Hypothesis Test of Two Means with Known Population Variance

The previous hypothesis tests have involved a single random variable. In some cases, samples are obtained from two different populations with means μ_1 and μ_2 and variances σ_1^2 and σ_2^2 and if $\bar{X}_1$ and $\bar{X}_2$ are the sample means of two independent random samples of sizes n_1 and n_2 from these populations.

In this section we consider statistical inferences on the difference in means of two normal distributions, where the variances σ_1^2 and σ_2^2 are known. The procedure for this section is summarized as follows.

Step 1: Formulate hypotheses:

$H_0: \mu_1 - \mu_2 = \Delta$

One of the three alternative hypotheses may be selected:

$H_1: \mu_1 - \mu_2 \neq \Delta$	$two-tailed\ tests$
$H_1: \mu_1 - \mu_2 < \Delta$	$one-tailed\ test (left-sided\ test)$
$H_1: \mu_1 - \mu_2 > \Delta$	$one-tailed\ test (right-sided\ test)$

Step 2: Select the Appropriate Model:

$$Z = \frac{\bar{X}_1 - \bar{X}_2 - \Delta}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \qquad\qquad where: \qquad \Delta = \mu_1 - \mu_2$$

Step 3: Select the Level of Significance:
A level of significance α must be selected for a hypothesis test.

Step 4: Compute Estimate of the Test Statistic (Z).

Step 5: Define the Region of Rejection:
The region of rejection depends on the level of significance.

If H$_1$ is	Then the region of rejection is
$\mu_1 - \mu_2 \neq \Delta$	$all\ Z < -Z_{\alpha/2}$ or $z > Z_{\alpha/2}$
$\mu_1 - \mu_2 < \Delta$	$all\ Z < -Z_{\alpha}$
$\mu_1 - \mu_2 > \Delta$	$all\ Z > Z_{\alpha}$

Step 6: Select the Appropriate Hypothesis.

Example:

To compare higher school seniors knowledge events in two different school districts sample of 60 seniors from each district where given a special test. If those of the first district obtained an average score at 78 with a standard deviation of 5. While those of the second district obtained an average score 74 with standard deviation of 4. Test at the level of significance α=0.05 whether the difference between the two means is significance.

Solution:

$H_0: \mu_1 - \mu_2 = 0 \ (\mu_1 = \mu_2)$ *Null Hypothesis*
$H_1: \mu_1 - \mu_2 \neq 0 \ (\mu_1 \neq \mu_2)$ *Alternative Hypothesis*
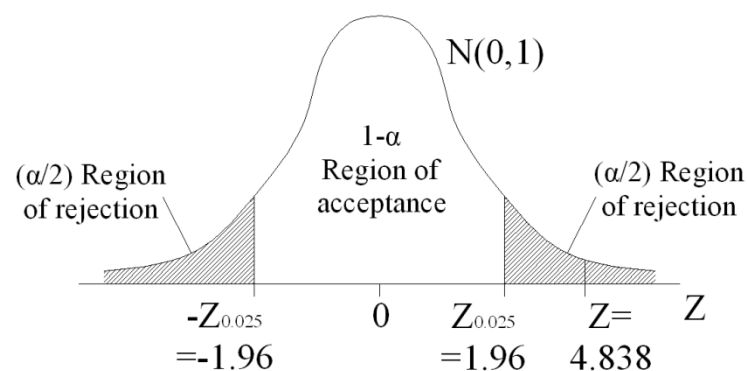
$\alpha = 0.05$

The test method is a two-proportion Z-test.

$$Z = \frac{\bar{X}_1 - \bar{X}_2 - \Delta}{\sqrt{\dfrac{\sigma_1^2}{n_1} + \dfrac{\sigma_2^2}{n_2}}}$$

$$Z = \frac{78 - 74 - 0}{\sqrt{\dfrac{5^2}{60} + \dfrac{4^2}{60}}} = 4.838$$

The Z-critical value corresponding to α = 0.05 is ±1.96

N(0,1)

(α/2) Region of rejection   1-α Region of acceptance   (α/2) Region of rejection

-$Z_{0.025}$ =-1.96   0 $Z_{0.025}$ =1.96   Z= 4.838   Z

∴ reject the null hypothesis H_0 and accept the alternative hypothesis H_1
There is real difference.

