Chapter One
Lecture (10)

1-17: Gradient, Divergence and Curl

Analytical mechanics
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1-17: Gradient, Divergence and Curl

The del operator (∇) in rectangular coordinate is given by:

$$\vec{\nabla} = \hat{\imath}\frac{d}{dx} + \hat{\jmath}\frac{d}{dy} + \hat{k}\frac{d}{dz}$$

The del operator (∇) is a vector that has no physical meaning or vector direction by itself

1-17-1: The gradient:

Let ϕ = (x, y, z) be a differentiable at any point, therefore the gradient of ϕ

$$\vec{\nabla}\phi = \hat{\imath}\frac{d\phi}{dx} + \hat{\jmath}\frac{d\phi}{dy} + \hat{k}\frac{d\phi}{dz}$$

The following computation formulas on gradient should be noted:

1- $\nabla(A+B) = \nabla A + \nabla B$

2- $\nabla(AB) = A\nabla B + B\nabla A$

3- $\nabla\left[\frac{A}{B}\right] = \frac{B\,\nabla A - A\nabla B}{B^2}$

4- $\nabla V^n = n V^{n-1} \nabla V$

1-17-2: The divergence:

Let $\vec{v}(x, y, z) = \hat{\imath}\, v_x + \hat{\jmath}\, v_y + \hat{k}\, v_z$ then the divergence of $\vec{v}$ (written div $\vec{v}$ or $\vec{\nabla} \cdot \vec{v}$) is:

$$\vec{\nabla}.\vec{v} = \frac{dv_x}{dx} + \frac{dv_y}{dy} + \frac{dv_z}{dz}$$

Note the following properties of the divergence of a vector

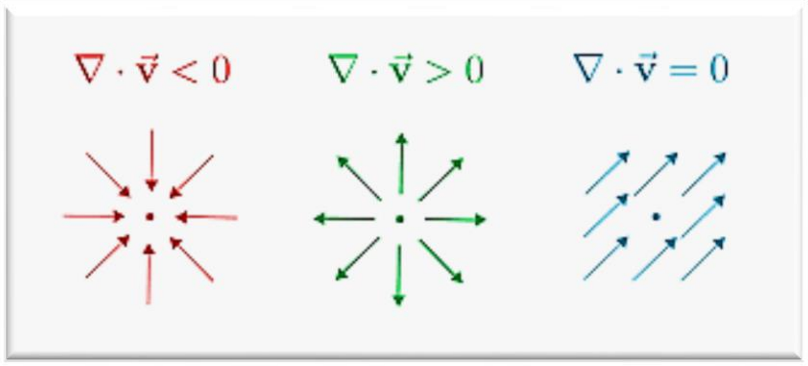
1- Its produces a scalar.

2- The divergence of a scalar V, div V make no sense.

3- $\nabla.(\mathbf{A}+\mathbf{B}) = \nabla.\mathbf{A} + \nabla.\mathbf{B}$

4- $\nabla.(V\mathbf{A}) = V\nabla.\mathbf{A} + \mathbf{A}\,.\,V\nabla$

5- $\vec{\nabla}.\vec{v} \neq \vec{v}.\vec{\nabla}$

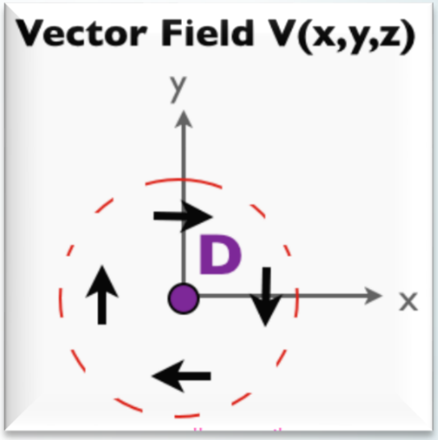


1-17-3: The curl:

The curl of the vector field $\vec{v}\,(x,y,z) = \hat{\imath}\, v_x + \hat{\jmath}\, v_y + \hat{k}\, v_z$

Is denoted by $\overrightarrow{curl v}$ or $\vec{\nabla} \times \vec{v}$ is:

$$\overrightarrow{\text{curl v}} = \vec{\nabla} \times \vec{v} = \begin{vmatrix} i & j & k \\ \dfrac{d}{d_x} & \dfrac{d}{d_y} & \dfrac{d}{d_z} \\ v_x & v_y & v_z \end{vmatrix}$$



Note the following properties of the curl of a vector

- The curl of a vector is another vector.
- The curl of a scalar V, $\nabla\times V$, makes no sense.
- $\nabla \times (\mathbf{A}+\mathbf{B}) = \nabla \times \mathbf{A} + \nabla \times \mathbf{B}$
- $\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}\,(\nabla \cdot \mathbf{B}) - \mathbf{B}\,(\nabla \cdot \mathbf{A}) + (\mathbf{B} \cdot \nabla)\,\mathbf{A} - (\mathbf{A} \cdot \nabla)\,\mathbf{B}$
- The divergence of the curl of a vector vanishes, that is, $\nabla \cdot (\nabla \times \mathbf{A}) = 0$
- The curl of the gradient of a scalar is vanishes that is, $\nabla \times \nabla V = 0$
- $\nabla \cdot (\nabla \mathrm{u}) = \nabla^2 u$ **Laplacian of u**

Note:

We can defined $d\vec{r}$ and $\vec{\nabla}$ in the

1- Cartesian Coordinate

$$d\vec{r} = \hat{\imath}dx + \hat{\jmath}dy + \hat{k}dz$$

$$\vec{\nabla} = \hat{\imath}\frac{d}{dx} + \hat{\jmath}\frac{d}{dy} + \hat{k}\frac{d}{dz}$$

2- Cylindrical Coordinate

$$d\vec{r} = \hat{e}_R dR + \hat{e}_\phi R d\phi + \hat{e}_z dz$$

$$\vec{\nabla} = \hat{e}_R\frac{d}{dR} + \hat{e}_\phi\frac{1}{R}\frac{d}{d\phi} + \hat{e}_z\frac{d}{dz}$$

3- Spherical Coordinate

$$d\vec{r} = \hat{e}_r dr + \hat{e}_\theta r d\theta + \hat{e}_\phi r sin\theta\, d\phi$$

$$\vec{\nabla} = \hat{e}_r\frac{d}{dr} + \hat{e}_\theta\frac{1}{r}\frac{d}{d\theta} + \hat{e}_\phi\frac{1}{r sin\theta}\frac{d}{d\phi}$$

Example 14:

The potential that represents an inverse square force is $V(r) = \frac{k}{r}$, where $r = (x^2 + y^2 + z^2)^{1/2}$. Using the definition $\vec{F} = -\vec{\nabla} V$, calculate the component of this force.

Solution:

$$\vec{F} = -\vec{\nabla} V = -\left(\hat{\imath}\frac{dV}{dx} + \hat{\jmath}\frac{dV}{dy} + \hat{k}\frac{dV}{dz}\right)$$

Since $V(r) = \frac{k}{r}$

$$\frac{dV}{dx} = -\frac{kx}{r^3}, \frac{dV}{dy} - \frac{ky}{r^3}, \text{ and } \frac{dV}{dz} = -\frac{kz}{r^3}$$

$$\vec{F} = \frac{k}{r^3}\left(\hat{\imath}x + \hat{\jmath}y + \hat{k}z\right) = \frac{k}{r^3}\vec{r}$$

Another solution by using spherical coordinate :

$$\vec{\nabla} = \hat{e}_r \frac{d}{dr} + \hat{e}_\theta \frac{1}{r}\frac{d}{d\theta} + \hat{e}_\phi \frac{1}{r\sin\theta}\frac{d}{d\phi}$$

$$\vec{F} = -\vec{\nabla} V = -k\hat{e}_r \frac{d}{dr}\left(\frac{1}{r}\right) + 0 + 0$$

$$\vec{F} = -k\hat{e}_r\left(-\frac{1}{r^2}\right)$$

$$\vec{F} = \frac{k}{r^2}\hat{e}_r$$

Example 15:

If $\phi = \frac{1}{r}$ where $r = (x^2 + y^2 + z^2)^{1/2}$. Show that $\vec{\nabla}\phi = \frac{-r}{r^3}$

Solution:

$$\nabla\Phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2 + y^2 + z^2)^{-1/2}$$

$$= \left(-\frac{1}{2}.2x\hat{i} - \frac{1}{2}.2y\hat{j} - \frac{1}{2}.2z\hat{k}\right)(x^2 + y^2 + z^2)^{-3/2}$$

$$= -(x\hat{i} + y\hat{j} + z\hat{k})(x^2 + y^2 + z^2)^{-\frac{3}{2}} = -\frac{r}{r^3}$$

Example 16:

Find a unit vector normal to the surface $xy^2 + xz = 1$ at point (-1,1,1).

