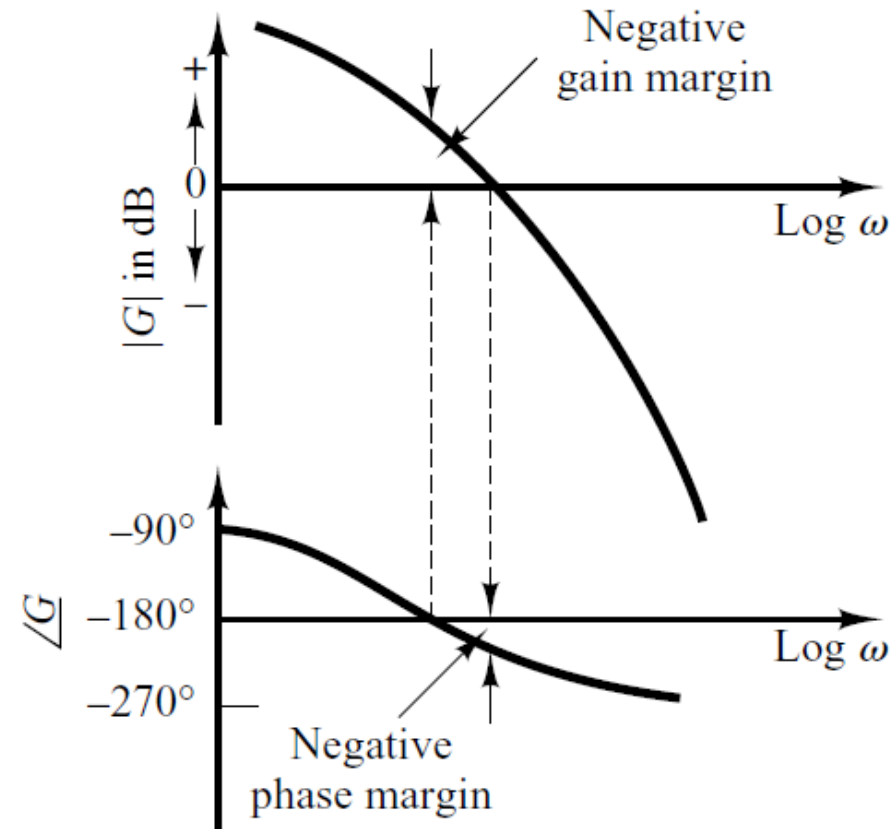


# Relative Stability الاستقرارية النسبية

## حد الكسب وحد الطور للمنظومات

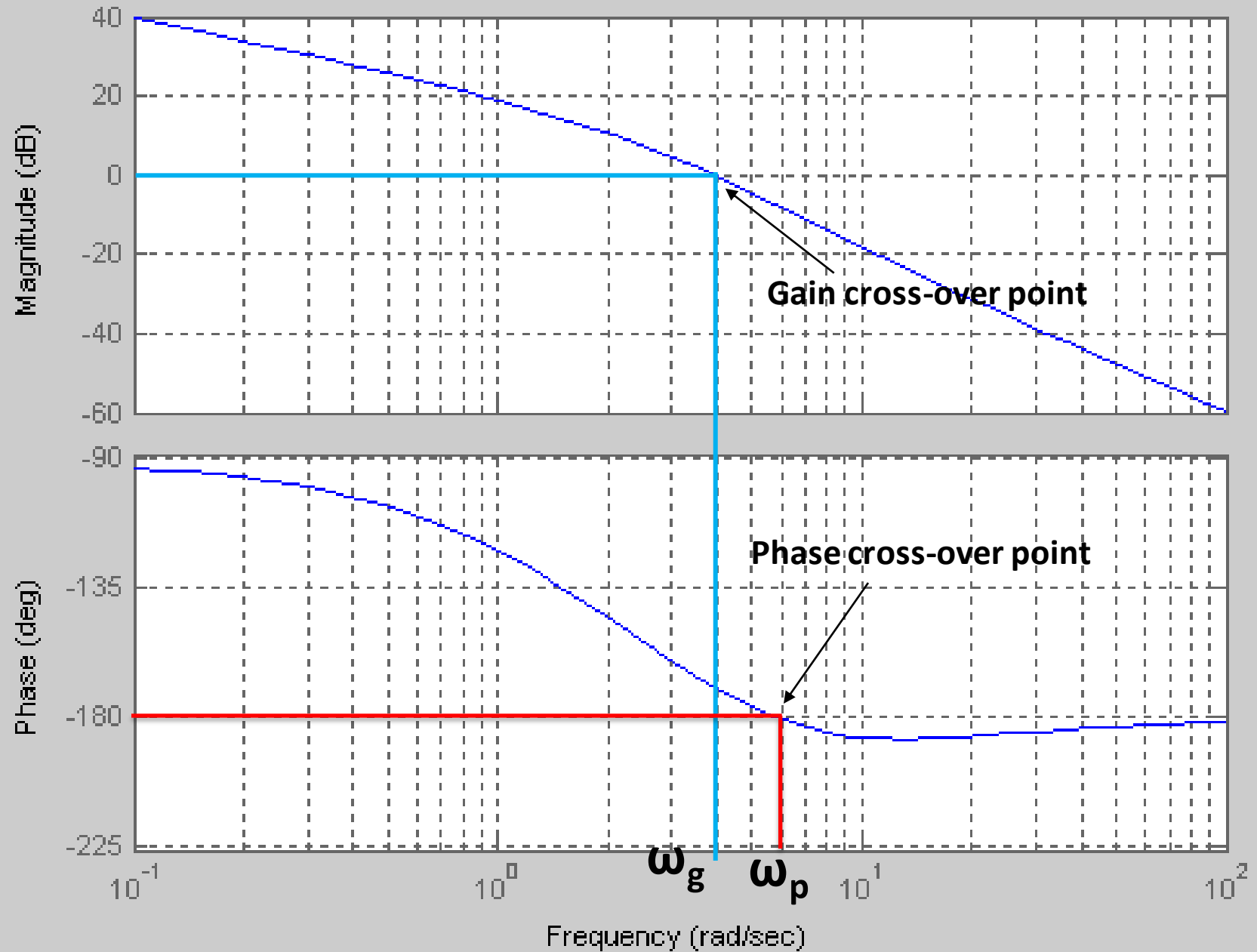


Stable system

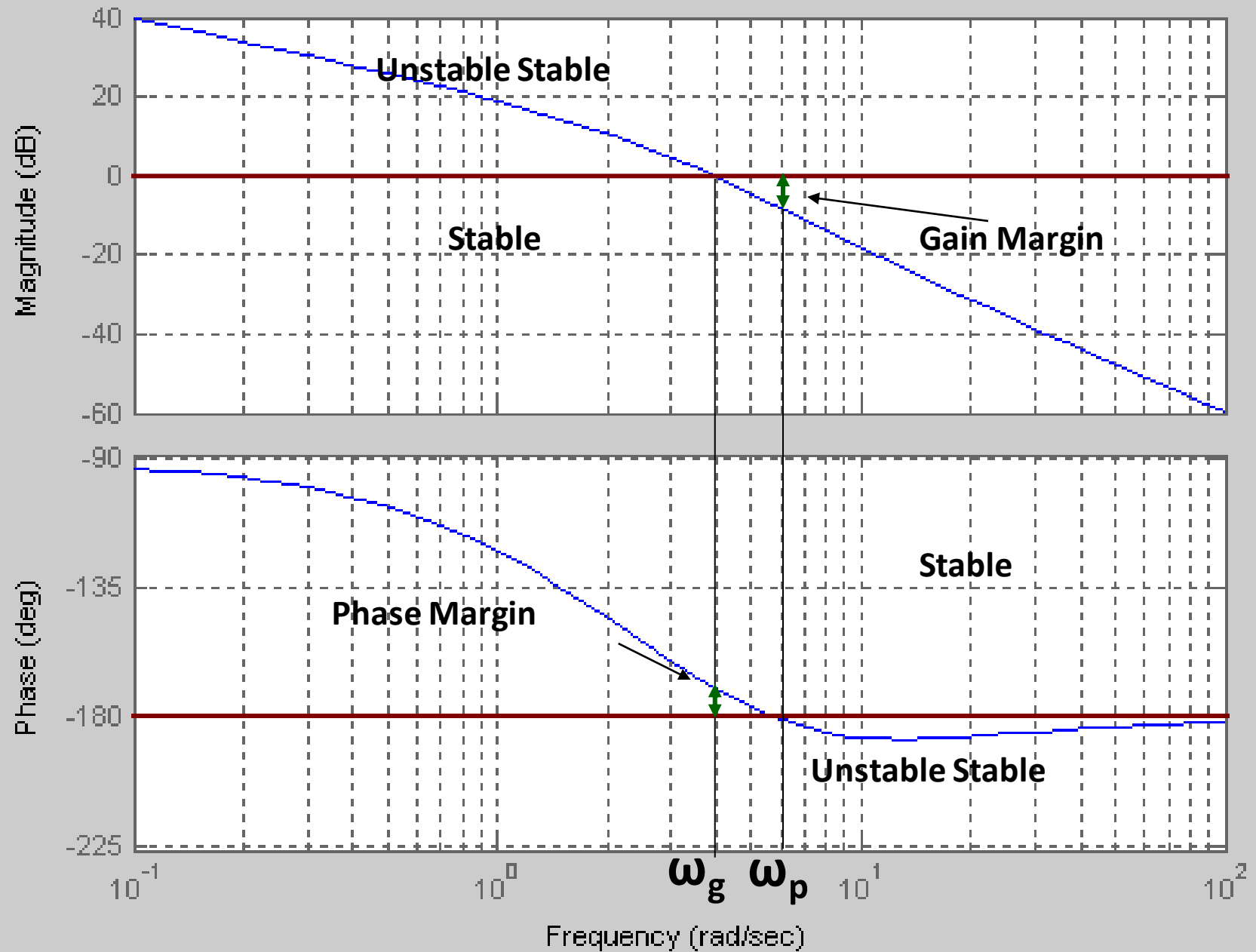


Unstable system

Bode Diagram

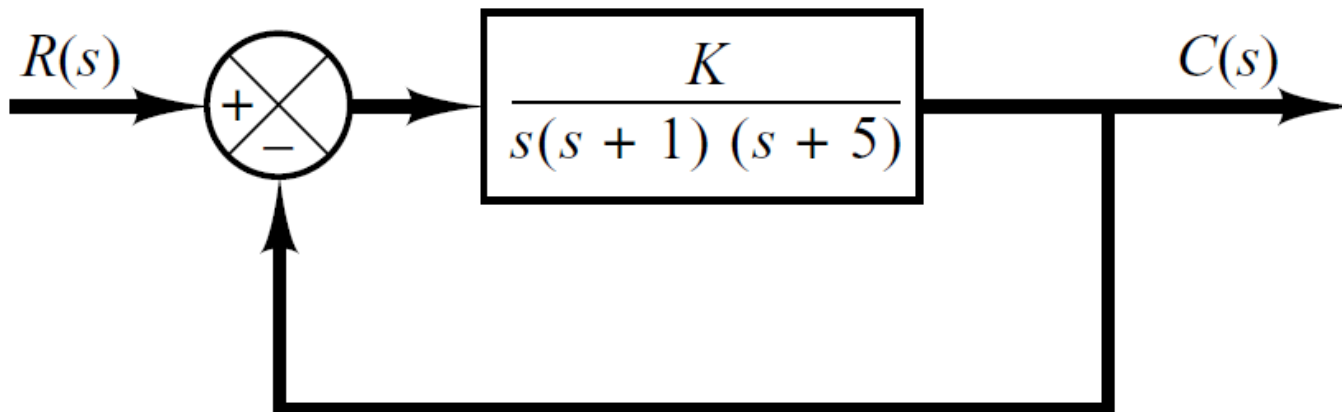


