مادة المقرر الدراسي

الرياضيات الهندسية-I

Engineering mathematics-I

المستوى الثاني ٢٠٢١/٢٠٢٠

اعداد

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" Partial Derivatives "

If $w = f(x,y)$

$$\frac{\partial w}{\partial x} = \lim_{\Delta x \to 0} \frac{\Delta w}{\Delta x} = \lim_{\Delta x \to 0} \frac{f[(x+\Delta x), y] - f(x,y)}{\Delta x}$$

keep y is constant

$$\frac{\partial w}{\partial y} = \lim_{\Delta y \to 0} \frac{\Delta w}{\Delta y} = \lim_{\Delta y \to 0} \frac{f[(x, (y+\Delta y))] - f(x,y)}{\Delta y}$$

keep ~~x~~ is constant

Ex: find $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ If $w = x^2 y - e^{xy} + \sin 5x^3 y$

Sol.

$$\frac{\partial w}{\partial x} = 2xy - y \cdot e^{xy} + \cos 5x^3 y * (15x^2 y)$$

$$\frac{\partial w}{\partial y} = x^2 - x e^{xy} + \cos 5x^3 y (5x^3)$$

Ex: If $w = \ln \sqrt{x^2+y^2}$ find $\frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}$?

$$\frac{\partial w}{\partial x} = \frac{1}{\sqrt{x^2+y^2}} * \frac{1}{2} * (x^2+y^2)^{-1/2} * 2x = \frac{x}{x^2+y^2}$$

$$\frac{\partial w}{\partial y} = \frac{1}{\sqrt{x^2+y^2}} * 1/2 * (x^2+y^2)^{-1/2} * 2y = \frac{y}{x^2+y^2}$$

Ex: $f(x,y,z,w) = x^2 e^{2y+3z} \cos 4w$

$$\frac{\partial f}{\partial x} = 2x\, e^{2y+3z} \cos 4w$$

$$\frac{\partial f}{\partial y} = x^2 * 2\, e^{2y+3z} \cdot \cos 4w$$

$$\frac{\partial f}{\partial z} = 3x^2\, e^{2y+3z} \cos 4w$$

$$\frac{\partial f}{\partial w} = x^2\, e^{2y+3z} \cdot (-\sin 4w) * 4$$

$$\frac{\partial f}{\partial w} = -4\, x^2\, e^{2y+3z} \sin 4w$$

Ex: find f_x and f_y if $f(x,y) = \dfrac{2y}{y+\cos x}$

$$f_x = \frac{(y+\cos x) * 0 - 2y * (-\sin x)}{(y+\cos x)^2} = \frac{2y \sin x}{(y+\cos x)^2}$$

$$f_y = \frac{(y+\cos x) * 2 - 2y * 1}{(y+\cos x)^2} = \frac{\cancel{2y} + 2\cos x - \cancel{2y}}{(y+\cos x)^2}$$

$$f_y = \frac{2\cos x}{(y+\cos x)^2}$$

Ex: find f_x & f_y for the functions:

a. $f(x,y) = \sin^2(x-3y)$

$$f_x = 2\sin(x-3y) * \cos(x-3y) * 1$$

$$f_y = 2\sin(x-3y)\cos(x-3y) * -3$$

$$f_y = -6\sin(x-3y)\cos(x-3y)$$

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b. $f(x,y) = (x^3 + \frac{y}{2})^{2/3}$

$$f_x = \frac{2}{\cancel{3}} * \frac{1}{\sqrt[3]{x^3+\frac{y}{2}}} * \cancel{3}x^2 = \frac{2x^2}{\sqrt[3]{x^3+\frac{y}{2}}}$$

$$f_y = \frac{\cancel{2}}{3} * \left(x^3+\frac{y}{2}\right)^{2/3 - 1 = -\frac{1}{3}} * \frac{1}{\cancel{2}}$$