Solution:

$$\nabla(xy^2 + xz) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(xy^2 + xz)$$

$$= (y^2 + z)\hat{i} + (2xy)\hat{j} + x\hat{k}$$

$$= 2\hat{i} - 2\hat{j} - \hat{k},\ at(-1, 1, 1)$$

a unit vector $= \frac{vector}{its\ magnitude}$

$$(2\hat{i} - 2\hat{j} - \hat{k})[(2)^2 + (-2)^2 + (-1)^2]^{-1/2}$$

$$= \frac{2}{3}\hat{i} - \frac{2}{3}\hat{j} - \frac{1}{3}\hat{k}$$

Example 17:

a: Show that $F = (2xy + z^2)\hat{\imath} + x^2\hat{\jmath} + 2xz\hat{k}$ is conservative force field

b: Find the scalar field

c: Find the work done in moving a unit mass in the field from point (1,0,1) to (2,1,-1)scalar field

Solution

(a) It is sufficient to show that Curl $\boldsymbol{F} = 0$

$$\nabla \times \boldsymbol{F} = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2xy + z^2 & x^2 & 2xz \end{vmatrix}$$

$$= \hat{\imath}.0 - \hat{\jmath}(2z - 2z) + \hat{k}(2x - 2x) = 0$$

(b) $d\Phi = F.\,d\boldsymbol{r} = ((2xy + z^2)\hat{\imath} + x^2\hat{\jmath} + 2xz\hat{k}).(\hat{\imath}dx + \hat{\jmath}dy + \hat{k}dz)$

$$= (2xy + z^2)\,dx + x^2\,dy + 2xzdz$$

$$= (2xydx + x^2\,dy) + (z^2\,dx + 2xzdz)$$

$$= d(x^2y) + d(z^2x) = d(x^2y + xz^2)$$

Therefore $\Phi = x^2y + xz^2 + \text{constant}$

(c) Work done $= \Phi_2 - \Phi_1 = 5.0$

Example 18:

If $\phi = x^2yz^3$ and $\mathbf{A} = xz\hat{\imath} - y^2\hat{\jmath} + 2x^2y\hat{k}$, find:

a- $\nabla\phi$ **b-** $\nabla.\mathbf{A}$ **c-** $\nabla\times\mathbf{A}$ **d-** $\text{div}(\phi\mathbf{A})$ **e-** $\text{curl}\,(\phi\mathbf{A})$

Solution

$$(a)\quad \nabla\phi = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k} = \frac{\partial}{\partial x}(x^2yz^3)\mathbf{i} + \frac{\partial}{\partial y}(x^2yz^3)\mathbf{j} + \frac{\partial}{\partial z}(x^2yz^3)\mathbf{k}$$

$$= 2xyz^3\mathbf{i} + x^2z^3\mathbf{j} + 3x^2yz^2\mathbf{k}$$

$$(b)\quad \nabla\cdot\mathbf{A} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\cdot(xz\mathbf{i} - y^2\mathbf{j} + 2x^2y\mathbf{k})$$

$$= \frac{\partial}{\partial x}(xz) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(2x^2y) = z - 2y$$

$$(c)\quad \nabla\times\mathbf{A} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\times(xz\mathbf{i} - y^2\mathbf{j} + 2x^2y\mathbf{k})$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xz & -y^2 & 2x^2y \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial y}(2x^2y) - \frac{\partial}{\partial z}(-y^2)\right)\mathbf{i} + \left(\frac{\partial}{\partial z}(xz) - \frac{\partial}{\partial x}(2x^2y)\right)\mathbf{j} + \left(\frac{\partial}{\partial x}(-y^2) - \frac{\partial}{\partial y}(xz)\right)\mathbf{k}$$

$$= 2x^2\mathbf{i} + (x - 4xy)\mathbf{j}$$

$$\phi\mathbf{A} = x^3yz^4\hat{\imath} - x^2y^3z^3\hat{\jmath} + 2x^4y^2z^3\hat{k}$$

$$(d)\quad \text{div}\,(\phi\mathbf{A}) = \nabla\cdot(\phi\mathbf{A}) = \nabla\cdot(x^3yz^4\mathbf{i} - x^2y^3z^3\mathbf{j} + 2x^4y^2z^3\mathbf{k})$$

$$= \frac{\partial}{\partial x}(x^3yz^4) + \frac{\partial}{\partial y}(-x^2y^3z^3) + \frac{\partial}{\partial z}(2x^4y^2z^3)$$

$$= 3x^2yz^4 - 3x^2y^2z^3 + 6x^4y^2z^2$$

$$(e)\quad \text{curl}\,(\phi\mathbf{A}) = \nabla\times(\phi\mathbf{A})$$

$$= \nabla\times(x^3yz^4\mathbf{i} - x^2y^3z^3\mathbf{j} + 2x^4y^2z^3\mathbf{k})$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^3yz^4 & -x^2y^3z^3 & 2x^4y^2z^3 \end{vmatrix}$$

$$= (4x^4yz^3 + 3x^2y^3z^2)\mathbf{i} + (4x^3yz^3 - 8x^3y^2z^3)\mathbf{j} - (2xy^3z^3 + x^3z^4)\mathbf{k}$$

Example 19:

Prove $\nabla.(\phi \mathbf{A}) = (\nabla\phi) \,.\, \mathbf{A} + \phi\,(\nabla.\, \mathbf{A})$, where $\mathbf{A} = \mathbf{i}A_1 + \mathbf{j}\,A_2 + \mathbf{k}\,A_3$

Solution

$$\nabla \cdot (\phi \mathbf{A}) = \nabla \cdot (\phi A_1 \mathbf{i} + \phi A_2 \mathbf{j} + \phi A_3 \mathbf{k})$$

$$= \frac{\partial}{\partial x}(\phi A_1) + \frac{\partial}{\partial y}(\phi A_2) + \frac{\partial}{\partial z}(\phi A_3)$$

$$= \frac{\partial \phi}{\partial x} A_1 + \frac{\partial \phi}{\partial y} A_2 + \frac{\partial \phi}{\partial z} A_3 + \phi\left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right)$$

$$= \left(\frac{\partial \phi}{\partial x}\mathbf{i} + \frac{\partial \phi}{\partial y}\mathbf{j} + \frac{\partial \phi}{\partial z}\mathbf{k}\right) \cdot (A_1 \mathbf{i} + A_2 \mathbf{j} + A_3 \mathbf{k})$$

$$+ \phi\left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right) \cdot (A_1 \mathbf{i} + A_2 \mathbf{j} + A_3 \mathbf{k})$$

$$= (\nabla \phi) \cdot \mathbf{A} + \phi(\nabla \cdot \mathbf{A})$$

Example 20:

Prove that div curl **A** = 0 , where **A** = **i**A_1 + **j** A_2 + **k** A_3

Solution

$$\text{div curl } \mathbf{A} = \nabla \cdot (\nabla \times \mathbf{A}) = \nabla \cdot \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_1 & A_2 & A_3 \end{vmatrix}$$

$$= \nabla \cdot \left[\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} \right) \mathbf{i} + \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x} \right) \mathbf{j} + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \right) \mathbf{k} \right]$$

$$= \frac{\partial}{\partial x} \left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} \right) + \frac{\partial}{\partial y} \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x} \right) + \frac{\partial}{\partial z} \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \right)$$

$$= \frac{\partial^2 A_3}{\partial x \partial y} - \frac{\partial^2 A_2}{\partial x \partial z} + \frac{\partial^2 A_1}{\partial y \partial z} - \frac{\partial^2 A_3}{\partial y \partial x} + \frac{\partial^2 A_2}{\partial z \partial x} - \frac{\partial^2 A_1}{\partial z \partial y}$$

$$= 0$$

Chapter One
Lecture (9)

1-15: Velocity and Acceleration in Cylindrical Coordinates

1-16: Velocity and Acceleration in Spherical Coordinates

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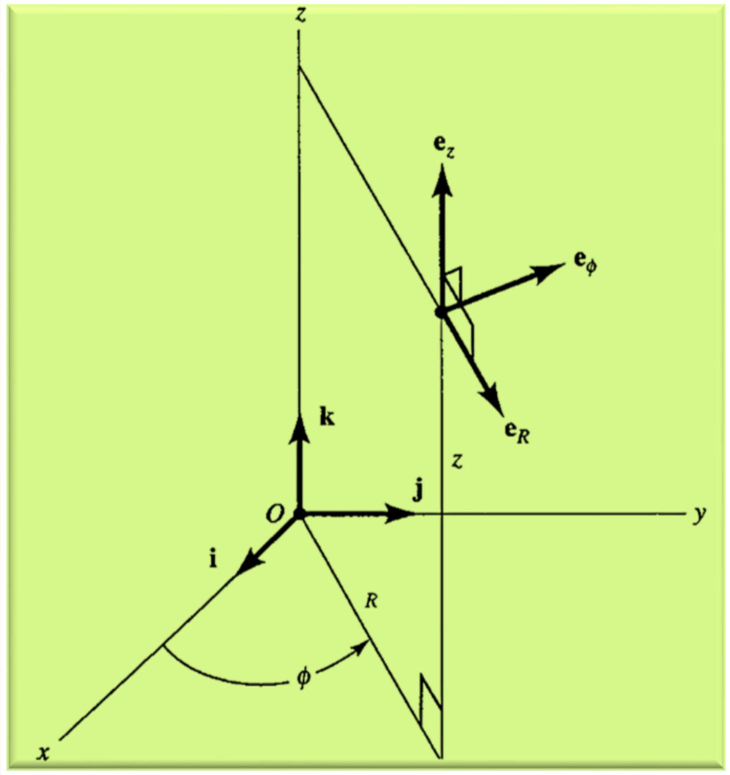
1-15: Velocity and Acceleration in Cylindrical Coordinates

The position of a particle in the case of three-dimensional motion can be described in cylindrical coordinates R, ϕ, z. The position vector is then written as:

$$\mathbf{r} = R\mathbf{e}_R + Z\mathbf{e}_z \quad \text{----------------} \quad 1$$

where e_R is a unit radial vector in the xy plane and e_z is the unit vector in the z direction. A third unit vector e_ϕ is needed so that the three e_R e_ϕ e_z vectors constitute a right-handed triad, as illustrated in Figure. We note that **k**= $\boldsymbol{e}_z$.