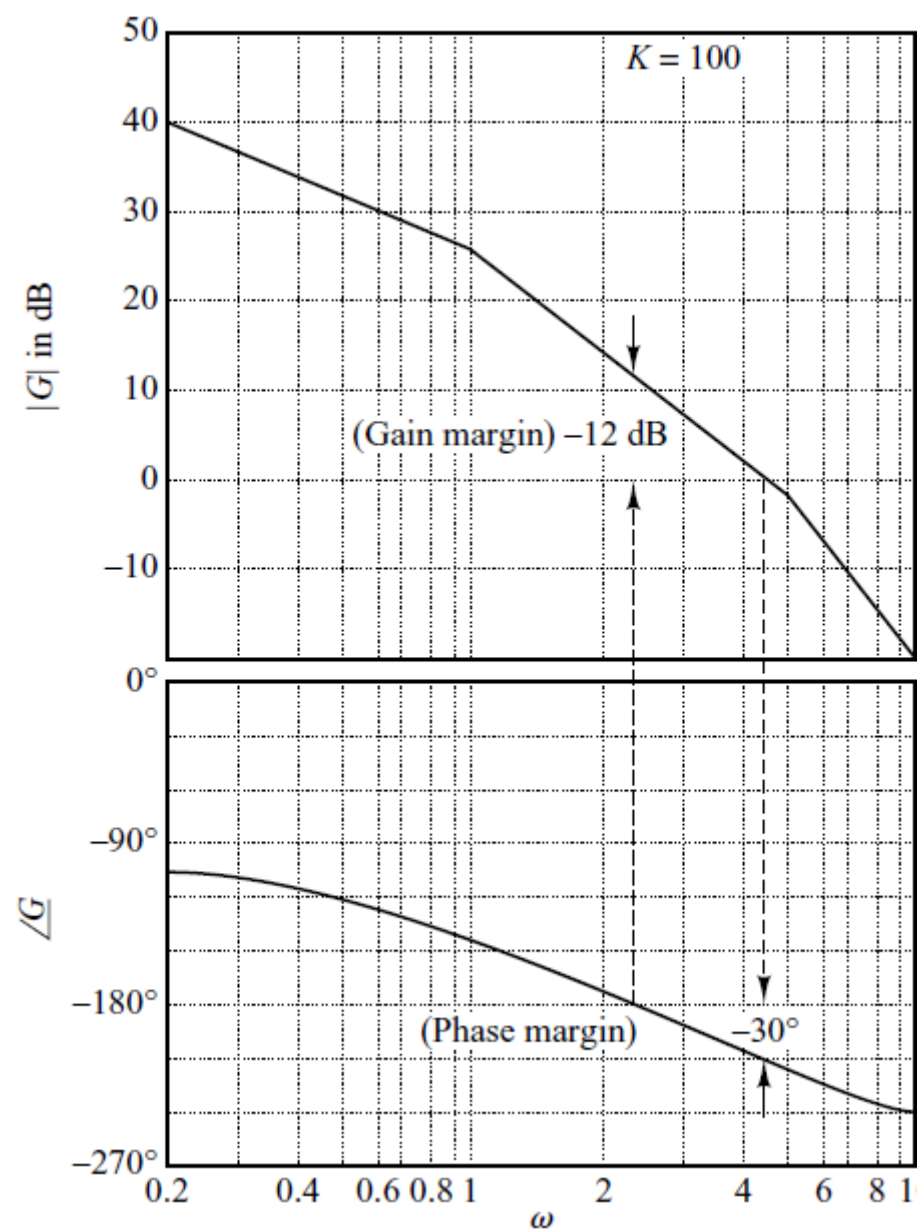
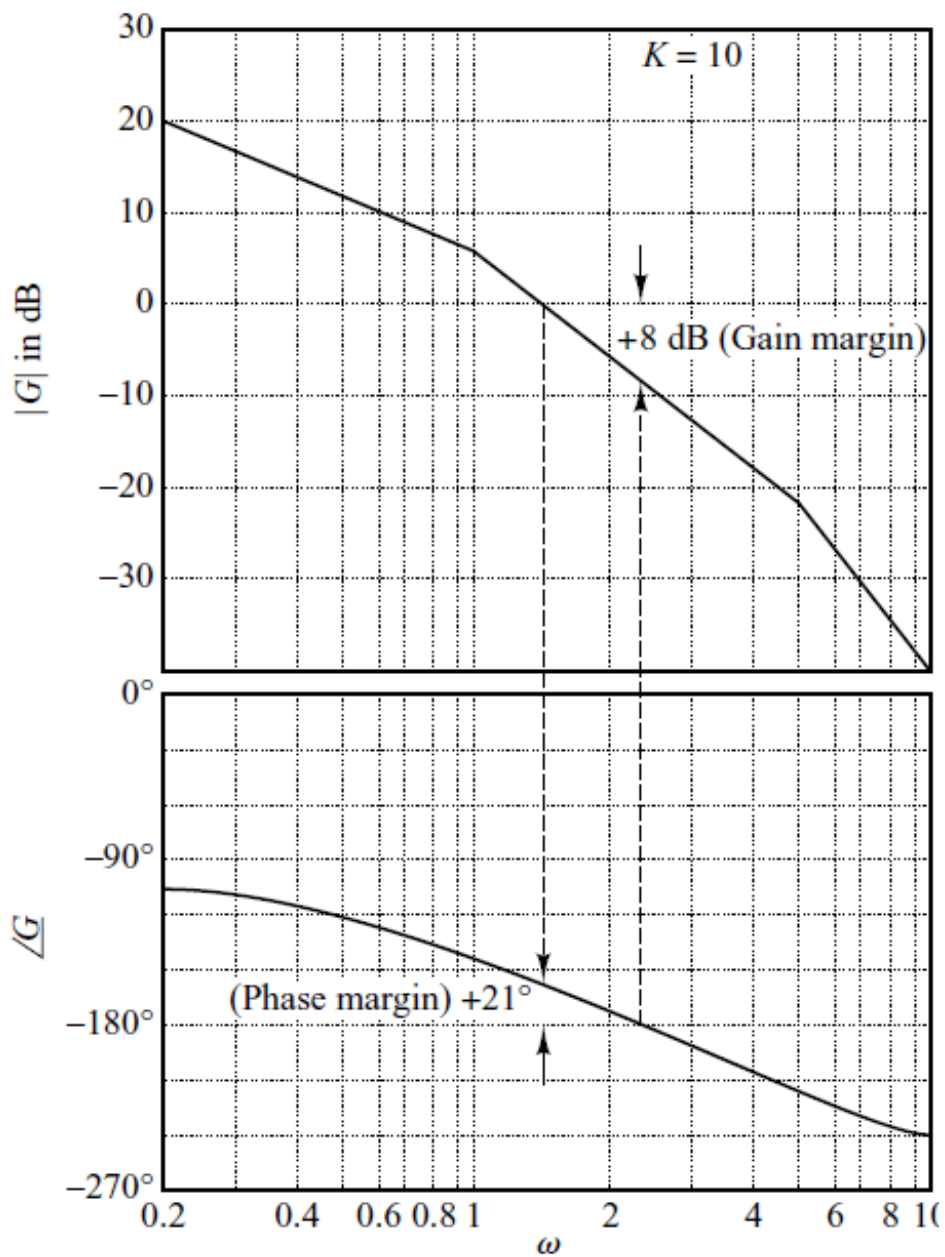
Bode Diagram



# Example#3

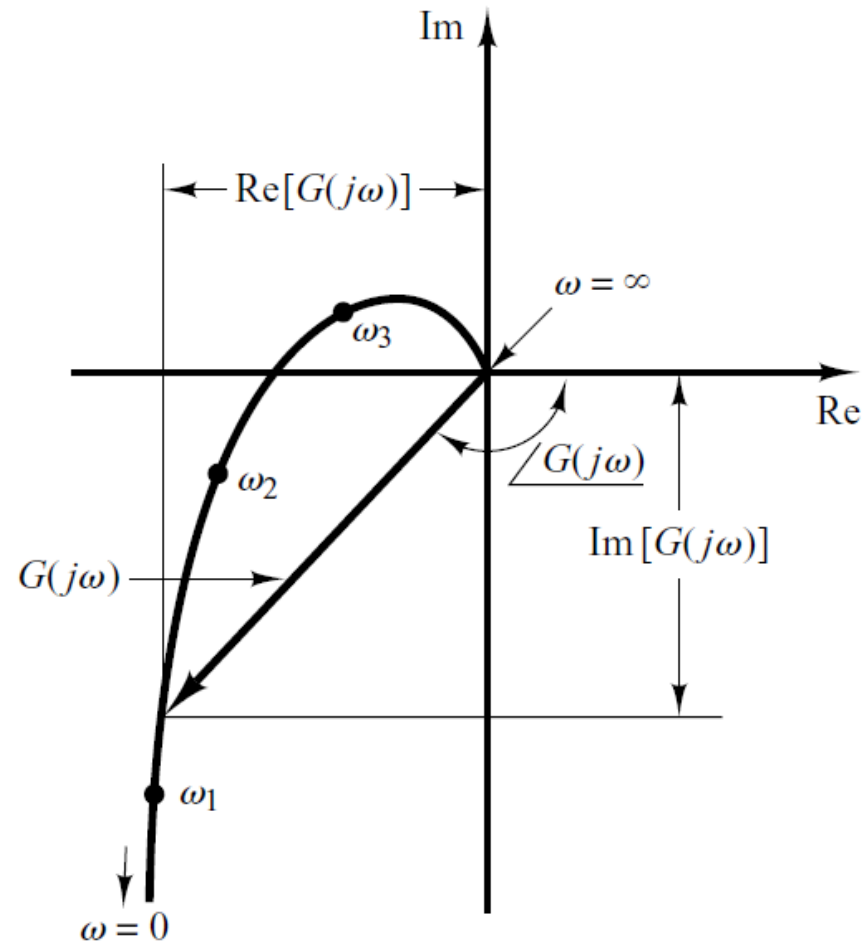
- Obtain the phase and gain margins of the system shown in following figure for the two cases where  $K=10$  and  $K=100$ .





