Chapter one

Prerequisites for Calculus
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1.1 functions الدوال

- تعريف الدالة function على الاعداد الحقيقة
- ايجاد اوسع مجال Domain
- ايجاد المدى range
- رسم الدالة graph the function
- الدوال الفردية والدوال الزوجية ، التناظر مع محور الصادات مع محور السينات مع نقطة الاصل
- الدوال المعرفة على شكل قطع function defined in pieces
- التزحيف لرسم الدوال shifting a graph of the functions
- التحجيم والانعكاس لرسم الدوال Scaling and reflecting a graph of the functions

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تعريف الدالة function على الاعداد الحقيقة

A function is mapping denoted by f defined from a set called domain of f denoted by D(f) and assigns a *unique* value f(x) in codomain for every x∈ $D(f)$

FIGURE 1.1 A diagram showing a function as a kind of machine.

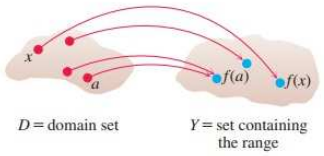


FIGURE 1.2 A function from a set *D* to a set *Y* assigns a unique element of *Y* to each element in *D*.

الدالة هي علاقة بين مجموعتين المجموعة الاولى تكون الاعداد الحقيقية او مجموعة جزئية منها تسمى المجال domain الى مجموعة اخر ايضا مجموعة الاعداد الحقيقة تسمى المجال المقابل codomain تكتب بالصيغة التالية

y=f(x)

العنصر x ينتمي الى المجال domain

مجموعة صور العناصر f(x) تسمى المدى Range

يسمى x المتغير المستقل independent variable

يسمى y المتغير المعتمد، dependent variable

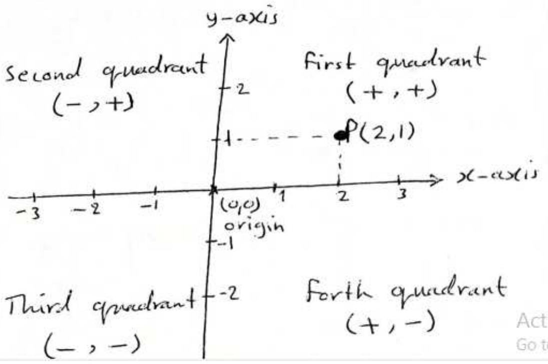
Graph the function

Cartesian coordinates and Graphs in the plane:

The cartesian coordinates are two Lines

Horizontal line is called the x-axis

Vertical line is called the y-axis



Graphs of Functions

If f is a function with domain D, its **graph** consists of the points in the Cartesian plane whose coordinates are the input-output pairs for f. In set notation, the graph is

$$\{(x, f(x)) \mid x \in D\}.$$

The graph of the function $f(x) = x + 2$ is the set of points with coordinates (x, y) for which $y = x + 2$. Its graph is the straight line sketched in Figure 1.3.

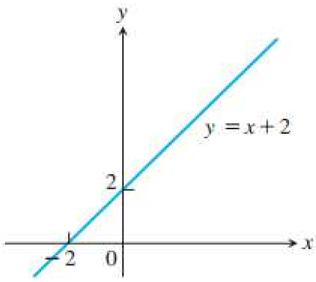


FIGURE 1.3 The graph of $f(x) = x + 2$ is the set of points (x, y) for which y has the value $x + 2$.

EXAMPLE 2 Graph the function $y = x^2$ over the interval $[-2, 2]$.

Solution Make a table of xy-pairs that satisfy the equation $y = x^2$. Plot the points (x, y) whose coordinates appear in the table, and draw a *smooth* curve (labeled with its equation) through the plotted points (see Figure 1.5).

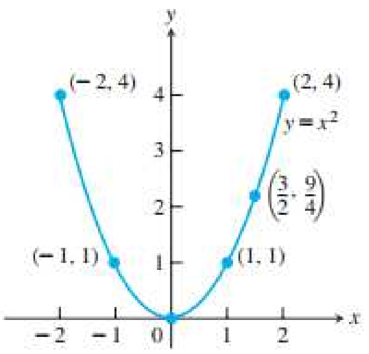


FIGURE 1.5 Graph of the function in Example 2.

x	$y = x^2$
-2	4
-1	1
0	0
1	1
$\frac{3}{2}$	$\frac{9}{4}$
2	4

ايجاد اوسع مجال Domain للدالة

EXAMPLE 1 Verify the natural domains and associated ranges of some simple functions. The domains in each case are the values of x for which the formula makes sense.

Function	Domain (x)	Range (y)
$y = x^2$	$(-\infty, \infty)$	$[0, \infty)$
$y = 1/x$	$(-\infty, 0) \cup (0, \infty)$	$(-\infty, 0) \cup (0, \infty)$
$y = \sqrt{x}$	$[0, \infty)$	$[0, \infty)$
$y = \sqrt{4 - x}$	$(-\infty, 4]$	$[0, \infty)$
$y = \sqrt{1 - x^2}$	$[-1, 1]$	$[0, 1]$

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كما لاحضنا في المثال السابق انه لايمكن ان نتكهن باوسع مجال للدالة مالم نعرف كيف ايجاد اوسع مجال domain كذلك كيف ايجاد المدى لها range لذلك سنتعرف على انواع الدوال كما موضح ادناه

Common Functions

A variety of important types of functions are frequently encountered in calculus.

Linear Functions A function of the form $f(x) = mx + b$, where m and b are fixed constants, is called a **linear function**. Figure 1.14a shows an array of lines $f(x) = mx$. Each of these has $b = 0$, so these lines pass through the origin. The function $f(x) = x$ where $m = 1$ and $b = 0$ is called the **identity function**. Constant functions result when the slope is $m = 0$ (Figure 1.14b).