Example:

    A product developer is interested in reducing the drying time of a primer paint. Two formulations of the paint are tested; formulation 1 is the standard chemistry, and formulation 2 has a new drying ingredient that should reduce the drying time. From experience, it is known that the standard deviation of drying time is 8 minutes, and this inherent variability should be unaffected by the addition of the new ingredient. Ten specimens are painted with formulation 1, and another 10 specimens are painted with formulation 2; the 20 specimens are painted in random order. The two sample average drying times are 121 minutes and 112 minutes, respectively. What conclusions can the product developer draw about the effectiveness of the new ingredient, using α=0.05?

Solution:

$H_0: \mu_1 - \mu_2 = 0 \ (\mu_1 = \mu_2)$ *Null Hypothesis*
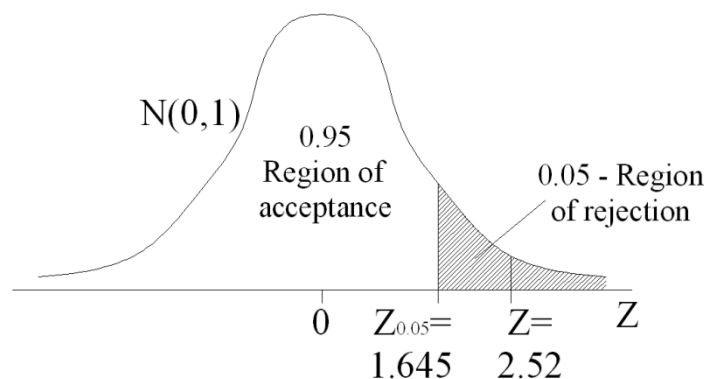$H_1: \mu_1 > \mu_2$ *Alternative Hypothesis*

$\alpha = 0.05$

The test method is a two-proportion Z-test.

$$Z = \frac{\bar{X}_1 - \bar{X}_2 - \Delta}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$$

$$Z = \frac{121 - 112 - 0}{\sqrt{\frac{8^2}{10} + \frac{8^2}{10}}} = 2.52$$

    The Z-critical value corresponding to $\alpha = 0.05$ is ± 1.645

∴ reject the null hypothesis H_0 and accept the alternative hypothesis H_1 at the α=0.05 level and conclude that adding the new ingredient to the paint significantly reduces the drying time.

Test of Hypothesis

Hypothesis testing is a formal procedure for using statistical concepts and measure in performing decision making. The following six steps can be used to make a statistical analysis of a hypothesis:

(1) Formulate hypotheses.
(2) Select the appropriate statistical model that identifies the test statistic.
(3) Specify the level of significance.
(4) Collect a sample of data and compute an estimate of the test statistic.
(5) Define the region of rejection for the test statistic.
(6) Select the appropriate hypothesis.

Step 1:

The first step is to formulate two or more hypotheses for testing. The hypotheses are usually statements indicating that a random variable has some specific distribution or that a population parameter has a specific value.

For example, suppose that our interest focuses on the mean burning rate (a parameter of this distribution). Specifically, we are interested in deciding whether or not the mean burning rate is 50 centimeters per second. We may express this formally as:

$$H_0: \mu = 50 \ cm/s$$
$$H_1: \mu \neq 50 \ cm/s$$

The statement $H_0: \mu = 50 \ cm/s$ is called the **null hypothesis**, and the statement
$H_1: \mu \neq 50 \ cm/s$ is called the **alternative hypothesis**. Since the alternative hypothesis specifies values of that could be either greater or less than 50 cm/s, it is called a **two-sided alternative hypothesis**. In some situations, we may wish to formulate a **one-sided alternative hypothesis**, as in

$$H_0: \mu = 50 \ cm/s \qquad or \qquad H_0: \mu = 50 \ cm/s$$
$$H_1: \mu < 50 \ cm/s \qquad\qquad\qquad H_1: \mu > 50 \ cm/s$$

The null and alternative hypotheses should be expressed both in mathematical terms. Thus, when a statistical analysis of sampled data

suggests that the null hypothesis should be rejected, the alternative hypothesis should be accepted.