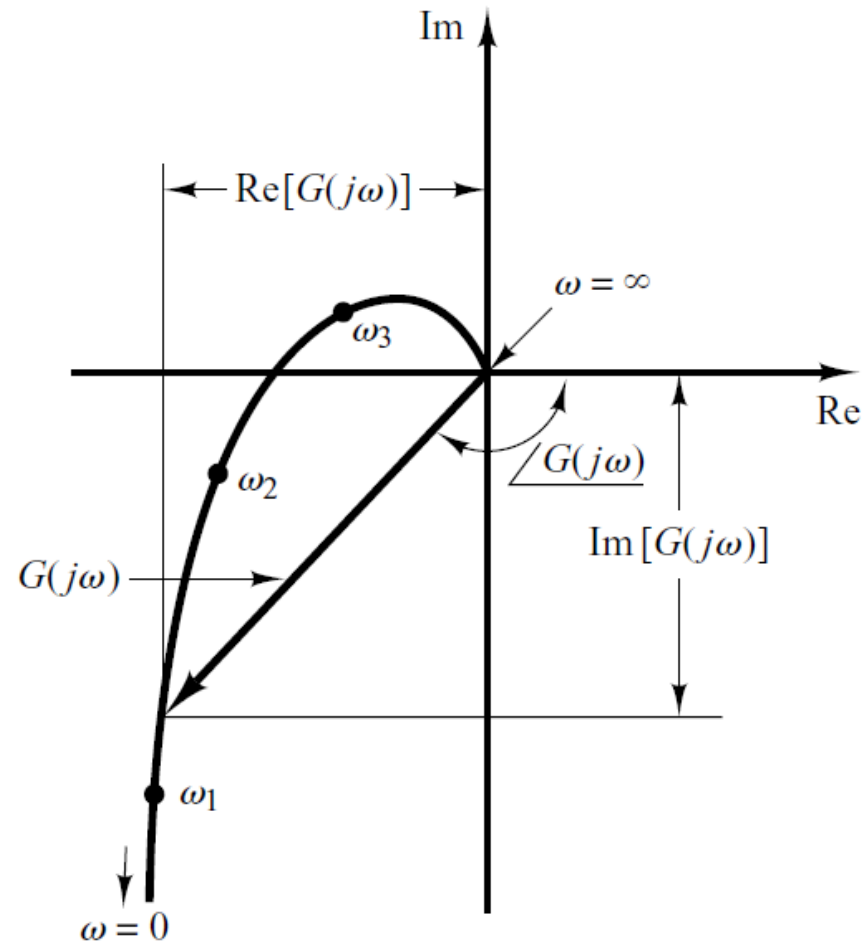
# Nyquist Plot (Polar Plot)

- The polar plot of a sinusoidal transfer function  $G(j\omega)$  is a plot of the magnitude of  $G(j\omega)$  versus the phase angle of  $G(j\omega)$  on polar coordinates as  $\omega$  is varied from zero to infinity.
- Thus, the polar plot is the locus of vectors  $|G(j\omega)| \angle G(j\omega)$  as  $\omega$  is varied from zero to infinity.



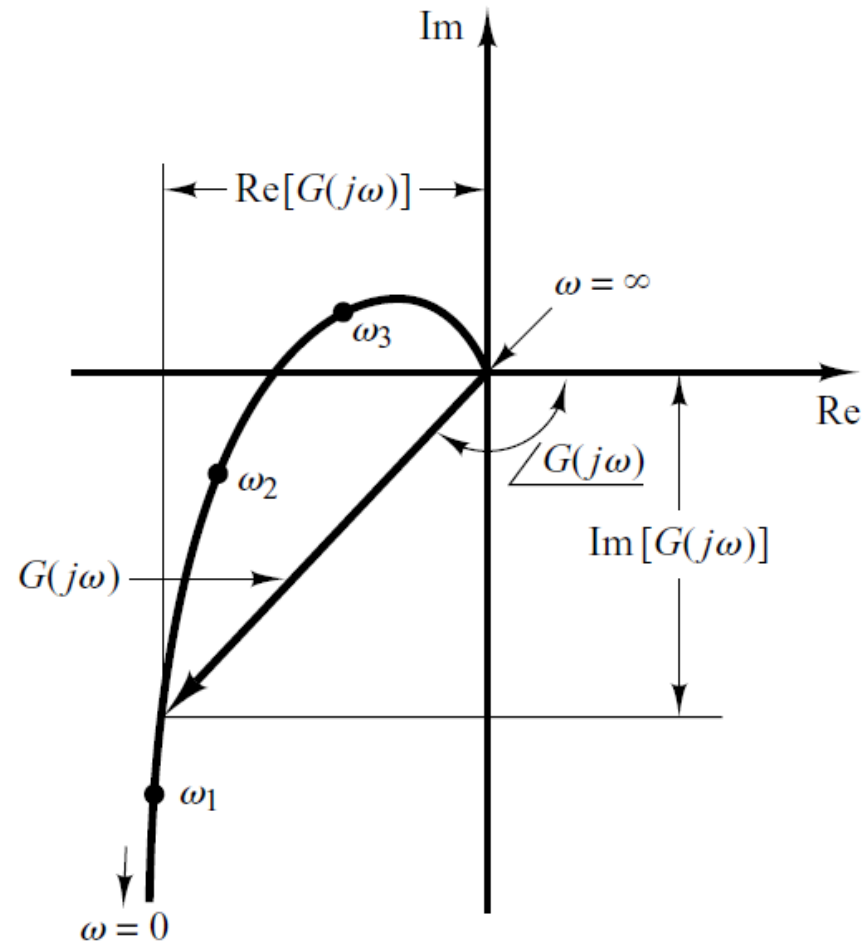
# Nyquist Plot (Polar Plot)

- Each point on the polar plot of  $G(j\omega)$  represents the terminal point of a vector at a particular value of  $\omega$ .
- The projections of  $G(j\omega)$  on the real and imaginary axes are its real and imaginary components.



# Nyquist Plot (Polar Plot)

- An advantage in using a polar plot is that it depicts the frequency response characteristics of a system over the entire frequency range in a single plot.
- One disadvantage is that the plot does not clearly indicate the contributions of each individual factor of the open-loop transfer function.





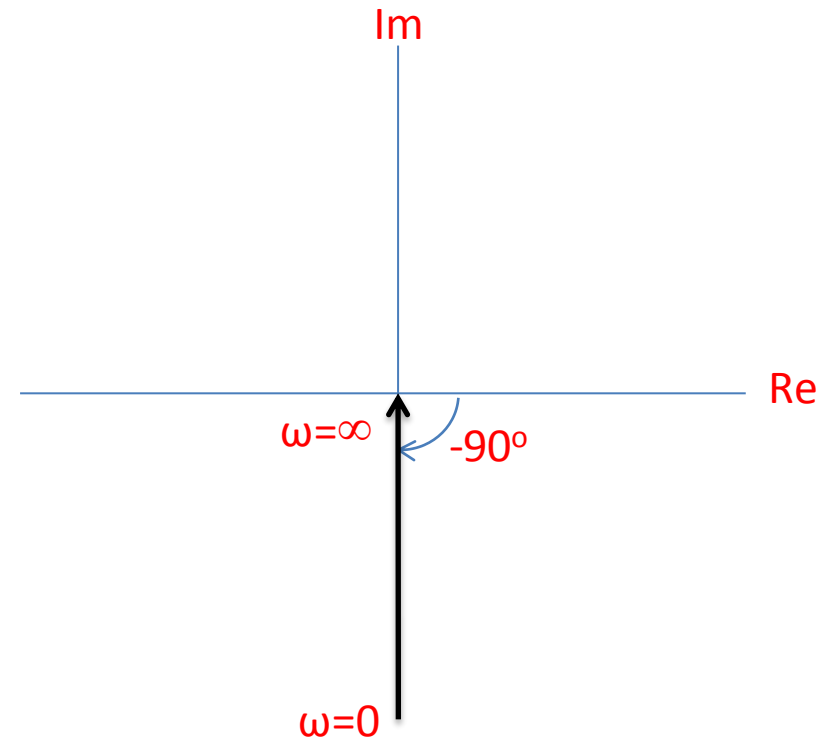
# Nyquist Plot of Integral and Derivative Factors

- The polar plot of  $G(j\omega)=1/j\omega$  is the negative imaginary axis, since

$$G(j\omega) = \frac{1}{j\omega}$$

$$G(j\omega) = \frac{1}{j\omega} \times \frac{-j\omega}{-j\omega} = -j \frac{1}{\omega}$$