مجال الدالة domain خط المستقيم هو كل الاعداد الحقيقية R

ومدى الدالة Range خط المستقيم هو كل الاعداد الحقيقية R

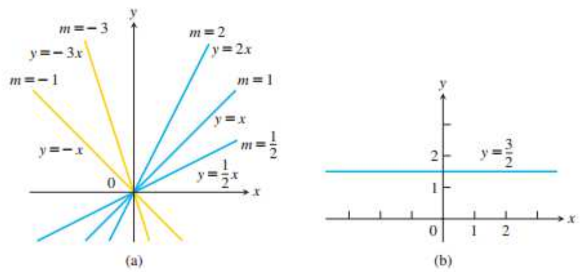

FIGURE 1.14 (a) Lines through the origin with slope m. (b) A constant function with slope $m = 0$.

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Power Functions A function $f(x) = x^a$, where a is a constant, is called a **power function**. There are several important cases to consider.

(a) $f(x) = x^a$ with $a = n$, a positive integer.

The graphs of $f(x) = x^n$, for $n = 1, 2, 3, 4, 5$, are displayed in Figure 1.15.

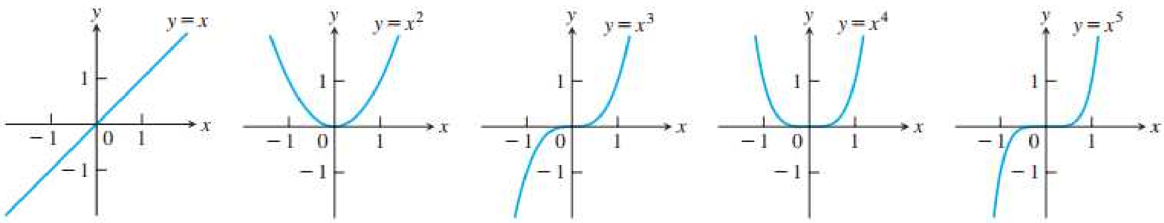


FIGURE 1.15 Graphs of $f(x) = x^n$, $n = 1, 2, 3, 4, 5$, defined for $-\infty < x < \infty$.

(b) $f(x) = x^a$ with $a = -1$ or $a = -2$.

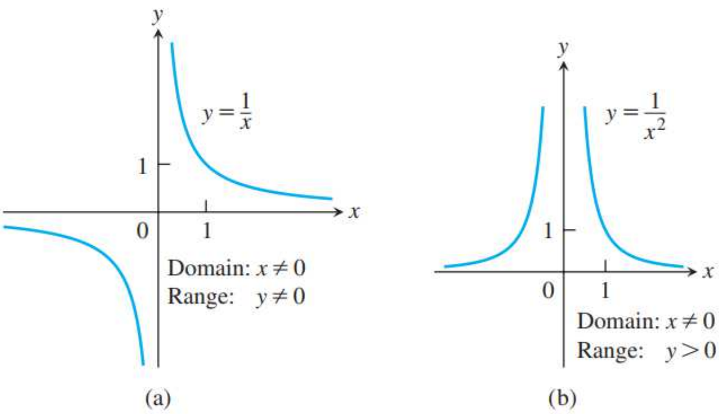


FIGURE 1.16 Graphs of the power functions $f(x) = x^a$. (a) $a = -1$, (b) $a = -2$.

(c) $a = \frac{1}{2}, \frac{1}{3}, \frac{3}{2}$, and $\frac{2}{3}$.

The functions $f(x) = x^{1/2} = \sqrt{x}$ and $g(x) = x^{1/3} = \sqrt[3]{x}$ are the **square root** and **cube root** functions, respectively. The domain of the square root function is $[0, \infty)$, but the cube root function is defined for all real x. Their graphs are displayed in Figure 1.17, along with the graphs of $y = x^{3/2}$ and $y = x^{2/3}$. (Recall that $x^{3/2} = (x^{1/2})^3$ and $x^{2/3} = (x^{1/3})^2$.)

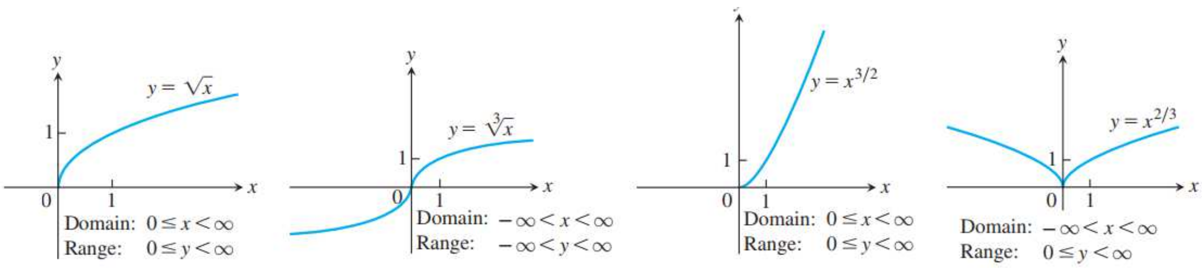


FIGURE 1.17 Graphs of the power functions $f(x) = x^a$ for $a = \frac{1}{2}, \frac{1}{3}, \frac{3}{2}$, and $\frac{2}{3}$.

