Chapter 5
Integration

Finite Sums

Sigma Notation for Finite Sums

The symbol $\sum_{k=1}^{n} a_k$ denotes the sum of the n terms $a_1 + a_2 + \dots + a_{n-1} + a_n$.

Example-1

1. $\sum_{k=1}^{2} a_k = a_1 + a_2$
2. $\sum_{k=1}^{4} a_k = a_1 + a_2 + a_3 + a_4$
3. $\sum_{k=1}^{n} a_k = a_1 + a_2 + \dots + a_{n-1} + a_n$

Example-2

1. $\sum_{k=1}^{5} k = 1 + 2 + 3 + 4 + 5 = 15$

2. $\sum_{k=1}^{3} (-1)^k k = (-1)^1(1) + (-1)^2(2) + (-1)^3(3) = -2$

3. $\sum_{k=1}^{2} \frac{k}{k+1} = \frac{1}{1+1} + \frac{2}{2+1} = \frac{1}{2} + \frac{2}{3} = \frac{7}{6}$

4. $\sum_{k=1}^{3}(k+4) = (1+4) + (2+4) + (3+4) = 5+6+7 = 18$

OR

$= \sum_{k=1}^{3} k + \sum_{k=1}^{3} 4 = (1+2+3) + (4+4+4) = 6+12 = 18$

Example-3

Find the value of $\sum_{k=1}^{3} \sin(\frac{k\pi}{2})$

Solution:

$\sum_{k=1}^{3} \sin\left(\frac{k\pi}{2}\right) = \sin\left(\frac{1.\pi}{2}\right) + \sin\left(\frac{2.\pi}{2}\right) + \sin\left(\frac{3.\pi}{2}\right) = 1 + 0 + (-1) = 0$

H.W

P.293

1,3,5,7,9,11,13,15

Definite Integral

Area under the curve

Let y= f(x) be a continuous function defined on a closed interval [a,b].

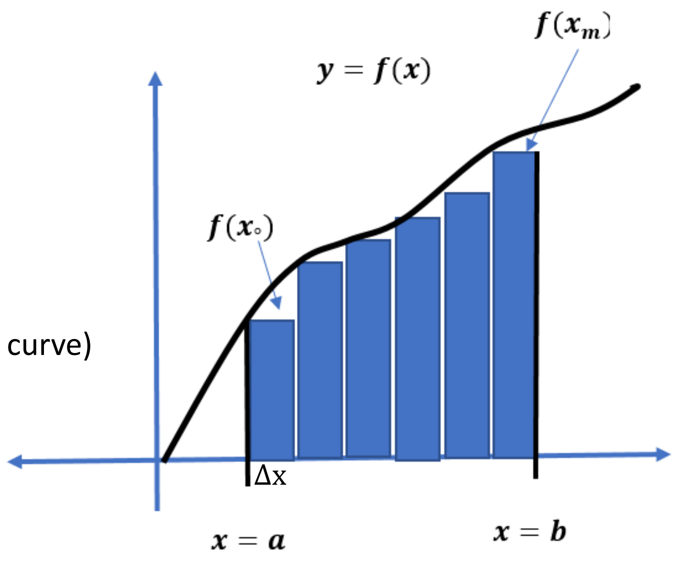
$A1=f(x_0)\Delta x$ (area of first rectangle)
$A2=f(x_1)\Delta x$ (area of second rectangle)

⋮
$An=f(x_{n-1})\Delta x$

$A_a^b \cong \sum_{k=1}^{n} f(x_{k-1})\Delta x$

$A_a^b = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_{k-1})\Delta x$

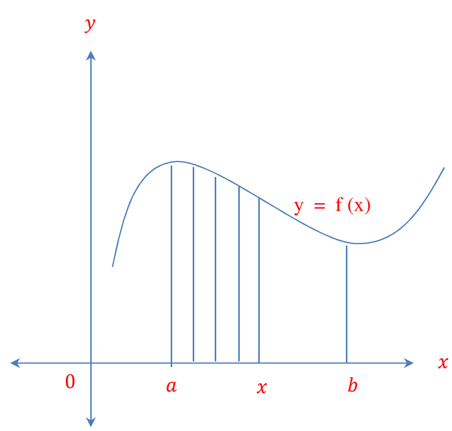
$A_a^b = \int_a^b f(x)dx$ (area under the curve)