In polar form  $G(j\omega) = \frac{1}{\omega} \angle -90^\circ$

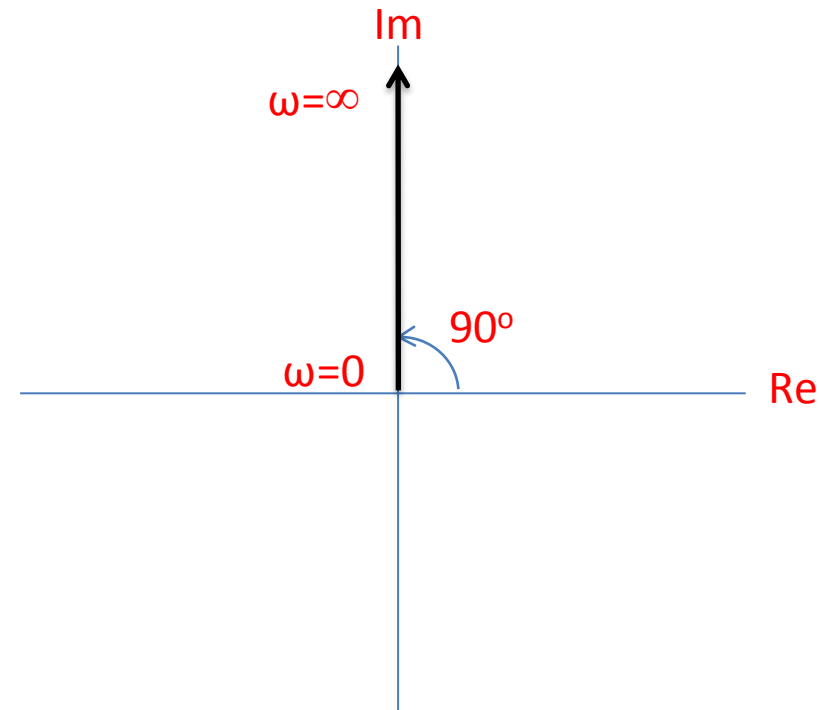


# Nyquist Plot of Integral and Derivative Factors

- The polar plot of  $G(j\omega)=j\omega$  is the positive imaginary axis, since

$$G(j\omega) = j\omega$$

In polar form  $G(j\omega) = \omega \angle 90^\circ$

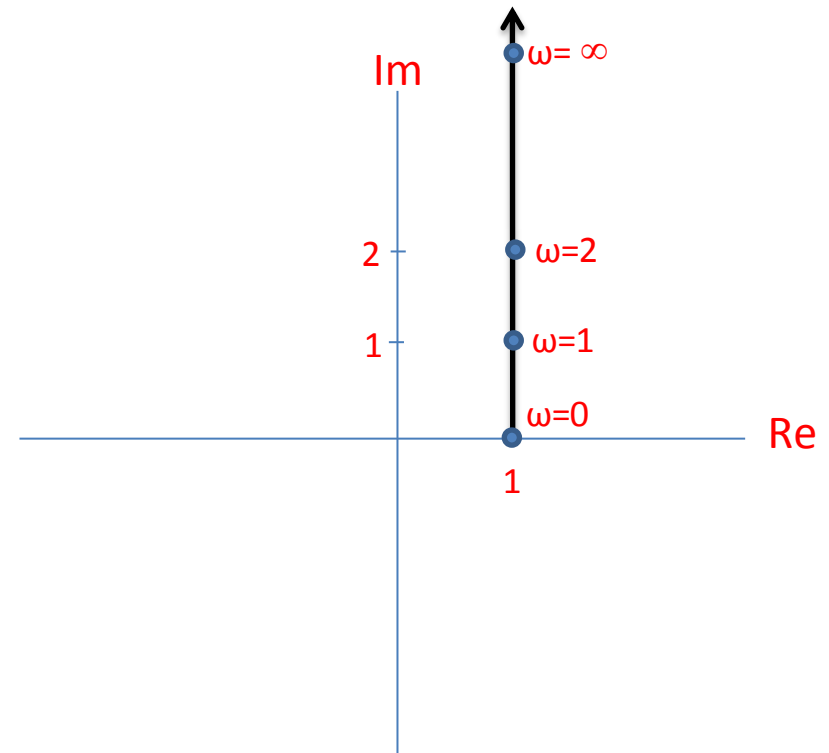


# Nyquist Plot of First Order Factors

- The polar plot of first order factor in numerator is

$$G(j\omega) = j\omega + 1$$

| $\omega$ | Re | Im       |
|----------|----|----------|
| 0        | 1  | 0        |
| 1        | 1  | 1        |
| 2        | 1  | 2        |
| $\infty$ | 1  | $\infty$ |



# Nyquist Plot of First Order Factors

- The polar plot of first order factor in denominator is

$$G(j\omega) = \frac{1}{j\omega + 1}$$

$$G(j\omega) = \frac{1}{j\omega + 1} \times \frac{1 - j\omega}{1 - j\omega}$$

$$G(j\omega) = \frac{1 - j\omega}{1 + \omega^2}$$

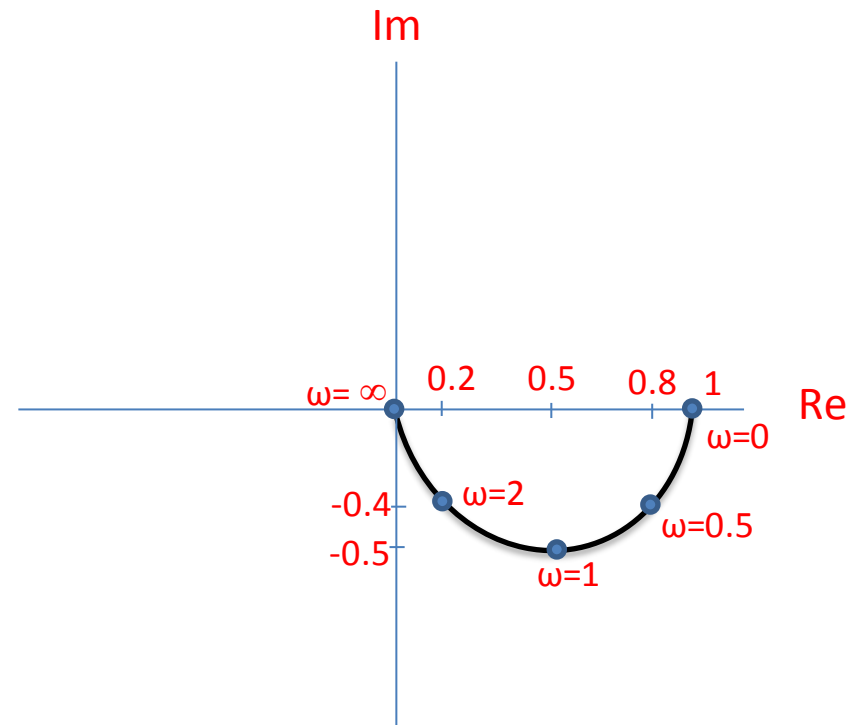
$$G(j\omega) = \frac{1}{1 + \omega^2} - j \frac{\omega}{1 + \omega^2}$$

| $\omega$ | Re  | Im   |
|----------|-----|------|
| 0        | 1   | 0    |
| 0.5      | 0.8 | 0.4  |
| 1        | 1/2 | -1/2 |
| 2        | 1/5 | -2/5 |
| $\infty$ | 0   | 0    |

# Nyquist Plot of First Order Factors

- The polar plot of first order factor in denominator is

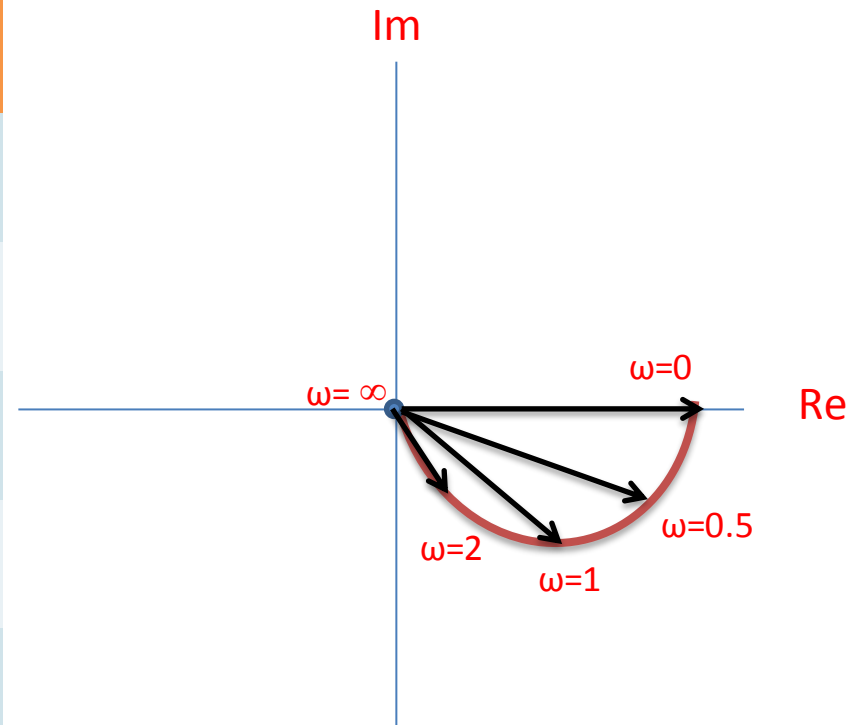
| $\omega$ | Re  | Im   |
|----------|-----|------|
| 0        | 1   | 0    |
| 0.5      | 0.8 | -0.4 |
| 1        | 0.5 | -0.5 |
| 2        | 0.2 | -0.4 |
| $\infty$ | 0   | 0    |



# Nyquist Plot of First Order Factors

- The polar plot of first order factor in denominator is

| $\omega$ | Re  | Im   | $ G(j\omega) $ | $\angle G(j\omega)$ |
|----------|-----|------|----------------|---------------------|
| 0        | 1   | 0    | 1              | $0^\circ$           |
| 0.5      | 0.8 | -0.4 | 0.9            | $-26^\circ$         |
| 1        | 0.5 | -0.5 | 0.7            | $-45^\circ$         |
| 2        | 0.2 | -0.4 | 0.4            | $-63^\circ$         |
| $\infty$ | 0   | 0    | 0              | $-90^\circ$         |



# Example#1

- Draw the polar plot of following open loop transfer function.

**Solution**

$$G(s) = \frac{1}{s(s+1)}$$

Put  $s = j\omega$

$$G(j\omega) = \frac{1}{j\omega(j\omega+1)}$$

$$G(j\omega) = \frac{1}{-\omega^2 + j\omega}$$

$$G(j\omega) = \frac{1}{-\omega^2 + j\omega} \times \frac{-\omega^2 - j\omega}{-\omega^2 - j\omega}$$

$$G(j\omega) = \frac{-\omega^2 - j\omega}{\omega^4 + \omega^2}$$

# Example#1

$$G(j\omega) = \frac{-\omega^2 - j\omega}{\omega^4 + \omega^2}$$

$$G(j\omega) = \frac{-\omega^2}{\omega^4 + \omega^2} - j \frac{\omega}{\omega^4 + \omega^2}$$

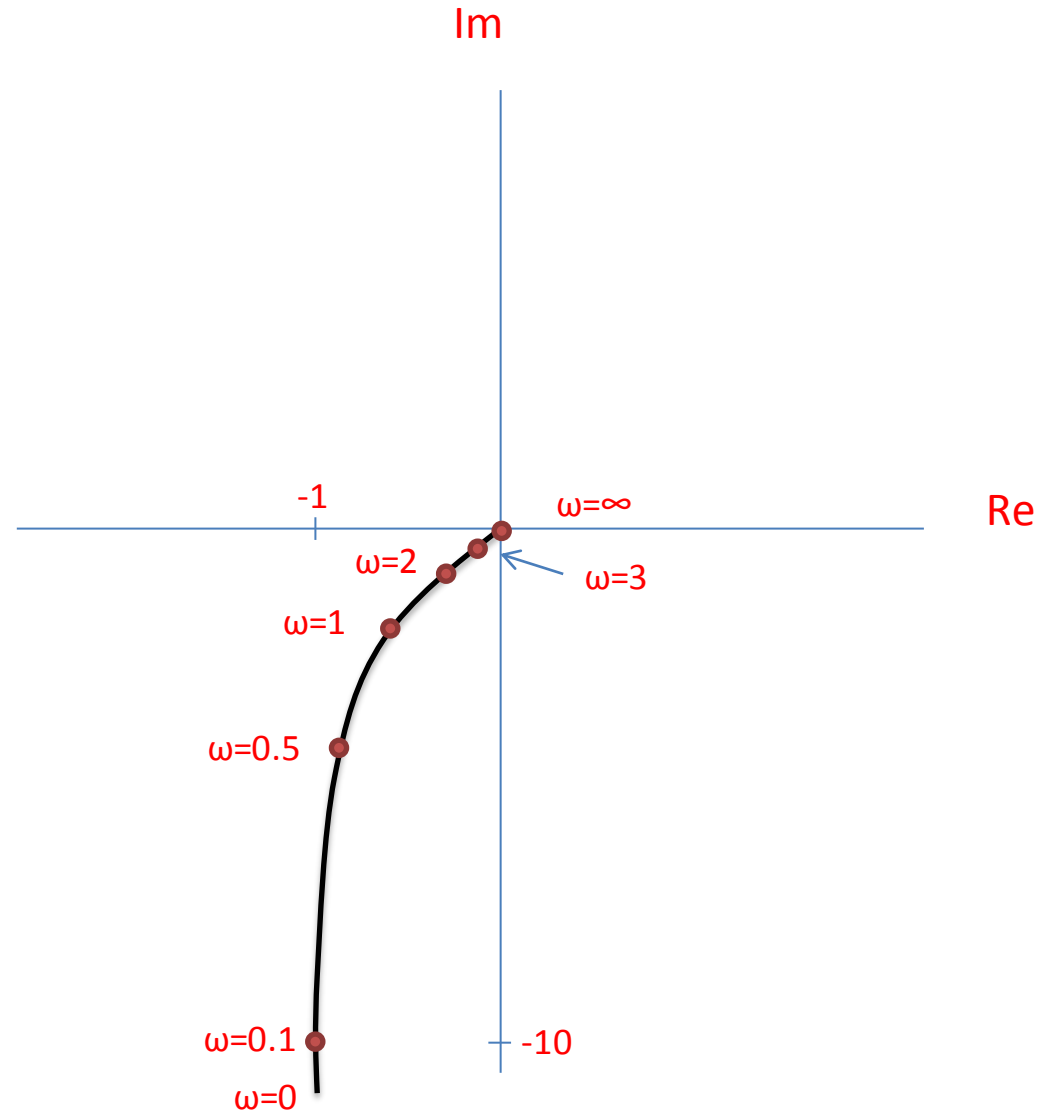
$$G(j\omega) = \frac{-1}{\omega^2 + 1} - j \frac{1}{\omega(\omega^2 + 1)}$$

| $\omega$ | Re       | Im       |
|----------|----------|----------|
| 0        | $\infty$ | $\infty$ |
| 0.1      | -1       | -10      |
| 0.5      | -0.8     | -1.6     |
| 1        | -0.5     | -0.5     |
| 2        | -0.2     | -0.1     |
| 3        | -0.1     | -0.03    |
| $\infty$ | 0        | 0        |