Polynomials A function p is a **polynomial** if

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where n is a nonnegative integer and the numbers $a_0, a_1, a_2, \ldots, a_n$ are real constants (called the **coefficients** of the polynomial). All polynomials have domain $(-\infty, \infty)$. If the

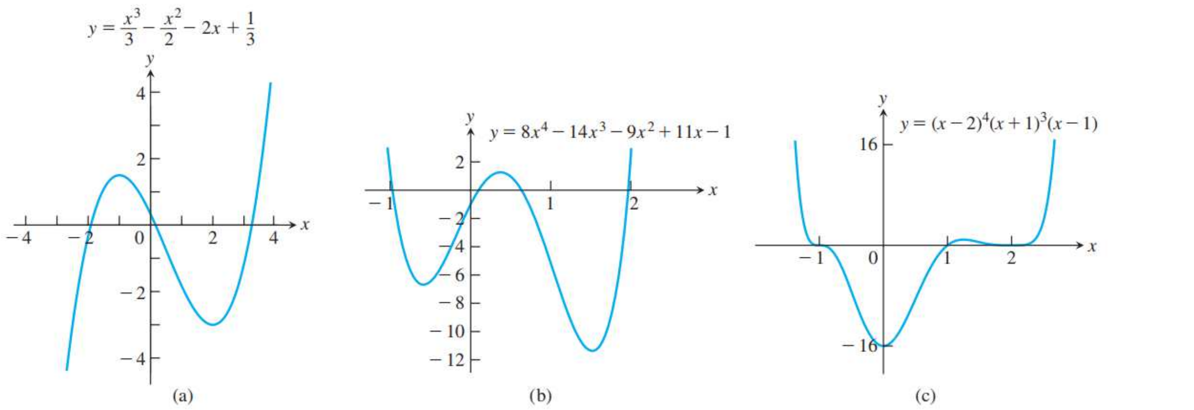
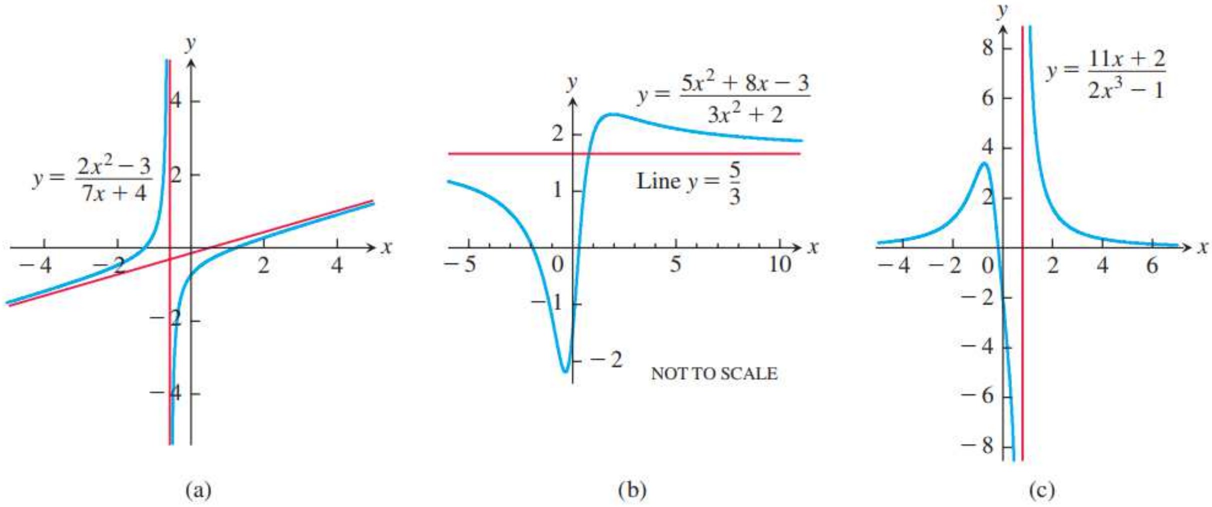


FIGURE 1.18 Graphs of three polynomial functions.

Rational Functions A **rational function** is a quotient or ratio $f(x) = p(x)/q(x)$, where p and q are polynomials. The domain of a rational function is the set of all real x for which $q(x) \neq 0$. The graphs of several rational functions are shown in Figure 1.19.



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Algebraic Functions Any function constructed from polynomials using algebraic operations (addition, subtraction, multiplication, division, and taking roots) lies within the

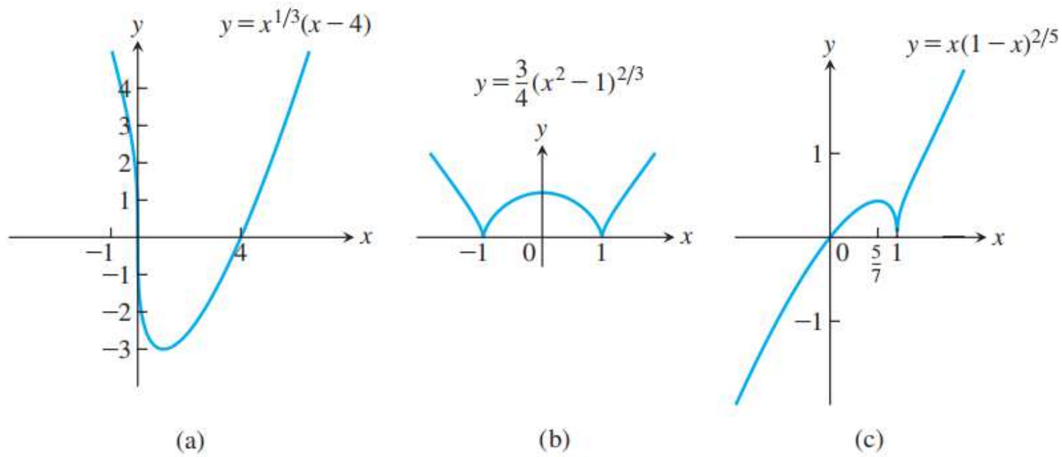

FIGURE 1.20 Graphs of three algebraic functions.