* Definite Integral :

$$A_a^x = \int_a^x f(x)\ dx$$

$$A_a^b = \int_a^b f(x)\ dx$$
$$= F(b) - F(a)$$



* Rules of definite integrals:

1) $\int_b^a f(x)\ dx = -\int_a^b f(x)\ dx$

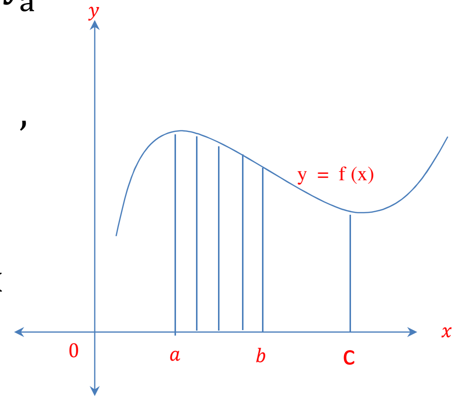
2) $\int_a^a f(x)\ dx = 0$

3) $\int_a^b k\, f(x)\ dx = k\int_a^b f(x)\ dx$, $\int_a^b - f(x)\ dx = -\int_a^b f(x)\ dx$

4) $\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

5) $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$,

$\int_b^c f(x)\,dx = \int_a^c f(x)\,dx - \int_a^b f(x)\,dx$



∗ Example (1) :

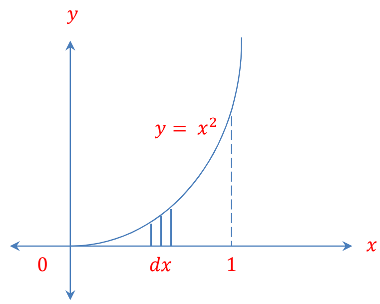
Find the area under the curve $y = x^2$ from $x = 0$ to $x = 1$.

Solution:

$A_0^1 = \int_0^1 y\,dx = \int_0^1 x^2\,dx$

$= \left[\frac{x^3}{3} \right]_0^1 = \frac{1}{3} - 0$

$A_0^1 = \frac{1}{3}\,\text{unit}^2$.



∗ Example (2) :

Find the area under one arch of $y = \cos x$.

Solution :

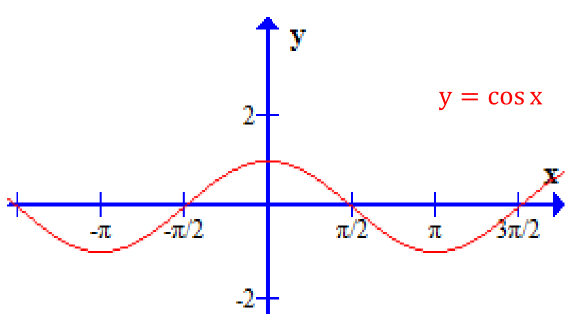
$A = \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \cos x\,dx$

$= [\sin x]_{-\pi/2}^{\pi/2}$

$= \sin\frac{\pi}{2} - \sin\frac{-\pi}{2}$

$= 1 - (-1)$

$= 2\ \text{unit}^2$.



* *H.W*

1 − Find the area under one arch of $y = \cos 3x$.

2 − Find the area under one arch of $y = \sin(x/2)$.

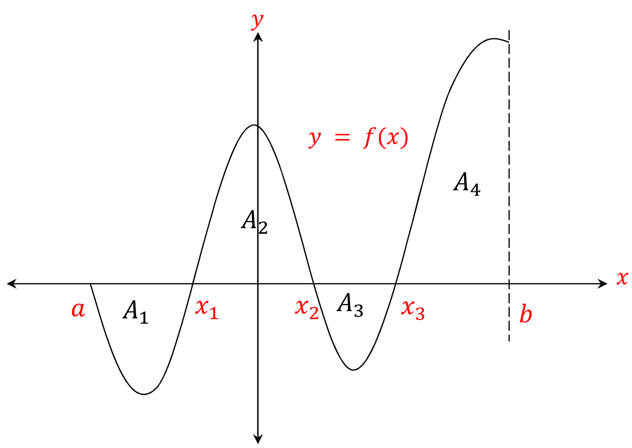
*** If f has both positive and negative value on [a , b] .**