$$f_y = \frac{1}{3\sqrt[3]{x^3+\frac{y}{2}}}$$

Ex: Find f_x & f_y & f_z for $f(x,y,z) = \sin^{-1}(xyz)$

$$f_x = \frac{1}{\sqrt{1-(xyz)^2}} * yz = \frac{yz}{\sqrt{1-x^2y^2z^2}}$$

$$f_y = \frac{1}{\sqrt{1-(xyz)^2}} * xz = \frac{xz}{\sqrt{1-x^2y^2z^2}}$$

$$f_z = \frac{1}{\sqrt{1-(xyz)^2}} * xy = \frac{xy}{\sqrt{1-x^2y^2z^2}}$$

Ex: Find the partial derivative of the function with respect to each variable:-

a. $f(t,\alpha) = \cos(2\pi t - \alpha)$

$f_t = -\sin(2\pi t - \alpha) * 2\pi$

$f_t = -2\pi \sin(2\pi t - \alpha)$

$f_\alpha = -\sin(2\pi t - \alpha) * -1$

$f_\alpha = \sin(2\pi t - \alpha)$

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b. $g(r,\theta,z) = r(1-\cos\theta) - z$

$g_r = (1-\cos\theta)$

$g_\theta = -r*(-\sin\theta) = r\sin\theta$

$g_z = -1$

Ex: Let $f(x,y) = 2x+3y-4$ find the slop of the line tangent to this surface at the point $(2,-1)$ and lying in the a. plane $x=2$, b. plane $y=-1$.

a. at plane $x=2 \Rightarrow f(x,y) = \cancel{4}+3y\cancel{-4} = 3y \Rightarrow f_y = 3$

$f_y(2,-1) = m = 3$.

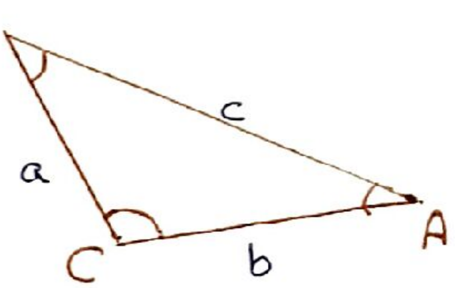
b. at plane $y=-1 \Rightarrow f(x,y) = 2x-3-4 \Rightarrow f(x,y) = 2x-7$

$f_x = 2 \Rightarrow f_x(2,-1) = m = 2$

* Express a (explicity and implicity) derivative "

Ex: for the traingle shown below find:

a- Express A Implicitly as a function of a, b and c and calculate $\partial A/\partial a$ and $\partial A/\partial b$



$$a^2 = b^2 + c^2 - 2bc\cos A$$

To find $\partial A/\partial a$

$$2a = 0 + 0 - 2bc(-\sin A) * \frac{\partial A}{\partial a}$$

$$\frac{\partial A}{\partial a} = \frac{\cancel{2}a}{\cancel{2}bc\sin A} = \frac{a}{bc\sin A}$$

To find $\partial A/\partial b$

$$0 = 2b + 0 - \left[2c\cos A + 2bc * (-\sin A)\frac{\partial A}{\partial b}\right]$$

$$0 = 2b - 2c\cos A + 2bc\sin A\frac{\partial A}{\partial b}$$

$$\frac{\partial A}{\partial b} = \frac{c\cos A - b}{bc\sin A}$$

b Express a implicity as a function of A, b and B and Calculate $\partial a/\partial A$, $\partial a/\partial B$

$$a^2 = b^2 + c^2 - 2bc\cos A$$

or

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

To find $\partial a/\partial A$

$$\frac{\sin A * \frac{\partial a}{\partial A} - a * \cos A}{\sin^2 A} = 0 \Longrightarrow \sin A \frac{\partial a}{\partial A} - a\cos A = 0$$

$$\therefore \frac{\partial a}{\partial A} = \frac{a\cos A}{\sin A}$$

To find $\partial a/\partial B$

$$\frac{1}{\sin A} \cdot \frac{\partial a}{\partial B} = b(-\csc B \cot B)$$

$$\therefore \frac{\partial a}{\partial B} = -b\csc B \cot B \sin A$$

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note: from equ. $a^2 = b^2 + c^2 - 2bc\cos A$

$$\cos A = \frac{a^2 - b^2 - c^2}{2bc} \Rightarrow A = \cos^{-1} \frac{a^2 - b^2 - c^2}{2bc}$$

when we find $\frac{\partial A}{\partial b}$ or $\frac{\partial A}{\partial a}$ or $\frac{\partial A}{\partial c}$

this called Explicity dervative.