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Trigonometric Functions The six basic trigonometric functions are reviewed in Section 1.3. The graphs of the sine and cosine functions are shown in Figure 1.21.

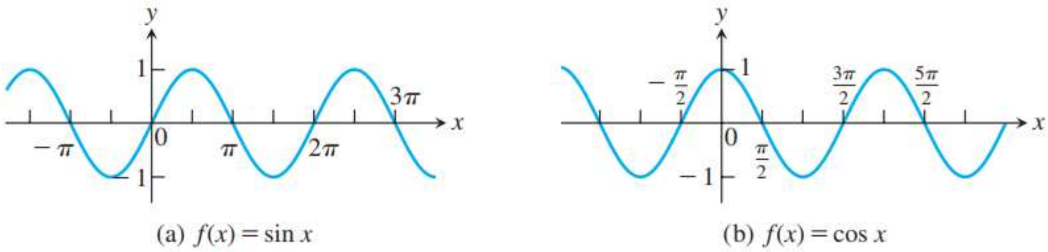

FIGURE 1.21 Graphs of the sine and cosine functions.

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ملخص لأوسع مجال Domain

- أوسع مجال لمتعداد الحدود polinomail هو كل الاعداد الحقيقة R
- أوسع مجال للدوال الجذرية $y = \sqrt[n]{f(x)}$ يعتمد على الأس n

اذا كان الاس n فردي فأنه اوسع مجال للدالة هو كل الاعداد الحقيقية R

اما اذا كان الاس n زوجي فأنه اوسع مجال للدالة يكون بأخذ داخل الدالة أكبر اويساوي الصفر $f(x) \geq 0$

- اذا كانت الدالة كسرية $y = \frac{f(x)}{g(x)}$ فأن اوسع مجال للدالة هو كل الاعداد الحقيقية ما عدا القيم التي تجعل المقام تساوي صفر $R\backslash\{g(x) = 0\}$
- اذا كانت الدالة كسرية وجذرية في ان واحد مثل $y = \frac{1}{\sqrt{g(x)}}$ فأن اوسع مجال لها هو بأخذ داخل الجذر أكبر من الصفر دون اخذ المساواة لان المساوة تجعل المقام تساوي صفر $g(x) > 0$.

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ملخص لأيجاد المدى Range

- المدى لدالة الخط المستقيم هو كل الاعداد الحقيقية
- اما المدى لمتعداد الحدود polinomail هو فغالباً ما يتم تحديده من خلال الرسم
- المجال للدوال الجذرية $y = \sqrt[n]{f(x)}$ هو كل القيم الحقيقية التي أكبر من الصفر $x \geq 0$
- اذا كانت الدالة كسرية $y = \frac{f(x)}{g(x)}$ وللعدد محدود من الدوال اذا استطعنا ان نكتب x بدلالة y أي ان نحول الدالة الى الصيغة $x = \frac{h(y)}{P(y)}$ فأن مجال الدالة الجديد يمثل المدى للدالة القديمة
- وبصورة عامة غالباً ما يأخذ المدى Range من الرسم.

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Example: Find the domain of the following functions

① $f(x) = x^2 + 1$

② $f(x) = \frac{1}{x^2 - 1}$

③ $f(x) = \sqrt{2x - 1}$

Solution ① Domain of $f(x) = x^2 + 1$ is $\mathbb{R}$, Since $f(x)$ is a polynomial

that is, $D: -\infty < x < \infty$

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② Since $f(x)$ is rational function, then denominator $\neq 0$

$x^2-1=0 \Rightarrow (x-1)(x+1)=0$

$\Rightarrow$ either $x=1$ or $x=-1$

$\therefore$ Domain of f is $R\setminus\{1,-1\}$

that is, $D: x\neq 1$ and $x\neq -1$

③ Since $f(x)$ is even root, then

$2x-1\geq 0$

$2x\geq 1$

$x\geq \frac{1}{2}$

$\therefore$ Domain: $x\geq\frac{1}{2}$ or $D:[\frac{1}{2},\infty)$

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Example: Find the range of the following functions

① $f(x)=(x-4)^2+5$

② $f(x)=\sqrt{3x+2}-1$

③ $f(x)=\frac{3}{x-2}$

Solution ① we see that $(x-4)^2\geq 0$ for all value of x

$(x-4)^2+5\geq 5$

$f(x)\geq 5$

$\therefore$ the range $y\geq 5$

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② Since we have a square root the range is $[-1, \infty)$

or $\sqrt{3x+2} \geq 0$

$\sqrt{3x+2} - 1 \geq -1$

$f(x) \geq -1$

$\Rightarrow$ $y \geq -1$, means $[-1, \infty)$ which is the range.

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③ $y = \dfrac{3}{x-2}$

$(x-2)y = 3$

$xy - 2y = 3$

$xy = 3 + 2y$

$x = \dfrac{3+2y}{y}$, the domain of this function is $y \neq 0$

$\therefore$ the range of $y = \dfrac{3}{x-2}$ is $R \setminus \{0\}$.

or $y \neq 0$.

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Intervals : (Finite Intervals) :

Open interval
$a < x < b$
or (a, b)

closed interval
$a \leq x \leq b$
or $[a, b]$

half – opened interval
$a \leq x < b$
or $[a, b)$

half – opened interval
$a < x \leq b$
or $(a, b]$

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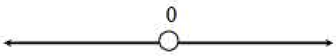
Intervals : (Infinite Intervals) :



$-\infty < x < \infty$
or $(-\infty, \infty)$

$a \leq x < \infty$
or $[a, \infty)$

$a < x < \infty$
or (a, ∞)

$-\infty < x \leq b$
or $(-\infty, b]$

$-\infty < x < b$
or $(-\infty, b)$

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EXAMPLE 1 Verify the natural domains and associated ranges of some simple functions. The domains in each case are the values of x for which the formula makes sense.