Step 2:

- To develop the two hypotheses, it is necessary to develop a test statistic reflecting the difference suggested by the alternative hypothesis.
- The computed value of a test statistic varies from one sample to the next.
- The test statistic is a random variable and has a sampling distribution.
- A hypothesis test should be based on a theoretical model that defines the distribution function of the test statistic and the parameters of the sampling distribution.
- The test statistic reflects the hypotheses and the data that are usually available.

For example:

$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \qquad , \qquad t = \frac{\bar{X} - \mu}{S/\sqrt{n}}$$

Step 3:

Significance level provides a probabilistic framework for accepting or rejecting the null hypothesis, and consequently making a decision.

Types of errors in hypothesis testing:

- ❖ Type I error: reject H_0 when in fact H_0 is true.
- ❖ Type II error: accept H_0 when in fact H_0 is false.

Decision	Situation	
	H_0 is true	H_0 is not true
Accept H_0	Correct decision	Incorrect decision: **(Type II error)**
Reject H_0	Incorrect decision: **(Type I error)**	Correct decision

The level of significance represents the probability of making a Type I error and is denoted by **α**. The probability of Type II error is denoted by **β**. Values of **0.05** and **0.01** for α are commonly selected.

> *Note:* As α decreases (good)
>
> β increases (Not good) and hence

Step 4:

After obtaining the necessary data, the sample is used to compute an estimate of the test statistic.

Step 5:

The region of rejection consists of those values of the test statistic that would be unlikely to occur when the null hypothesis is in fact true. On the other hand, the region of acceptance consists of those values of the test statistic that would be accepted when the null hypothesis is in the fact true.

The region of rejection is usually represented by one or both tails of the distribution of the test statistic.

Step 6:

The null hypothesis is rejected when the computed value lies in the region of rejection. Rejection of the null hypothesis implies acceptance of the alternative hypothesis.

Hypothesis Tests of Means:

(1)   Test of the Mean with Known Population Variance:

Step 1: Formulate hypotheses:

$$H_0: \mu = \mu_0$$

One of the three alternative hypotheses may be selected:

$$H_1: \mu \neq \mu_0 \qquad two - tailed\ tests$$
$$H_1: \mu < \mu_0 \qquad one - tailed\ test(left - sided\ test)$$
$$H_1: \mu > \mu_0 \qquad one - tailed\ test\ (right - sided\ test)$$

Step 2: Select the Appropriate Model:

For a random sample of size n, the sample mean $\bar{X}$ can be used in calculating the value of the test statistic Z as

$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \qquad \{normal\ population\ with\ known\ variance\}$$

$$or \qquad Z = \frac{\bar{X} - \mu}{S/\sqrt{n}} \qquad \{for\ large\ sample(n > 30)and\ \sigma\ is\ unknown\ \}$$

Step 3: Select the Level of Significance:
   A level of significance α must be selected for a hypothesis test.

Step 4: Compute Estimate of the Test Statistic (Z).
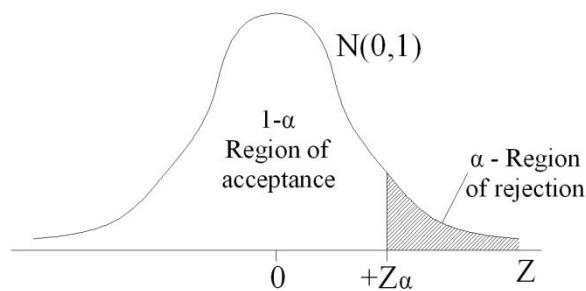
Step 5: Define the Region of Rejection:
   For the standard normal distribution, the level of significance is the only characteristic required to determine the critical value of the test statistic.