$$A_1 = -\int_{a}^{x_1} f(x)\, dx$$

$$A_2 = \int_{x_1}^{x_2} f(x)\, dx$$

$$A_3 = -\int_{x_2}^{x_3} f(x)\, dx$$

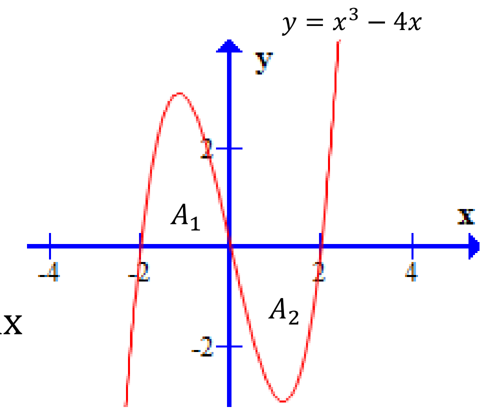
$$A_4 = \int_{x_3}^{b} f(x)\, dx$$



$$\text{Total area} = |A_1| + |A_2| + |A_3| + |A_4| \;.$$

* **Example** : Find the area of the region between the x − axis and the curve $y = x^3 - 4x$, $-2 \le x \le 2$.

Solution :



$y = x^3 - 4x = 0 \;\Rightarrow\; x(x^2 - 4) = 0$

$x = 0$, $x = \mp 2$

$$\int_{-2}^{2} y \; dx = \int_{-2}^{0} (x^3 - 4x) \; dx + \int_{0}^{2} (x^3 - 4x) dx$$

$$= \left[\frac{x^4}{4} - \frac{4x^2}{2}\right]_{-2}^{0} + \left[\frac{x^4}{4} - \frac{4x^2}{2}\right]_{0}^{2}$$

$$\Rightarrow \left[(0-0) - \left(\frac{16}{4} - 2(4)\right)\right] + \left[\left(\frac{16}{4} - 2(4)\right) - (0-0)\right]$$

$$= [-4 + 8] + [4 - 8] = |+4| + |-4| = 8 \text{ units}^2 \; .$$

* **Indefinite Integral** :

$$\int f(x) \; dx = F(x) + C$$

where C is an arbitrary constant (constant of integration).

* **Integration formulas**:

1 - $\int x^n \; dx = \frac{x^{n+1}}{n+1} + C \qquad n \ne 1$.

2 - $\int x^{-1} \; dx = \ln x + C$.

3 - $\int \sin x \; dx = -\cos x + C$.

4 - $\int \cos x \; dx = \sin x + C$.

5 - $\int \sec^2 x \; dx = \tan x + C$.

6 - $\int \csc^2 x \; dx = -\cot x + C$.

7 - $\int \sec x \tan x \; dx = \sec x + C$.

8 - $\int \csc x \; \cot x = -\csc x + C$.

H. W . P. 316 :
1 , 5 , 6 ,7 , 9 , 13 ,16 , 17, 19 , 21, 22

$*$ Example (1): $\int (x^5 - 3x^3 + 2x^2 - x + 1)\, dx$

$$= \frac{x^6}{6} - \frac{3}{4}x^4 + 2\frac{x^3}{3} - \frac{x^2}{2} + x + C$$

$*$ Example (2): $\int \sin 3x \; dx$

$$= \frac{-\cos 3x}{3} + C$$

$*$ Example (3): $\int \cos \frac{x}{2} \, dx$

$$= 2 \sin \frac{x}{2} + C$$

$*$ Example (4): $\int 5 \sec 2x \; \tan 2x \; dx$

$$= \frac{5}{2} \sec 2x + C$$

$*$ Example (5): $\int \sin^2 x \; dx$

Solution :

$$= \int \frac{1}{2} (1 - \cos 2x) \; dx$$

$$= \frac{1}{2} [\, x - \frac{\sin 2x}{2} \,] + C$$

$*$ Example (6): $\int \cos^2 x \; dx$

Solution :

$$= \int \frac{1}{2} (1 + \cos 2x) \; dx$$

$$= \frac{1}{2} [\, x + \frac{\sin 2x}{2}] + C$$

$*$ **Example (7) :** Solve the following initial value problem :

$\frac{d^2y}{dx^2} = 6x - 2$, initial condition $\frac{dy}{dx} = 0$ and $y = 10$ when $x = 1$.