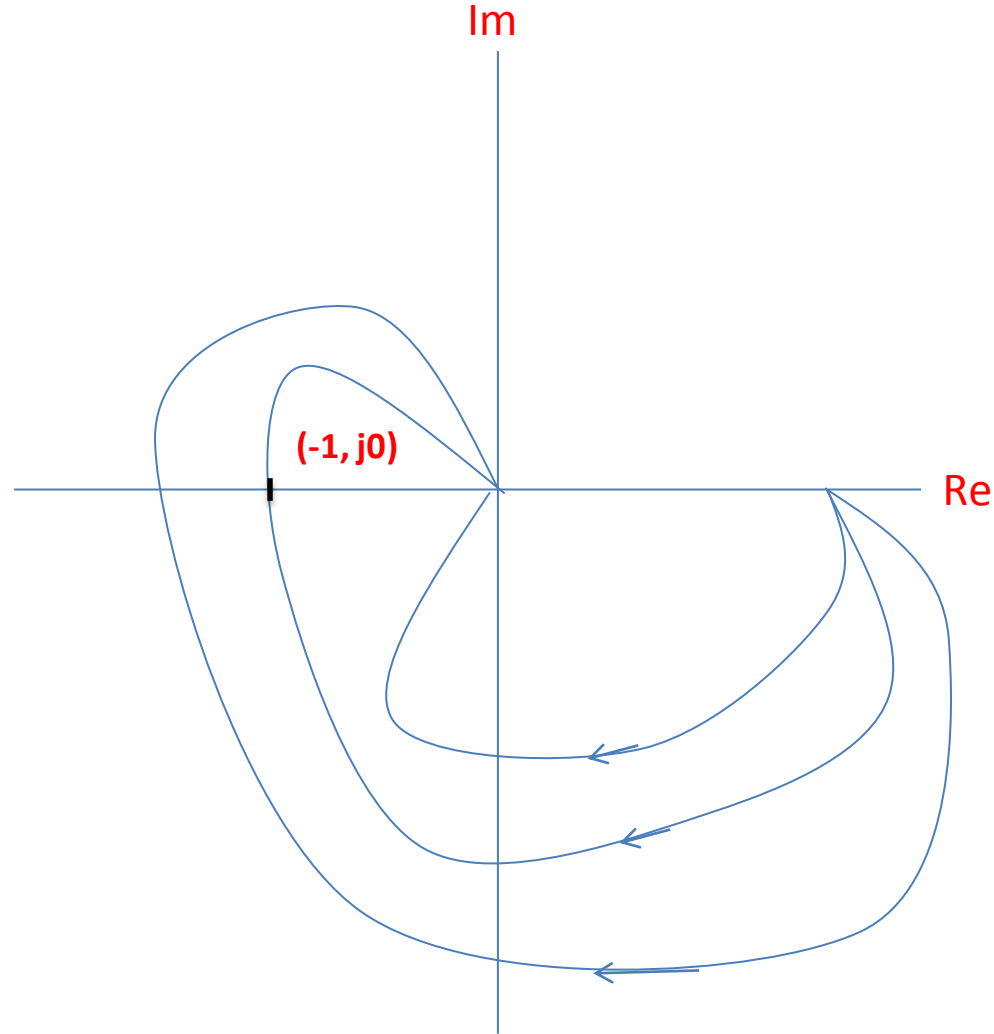
# Example#1

| $\omega$ | Re       | Im       |
|----------|----------|----------|
| 0        | $\infty$ | $\infty$ |
| 0.1      | -1       | -10      |
| 0.5      | -0.8     | -1.6     |
| 1        | -0.5     | -0.5     |
| 2        | -0.2     | -0.1     |
| 3        | -0.1     | -0.03    |
| $\infty$ | 0        | 0        |

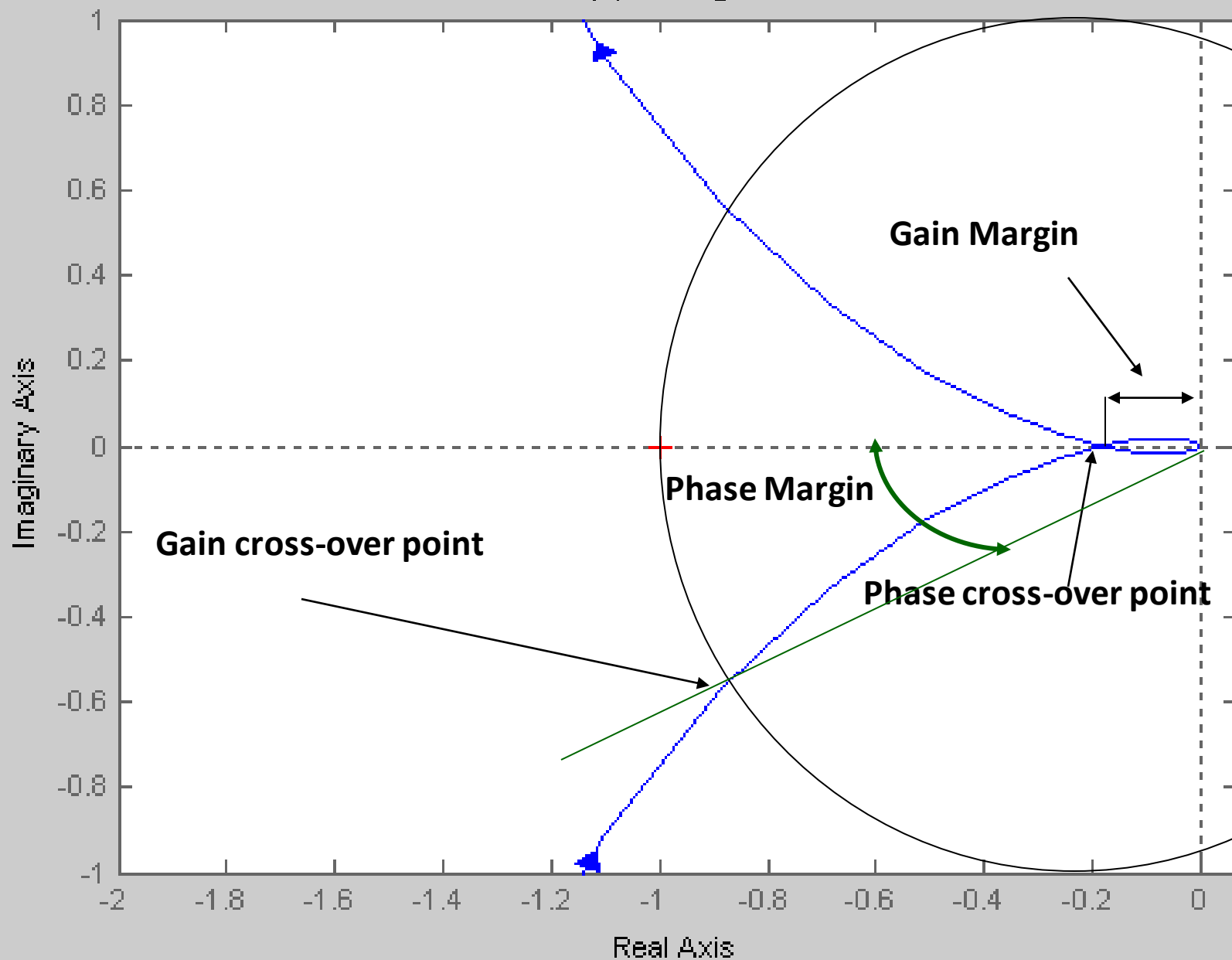


# Nyquist Stability Criterion

- The Nyquist stability criterion determines the stability of a closed-loop system from its open-loop frequency response and open-loop poles.
- *A minimum phase closed loop system will be stable if the Nyquist plot of open loop transfer function does not encircle  $(-1, j0)$  point.*