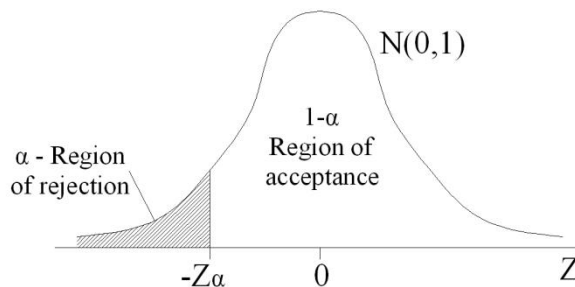| If H_1 is	Then the region of rejection is
$\mu \neq \mu_0$	$all\ Z < -Z_{\alpha/2}$ or $Z > Z_{\alpha/2}$
$\mu < \mu_0$	$all\ Z < -Z_\alpha$
$\mu > \mu_0$	$all\ Z > Z_\alpha$

$H_0: \mu = \mu_0$

$H_1: \mu \neq \mu_0$
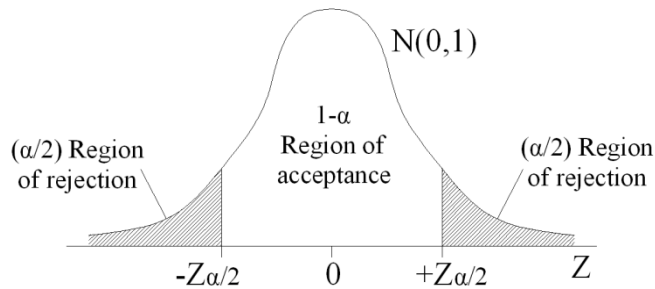
$H_0: \mu = \mu_0$

$H_1: \mu > \mu_0$

$H_0: \mu = \mu_0$

$H_1: \mu < \mu_0$

Step 6: Select the Appropriate Hypothesis.

Example:

A random sample of 100 recorded deaths in the United States during the past year showed an average life span of 71.8 years. Assuming a population standard deviation of 8.9 years, does this seem to indicate that the: mean life span today is greater than 70 years? Use a 0.05 level of significance.

Solution:

$H_0: \mu = 70 \text{ years}$ *Null Hypothesis*
$H_1: \mu > 70 \text{ years}$ *Alternative Hypothesis*

$\alpha = 0.05$

$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} = \frac{71.8 - 70}{8.9/\sqrt{100}} = 2.02$$

Right Tailed Z-Test:
$Z_\alpha = Z_{0.05} = +1.645$

$\therefore Z_\alpha < Z$
$\therefore$ reject the null hypothesis H_0 and accept the alternative hypothesis H_1

As a result, the evidence in favor of H_1 is even stronger than that suggested by a 0.05 level of significance.

Example: 50 smokers were questioned about the number of hours they sleep each day. We want to test the hypothesis that the smokers need *less sleep than* the general public which needs an average of 7.7 hours of sleep. We follow the steps below.

Compute a rejection region for a significance level of 0.05.
If the sample mean is 7.5 and the population standard deviation is 0.5, what can you conclude?

Solution:

$H_0: \mu = 7.7 \ hours$ *Null Hypothesis*
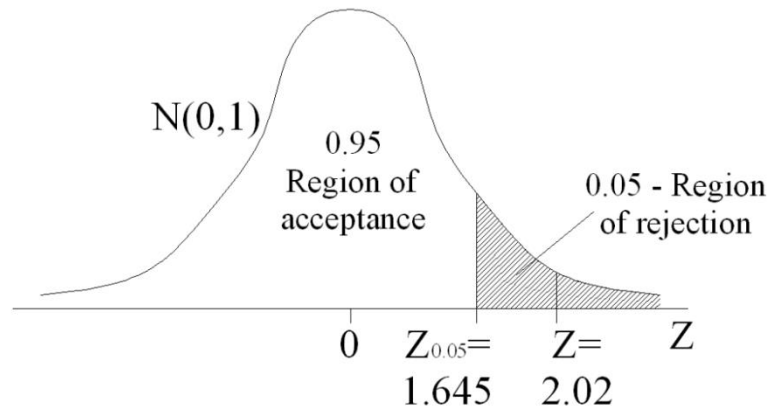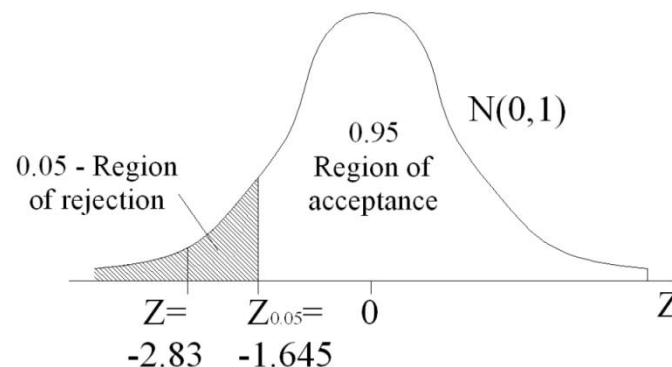$H_1: \mu < 7.7 \ hours$ *Alternative Hypothesis*