Second-Order Partial Derivatives

- when we differentiate a function $f(x,y)$ twice, we produce its second derivatives. these derivatives are usally denoted by:

$$\frac{\partial^2 f}{\partial x^2} \text{ or } f_{xx} \quad , \quad \frac{\partial^2 f}{\partial y^2} \text{ or } f_{yy}$$

$$\frac{\partial^2 f}{\partial x \partial y} \text{ or } f_{yx} \quad , \quad \frac{\partial^2 f}{\partial y \partial x} \text{ or } f_{xy}$$

- The defining equations are:

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) \quad , \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

Ex: If $f(x,y) = x\cos y + y e^{x}$ find the second-order derivative $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial y \partial x}$, $\frac{\partial^2 f}{\partial y^2}$ and $\frac{\partial^2 f}{\partial x \partial y}$

Sol.

$$\frac{\partial^{\cancel{2}} f}{\partial x^{\cancel{2}}} = \frac{\partial}{\partial x}(x\cos y + y e^{x})$$

$$\frac{\partial f}{\partial x} = \cos y + y e^{x}$$

$$, \quad \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x\cos y + y e^{x})$$

$$, \quad \frac{\partial f}{\partial y} = -x\sin y + e^{x}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

$$= \frac{\partial}{\partial y}(\cos y + y e^{x})$$

$$\boxed{\frac{\partial^2 f}{\partial y \partial x} = -\sin y + e^{x}}$$

$$, \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}(-x\sin y + e^{x})$$

$$\boxed{\frac{\partial^2 f}{\partial x \partial y} = -\sin y + e^{x}}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(\cos y + y e^x\right) = \underline{\underline{y e^x}}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(-x \sin y + e^x\right) = \underline{\underline{-x \cos y}}$$

- note: If $f(x,y)$ and its partial derivatives f_x, f_y, f_{xy} and f_{yx} are defined throughout an open region containing a point (a,b) and are all continuous at (a,b) then

$$\boxed{f_{xy}(a,b) = f_{yx}(a,b)}$$

Ex: Find all the second-order partial derivatives of the functions for the following below:

a) $S(x,y) = \tan^{-1}\left(\frac{y}{x}\right)$

$$\frac{x * 0 - y * 1}{x^2}$$

$$\frac{\partial S'}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2} * \frac{-y}{x^2} = \frac{-y}{x^2 + \frac{\cancel{x^2} * y^2}{\cancel{x^2}}} = \frac{-y}{x^2+y^2}$$

$$\boxed{\frac{\partial^2 S'}{\partial x^2}} = \frac{\partial}{\partial x}\left(\frac{-y}{x^2+y^2}\right) = \frac{(x^2+y^2) * 0 - (-y) * 2x}{(x^2+y^2)^2} = \boxed{\frac{2xy}{(x^2+y^2)^2}}$$

$$\frac{\partial S'}{\partial y} = \frac{1}{1+\left(\frac{y}{x}\right)^2} * \frac{1}{x} = \frac{x}{x^2+y^2} \leftarrow \qquad \frac{1}{x + \cancel{x} * \frac{y^2}{x\cancel{x}}} = \frac{1}{\frac{x^2+y^2}{x}} = \frac{x}{x^2+y^2}$$

$$\boxed{\frac{\partial^2 S'}{\partial y^2}} = \frac{\partial S'}{\partial y}\left(\frac{x}{x^2+y^2}\right) = \frac{(x^2+y^2)*0 - x * 2y}{(x^2+y^2)^2} = \boxed{\frac{-2xy}{(x^2+y^2)^2}}$$

$$\frac{\partial^2 S'}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial S'}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) = \frac{(x^2+y^2)*1 - x*2x}{(x^2+y^2)^2} = \frac{x^2+y^2-2x^2}{(x^2+y^2)^2}$$

$$\boxed{\frac{\partial^2 S'}{\partial x \partial y} = \frac{y^2-x^2}{(x^2+y^2)^2}}$$

$$\frac{\partial^2 S'}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial S'}{\partial x}\right) = \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right) = \frac{(x^2+y^2)*-1 - (-y*2y)}{(x^2+y^2)^2}$$

$$\boxed{\frac{\partial^2 S'}{\partial y \partial x} = \frac{y^2-x^2}{(x^2+y^2)^2}}$$