Nyquist Diagram



# Nichol's Chart (Log-magnitude vs Phase Plot)

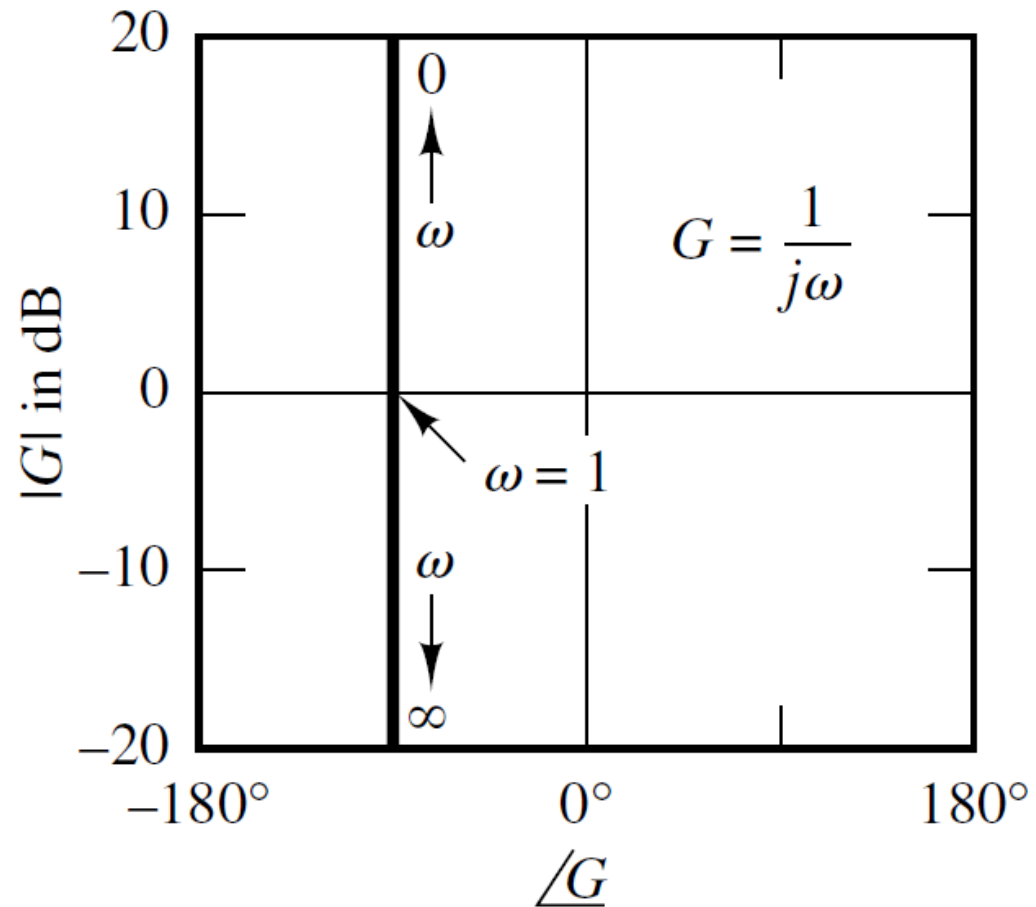
- Another approach to graphically portraying the frequency-response characteristics is to use the log-magnitude-versus-phase plot.
- Which is a plot of the logarithmic magnitude in decibels versus the phase angle for a frequency range of interest.
- In the manual approach the log-magnitude-versus-phase plot can easily be constructed by reading values of the log magnitude and phase angle from the Bode diagram.
- Advantages of the log-magnitude-versus-phase plot are that the relative stability of the closed-loop system can be determined quickly and that compensation can be worked out easily.

# Nichol's Chart of simple transfer functions

$$G(j\omega) = \frac{1}{j\omega}$$

$$G(j\omega) = \frac{1}{\omega} \angle -90^\circ$$

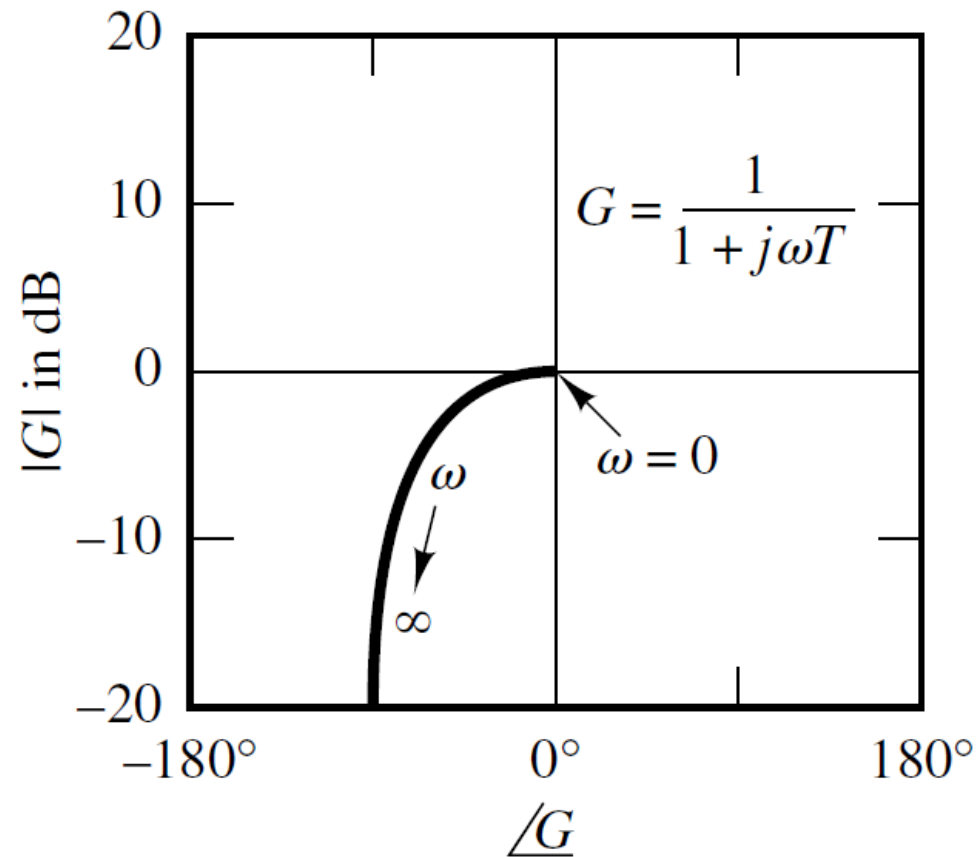
| $\omega$ | db        | $\phi$      |
|----------|-----------|-------------|
| 0        | $\infty$  | $-90^\circ$ |
| 0.5      | 6         | $-90^\circ$ |
| 1        | 0         | $-90^\circ$ |
| 2        | -6        | $-90^\circ$ |
| $\infty$ | $-\infty$ | $-90^\circ$ |



# Nichol's Chart of simple transfer functions

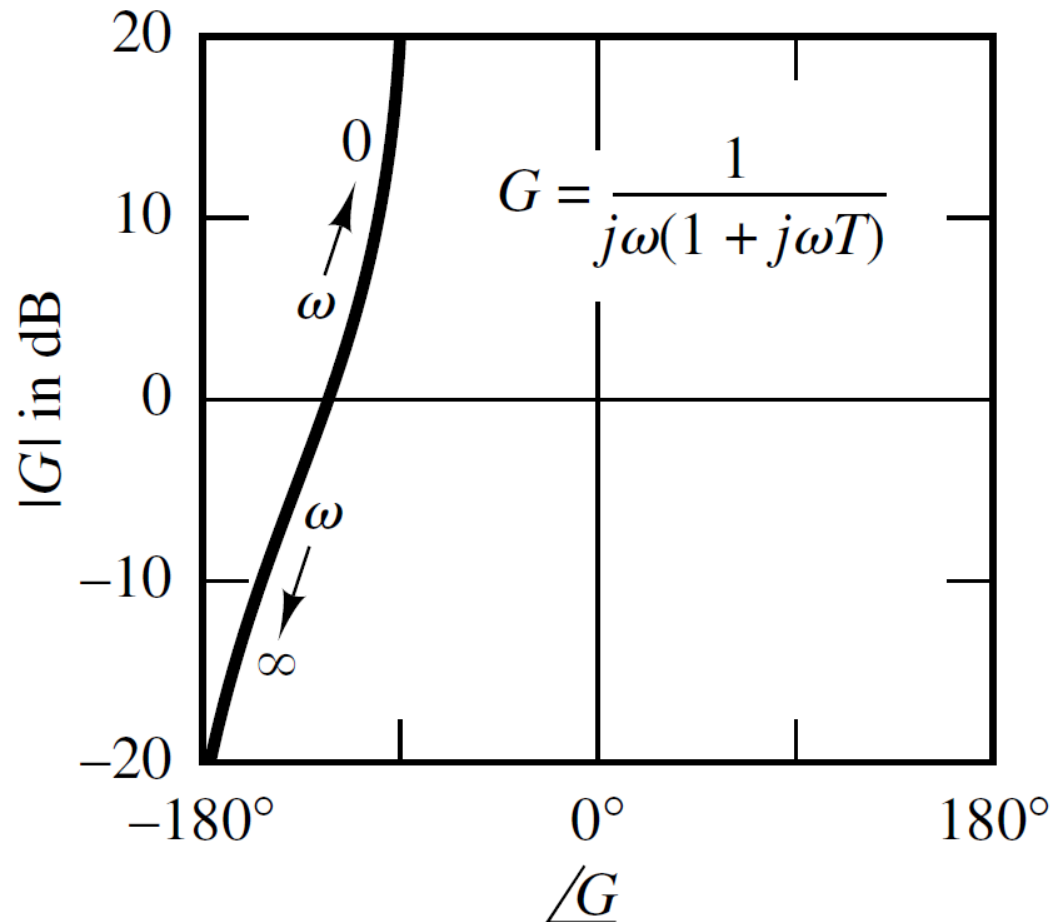
$$G(j\omega) = \frac{1}{j\omega T + 1}$$