Function	Domain (x)	Range (y)
$y = x^2$	$(-\infty, \infty)$	$[0, \infty)$
$y = 1/x$	$(-\infty, 0) \cup (0, \infty)$	$(-\infty, 0) \cup (0, \infty)$
$y = \sqrt{x}$	$[0, \infty)$	$[0, \infty)$
$y = \sqrt{4 - x}$	$(-\infty, 4]$	$[0, \infty)$
$y = \sqrt{1 - x^2}$	$[-1, 1]$	$[0, 1]$

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Even Functions and Odd Functions: Symmetry

The graphs of *even* and *odd* functions have special symmetry properties.

DEFINITIONS A function $y = f(x)$ is an

even function of x if $f(-x) = f(x)$,
odd function of x if $f(-x) = -f(x)$,

for every x in the function's domain.

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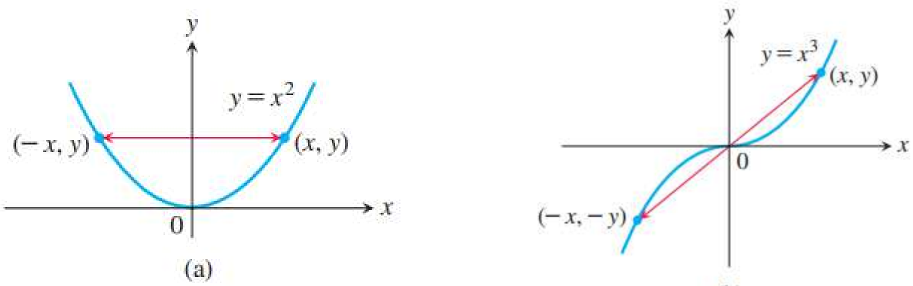
The graph of an even function is **symmetric about the *y*-axis**. Since $f(-x) = f(x)$, a point (x, y) lies on the graph if and only if the point $(-x, y)$ lies on the graph (Figure 1.12a). A reflection across the *y*-axis leaves the graph unchanged.

The graph of an odd function is **symmetric about the origin**. Since $f(-x) = -f(x)$, a point (x, y) lies on the graph if and only if the point $(-x, -y)$ lies on the graph (Figure 1.12b). Equivalently, a graph is symmetric about the origin if a rotation of 180° about the origin leaves the graph unchanged. Notice that the definitions imply that both x and $-x$ must be in the domain of f.



(a) (b)

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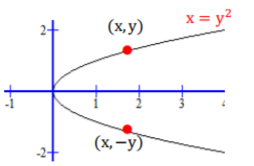
* Symmetry tests for graphs :

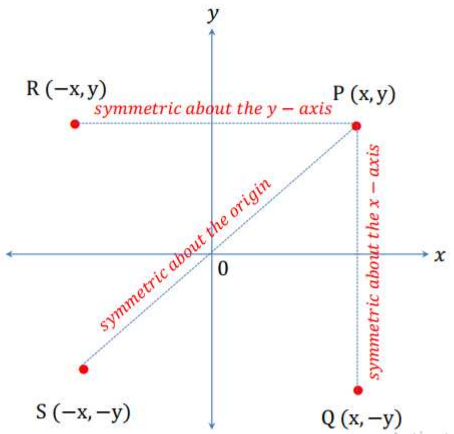
-symmetry about the x-axis :
If the point (x, y) lies on the graph then the point $(x, -y)$ lies on the graph.



Symmetric points :
$p(x, y)$ and $Q(x, -y)$ symmyric about the $x-$ axis.

$P(x, y)$ and $R(-x, y)$ symmyric about the $y-$ axis.

$P(x, y)$ and $S(-x, -y)$ symmyric about the origin .



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EXAMPLE 8 Here are several functions illustrating the definitions.

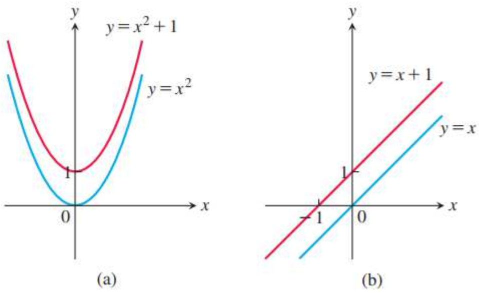
$f(x) = x^2$ — Even function: $(-x)^2 = x^2$ for all x; symmetry about y-axis. So $f(-3) = 9 = f(3)$. Changing the sign of x does not change the value of an even function.

$f(x) = x^2 + 1$ — Even function: $(-x)^2 + 1 = x^2 + 1$ for all x; symmetry about y-axis (Figure 1.13a).

$f(x) = x$ — Odd function: $(-x) = -x$ for all x; symmetry about the origin. So $f(-3) = -3$ while $f(3) = 3$. Changing the sign of x changes the sign of an odd function.

$f(x) = x + 1$ — Not odd: $f(-x) = -x + 1$, but $-f(x) = -x - 1$. The two are not equal.

Not even: $(-x) + 1 \neq x + 1$ for all $x \neq 0$ (Figure 1.13b). ■



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Piecewise-Defined Functions

Sometimes a function is described in pieces by using different formulas on different parts of its domain. One example is the **absolute value function**

$$|x| = \begin{cases} x, & x \geq 0 \quad \text{First formula} \\ -x, & x < 0 \quad \text{Second formula} \end{cases}$$

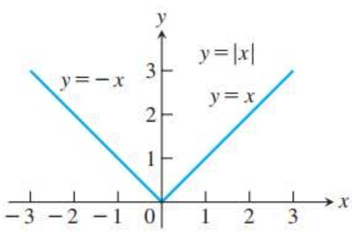

FIGURE 1.8 The absolute value function has domain $(-\infty, \infty)$ and range $[0, \infty)$.