(4)

b) $w = \underset{1}{x}\ \underset{2}{\sin(x^2y)}$

Sol.

$$\frac{\partial w}{\partial x} = x * \cos(x^2y) * 2xy + \sin(x^2y) * 1 = 2x^2y\cos x^2y + \sin x^2y$$

$$\frac{\partial^2 w}{\partial x^2} = \frac{\partial}{\partial x}\left(2x^2y\cos x^2y + \sin x^2y\right) = 2x^2y * (-\sin x^2y) * 2xy + \cos x^2y * 4xy + \cos x^2y * 2xy$$

$$\therefore \frac{\partial^2 w}{\partial x^2} = -4x^3y^2 \sin x^2y + 6xy\cos x^2y$$

$$\frac{\partial^2 w}{\partial y\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial x}\right) = \frac{\partial}{\partial y}\left(2x^2y\cos x^2y + \sin x^2y\right)$$

$$= 2x^2y * (-\sin x^2y) * x^2 + \cos x^2y * 2x^2 + \cos x^2y * (x^2)$$

$$\therefore \frac{\partial^2 w}{\partial y\partial x} = -2x^4y\sin x^2y + 3x^2\cos x^2y$$

$$\frac{\partial w}{\partial y} = x\cos x^2y * x^2 = x^3\cos x^2y$$

$$\frac{\partial^2 w}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial y}\right) = \frac{\partial}{\partial y}\left(x^3\cos x^2y\right) = x^3(-\sin x^2y) * x^2$$

$$\therefore \frac{\partial^2 w}{\partial y^2} = -x^5\sin x^2y$$

$$\frac{\partial^2 w}{\partial x\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial y}\right) = \frac{\partial}{\partial x}\left(x^3\cos x^2y\right) = x^3 * (-\sin x^2y) * 2xy + \cos x^2y * 3x^2$$

$$\therefore \frac{\partial^2 w}{\partial x\partial y} = -2x^4y\sin x^2y + 3x^2\cos x^2y$$

Ex: verify that $\omega_{xy} = \omega_{yx}$ for the following:

$$\omega = \ln(2x+3y)$$

Sol.

$$\frac{\partial \omega}{\partial x} = \frac{1}{2x+3y} * 2 = \frac{2}{2x+3y}$$

$$\frac{\partial^2 \omega}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial \omega}{\partial x}\right) = \frac{\partial}{\partial y}\left(\frac{2}{2x+3y}\right) = \frac{(2x+3y)*0 - 2*3}{(2x+3y)^2}$$

$$\therefore \frac{\partial^2 \omega}{\partial y \partial x} = \frac{-6}{(2x+3y)^2}$$

$$\frac{\partial \omega}{\partial y} = \frac{1}{2x+3y} * 3 = \frac{3}{2x+3y}$$

$$\frac{\partial^2 \omega}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial \omega}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{3}{2x+3y}\right) = \frac{(2x+3y)*0 - 3*2}{(2x+3y)^2}$$

$$\therefore \frac{\partial^2 \omega}{\partial x \partial y} = \frac{-6}{(2x+3y)^2}$$

$$\therefore \omega_{xy} = \omega_{yx}$$

" Partial Derivatives of still higher Order "

*- Third and fourth-order derivatives denoted by symbols like:

$$\frac{\partial^3 f}{\partial x \partial y^2} = f_{yyx} \quad , \quad \frac{\partial^4 f}{\partial x^2 \partial y^2} = f_{yyxx}$$

Ex: Find f_{yxyz} if $f(x,y,z) = 1 - 2xy^2z + x^2y$

$f_y = -2x \cdot (2y) \cdot z + x^2 \Rightarrow f_y = -4xyz + x^2$

$f_{yx} = -4yz + 2x$

$f_{yxy} = -4z$

$f_{yxyz} = -4$

"The chain Rule"