$$G(j\omega) = \frac{1}{\sqrt{T^2\omega^2 + 1}} \angle -\tan^{-1}(\omega T)$$

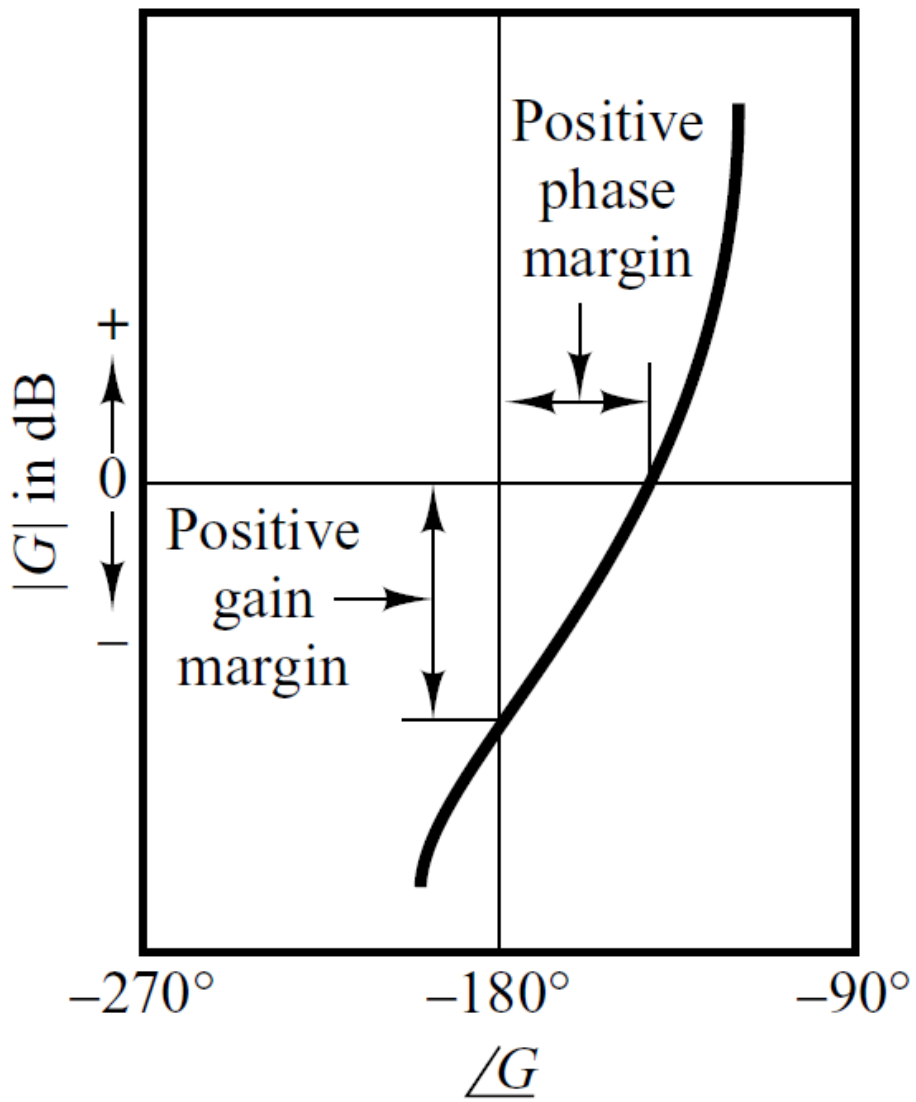


# Nichol's Chart of simple transfer functions

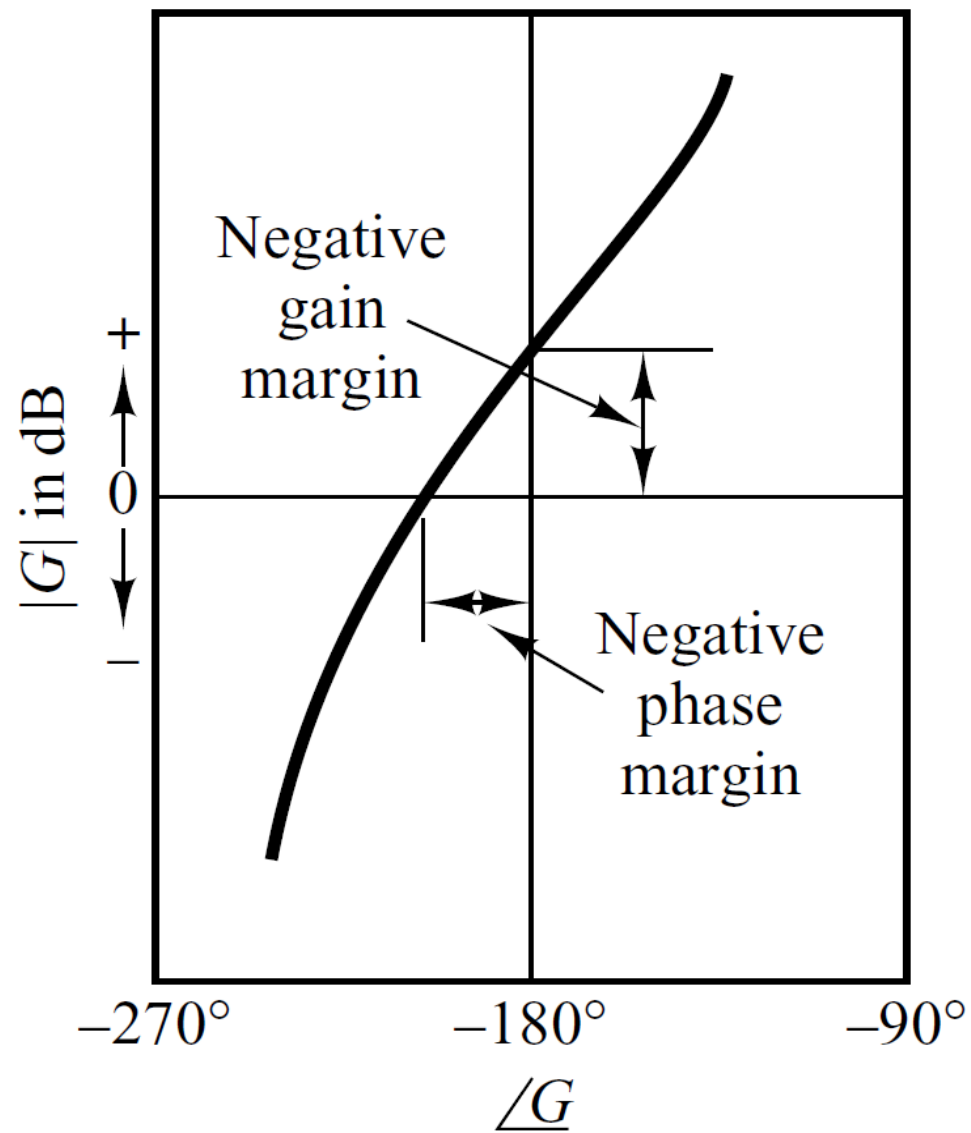
$$G(j\omega) = \frac{1}{j\omega(j\omega T + 1)}$$



# Relative Stability

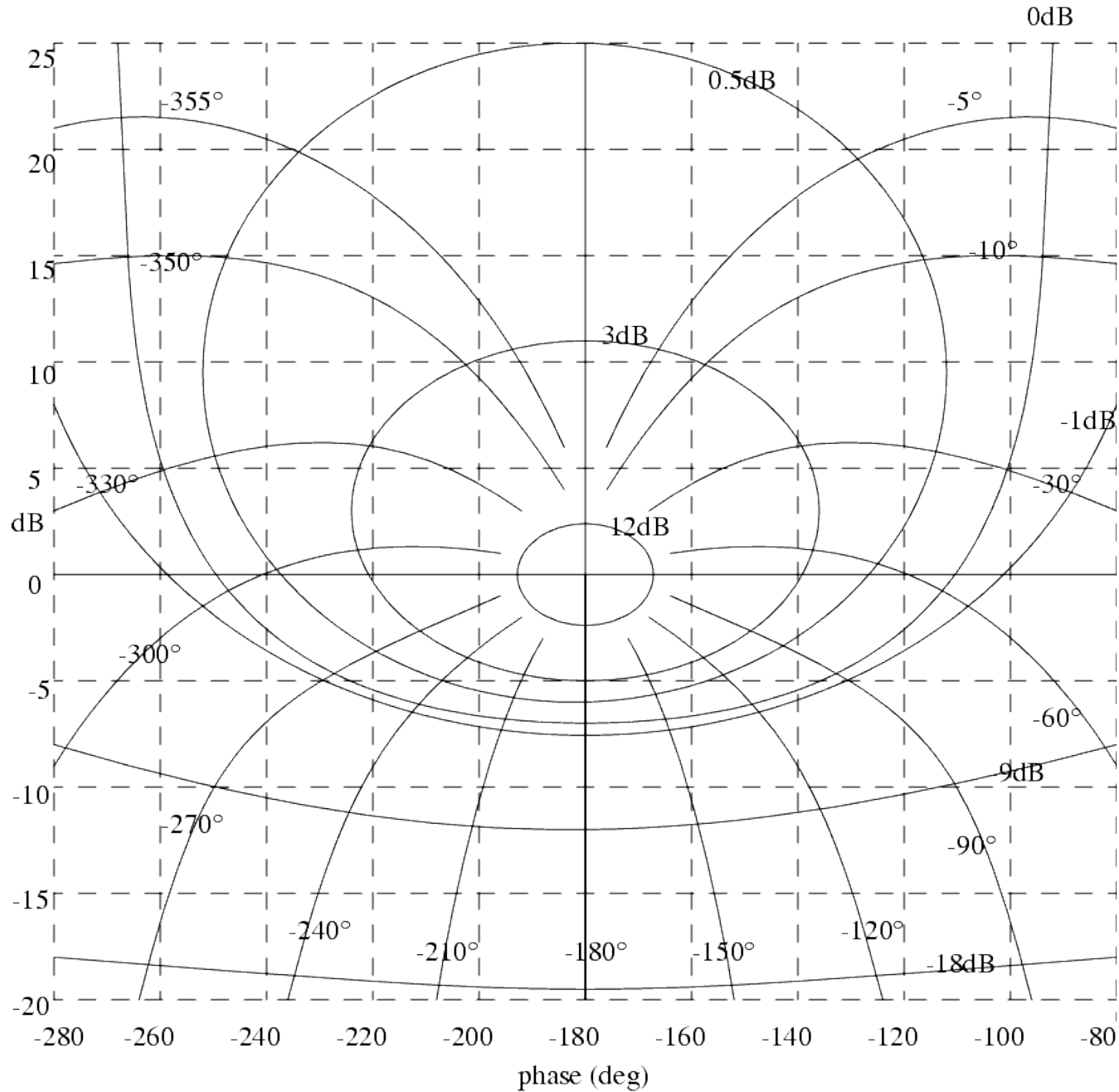


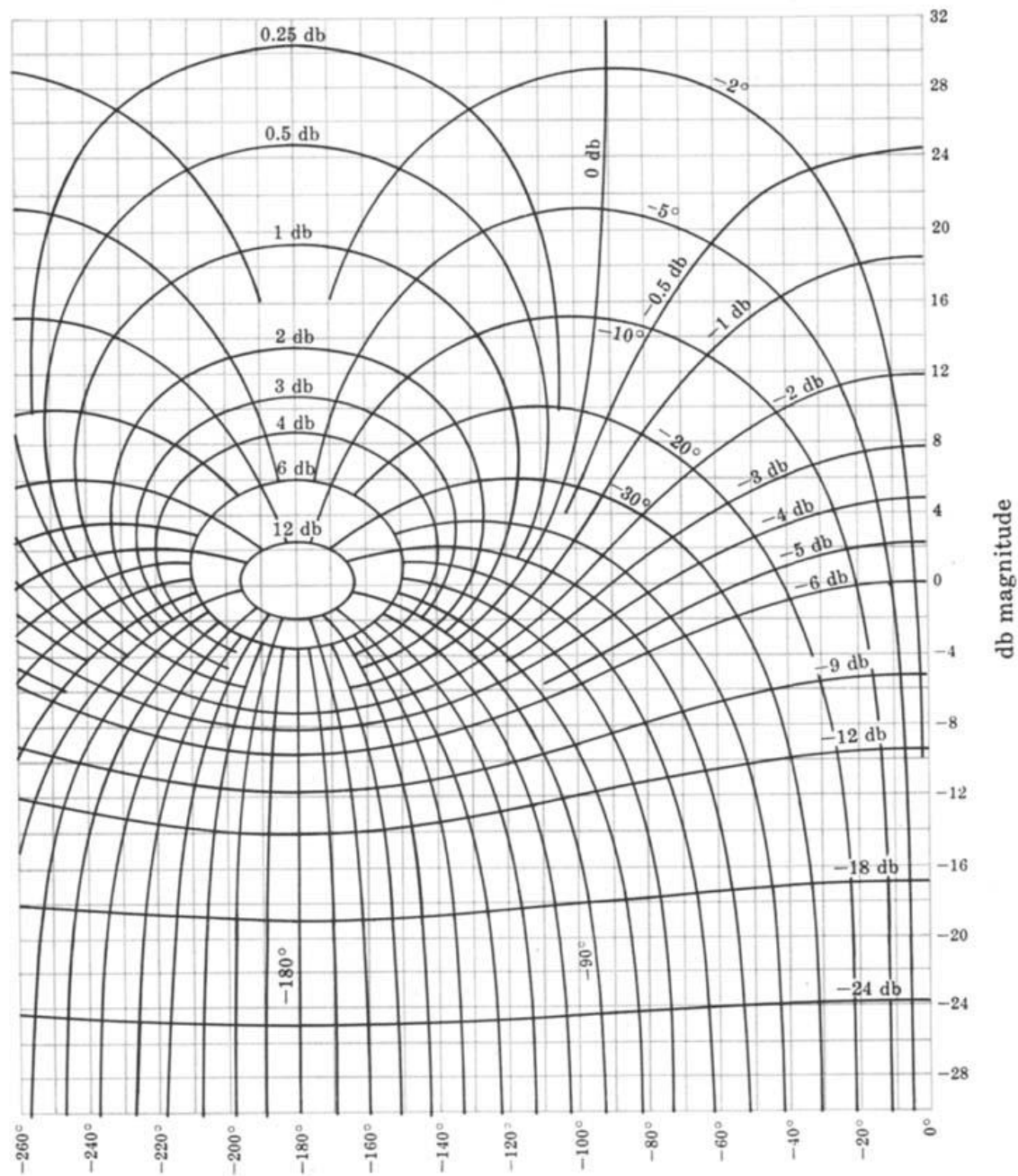
Stable system



Unstable system







# Example#1

- Draw the Nichol's Chart of following open loop transfer function and obtain the Gain Margin and Phase Margin.