Solution :

$$\frac{dy}{dx} = 3x^2 - 2x + C_1$$

$$0 = 3(1) - 2(1) + C_1 \rightarrow C_1 = -1 .$$

$$\frac{dy}{dx} = 3x^2 - 2x - 1 .$$

$$y = x^3 - x^2 - x + C_2$$

$$10 = 1 - 1 - 1 + C_2$$

$$C_2 = 11$$

$$y = x^3 - x^2 - x + 11 .$$

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- 5
- 10
- 15
- 21
- 26
- 37
- 41
- 44

$*$ Integration by substitution:

$*$ **Example (1) :** $\int (x + 2)^5 \, dx$

Solution:

let $u = x + 2$, $du = dx$.

$$= \int (u)^5 \, du$$

$$= \frac{u^6}{6} + C$$

$$= \frac{1}{6} (x + 2)^6 + C .$$

∗ Example (2): $\int \sqrt{4x - 1}\ dx$

Solution :

let $u = 4x - 1$, $du = 4dx \Rightarrow dx = \frac{du}{4}$

$\therefore \int \sqrt{4x - 1}\ dx$

$= \int \sqrt{u} \cdot \frac{du}{4}$

$= \frac{1}{4} \int u^{1/2}\ du$

$= \frac{1}{4} \cdot \frac{2}{3} u^{3/2} + C$

$= \frac{1}{6} (4x - 1)^{3/2} + C$.

∗ Example (3): $\int \cos(7x + 5)\ dx$

Solution :

let $u = 7x + 5 \Rightarrow du = 7dx \Rightarrow dx = \frac{du}{7}$.

$\therefore \int \cos(7x + 5)\ dx = \int \cos u \cdot \frac{du}{7}$

$= \frac{1}{7} \sin u + C$

$= \frac{1}{7} \sin(7x + 5) + C$.

Example (4) : $\int x^2 \sin x^3 \, dx$

Solution:

let $u = x^3$

$du = 3x^2 dx$

$x^2 dx = \frac{du}{3}$

$$\therefore \int x^2 \sin x^3 \, dx = \int \sin u \cdot \frac{du}{3}$$

$$= -\frac{1}{3} \cos u + C$$

$$= -\frac{1}{3} \cos x^3 + C \ .$$

∗ Example (5) : $\int \frac{dx}{\cos^2 2x}$

Solution:

$$= \int \sec^2 2x \, dx$$

$$= \frac{\tan 2x}{2} + C \ .$$

∗ Example (6) : $\int (x^2 + 2x + 3)(x + 1) \, dx$

Solution :

$$= \frac{1}{2} \cdot \frac{(x^2 + 2x + 3)^2}{2} + C \ .$$

∗ Example (7): $\int (x^2 + 2x + 3)^3 \; (x+1)\, dx$

Solution : $= \frac{1}{2} \cdot \frac{(x^2 + 2x + 3)^4}{4} + C$.

∗ Example (8) : $\int \sin^4 x \; \cos x \; dx$

Solution : $= \frac{\sin^5 x}{5} + C$.

∗ Example (9) : $\int \sin x \; \cos x \; dx$

Solution : $= \frac{\sin^2 x}{2} + C$, or $= \frac{-\cos^2 x}{2} + C$.

∗ Example (10): $\int \frac{2z}{\sqrt[3]{z^2 + 1}} \, dz$

Solution : $= \int 2z \; (z^2 + 1)^{-1/3} \, dz = \frac{3}{2} \; (z^2 + 1)^{2/3} + C$.

∗ Example (11) : $\int_0^{\pi/4} \tan x \; \sec^2 x \; dx$

Solution :

$$= \left[\frac{\tan^2 x}{2} \right]_0^{\pi/4}$$

$$= \frac{1}{2} \left[\left(\tan^2\left(\frac{\pi}{4}\right) - \tan^2 0 \right) \right] = \frac{1}{2} (1 - 0) = \frac{1}{2}$$

or : $\int_0^{\pi/4} \tan x \; \sec x \, . \sec x \; dx$

$$= \left[\frac{\sec^2 x}{2} \right]_0^{\pi/4} = \frac{1}{2} \left[\sec^2 \left(\frac{\pi}{4}\right) - \sec^2 0 \right]$$

$$= \frac{1}{2} \left[\left(\sqrt{2} \right)^2 - 1 \right] = \frac{1}{2} \; (2 - 1) = \frac{1}{2}$$

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- 3
- 6
- 12
- 16
- 17
- 38
- 39
- 40
- 48
- 49