- If $\omega = f(x,y)$ is differentiable and $x = x(t)$, $y = y(t)$ are differentiable functions of (t), then:

$$\boxed{\frac{\partial \omega}{\partial t} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial t}}$$

- If $\omega = f(x,y,z)$ is differentiable and x, y, z are diff. functions of (t) then:

$$\boxed{\frac{\partial \omega}{\partial t} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial t} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial t}}$$

- If $w = f(x,y,z)$, $x = g(r,s)$, $y = h(r,s)$ and $z = k(r,s)$ if all four functions are differentiable, then:

$$\frac{\partial w}{\partial r} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial r} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial r}$$

$$\frac{\partial w}{\partial s} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial s} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial s}$$

Ex: by using chain rule express $\frac{\partial w}{\partial t}$ as a function of (t) for the following below then find $\frac{\partial w}{\partial t}$ at the given value of (t).

1) $w = 2ye^{x} - \ln z$, $x = \ln(t^{2}+1)$, $y = \tan^{-1} t$, $z = e^{t}$, $t=1$

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial t} + \frac{\partial w}{\partial z} * \frac{\partial z}{\partial t}$$

$$\frac{\partial w}{\partial x} = 2ye^{x} \quad , \quad \frac{\partial w}{\partial y} = 2e^{x} \quad , \quad \frac{\partial w}{\partial z} = \frac{-1}{z}$$

$$\frac{\partial x}{\partial t} = \frac{1}{t^{2}+1} * 2t = \frac{2t}{t^{2}+1}$$

$$\frac{\partial y}{\partial t} = \frac{1}{t^{2}+1} \quad , \quad \frac{\partial z}{\partial t} = e^{t}$$

$$\therefore \frac{\partial w}{\partial t} = 2ye^{x} * \frac{2t}{t^{2}+1} + 2e^{x} * \frac{1}{t^{2}+1} + \frac{-1}{z} * e^{t}$$

$$\frac{\partial w}{\partial t} = 4ye^{x}\left(\frac{t}{t^{2}+1}\right) + \frac{2e^{x}}{t^{2}+1} - \frac{e^{t}}{z}$$

To find $\frac{\partial w}{\partial t}$ at $t=1$

$$\frac{\partial w}{\partial t} = 4\tan^{-1} t * e^{\cancel{\ln(t^{2}+1)}} * \frac{\cancel{2}t}{\cancel{t^{2}+1}} + \frac{2\cancel{(t^{2}+1)}}{\cancel{t^{2}+1}} - \frac{\cancel{e^{t}}}{\cancel{e^{t}}}$$

$$\frac{\partial w}{\partial t} = 4t\tan^{-1} t + 1 \qquad \text{at } t=1$$

$$\frac{\partial w}{\partial t} = 4*1*\underbrace{\tan^{-1}(1)}_{\frac{\pi}{4}} + 1 = \pi + 1$$

2. $w = \frac{x}{z} + \frac{y}{z}$, $x = \cos^2 t$, $y = \sin^2 t$, $z = 1/t$, $t = 3$

$w = \frac{x+y}{z}$

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial t} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial t}$$

$$\frac{\partial w}{\partial x} = \frac{1}{z}, \quad \frac{\partial w}{\partial y} = \frac{1}{z}, \quad \frac{\partial w}{\partial z} = \frac{z * 0 - (x+y) * 1}{z^2} = \frac{-(x+y)}{z^2}$$

$$\frac{\partial x}{\partial t} = 2 \cos t (-\sin t) = -2 \cos t \sin t$$

$$\frac{\partial y}{\partial t} = 2 \sin t \cos t, \quad \frac{\partial z}{\partial t} = \frac{-1}{t^2}$$

$$\therefore \frac{\partial w}{\partial t} = \cancel{\frac{-1}{z} * 2 \cos t \sin t} + \cancel{\frac{2}{z} \sin t \cos t} - \frac{x+y}{z^2} * \frac{-1}{t^2}$$

$$\boxed{\frac{\partial w}{\partial t} = \frac{x+y}{z^2 t^2}}$$

To find $\frac{\partial w}{\partial t}$ at $t = 3$

$$\frac{\partial w}{\partial t} = \frac{\cos^2 t + \sin^2 t}{\cancel{\frac{1}{t^2}} * \cancel{t^2}} = \frac{1}{1} = 1$$