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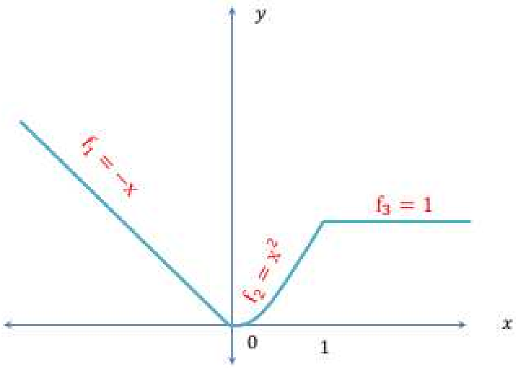
∗ **Example (1) :** Graph :

$$y = f(x) = \begin{cases} -x & x \le 0 \\ x^2 & 0 \le x \le 1 \\ 1 & x > 1 \end{cases}$$

Solution :



EXERCISES 1.1

Functions

In Exercises 1–6, find the domain and range of each function.

1. $f(x) = 1 + x^2$
2. $f(x) = 1 - \sqrt{x}$
3. $F(x) = \sqrt{5x + 10}$
4. $g(x) = \sqrt{x^2 - 3x}$
5. $f(t) = \dfrac{4}{3 - t}$
6. $G(t) = \dfrac{2}{t^2 - 16}$

Functions and Graphs

Find the natural domain and graph the functions in Exercises 15–20.

15. $f(x) = 5 - 2x$
16. $f(x) = 1 - 2x - x^2$
17. [illegible]
18. $g(x) = \sqrt{-x}$
19. [illegible]
20. [illegible]
21. Find the domain of $y = \dfrac{x + 3}{4 - \sqrt{x^2 - 9}}$.
22. Find the range of $y = 2 + \sqrt{9 + x^2}$.

Piecewise-Defined Functions

Graph the functions in Exercises 25–28.

25. $f(x) = \begin{cases} x, & 0 \le x \le 1 \\ 2 - x, & 1 < x \le 2 \end{cases}$
26. $g(x) = \begin{cases} 1 - x, & 0 \le x \le 1 \\ 2 - x, & 1 < x \le 2 \end{cases}$
27. $F(x) = \begin{cases} 4 - x^2, & x \le 1 \\ x^2 + 2x, & x > 1 \end{cases}$
28. $G(x) = \begin{cases} 1/x, & x < 0 \\ x, & 0 \le x \end{cases}$

In Exercises 47–58, say whether the function is even, odd, or neither. Give reasons for your answer.

47. $f(x) = 3$

48. $f(x) = x^{-5}$

49. $f(x) = x^2 + 1$

50. $f(x) = x^2 + x$

51. $g(x) = x^3 + x$

52. $g(x) = x^4 + 3x^2 - 1$

53. $g(x) = \dfrac{1}{x^2 - 1}$

54. $g(x) = \dfrac{x}{x^2 - 1}$

55. $h(t) = \dfrac{1}{t - 1}$

56. $h(t) = |t^3|$

57. $h(t) = 2t + 1$

58. $h(t) = 2|t| + 1$

59. $\sin 2x$

60. $\sin x^2$

61. $\cos 3x$

62. $1 + \cos x$

Shifting a Graph of a Function

A common way to obtain a new function from an existing one is by adding a constant to each output of the existing function, or to its input variable. The graph of the new function is the graph of the original function shifted vertically or horizontally, as follows.

Shift Formulas

Vertical Shifts

$y = f(x) + k$ Shifts the graph of f *up* k units if $k > 0$
Shifts it *down* $|k|$ units if $k < 0$

Horizontal Shifts

$y = f(x + h)$ Shifts the graph of f *left* h units if $h > 0$
Shifts it *right* $|h|$ units if $h < 0$

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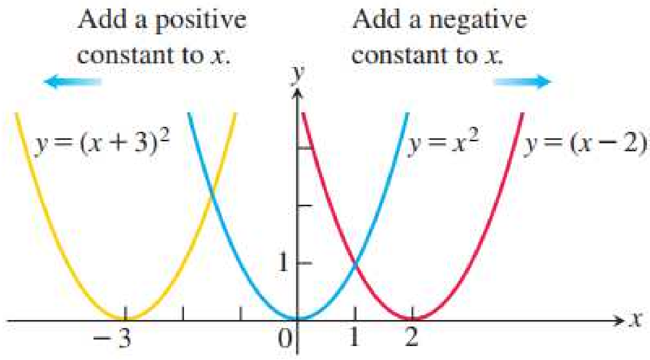
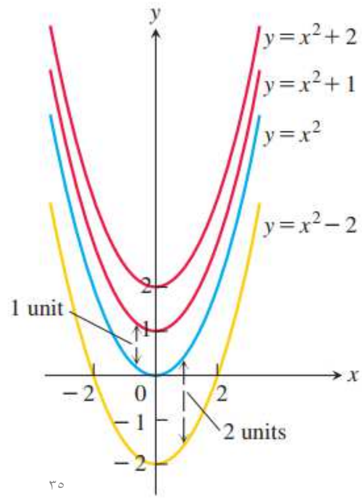
EXAMPLE 3

(a) Adding 1 to the right-hand side of the formula $y = x^2$ to get $y = x^2 + 1$ shifts the graph up 1 unit (Figure 1.29).