$\alpha = 0.05$

$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} = \frac{7.5 - 7.7}{0.5/\sqrt{50}} = -2.83$$

Left Tailed Z-Test:

$$Z_\alpha = Z_{0.05} = -1.645$$

$\because Z < Z_\alpha$
$\therefore$ reject the null hypothesis H_0 and accept the alternative hypothesis H_1

We can conclude that smokers need less sleep by a 0.05 level of significance.

Example:

Suppose that we want to test the hypothesis with a significance level of 0.05 that the climate has *changed* since industrialization. Suppose that the mean temperature throughout history is 50 degrees. During the last 40 years, the mean temperature has been 51 degrees and suppose the population standard deviation is 2 degrees. What can we conclude?

Solution:

$H_0: \mu = 50 \; degrees$ *Null Hypothesis*

$H_1: \mu \neq 50 \; degrees$ *Alternative Hypothesis*

$\alpha/2 = 0.05/2 = 0.025$

$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} = \frac{51 - 50}{2/\sqrt{40}} = 3.16$$

Two Tailed Z-Test:

$Z_{\alpha/2} = Z_{0.025}(1 - 0.025 = 0.975) = +1.96$

$Z_{\alpha/2} = Z_{0.025} = -1.96$
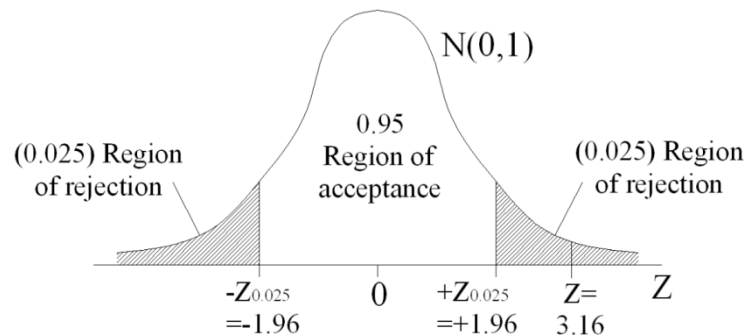
$\because Z > +Z_\alpha$

$\therefore$ reject the null hypothesis H_0 and accept the alternative hypothesis H_1

We can conclude that there has been a change in temperature by a 0.05 level of significance.

(2):Test of the Mean with *Unknown* Population Variance:

Step 1: Formulate hypotheses:

$H_0: \mu = \mu_0$

One of the three alternative hypotheses may be selected:

$H_1: \mu \neq \mu_0$ $two - tailed \; tests$

$H_1: \mu < \mu_0$ $one - tailed \; test(left - sided \; test)$

$H_1: \mu > \mu_0$ $one - tailed \; test(right - sided \; test)$

Step 2: Select the Appropriate Model:

For a random sample of size n, the sample mean $\bar{X}$ and standard deviation S can be used in calculating the value of the test statistic t as

$$t = \frac{\bar{X} - \mu}{S/\sqrt{n}}$$

$where: S = sample\ standard\ deviation.$

$$S = \pm \sqrt{\frac{\sum_{i=1}^{n} f_i X_i^2 - (\sum_{i=1}^{n} f_i X_i)^2/n}{n-1}}$$

Step 3: Select the Level of Significance:
A level of significance α must be selected for a hypothesis test.

Step 4: Compute Estimate of the Test Statistic (t).