$\therefore \frac{\partial w}{\partial t}$ at any $t = \underline{1}$

Ex: Find $\partial z/\partial u$ and $\partial z/\partial v$ for the following.

$$z = 4e^x \ln y \quad , \quad x = \ln(u\cos v) \quad , \quad y = u \sin v$$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial u} = 4e^x \ln y * \frac{1}{u\cancel{\cos v}} * \cancel{\cos v} + 4e^x * \frac{1}{y} * \sin v$$

$$\therefore \frac{\partial z}{\partial u} = \frac{4e^x \ln y}{u} + \frac{4e^x \sin v}{y}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v}$$

$$= 4e^x \ln y * \frac{1}{\cancel{u}\cos v} * \cancel{u} * (-\sin v) + 4e^x * \frac{1}{y} * u\cos v$$

$$\therefore \frac{\partial z}{\partial v} = -4e^x \ln y \tan v + \frac{4e^x . u . \cos v}{y}$$

"Maximum and Minimum & saddle Point"

= Let $f(x,y)$ be defined on a region R containing the point (a,b) then.

① $f(a,b)$ is a local Maximum value of f if $f(a,b) \geq f(x,y)$ for all domain Points (x,y) in an open disk centered at (a,b).

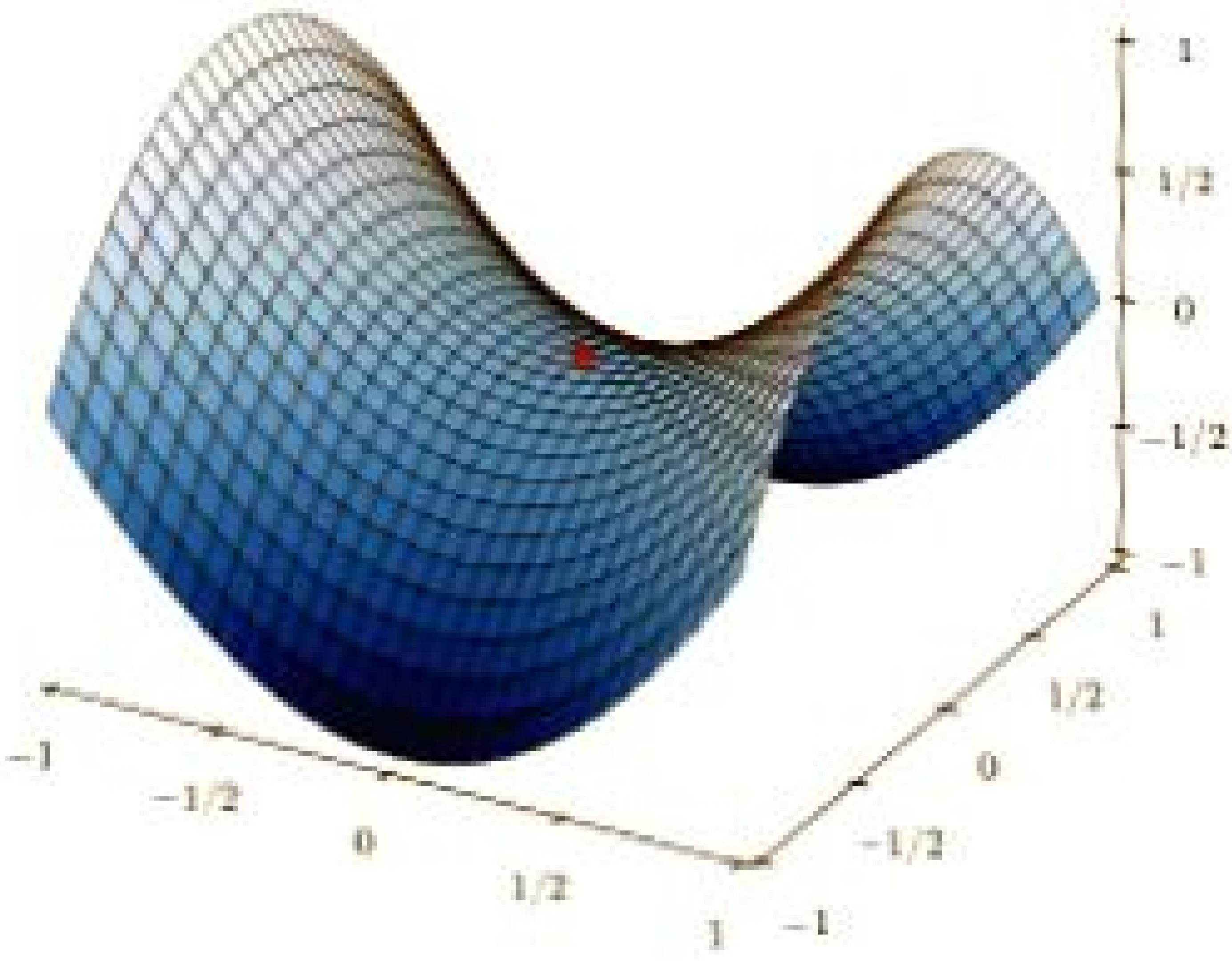
② $f(a,b)$ is a Local Minimum value of f if $f(a,b) \leq f(x,y)$ for all domain Points in an open disk centered at (a,b).