As before the velocity and acceleration vectors can be found by differentiating equation (1). In the same way as we have done before for the plane polar coordinate shows that:

$$\frac{d\hat{e}_R}{dt} = \dot{\phi}\,\hat{e}_\phi \quad and \quad \frac{d\hat{e}_\phi}{dt} = -\dot{\phi}\,\hat{e}_R$$

The unit vector $\boldsymbol{e}_z$ does not change in direction, so its time derivative is zero.

$$so \quad \frac{d\hat{e}_z}{dt} = 0$$

$$\vec{v} = \frac{d\vec{r}}{dt} = \dot{R}\hat{e}_R + R\frac{d\hat{e}_R}{dt} + \dot{z}\hat{e}_z + z\frac{d\hat{e}_z}{dt}$$

Substituting for the time derivative, we get:

$$\vec{v} = \dot{R}\hat{e}_R + R\dot{\phi}\,\hat{e}_\phi + \dot{z}\hat{e}_z \text{ ------------- } 2$$

$$\vec{a} = \ddot{R}\hat{e}_R + \dot{R}\frac{d\hat{e}_R}{dt} + (\dot{R}\dot{\phi} + R\ddot{\phi})\hat{e}_\phi + R\dot{\phi}\frac{d\hat{e}_\phi}{dt} + \ddot{z}\,\hat{e}_z$$

Substituting for the time derivative, we get:

$$\vec{a} = \ddot{R}\hat{e}_R + \dot{R}\,\dot{\phi}\hat{e}_\phi + \dot{R}\dot{\phi}\hat{e}_\phi + R\ddot{\phi}\hat{e}_\phi - R\dot{\phi}^2\hat{e}_R + \ddot{z}\,\hat{e}_z$$

$$\vec{a} = (\ddot{R} - R\dot{\phi}^2)\hat{e}_R + (2\dot{R}\dot{\phi} + R\ddot{\phi})\hat{e}_\phi + \ddot{z}\hat{e}_z \text{ ------------- } 3$$

An alternative way of obtaining the derivative of the unit vectors is to differentiating the relationships between the fixed unit triad **i j k** and the rotated $\boldsymbol{e}_R\ \boldsymbol{e}_\phi\ \boldsymbol{e}_z$.

The relationship between the unit vectors of Cartesian and cylindrical coordinate

$$\begin{pmatrix} \hat{e}_R \\ \hat{e}_\phi \\ \hat{e}_z \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{pmatrix}$$

The relationships between the fixed unit triad **i j k** and the rotated triad $\boldsymbol{e}_R\ \boldsymbol{e}_\phi\ \boldsymbol{e}_z$ is:

$$\hat{e}_R = \hat{i}\cos\phi + \hat{j}\sin\phi$$

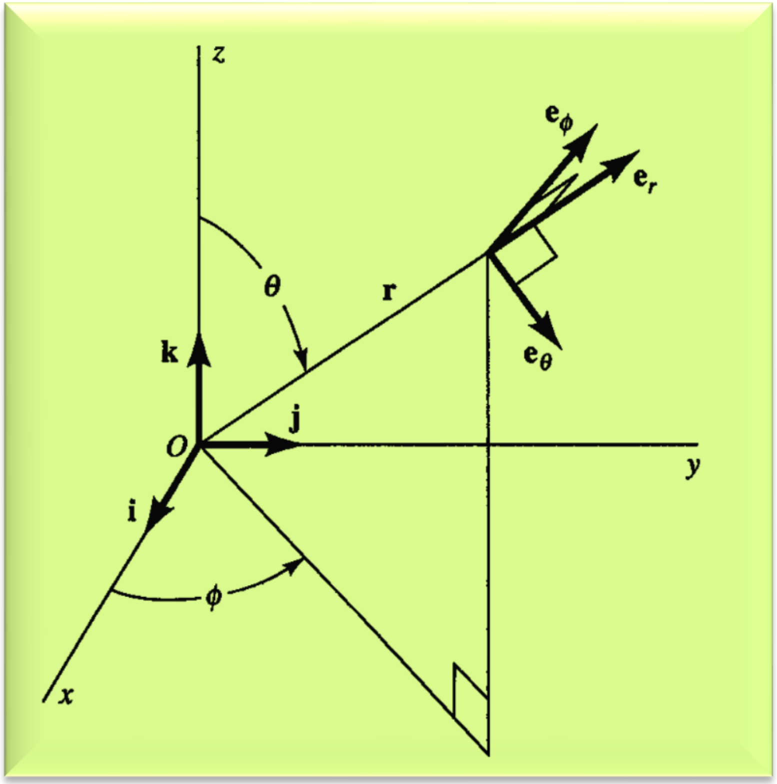
$$\hat{e}_\phi = -\hat{i}\sin\phi + \hat{j}\cos\phi$$

$$\hat{e}_z = \hat{k}$$

1-16: Velocity and Acceleration in Spherical Coordinates

When spherical coordinates r, θ, ϕ are employed to describe the position of a particle, the position vector is written as the product of the radial distance r and the unit radial vector $\boldsymbol{e}_r$, as with plane polar coordinates. Thus,

$$\mathbf{r} = r\mathbf{e}_r$$



The direction of $\boldsymbol{e}_r$ is now specified by the two angles θ and ϕ. We introduce two more unit vectors, $\boldsymbol{e}_\theta$ and $\boldsymbol{e}_\phi$, as shown in Figure

The velocity vector

$$\vec{\mathrm{v}} = \frac{\mathrm{d}}{\mathrm{dt}}\vec{\mathrm{r}} = \dot{\mathrm{r}}\hat{\mathrm{e}}_{\mathrm{r}} + r\frac{d\hat{e}_r}{dt} \ldots\ldots. (2)$$

To express the derivative d$\boldsymbol{e}_r$/dt in terms of the unit vectors in the rotated triad. we can derive relationships between the **i j k** and $\boldsymbol{e}_r$ $\boldsymbol{e}_\theta$ $\boldsymbol{e}_\phi$ triads. For example, because any vector can be expressed in terms of its projections on to the x, y, z, coordinate axes

$$\mathbf{e}_r = \mathbf{i}(\mathbf{e}_r \cdot \mathbf{i}) + \mathbf{j}(\mathbf{e}_r \cdot \mathbf{j}) + \mathbf{k}(\mathbf{e}_r \cdot \mathbf{k})$$

$\boldsymbol{e}_r.\boldsymbol{i}$ is the projection of the unit vector $\boldsymbol{e}_r$ directly onto the unit vector **i**. and it is equal to cos α, (as shown in equation)

$$B\cos\theta = \frac{\mathbf{B}\cdot\mathbf{A}}{A} = \mathbf{B}\cdot\mathbf{n}$$

the cosine of the angle between those two unit vectors.

We need to express this dot product in terms of θ and ϕ, not a. We can obtain the desired relation by making two successive projections to get to the x-axis. First project $\boldsymbol{e}_r$ onto the xy plane, and then project from there onto the x-axis. The first projection gives us a factor of sin θ, while the second yields a factor of cos ϕ. The magnitude of the projection obtained in this way is the desired dot product:

$$\hat{e}_r \cdot \hat{i} = \sin\theta\cos\phi, \quad \hat{e}_r \cdot \hat{j} = \sin\theta\sin\phi \quad and \quad \hat{e}_r \cdot \hat{k} = \cos\theta$$

The relationships for $\boldsymbol{e}_\theta$ and $\boldsymbol{e}_\phi$ can be obtained as above, yielding the desired relations

$$\mathbf{e}_r = \mathbf{i}\sin\theta\cos\phi + \mathbf{j}\sin\theta\sin\phi + \mathbf{k}\cos\theta$$
$$\mathbf{e}_\theta = \mathbf{i}\cos\theta\cos\phi + \mathbf{j}\cos\theta\sin\phi - \mathbf{k}\sin\theta \qquad \text{---------} \; 3$$
$$\mathbf{e}_\phi = -\mathbf{i}\sin\phi + \mathbf{j}\cos\phi$$

which express the unit vectors of the rotated triad in terms of the fixed triad **i j k**.

Differentiate the first equation with respect to time. The result is

$$\frac{d\hat{e}_r}{dt} = \hat{\imath}(\dot{\theta}\cos\theta\cos\phi - \dot{\phi}\sin\theta\sin\phi) + \hat{\jmath}(\dot{\theta}\cos\theta\sin\phi + \dot{\phi}\sin\theta\cos\phi) - \hat{k}\dot{\theta}\sin\theta$$

By using the expressions for $\boldsymbol{e}_\theta$ and $\boldsymbol{e}_\phi$ in Equation 3 we find that the above equation reduces to

$$\frac{d\hat{e}_r}{dt} = \hat{e}_\phi\dot{\phi}\sin\theta + \hat{e}_\theta\dot{\theta}$$

The other two derivatives are found through a similar procedure. The results are:

$$\frac{d\mathbf{e}_\theta}{dt} = -\mathbf{e}_r\dot{\theta} + \mathbf{e}_\phi\dot{\phi}\cos\theta$$

$$\frac{d\mathbf{e}_\phi}{dt} = -\mathbf{e}_r\dot{\phi}\sin\theta - \mathbf{e}_\theta\dot{\phi}\cos\theta$$

To find **v**, we insert the expression for d$\boldsymbol{e}_r$/dt into Equation 2 The final result is:

$$\mathbf{v} = \mathbf{e}_r \dot{r} + \mathbf{e}_\phi r\dot{\phi}\sin\theta + \mathbf{e}_\theta r\dot{\theta}$$

giving the velocity vector in terms of its components in the rotated triad $\boldsymbol{e}_R\ \boldsymbol{e}_\phi\ \boldsymbol{e}_z$.