$$G(s) = \frac{1}{s(s+1)(s+3)}$$

**Solution**

$$G(j\omega) = \frac{1}{j\omega(j\omega+1)(j\omega+3)}$$

$$G(j\omega) = \frac{1}{-4\omega^2 - j(\omega^3 - 3\omega)}$$

$$G(j\omega) = \frac{1}{-4\omega^2 - j(\omega^3 - 3\omega)} \times \frac{-4\omega^2 + j(\omega^3 - 3\omega)}{-4\omega^2 + j(\omega^3 - 3\omega)}$$

# Example#1

$$G(j\omega) = \frac{1}{-4\omega^2 - j(\omega^3 - 3\omega)} \times \frac{-4\omega^2 + j(\omega^3 - 3\omega)}{-4\omega^2 + j(\omega^3 - 3\omega)}$$

$$G(j\omega) = \frac{-4\omega^2 + j(\omega^3 - 3\omega)}{-16\omega^4 - (\omega^3 - 3\omega)^2}$$

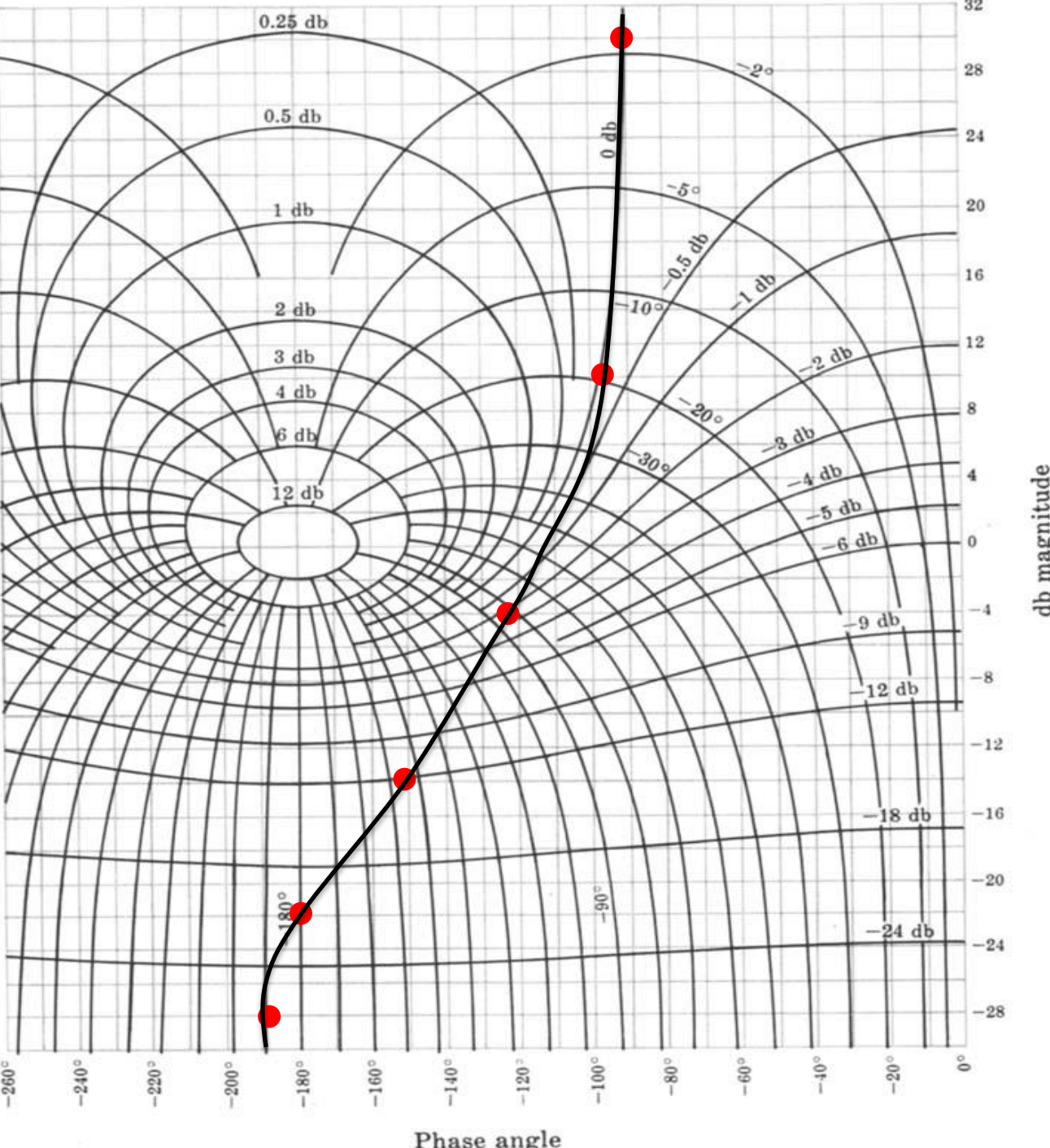
$$G(j\omega) = \frac{-4\omega^2}{-16\omega^4 - (\omega^3 - 3\omega)^2} + j \frac{(\omega^3 - 3\omega)}{-16\omega^4 - (\omega^3 - 3\omega)^2}$$

# Example#1

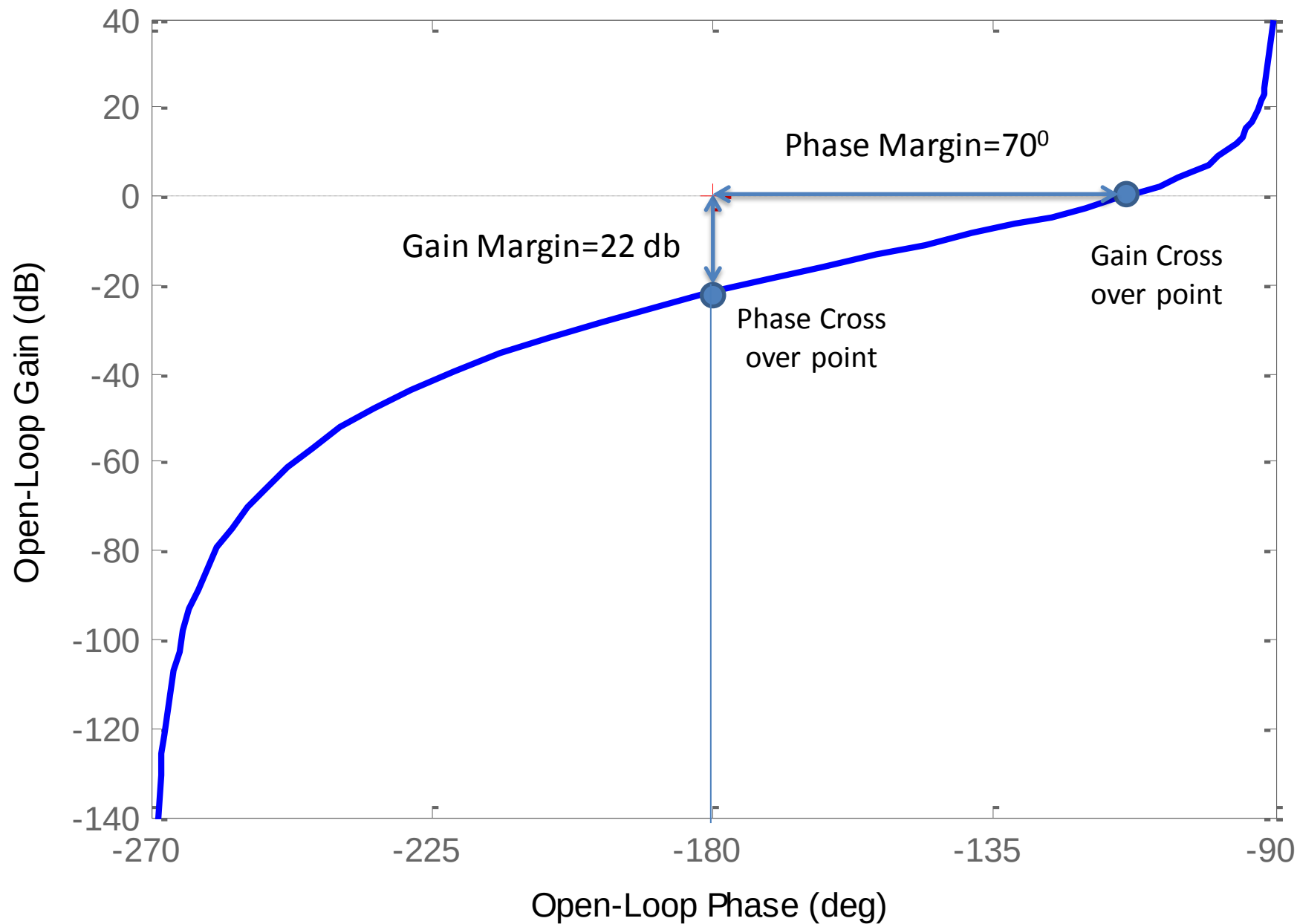
$$G(j\omega) = \frac{-4}{16\omega^2 - (\omega^2 - 3)^2} + j \frac{(\omega^2 - 3)}{16\omega^3 - \omega(\omega^2 - 3)^2}$$

| $\omega$ | $ G(j\omega) _{db}$ | $\angle G(j\omega)$ |
|----------|---------------------|---------------------|
| 0.01     | 30                  | -90°                |
| 0.1      | 10.3                | -97.5°              |
| 0.5      | -4.4                | -125°               |
| 1        | -14                 | -153°               |
| 2        | -22                 | -180°               |
| 10       | -26                 | -189°               |

| $\omega$ | $ G(j\omega) _{db}$ | $\angle G(j\omega)$ |
|----------|---------------------|---------------------|
| 0.01     | 30                  | $-90^\circ$         |
| 0.1      | 10.3                | $-97.5^\circ$       |
| 0.5      | -4.4                | $-125^\circ$        |
| 1        | -14                 | $-153^\circ$        |
| 2        | -22                 | $-180^\circ$        |
| 10       | -26                 | $-189^\circ$        |



Nichols Chart



Nichols Chart