= If $f(x,y)$ has a local maximum or minimum value at interior point (a,b) of its domain and if the first partial derivatives exist there, then $f_x(a,b)=0$ and $f_y(a,b)=0$

= An interior point of the domain of a function $f(x,y)$ where both f_x and f_y are zero or where one or both of f_x and f_y do not exist is a critical Point. of f.

= A differentiable function $f(x,y)$ has a saddle point at a critical Point (a,b) if in every open disk centered at (a,b) there are domain Points (x,y) where $f(x,y) > f(a,b)$ and domain Points (x,y) where $f(x,y) < f(a,b)$. the corresponding Point $(a,b,f(a,b))$ on the surface $z=f(x,y)$ is called saddle point of the surface.

= In mathematics, a saddle point is a point on the surface of the graph of a function where the slopes (derivatives) in orthogonal direction are all zero (a critical Point), but which is not a local extremum of the function. An example of a saddle Point shown on the right is when there is a critical Point with a relative minimum along one axial direction (between Peaks) and at a relative max. alonge the crossing axis.


1
1/2
0
−1/2
−1
−1
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1
1
1/2
0
−1/2
−1

(*) Suppose that $f(x,y)$ and its first and second partial derivatives are continuous throughout a disk centered at (a,b) and that $f_x(a,b) = f_y(a,b) = 0$ Then.

1. f has a local Maximum at (a,b) if $\boxed{f_{xx} < 0 \text{ and } f_{xx}f_{yy} - f_{xy}^2 > 0}$ at (a,b).

2. f has a local Minimum at (a,b) if $\boxed{f_{xx} > 0 \text{ and } f_{xx}f_{yy} - f_{xy}^2 > 0}$ at (a,b).

3. f has a Saddle Point at (a,b) if $\boxed{f_{xx}f_{yy} - f_{xy}^2 < 0}$ at (a,b)

4. the test is inconclusive at (a,b) if $\boxed{f_{xx}f_{yy} - f_{xy}^2 = 0}$ at (a,b)
In this case we must find some other way to determine the behaviors of f at (a,b).

Ex: Find all the maximum, local minimum, and saddle points of the functions below:

① $f(x,y) = \dfrac{1}{x^2+y^2-1} \Rightarrow f_x = \dfrac{-2x}{(x^2+y^2-1)^2} = 0 \Rightarrow \boxed{x=0}$

$f_y = \dfrac{-2y}{(x^2+y^2-1)^2} = 0 \Rightarrow \boxed{y=0}$

$f(0,0) = -1$ @ $(0,0)$ is the critical point

$$f_{xx} = \frac{(x^2+y^2-1)^2(-2) + 2\cancel{x} * 2(x^2+y^2-1) * (2x)}{(x^2+y^2-1)^4} \quad @ (0,0)$$

f_{xx} at $x=0$, $y=0$: $= \dfrac{1(-2)+0}{1} = -2$