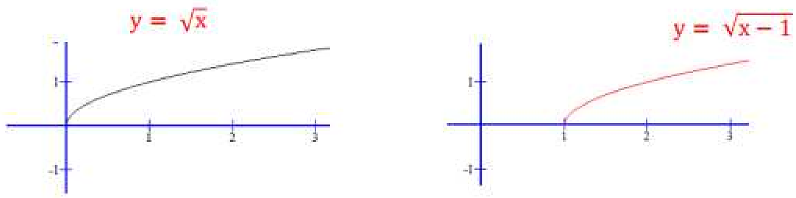
(b) Adding -2 to the right-hand side of the formula $y = x^2$ to get $y = x^2 - 2$ shifts the graph down 2 units (Figure 1.29).

(c) Adding 3 to x in $y = x^2$ to get $y = (x + 3)^2$ shifts the graph 3 units to the left, while adding -2 shifts the graph 2 units to the right (Figure 1.30).

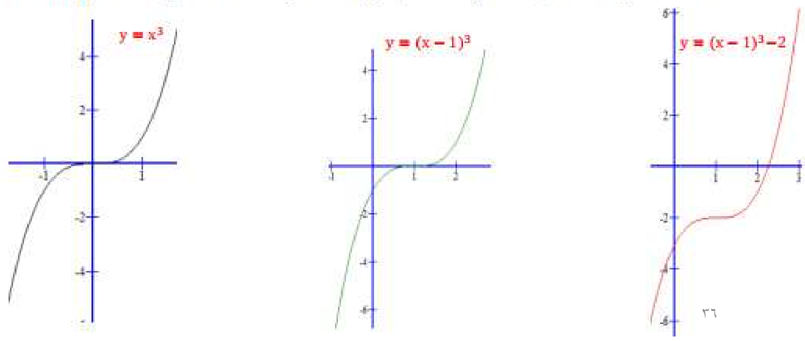




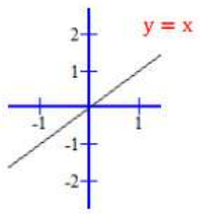
∗ **Example :** $y = \sqrt{x-1}$:

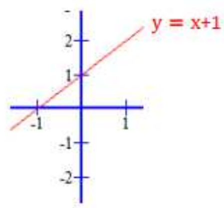


∗ **Example :** $y + 2 = (x-1)^3$, $y = (x-1)^3 - 2$

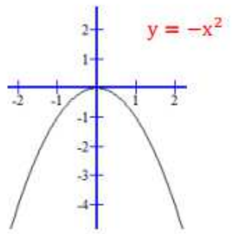


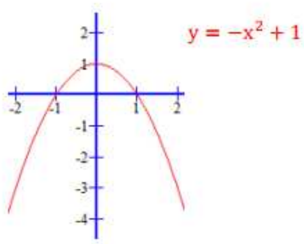
* **Example:** $\mathbf{y = x + 1}$:





* **Example:** $\mathbf{y = -x^2 + 1}$:





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Scaling and Reflecting a Graph of a Function

To scale the graph of a function $y = f(x)$ is to stretch or compress it, vertically or horizontally. This is accomplished by multiplying the function f, or the independent variable x, by an appropriate constant c. Reflections across the coordinate axes are special cases where $c = -1$.

Vertical and Horizontal Scaling and Reflecting Formulas

For $c > 1$, the graph is scaled:

$y = cf(x)$	Stretches the graph of f vertically by a factor of c.
$y = \frac{1}{c}f(x)$	Compresses the graph of f vertically by a factor of c.
$y = f(cx)$	Compresses the graph of f horizontally by a factor of c.
$y = f(x/c)$	Stretches the graph of f horizontally by a factor of c.

For $c = -1$, the graph is reflected:

$y = -f(x)$	Reflects the graph of f across the x-axis.
$y = f(-x)$	Reflects the graph of f across the y-axis.

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EXAMPLE 4 Here we scale and reflect the graph of $y = \sqrt{x}$.

(a) Vertical: Multiplying the right-hand side of $y = \sqrt{x}$ by 3 to get $y = 3\sqrt{x}$ stretches the graph vertically by a factor of 3, whereas multiplying by $1/3$ compresses the graph vertically by a factor of 3 (Figure 1.32).

(b) Horizontal: The graph of $y = \sqrt{3x}$ is a horizontal compression of the graph of $y = \sqrt{x}$ by a factor of 3, and $y = \sqrt{x/3}$ is a horizontal stretching by a factor of 3 (Figure 1.33). Note that $y = \sqrt{3x} = \sqrt{3}\sqrt{x}$ so a horizontal compression *may* correspond to a vertical stretching by a different scaling factor. Likewise, a horizontal stretching may correspond to a vertical compression by a different scaling factor.

(c) Reflection: The graph of $y = -\sqrt{x}$ is a reflection of $y = \sqrt{x}$ across the x-axis, and $y = \sqrt{-x}$ is a reflection across the y-axis (Figure 1.34). ■

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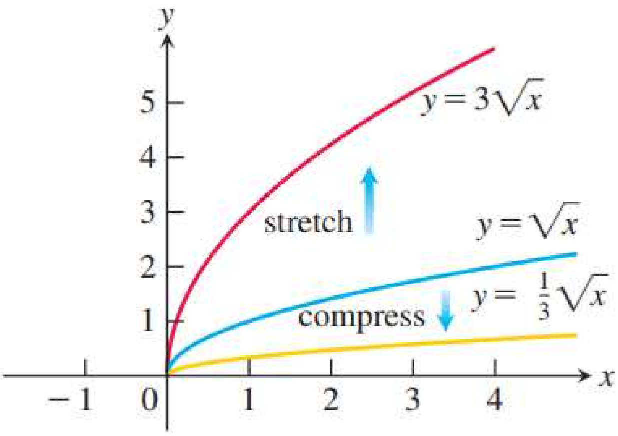

FIGURE 1.32 Vertically stretching and compressing the graph $y = \sqrt{x}$ by a factor of 3 (Example 4a).