Step 5: Define the Region of Rejection:
The region of rejection depends on the level of significance, the degrees of freedom *(n-1)*, and the statement of the alternative hypothesis.

If H_1 is	Then the region of rejection is
$\mu \neq \mu_0$	all $t < -t_{\alpha/2,n-1}$ or $t > t_{\alpha/2,n-1}$
$\mu < \mu_0$	all $t < -t_{\alpha,n-1}$
$\mu > \mu_0$	all $t > t_{\alpha,n-1}$

Step 6: Select the Appropriate Hypothesis.

T-Distribution Table (One Tail)

DF	A = 0.1	0.05	0.025	0.01	0.005	0.001	0.0005
∞	$t_a = 1.282$	1.645	1.960	2.326	2.576	3.091	3.291
1	3.078	6.314	12.706	31.821	63.656	318.289	636.578
2	1.886	2.920	4.303	6.965	9.925	22.328	31.600
3	1.638	2.353	3.182	4.541	5.841	10.214	12.924
4	1.533	2.132	2.776	3.747	4.604	7.173	8.610
5	1.476	2.015	2.571	3.365	4.032	5.894	6.869

6	1.440	1.943	2.447	3.143	3.707	5.208	5.959
7	1.415	1.895	2.365	2.998	3.499	4.785	5.408
8	1.397	1.860	2.306	2.896	3.355	4.501	5.041
9	1.383	1.833	2.262	2.821	3.250	4.297	4.781
10	1.372	1.812	2.228	2.764	3.169	4.144	4.587
11	1.363	1.796	2.201	2.718	3.106	4.025	4.437
12	1.356	1.782	2.179	2.681	3.055	3.930	4.318
13	1.350	1.771	2.160	2.650	3.012	3.852	4.221
14	1.345	1.761	2.145	2.624	2.977	3.787	4.140
15	1.341	1.753	2.131	2.602	2.947	3.733	4.073
16	1.337	1.746	2.120	2.583	2.921	3.686	4.015
17	1.333	1.740	2.110	2.567	2.898	3.646	3.965
18	1.330	1.734	2.101	2.552	2.878	3.610	3.922
19	1.328	1.729	2.093	2.539	2.861	3.579	3.883
20	1.325	1.725	2.086	2.528	2.845	3.552	3.850
21	1.323	1.721	2.080	2.518	2.831	3.527	3.819
22	1.321	1.717	2.074	2.508	2.819	3.505	3.792
23	1.319	1.714	2.069	2.500	2.807	3.485	3.768
24	1.318	1.711	2.064	2.492	2.797	3.467	3.745
25	1.316	1.708	2.060	2.485	2.787	3.450	3.725
26	1.315	1.706	2.056	2.479	2.779	3.435	3.707
27	1.314	1.703	2.052	2.473	2.771	3.421	3.689
28	1.313	1.701	2.048	2.467	2.763	3.408	3.674
29	1.311	1.699	2.045	2.462	2.756	3.396	3.660
30	1.310	1.697	2.042	2.457	2.750	3.385	3.646
60	1.296	1.671	2.000	2.390	2.660	3.232	3.460

| 120 | 1.289 | 1.658 | 1.980 | 2.358 | 2.617 | 3.160 | 3.373 |
| 1000 | 1.282 | 1.646 | 1.962 | 2.330 | 2.581 | 3.098 | 3.300 |