To find the acceleration, we differentiate the above expression with respect to time. This gives

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \mathbf{e}_r\ddot{r} + \dot{r}\frac{d\mathbf{e}_r}{dt} + \mathbf{e}_\phi\frac{d(r\dot{\phi}\sin\theta)}{dt} + r\dot{\phi}\sin\theta\frac{d\mathbf{e}_\phi}{dt} + \mathbf{e}_\theta\frac{d(r\dot{\theta})}{dt} + r\dot{\theta}\frac{d\mathbf{e}_\theta}{dt}$$

Upon using the derivatives of the unit vectors, the above expression for the acceleration reduces to:

$$\mathbf{a} = (\ddot{r} - r\dot{\phi}^2\sin^2\theta - r\dot{\theta}^2)\mathbf{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2\sin\theta\cos\theta)\mathbf{e}_\theta$$
$$+ (r\ddot{\phi}\sin\theta + 2\dot{r}\dot{\phi}\sin\theta + 2r\dot{\theta}\dot{\phi}\cos\theta)\mathbf{e}_\phi$$

giving the acceleration vector in terms of its components in the rotated triad $\boldsymbol{e}_R\ \boldsymbol{e}_\phi\ \boldsymbol{e}_z$.

Example 13:

A bead slides on a wire bent into the form of a helix, the motion of the bead being given in cylindrical coordinates by R = b, $\phi = \omega t$, z = ct. Find the velocity and acceleration vectors as functions of time.

Solution:

Differentiating, we find $\dot{R} = \ddot{R} = 0$, $\dot{\phi} = \omega, \ddot{\phi} = 0, \dot{z} = c$, $\ddot{z} = 0$,So, from Equations of velocity and acceleration we have:

$$\vec{v} = \dot{R}\hat{e}_R + R\dot{\phi}\,\hat{e}_\phi + \dot{z}\hat{e}_z$$

$$\mathbf{v} = b\omega\mathbf{e}_\phi + c\mathbf{e}_z$$

$$\vec{a} = (\ddot{R} - R\dot{\phi}^2)\hat{e}_R + (2\dot{R}\dot{\phi} + R\ddot{\phi})\hat{e}_\phi + \ddot{z}\hat{e}_z$$

$$\mathbf{a} = -b\omega^2\mathbf{e}_R$$

Chapter One
Lecture (11)

Solved Problems (Chapter One)

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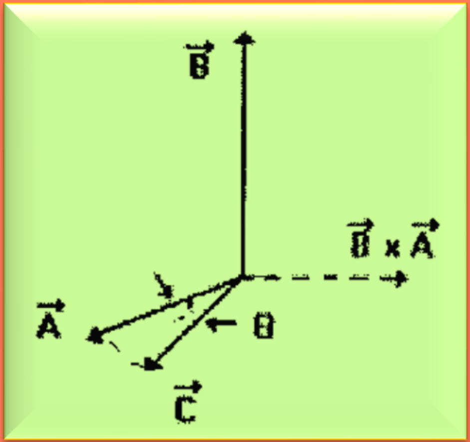
Problems of The Chapter One

1: Assume that two vectors **A** and **B** are known. Let **C** be an unknown vector such that **A. C** = **u** is a known quantity and **A** × **C** = **B**. Find **C** in terms of **A, B, u**, and the magnitude of **A**.

Solution:

$$B = |\vec{B}| = |\vec{A}\times\vec{C}| = AC\sin\theta \quad \Rightarrow \quad C_y = C\sin\theta = \frac{B}{A}$$

$$\vec{A}\cdot\vec{C} = AC\cos\theta = u \quad \Rightarrow \quad C_x = C\cos\theta = \frac{u}{A}$$



$$\vec{C} = \frac{\vec{A}}{A}C_x + \frac{\vec{B}\times\vec{A}}{|\vec{B}\times\vec{A}|}C_y = \frac{u}{A^2}\vec{A} + \frac{\vec{B}\times\vec{A}}{AB}\left(\frac{B}{A}\right)$$

$$= \frac{u}{A^2}\vec{A} + \frac{1}{A^2}\vec{B}\times\vec{A}$$

2: Find a unit vector normal to the plane containing the two vectors:

$$\vec{A} = 2\hat{i} + \hat{j} - \hat{k} \quad \text{and} \quad \vec{B} = \hat{i} - \hat{j} + 2\hat{k}$$

Solution:

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & -\hat{j} & \hat{k} \\ 2 & 1 & -1 \\ 1 & -1 & 2 \end{vmatrix} = \hat{i} - 5\hat{j} - 3\hat{k}$$

$$\left|\vec{A} \times \vec{B}\right| = \sqrt{1^2 + (-5)^2 + (-3)^2} = \sqrt{35}$$

$$\text{unit vector} = \frac{\text{the vector}}{\text{its magnitude}} \Rightarrow \hat{n} = \frac{\vec{A} \times \vec{B}}{\left|\vec{A} \times \vec{B}\right|}$$

$$\hat{n} = \frac{1}{\sqrt{35}}(\hat{i} - 5\hat{j} - 3\hat{k})$$

3: A racing car moves on a circle of constant radius b. If the speed of the car varies with time t according to the equation $v = ct$ where c is a positive constant, **show that** the angle between the velocity vector and the acceleration vector is 45° at time $t = \sqrt{b/c}$

Solution:

(**Hint:** At this time the tangential and normal components of the acceleration are equal in magnitude.)

$$\mathbf{v} = v\boldsymbol{\tau} = ct\boldsymbol{\tau}$$

$$\mathbf{a} = \dot{v}\boldsymbol{\tau} + \frac{v^2}{\rho}\mathbf{n} = c\,\boldsymbol{\tau} + \frac{c^2t^2}{b}\mathbf{n}$$

$$\text{at } t = \sqrt{\frac{b}{c}}, \quad \vec{v} = \hat{\tau}\sqrt{bc} \quad \text{and} \quad \vec{a} = c\hat{\tau} + c\hat{n}$$

$$\cos\theta = \frac{\vec{v}\cdot\vec{a}}{va} = \frac{c\sqrt{bc}}{\sqrt{bc}\sqrt{2c^2}} = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad \theta = 45^\circ$$

4: A small ball is fastened to a long rubber band and twirled around in such away that the ball moves in an elliptical path given by the equation **r**(t) = **i** b cos ωt +**j** 2b sin ωt , where b and ω are constants. **Find** the speed of the ball as a function of t. In particular, find *v* at t =0 and at t = π/2 ω, at which times the ball is, respectively, at its minimum and maximum distances from the origin.

Solution:

$$\vec{v}(t) = -\hat{i}b\omega\sin(\omega t) + \hat{j}2b\omega\cos(\omega t)$$

$$|\vec{v}| = \left(b^2\omega^2\sin^2\omega t + 4b^2\omega^2\cos^2\omega t\right)^{\frac{1}{2}} = b\omega\left(1 + 3\cos^2\omega t\right)^{\frac{1}{2}}$$

$$\vec{a}(t) = -\hat{i}b\omega^2\cos\omega t - \hat{j}2b\omega^2\sin\omega t$$

$$|\vec{a}| = b\omega^2\left(1 + 3\sin^2\omega t\right)^{\frac{1}{2}}$$

$$\text{at} \quad t = 0, \quad |\vec{v}| = 2b\omega; \quad \text{at} \quad t = \frac{\pi}{2\omega}, \quad |\vec{v}| = b\omega$$

5: A bee goes out from its hive in a spiral path given in plane polar coordinates by $r = be^{kt}$ $\theta = ct$ where b, k, and c are positive constants. **Show** that the angle between the velocity vector and the acceleration vector remains constant as the bee moves outward.

Solution:

(Hint: Find **v . a** / *v* a)

$$\vec{v} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta = bke^{kt}\hat{e}_r + bce^{kt}\hat{e}_\theta$$

$$\vec{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\hat{e}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{e}_\theta = b\left(k^2 - c^2\right)e^{kt}\hat{e}_r + 2bcke^{kt}\hat{e}_\theta$$

$$\cos\phi = \frac{\vec{v}\cdot\vec{a}}{va} = \frac{b^2k\left(k^2 - c^2\right)e^{2kt} + 2b^2c^2ke^{2kt}}{be^{kt}\left(k^2 + c^2\right)^{\frac{1}{2}} be^{kt}\left[\left(k^2 - c^2\right)^2 + 4c^2k^2\right]^{\frac{1}{2}}}$$

$$\cos\phi = \frac{k\left(k^2 + c^2\right)}{\left(k^2 + c^2\right)^{\frac{1}{2}}\left(k^2 + c^2\right)} = \frac{k}{\left(k^2 + c^2\right)^{\frac{1}{2}}}, \quad \text{a constant}$$

6:Prove that $\vec{v}\,.\,\vec{a} = v\dot{v}$ and, hence, that for a moving particle **v** and **a** are perpendicular to each other if the speed *v* is constant.