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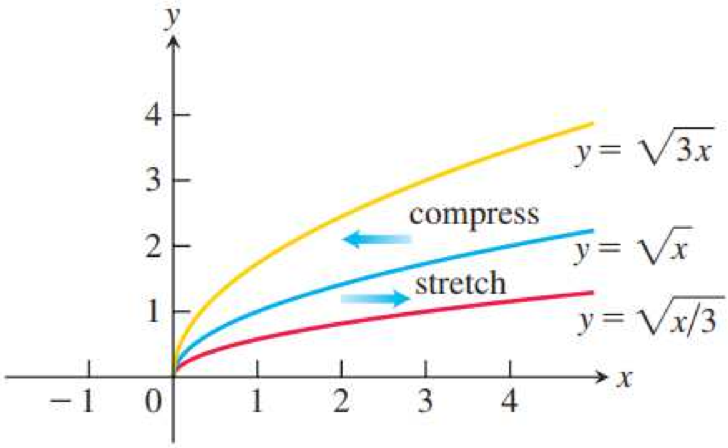

FIGURE 1.33 Horizontally stretching and compressing the graph $y = \sqrt{x}$ by a factor of 3 (Example 4b).

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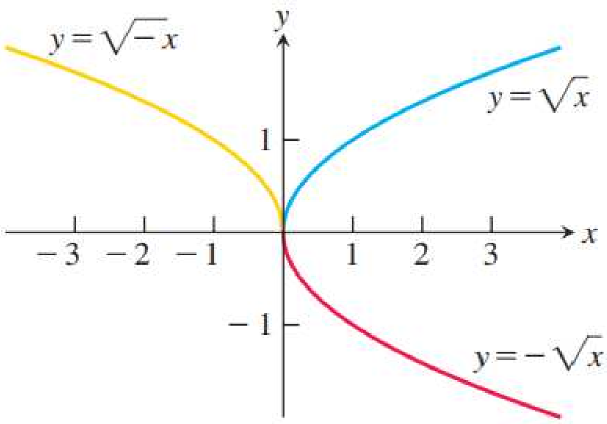

FIGURE 1.34 Reflections of the graph $y = \sqrt{x}$ across the coordinate axes (Example 4c).

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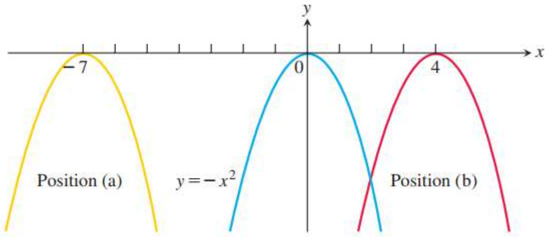
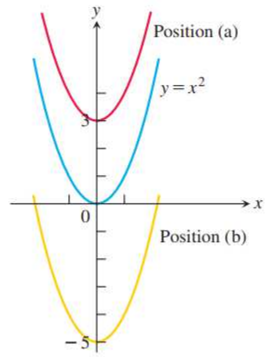
Shifting Graphs

23. The accompanying figure shows the graph of $y = -x^2$ shifted to two new positions. Write equations for the new graphs.



24. The accompanying figure shows the graph of $y = x^2$ shifted to two new positions. Write equations for the new graphs.



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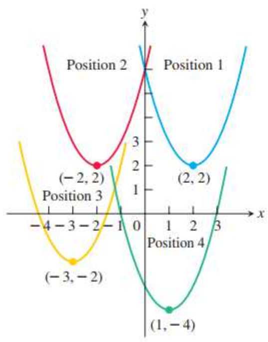
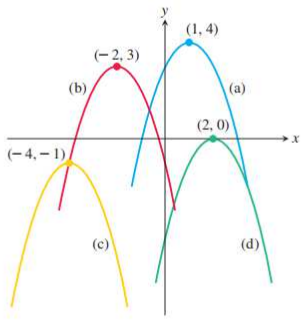
25. Match the equations listed in parts (a)–(d) to the graphs in the accompanying figure.

a. $y = (x - 1)^2 - 4$ **b.** $y = (x - 2)^2 + 2$

c. $y = (x + 2)^2 + 2$ **d.** $y = (x + 3)^2 - 2$



26. The accompanying figure shows the graph of $y = -x^2$ shifted to four new positions. Write an equation for each new graph.



Exercises 27–36 tell how many units and in what directions the graphs of the given equations are to be shifted. Give an equation for the shifted graph. Then sketch the original and shifted graphs together, labeling each graph with its equation.

27. $x^2 + y^2 = 49$ Down 3, left 2

28. $x^2 + y^2 = 25$ Up 3, left 4

29. $y = x^3$ Left 1, down 1

30. $y = x^{2/3}$ Right 1, down 1

31. $y = \sqrt{x}$ Left 0.81

32. $y = -\sqrt{x}$ Right 3

33. $y = 2x - 7$ Up 7

34. $y = \frac{1}{2}(x + 1) + 5$ Down 5, right 1

Vertical and Horizontal Scaling

Exercises 59–68 tell by what factor and direction the graphs of the given functions are to be stretched or compressed. Give an equation for the stretched or compressed graph.

59. $y = x^2 - 1$, stretched vertically by a factor of 3

60. $y = x^2 - 1$, compressed horizontally by a factor of 2

61. $y = 1 + \frac{1}{x^2}$, compressed vertically by a factor of 2