Two Tails T Distribution Table

DF	A = 0.2	0.10	0.05	0.02	0.01	0.002	0.001
∞	$t_a = 1.282$	1.645	1.960	2.326	2.576	3.091	3.291
1	3.078	6.314	12.706	31.821	63.656	318.289	636.578
2	1.886	2.920	4.303	6.965	9.925	22.328	31.600
3	1.638	2.353	3.182	4.541	5.841	10.214	12.924
4	1.533	2.132	2.776	3.747	4.604	7.173	8.610
5	1.476	2.015	2.571	3.365	4.032	5.894	6.869
6	1.440	1.943	2.447	3.143	3.707	5.208	5.959
7	1.415	1.895	2.365	2.998	3.499	4.785	5.408
8	1.397	1.860	2.306	2.896	3.355	4.501	5.041
9	1.383	1.833	2.262	2.821	3.250	4.297	4.781
10	1.372	1.812	2.228	2.764	3.169	4.144	4.587
11	1.363	1.796	2.201	2.718	3.106	4.025	4.437
12	1.356	1.782	2.179	2.681	3.055	3.930	4.318
13	1.350	1.771	2.160	2.650	3.012	3.852	4.221
14	1.345	1.761	2.145	2.624	2.977	3.787	4.140
15	1.341	1.753	2.131	2.602	2.947	3.733	4.073
16	1.337	1.746	2.120	2.583	2.921	3.686	4.015
17	1.333	1.740	2.110	2.567	2.898	3.646	3.965
18	1.330	1.734	2.101	2.552	2.878	3.610	3.922
19	1.328	1.729	2.093	2.539	2.861	3.579	3.883
20	1.325	1.725	2.086	2.528	2.845	3.552	3.850
21	1.323	1.721	2.080	2.518	2.831	3.527	3.819
22	1.321	1.717	2.074	2.508	2.819	3.505	3.792
23	1.319	1.714	2.069	2.500	2.807	3.485	3.768

24	1.318	1.711	2.064	2.492	2.797	3.467	3.745
25	1.316	1.708	2.060	2.485	2.787	3.450	3.725
26	1.315	1.706	2.056	2.479	2.779	3.435	3.707
27	1.314	1.703	2.052	2.473	2.771	3.421	3.689
28	1.313	1.701	2.048	2.467	2.763	3.408	3.674
29	1.311	1.699	2.045	2.462	2.756	3.396	3.660
30	1.310	1.697	2.042	2.457	2.750	3.385	3.646
60	1.296	1.671	2.000	2.390	2.660	3.232	3.460
120	1.289	1.658	1.980	2.358	2.617	3.160	3.373
8	1.282	1.645	1.960	2.326	2.576	3.091	3.291

Example:

Is the temperature required to damage a computer on the average *less than* 110 degrees? Because of the price of testing, twenty computers were tested to see what minimum temperature will damage the computer. The damaging temperature averaged 109 degrees with a sample standard deviation of 3 degrees.   Assume that the distribution of all computers' damaging temperatures is approximately normal. (use α = 0.05)

Solution:

$H_0: \mu = 110\ degrees$ *Null Hypothesis*
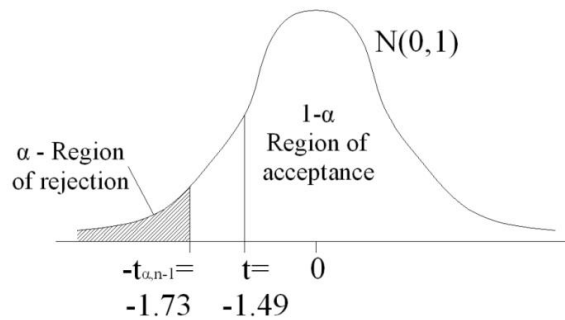$H_1: \mu < 110\ degrees$ *Alternative Hypothesis*
$\alpha = 0.05$
$degree\ of\ freedom = n - 1 = 20 - 1 = 19$

$$t = \frac{\bar{X} - \mu}{S/\sqrt{n}} = \frac{109 - 110}{3/\sqrt{20}} = -1.49$$

Left Tailed t-Test:

$$t_{\alpha, n-1} = t_{0.05, 19} = -1.729$$

Since $\quad t > t_{0.05,19}\ (-1.49 > -1.73)$

$\therefore$ accept the null hypothesis H_0 and reject the alternative hypothesis H_1

We conclude that there is insufficient evidence to suggest that the temperature required to damage a computer on the average less than 110 degrees.

Example:

Suppose the manufacturer claims that the mean lifetime of a light bulb is more than 10000 hours. In a sample of 30 light bulbs, it was found that they only last 9900 hours on average. Assume the sample standard deviation is 125 hours. At 0.05 significance level, can we reject the claim by the manufacturer?