Solution:

(***Hint***: Differentiate both sides of the equation $v \cdot v = v^2$ with respect to t. Note, $\dot{v}$ is not the same as $|a|$. It is the magnitude of the acceleration of the particle along its instantaneous direction of motion.)

$$\vec{v} \cdot \vec{v} = v^2$$

$$\frac{d\vec{v}}{dt} \cdot \vec{v} + \vec{v} \cdot \frac{d\vec{v}}{dt} = 2v\dot{v}$$

$$2\vec{v} \cdot \vec{a} = 2v\dot{v}$$

$$\vec{v} \cdot \vec{a} = v\dot{v}$$

7: Show that the tangential component of the acceleration of a moving particle is given by the expression $a_\tau = \frac{\mathbf{v} \cdot \mathbf{a}}{v}$ and the normal component is:

$$a_n = \left(a^2 - a_\tau^2\right)^{1/2} = \left[a^2 - \frac{(\mathbf{v} \cdot \mathbf{a})^2}{v^2}\right]^{1/2}$$

Solution:

$$\vec{v} = v\hat{\tau} \text{ and } \vec{a} = a_\tau \hat{\tau} + a_n \hat{n}$$

$$\vec{v} \cdot \vec{a} = v a_\tau, \text{ so } a_\tau = \frac{\vec{v} \cdot \vec{a}}{v}$$

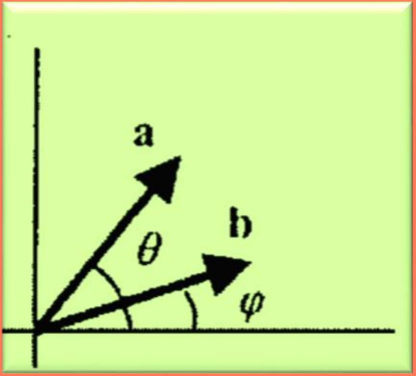
$$a^2 = a_\tau^2 + a_n^2, \text{ so } a_n = \left(a^2 - a_\tau^2\right)^{\frac{1}{2}}$$

8: Use vector algebra to derive the following trigonometric identities

$$\textbf{(a)}\ \cos(\theta-\phi)=\cos\theta\cos\phi+\sin\theta\sin\phi$$

$$\textbf{(b)}\ \sin(\theta-\phi)=\sin\theta\cos\phi-\cos\theta\sin\phi$$

Solution:



(a) $$a=\hat{i}\cos\theta+\hat{j}\sin\theta$$

$$b=\hat{i}\cos\varphi+\hat{j}\sin\varphi$$

$$a\cdot b=\cos(\theta-\varphi)=\left(\hat{i}\cos\theta+\hat{j}\sin\theta\right)\cdot\left(\hat{i}\cos\varphi+\hat{j}\sin\varphi\right)$$

$$\cos(\theta-\varphi)=\cos\theta\cos\varphi+\sin\theta\sin\varphi$$

(b) $$b\times a=\left|\hat{k}\right|\sin(\theta-\varphi)=\left|\left(\hat{i}\cos\theta+\hat{j}\sin\theta\right)\times\left(\hat{i}\cos\varphi+\hat{j}\sin\varphi\right)\right|$$

$$\sin(\theta-\varphi)=\sin\theta\cos\varphi-\cos\theta\sin\varphi$$

Chapter One
The Vectors (2)

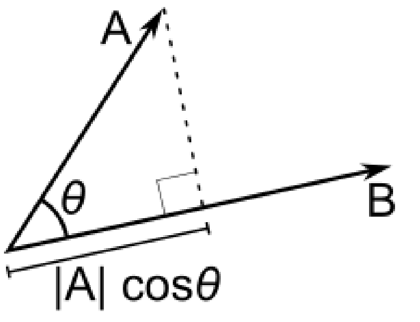
1.6: Dot or Scalar Product

1.7: Cross or Vector Product

Dot OR Scalar Product:

The dot or scalar product of two vectors **A** and **B**, denoted by A.B (read **A** dot **B**) is defined as the product of the magnitudes of *A* and *B* and the of the angle between them(projection of **A** on **B**). In symbols,

$$\mathbf{A} \cdot \mathbf{B} = AB \cos \theta; \quad 0 < \theta < \pi$$



Assuming that neither **A** nor **B** is the zero vector, an immediate consequence of the definition is that **A . B** = 0 if and only if **A** and **B** are perpendicular. Note that A . B is a scalar and not a vector.

The following laws are valid:

1. $\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}$ Commutative Law for Dot Products

2. $\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C}$ Distributive Law

3. $m(\mathbf{A} \cdot \mathbf{B}) = (m\mathbf{A}) \cdot \mathbf{B} = \mathbf{A} \cdot (m\mathbf{B}) = (\mathbf{A} \cdot \mathbf{B})m$,

where m is a scalar.

4. $i \cdot i = j \cdot j = k \cdot k = 1$; $i \cdot j = j \cdot k = k \cdot i = 0$

5. If $\mathbf{A} = A_1 i + A_2 j + A_3 k$ and $\mathbf{B} = B_1 i + B_2 j + B_3 k$, then

$$\mathbf{A} \cdot \mathbf{B} = A_1B_1 + A_2B_2 + A_3B_3$$

In particular, we can write:

$$\mathbf{A} \cdot \mathbf{B} = A_xB_x + A_yB_y + A_zB_z$$

6. The square of the magnitude of a vector **A** is given by the dot product of A with itself,

$$\mathbf{A}^2 = |\mathbf{A}|^2 = \mathbf{A}.\mathbf{A}$$

As **an example** of the dot product, suppose that an object under the action of a constant force undergoes a linear displacement **Δs**, (as shown in Figure 1-4) .By definition, the work ΔW done by the force is given by the product of the component of the force **F** in the direction of **Δs**, multiplied by the magnitude Δ*s* of the displacement; that is,

$$\Delta W = (F\cos\theta)\Delta s$$

where θ is the angle between **F** and **Δs**. But the expression on the right is just the dot product of **F** and **Δs**, that is,

$$\mathbf{\Delta W} = \mathbf{F}.\mathbf{\Delta s}$$

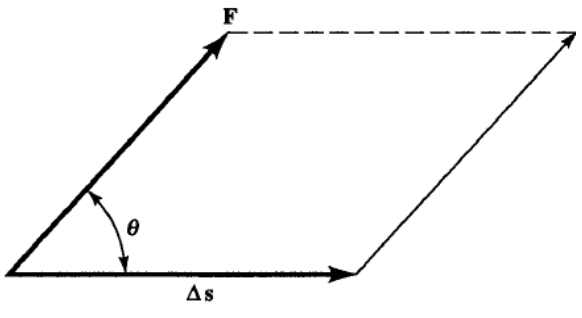


Fig.1-4

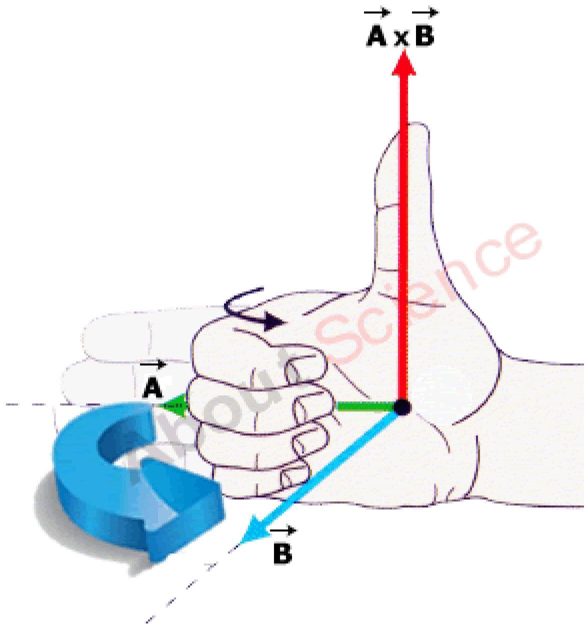
1-7: CROSS OR VECTOR PRODUCT:

The cross or vector product of **A** and **B** is a vector **C** = **A** × **B** (read **A** cross **B**). The magnitude of **A** × **B** is defined as the product of the magnitudes of **A** and **B** and the sine of the angle between them. The direction of the vector **C = A × B** is perpendicular to the plane of **A** and **B** and such that **A, B,** and **C** form a right-handed system.

In symbols,

$$\mathbf{A} \times \mathbf{B} = AB \sin \theta \mathbf{u}; \quad 0 \leq \theta \leq \pi$$

where **u** is a unit vector indicating the direction of **A × B**. If A = B or if A is parallel to B, then sin θ = 0 and A × B = 0.



4. $i \times i = j \times j = k \times k = 0$,

The following laws for cross product are shown:

1. $\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$ (Commutative Law for Cross Products Fails)

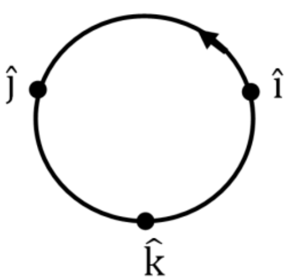
2. $\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C}$ Distributive Law

3. $m(\mathbf{A} \times \mathbf{B}) = (m\mathbf{A}) \times \mathbf{B} = \mathbf{A} \times (m\mathbf{B}) = (\mathbf{A} \times \mathbf{B})m$, where m is a scalar.

Also the following consequences of the definition are important:

4. $\mathbf{i} \times \mathbf{j} = \mathbf{k} = -\mathbf{j} \times \mathbf{i}$; $\mathbf{j} \times \mathbf{k} = \mathbf{i} = -\mathbf{k} \times \mathbf{j}$; $\mathbf{k} \times \mathbf{i} = \mathbf{j} = -\mathbf{i} \times \mathbf{k}$

This cyclic nature of the cross product can be emphasized by diagramming the multiplication table as shown in Figure



5. $|\mathbf{A} \times \mathbf{B}|$ = the area of a parallelogram with sides **A** and **B.**

6. If $\mathbf{A} \times \mathbf{B} = 0$ and neither **A** nor **B** is a null vector, then **A** and **B** are parallel.

7. If $\mathbf{A} = A_x i + A_y j + A_z k$ and $\mathbf{B} = B_x i + B_y j + B_z k$, then

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} i & j & k \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$\mathbf{A} \times \mathbf{B} = \mathbf{i}\begin{vmatrix} A_y & A_z \\ B_y & B_z \end{vmatrix} + \mathbf{j}\begin{vmatrix} A_z & A_x \\ B_z & B_x \end{vmatrix} + \mathbf{k}\begin{vmatrix} A_x & A_y \\ B_x & B_y \end{vmatrix}$$

The cross product expressed in **i, j ,k** form is:

$$\mathbf{A} \times \mathbf{B} = \mathbf{i}(A_y B_z - A_z B_y) + \mathbf{j}(A_z B_x - A_x B_z) + \mathbf{k}(A_x B_y - A_y B_x)$$

Let us calculate the magnitude of the cross product. We have

$$|\mathbf{A}\times\mathbf{B}|^2 = \left(A_x^2 + A_y^2 + A_z^2\right)\left(B_x^2 + B_y^2 + B_z^2\right) - (A_x B_x + A_y B_y + A_z B_z)^2$$

This can be reduced to:

$$|\mathbf{A}\times\mathbf{B}|^2 = \left(A_x^2 + A_y^2 + A_z^2\right)\left(B_x^2 + B_y^2 + B_z^2\right) - (A_x B_x + A_y B_y + A_z B_z)^2$$

from the definition of the dot product, the above equation may be written in the form:

$$|\mathbf{A}\times\mathbf{B}|^2 = A^2B^2 - (\mathbf{A}\cdot\mathbf{B})^2$$

Taking the square root of both sides of Equation above we can express the magnitude of the cross product as

$$|\mathbf{A}\times\mathbf{B}| = AB(1-\cos^2\theta)^{1/2} = AB\sin\theta$$

where θ is the angle between **A** and **B**.

A physical example of the cross product is the rotational moment (torque)

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$$

Let a force **F** act at a point P(x, y, z), as shown in Figure 1-5, and let the vector **OP** be designated by **r**; that is,

$$\mathbf{OP} = \mathbf{r} = \mathbf{i}x + \mathbf{j}y + \mathbf{k}z$$

The moment **N** of force, or the torque **N**, about a given point 0 is defined as the cross product **N= r x F**

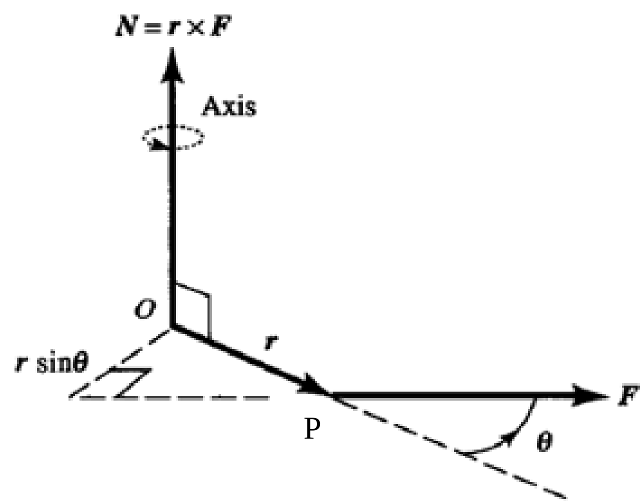
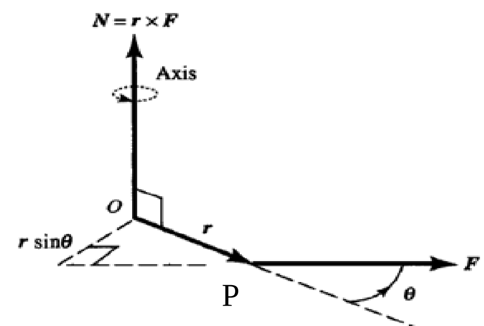


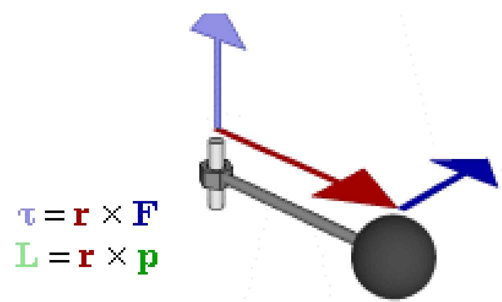
Figure 1-5: Illustration of the moment of a force about a point 0.

Thus, the moment of a force about a point is a having a magnitude and a direction. If a single force is applied at a point P on a body that is initially at rest and is free to turn about a fixed point 0 as a pivotمحور, then the body tends to rotate. The axis of this rotation is perpendicular to the force F, and it is also perpendicular to the line OP; therefore, the direction of the torque vector **N** is along the axis of rotation. The magnitude of the torque is given by:

$$|\mathrm{N}| = |\mathbf{r} \times \mathbf{F}| = r\,F\sin\theta$$



in which θ is the angle between **r** and **F**. Thus, |N| can be regarded as the product of the magnitude of the force and the quantity *r sin θ*, which is just the perpendicular distance from the line of action of the force to the point 0.



Representation of a given vector as the product of a scalar and a single unit vector:

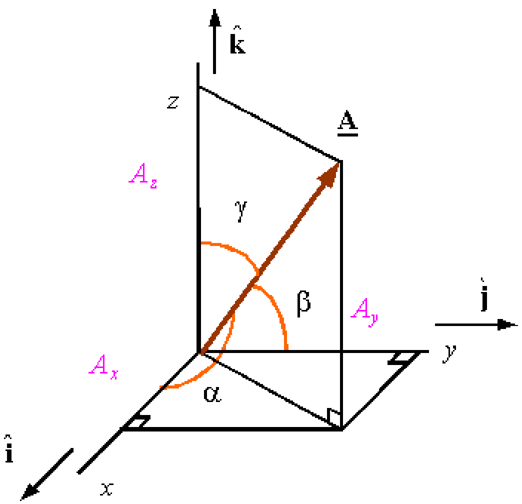
Consider the equation:

$$\mathbf{A} = \mathbf{i}\, A_x + \mathbf{j}\, A_x + \mathbf{k}\, A_x$$

Multiply and divide on the right by the magnitude of **A**

$$\mathbf{A} = A\left(\mathbf{i}\frac{A_x}{A} + \mathbf{j}\frac{A_y}{A} + \mathbf{k}\frac{A_z}{A}\right)$$

Now $A_x/A = \cos\alpha$, $A_y/A = \cos\beta$, and $\cos\gamma = A_z/A$ are the direction cosines of the vector A, and α, β, and γ are the direction angles. Thus, we can write are the direction cosines of vector A, and are the direction angles.



Thus we can write

$$A = A(i \cos\alpha + j \cos\beta + k \cos\gamma) = A(\cos\alpha, \cos\beta, \cos\gamma)$$

or $\mathbf{A} = \mathbf{n}A$

where **n** is a unit vector whose components are cosα ,cos β, and cos γ. Consider any other vector B. Clearly, the projection of B on A is just

$$B\cos\theta = \frac{\mathbf{B}\cdot\mathbf{A}}{A} = \mathbf{B}\cdot\mathbf{n}$$

where θ is the angle between A and B.

Example 1:

Two vectors $\vec{A} = 2\hat{i} + 3\hat{j} - \hat{k}$ *and* $\vec{B} = -\hat{i} + \hat{j} + 2\hat{k}$

1- Find the angle between the vectors

2- Find the vector product of the vectors

Solution:

1- $$\vec{A}.\vec{B} = A_x B_x + A_y B_y + A_z B_z = 2\times(-1) + 3\times 1 + (-1)\times 2 = -1$$

$$A = \left|\vec{A}\right| = \sqrt{2^2 + 3^2 + (-1)^2} = 3.74 \ units \quad B = \left|\vec{B}\right| = \sqrt{(-1)^2 + 1^2 + 2^2} = 2.45 \ units$$

and since **i . i = j . j = k . k =** 1;

i . j = j . k = k . i = 0

$$\vec{A}.\vec{B} = AB\cos\theta \quad \Rightarrow \quad \theta = \cos^{-1}\left(\frac{\vec{A}.\vec{B}}{AB}\right)$$

$$\theta = \cos^{-1}\left(\frac{-1}{3.74\times 2.45}\right) = \cos^{-1}(-0.109) = 96.3^0$$

2-

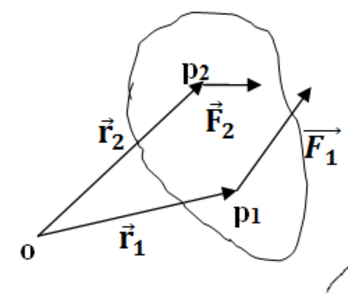
$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & -1 \\ -1 & 1 & 2 \end{vmatrix} = \hat{i}\,(6+1) - \hat{j}\,(4-1) + \hat{k}\,(2+3) = 7\hat{i} - 3\hat{j} + 5\hat{k}$$