Solution:

$H_0: \mu \geq 10000\ hours \qquad Null\ Hypothesis$
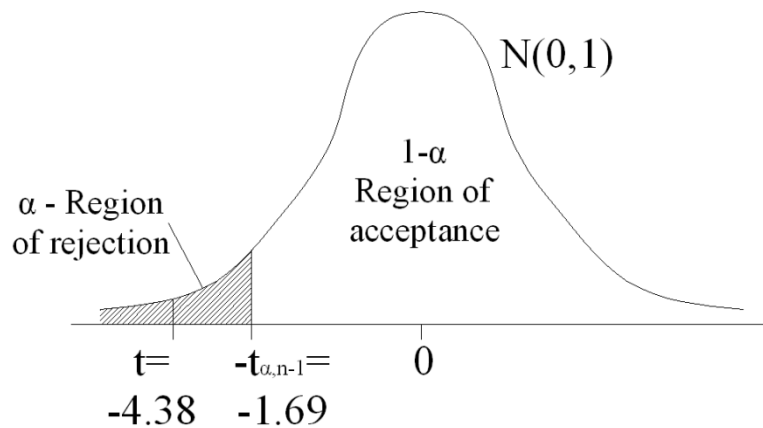$H_1: \mu < 10000\ hours \qquad Alternative\ Hypothesis$
$\alpha = 0.05$
$degree\ of\ freedom = n - 1 = 30 - 1 = 29$

$$t = \frac{\bar{X} - \mu}{S/\sqrt{n}} = \frac{9900 - 10000}{125/\sqrt{30}} = -4.3818$$

Left Tailed t-Test:

$t_{\alpha,n-1} = t_{0.05,29} = -1.699$

Since
$t < t_{0.05,19} \ (-4.38 < -1.69)$
∴ reject the null hypothesis H_0 and accept the alternative hypothesis H_1

At 0.05 significance level, we can reject the claim that mean lifetime of a light bulb is above 10000 hours.

Example:

    A Little League baseball coach wants to know if his team is representative of other teams in scoring runs. Nationally, the average number of runs scored by a Little League team in a game is 5.7. He chooses five games at random in which his team scored 5, 9, 4, 11, and 8 runs. Is it likely that his team's scores could have come from the national distribution? Assume an α=0.05.

Solution:

Because the team's scoring rate could be either higher than or lower than the national average, the problem calls for a two-tailed test. First, state the null and alternative hypotheses:

$H_0: \mu = 5.7 \ runs$ $Null \ Hypothesis$
$H_1: \mu \neq 5.7 \ runs$ $Alternative \ Hypothesis$
$\alpha = 0.05$
$degree \ of \ freedom = n - 1 = 5 - 1 = 4$
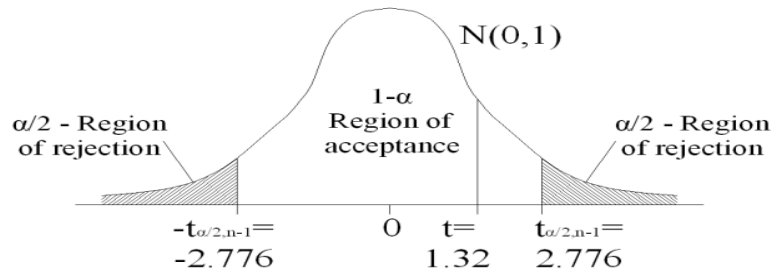$$\bar{X} = \frac{5 + 9 + 4 + 11 + 8}{5} = 7.4$$
$$S = \sqrt{\frac{307 - 1369/5}{5 - 1}} = 2.88$$

$$t = \frac{\bar{X} - \mu}{S/\sqrt{n}} = \frac{7.4 - 5.7}{2.88/\sqrt{5}} = 1.32$$

Two Tailed t-Test:

$$t_{\alpha/2, n-1} = t_{0.025,4} = 2.776$$

Since $-t_{0.025,4} < t < t_{0.025,4}$ $(-2.776 < 1.32 < 2.776)$

∴ accept the null hypothesis H_0 and reject the alternative hypothesis H_1

At 0.05 significance level, the mean of this team is equal to the population mean. The coach can conclude that his team fits in with the national distribution on runs scored.