Example 2:

A force $\vec{F}_1 = \hat{i} + \hat{j}$ was applied on a body at a point p_1 such that $o\vec{p}_1 = \vec{r}_1 = 2\hat{i} + \hat{j}$ and a second force $\vec{F}_2 = \hat{j} - \hat{k}$ was applied at a point $o\vec{p}_2 = \vec{r}_2 = \hat{i} + \hat{j} + \hat{k}$. Find: (a) the total moment $\vec{N}$

(b) the magnitude of $\vec{N}$ (c) the direction cosines of $\vec{N}$

Solution:



(a) $\vec{N}_1 = \vec{r}_1 \times \vec{F}_1$ and $\vec{N}_2 = \vec{r}_2 \times \vec{F}_2$

$$\vec{N}_1 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & 0 \\ 1 & 1 & 0 \end{vmatrix} = \hat{k} \qquad \vec{N}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 0 & 1 & -1 \end{vmatrix} = -2\hat{i} + \hat{j} + \hat{k}$$

The total moment is $\vec{N} = \vec{N}_1 + \vec{N}_2 = \hat{k} + (-2\,\hat{i} + \hat{j} + \hat{k}) = -2\,\hat{i} + \hat{j} + 2\,\hat{k}$

(b) $\vec{N} = \sqrt{4+1+4} = 3$

(c)

$$\cos\alpha = \frac{N_x}{N} = \frac{-2}{3} \quad , \cos\beta = \frac{N_y}{N} = \frac{1}{3} \quad \text{and} \quad \cos\gamma = \frac{N_z}{N} = \frac{2}{3}$$

1-9:Triple Products:

The expression

$$\mathbf{A}.\,(\mathbf{B} \times \mathbf{C})$$

is called the **scalar triple product** of **A**, **B**, and **C**. It is a **scalar** because it is the dot product of two vectors. Referring to the determinant expressions for the cross product, we see that the scalar triple product may be written:

$$\mathbf{A}.\,(\mathbf{B} \times \mathbf{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

Because the exchange of the terms of two rows or of two columns of a determinant changes its sign but not its absolute value, we can derive the following useful equation:

1- (A . B)C $\neq$ A(B . C) in general

2- A . (B × C) =B. (C × A) = C . (A× B) = volume of a parallelepiped having A, B, and C as edges, or the negative of this volume according as A, B, and C do or do not form a right handed system. Thus, the dot and the cross may be interchanged in the scalar triple product.

The expression

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$$

is called the **vector triple product**.

The following equation holds for the vector triple product:

1- $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) \neq (\mathbf{A} \times \mathbf{B}) \times \mathbf{C}$ (Associative Law for Cross Products Fails)

2- $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\, \mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\, \mathbf{C}$

$(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} = (\mathbf{A} \cdot \mathbf{C})\, \mathbf{B} - (\mathbf{B} \cdot \mathbf{C})\, \mathbf{A}$

Vector triple products are particularly useful in the study of rotating coordinate systems and rotations of rigid bodies.

Example 3:

Given the three vectors **A = 1, B = i—j,** and **C = k**, find

1- **A. (B × C)**

2- A× (B × C)

Solution:

1-

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \begin{vmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1(-1+0) = -1$$

2-

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B}) = (\mathbf{i}-\mathbf{j})0 - \mathbf{k}(1-0) = -\mathbf{k}$$

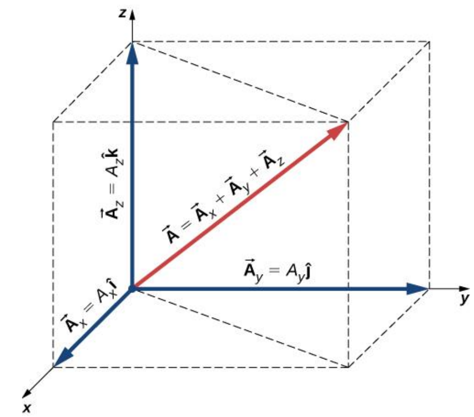
Chapter One
The Vectors (4)

1-10: Change of Coordinate System:

The Transformation Matrix:

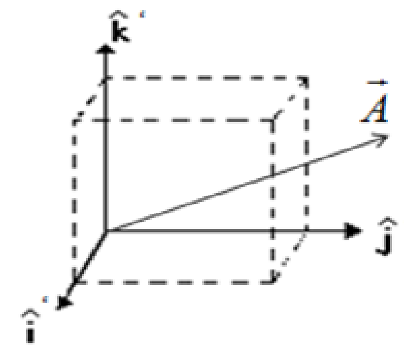
Analytical mechanics
Dr. Ali A. Mohammed Saleh

1-10:Change of Coordinate System: (The Transformation Matrix)



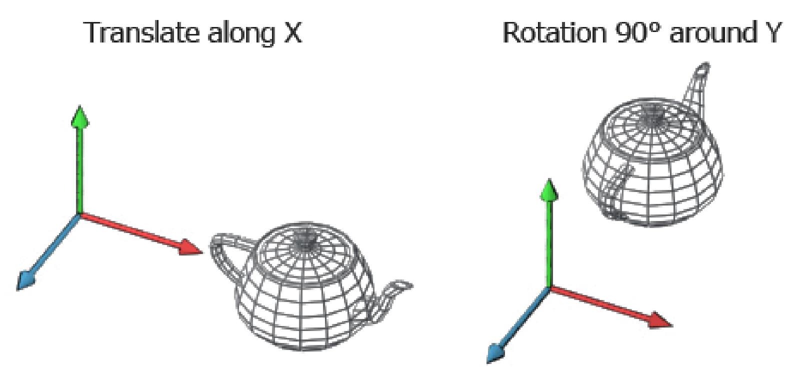
The rotation of a Cartesian coordinate system is an example of an orthogonal transformation. Consider the vector A expressed relative to the triad ijk:

$$\mathbf{A} = \mathbf{i}A_x + \mathbf{j}A_y + \mathbf{k}A_z$$



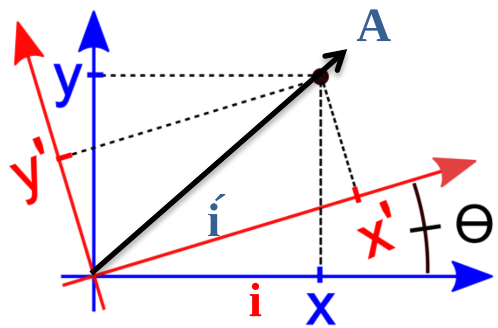
Relative to a new triad **i'j'k'** having a different orientation from that of **i j k**, the same vector **A** is expressed as:

$$\mathbf{A} = \mathbf{i}'A_{x'} + \mathbf{j}'A_{y'} + \mathbf{k}'A_{z'}$$



Now the dot product **A. i'** is just $A_{x'}$, that is, the projection of **A** on the unit vector **i'**. Thus, we may write

$$A_{x'} = \mathbf{A}\cdot\mathbf{i}' = (\mathbf{i}\cdot\mathbf{i}')A_x + (\mathbf{j}\cdot\mathbf{i}')A_y + (\mathbf{k}\cdot\mathbf{i}')A_z$$
$$A_{y'} = \mathbf{A}\cdot\mathbf{j}' = (\mathbf{i}\cdot\mathbf{j}')A_x + (\mathbf{j}\cdot\mathbf{j}')A_y + (\mathbf{k}\cdot\mathbf{j}')A_z$$
$$A_{z'} = \mathbf{A}\cdot\mathbf{k}' = (\mathbf{i}\cdot\mathbf{k}')A_x + (\mathbf{j}\cdot\mathbf{k}')A_y + (\mathbf{k}\cdot\mathbf{k}')A_z$$



The scalar products (**i . i'**), (**i . j'**), and so on are called the **coefficients of transformation**. They are equal to the direction cosines of the axes of the primed coordinate system relative to the unprimed system. The unprimed components are similarly expressed as:

$$A_x = \mathbf{A}\cdot\mathbf{i} = (\mathbf{i}'\cdot\mathbf{i})A_{x'} + (\mathbf{j}'\cdot\mathbf{i})A_{y'} + (\mathbf{k}'\cdot\mathbf{i})A_{z'}$$
$$A_y = \mathbf{A}\cdot\mathbf{j} = (\mathbf{i}'\cdot\mathbf{j})A_{x'} + (\mathbf{j}'\cdot\mathbf{j})A_{y'} + (\mathbf{k}'\cdot\mathbf{j})A_{z'}$$
$$A_z = \mathbf{A}\cdot\mathbf{k} = (\mathbf{i}'\cdot\mathbf{k})A_{x'} + (\mathbf{j}'\cdot\mathbf{k})A_{y'} + (\mathbf{k}'\cdot\mathbf{k})A_{z'}$$

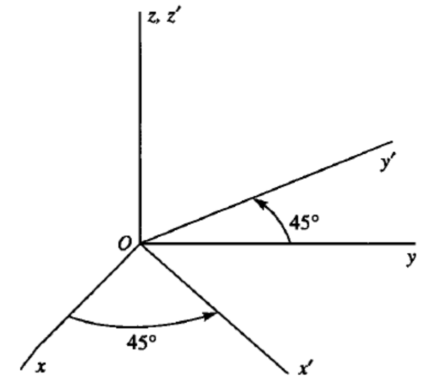
The equations of transformation are conveniently expressed in matrix notation. Thus,

The 3-by-3 matrix in Equation above is called the **transformation matrix**.

$$\begin{pmatrix} A_{x'} \\ A_{y'} \\ A_{z'} \end{pmatrix} = \begin{pmatrix} \mathbf{i}\cdot\mathbf{i}' & \mathbf{j}\cdot\mathbf{i}' & \mathbf{k}\cdot\mathbf{i}' \\ \mathbf{i}\cdot\mathbf{j}' & \mathbf{j}\cdot\mathbf{j}' & \mathbf{k}\cdot\mathbf{j}' \\ \mathbf{i}\cdot\mathbf{k}' & \mathbf{j}\cdot\mathbf{k}' & \mathbf{k}\cdot\mathbf{k}' \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$$

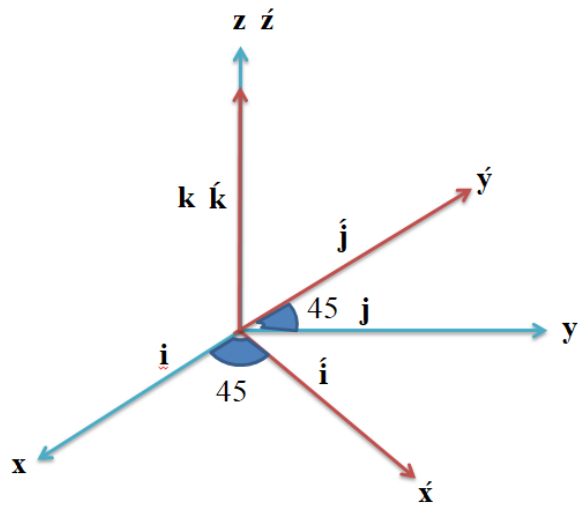
Example 4:

Express the vector **A = 3i + 2j + k** in terms of the triad **i'j'k'**, where the x'y'—axes are rotated 45° around the z-axis, with the z- and z'-axes coinciding, as shown in Figure. Referring to the figure, we have for the coefficients of transformation **i . i'** = cos 45° and so on,



we have from the figure:

$$\mathbf{i}\cdot\mathbf{i}' = 1/\sqrt{2} \qquad \mathbf{j}\cdot\mathbf{i}' = 1/\sqrt{2} \qquad \mathbf{k}\cdot\mathbf{i}' = 0$$
$$\mathbf{i}\cdot\mathbf{j}' = -1/\sqrt{2} \qquad \mathbf{j}\cdot\mathbf{j}' = 1/\sqrt{2} \qquad \mathbf{k}\cdot\mathbf{j}' = 0$$
$$\mathbf{i}\cdot\mathbf{k}' = 0 \qquad \mathbf{j}\cdot\mathbf{k}' = 0 \qquad \mathbf{k}\cdot\mathbf{k}' = 1$$



$$\begin{matrix} x \\ y \\ z \end{matrix} \begin{pmatrix} cos(\theta) & sin(\theta) & 0 \\ -sin(\theta) & cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\cos\left(\frac{\pi}{2} + x\right) = -\sin x$$
$$\cos\left(\frac{\pi}{2} - x\right) = \sin x$$

$$\begin{pmatrix} A'_x \\ A'_y \\ A'_z \end{pmatrix} = \begin{pmatrix} \hat{i}\cdot\hat{i}' & \hat{j}\cdot\hat{i}' & \hat{k}\cdot\hat{i}' \\ \hat{i}\cdot\hat{j}' & \hat{j}\cdot\hat{j}' & \hat{k}\cdot\hat{j}' \\ \hat{i}\cdot\hat{k}' & \hat{j}\cdot\hat{k}' & \hat{k}\cdot\hat{k}' \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$$

$$\begin{pmatrix} A'_x \\ A'_y \\ A'_z \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$$

We substitute the components of the vector **A = 3i + 2j + k**

$$\begin{pmatrix} A'_x \\ A'_y \\ A'_z \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} \Rightarrow \begin{pmatrix} A'_x \\ A'_y \\ A'_z \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$$

These give:

$$A_{x'} = \frac{3}{\sqrt{2}} + \frac{2}{\sqrt{2}} = \frac{5}{\sqrt{2}} \qquad A_{y'} = \frac{-3}{\sqrt{2}} + \frac{2}{\sqrt{2}} = \frac{-1}{\sqrt{2}} \qquad A_{z'} = 1$$

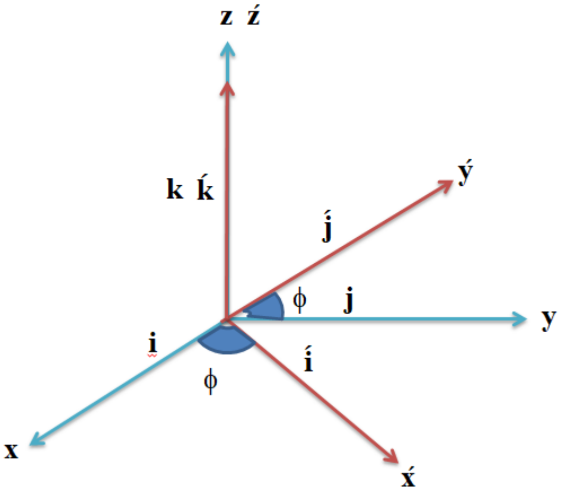
so that, in the primed system, the vector A is given by

$$\vec{A} = \frac{5}{\sqrt{2}}\hat{i}' - \frac{1}{\sqrt{2}}\hat{j}' + \hat{k}'$$

Example 5:

Find the transformation matrix for a rotation of the primed coordinate system through an angle φ about the z-axis. We have

$$\mathbf{i}\cdot\mathbf{i}' = \mathbf{j}\cdot\mathbf{j}' = \cos\phi$$
$$\mathbf{j}\cdot\mathbf{i}' = -\mathbf{i}\cdot\mathbf{j}' = \sin\phi$$
$$\mathbf{k}\cdot\mathbf{k}' = 1$$

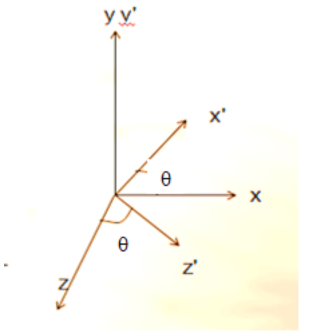


and all other dot products are zero; hence, the transformation matrix is:

$$\begin{pmatrix} A'_x \\ A'_y \\ A'_z \end{pmatrix} = \begin{pmatrix} \hat{i}\cdot\hat{i}' & \hat{j}\cdot\hat{i}' & \hat{k}\cdot\hat{i}' \\ \hat{i}\cdot\hat{j}' & \hat{j}\cdot\hat{j}' & \hat{k}\cdot\hat{j}' \\ \hat{i}\cdot\hat{k}' & \hat{j}\cdot\hat{k}' & \hat{k}\cdot\hat{k}' \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} \Rightarrow \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The transformation matrix for a rotation about a different coordinate axis-say, the y-axis through an angle (θ) given by the matrix

$$\begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix}$$



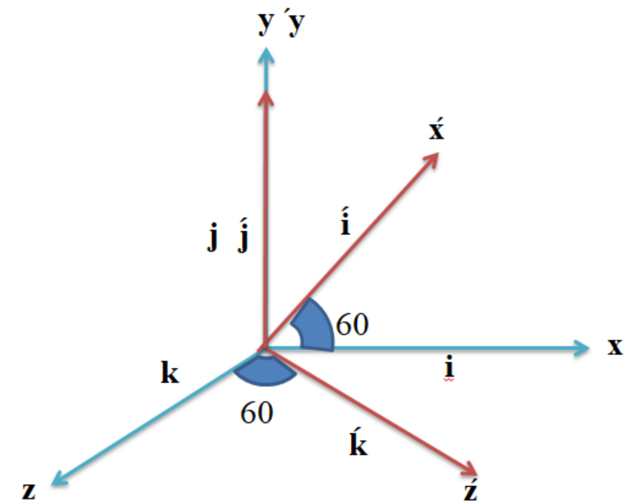
Consequently, the matrix for the combination of two rotations, the first being about the z-axis (angle ϕ) and the second being about the new y' -axis (angle (θ), is given by the matrix product

$$\underbrace{\begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix}}_{\text{y-axis}} \underbrace{\begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\text{z-axis}} = \begin{pmatrix} \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \\ \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \end{pmatrix}$$

Now matrix multiplication is, in general, noncommutative; therefore, we might expect that the result would be different if the order of the rotations, and, therefore, the order of the matrix multiplication, were reversed.

Example6:

Express the vector $\hat{i}+\hat{j}$ in terms of the triad $\hat{i}'\ \hat{j}'\ \hat{k}'$ where the x' z' axes are rotated 60 degree around the y axis, and the y' axes coinciding.



Solution: From the figure . we can see that

$$\hat{i}\cdot\hat{i}' = \hat{k}\cdot\hat{k}' = \cos 60^0 = \frac{1}{2}$$

$$\hat{i}\cdot\hat{k}' = \sin 60^0 = \frac{\sqrt{3}}{2} \qquad \hat{i}\cdot\hat{j}' = \cos 90 = 0$$

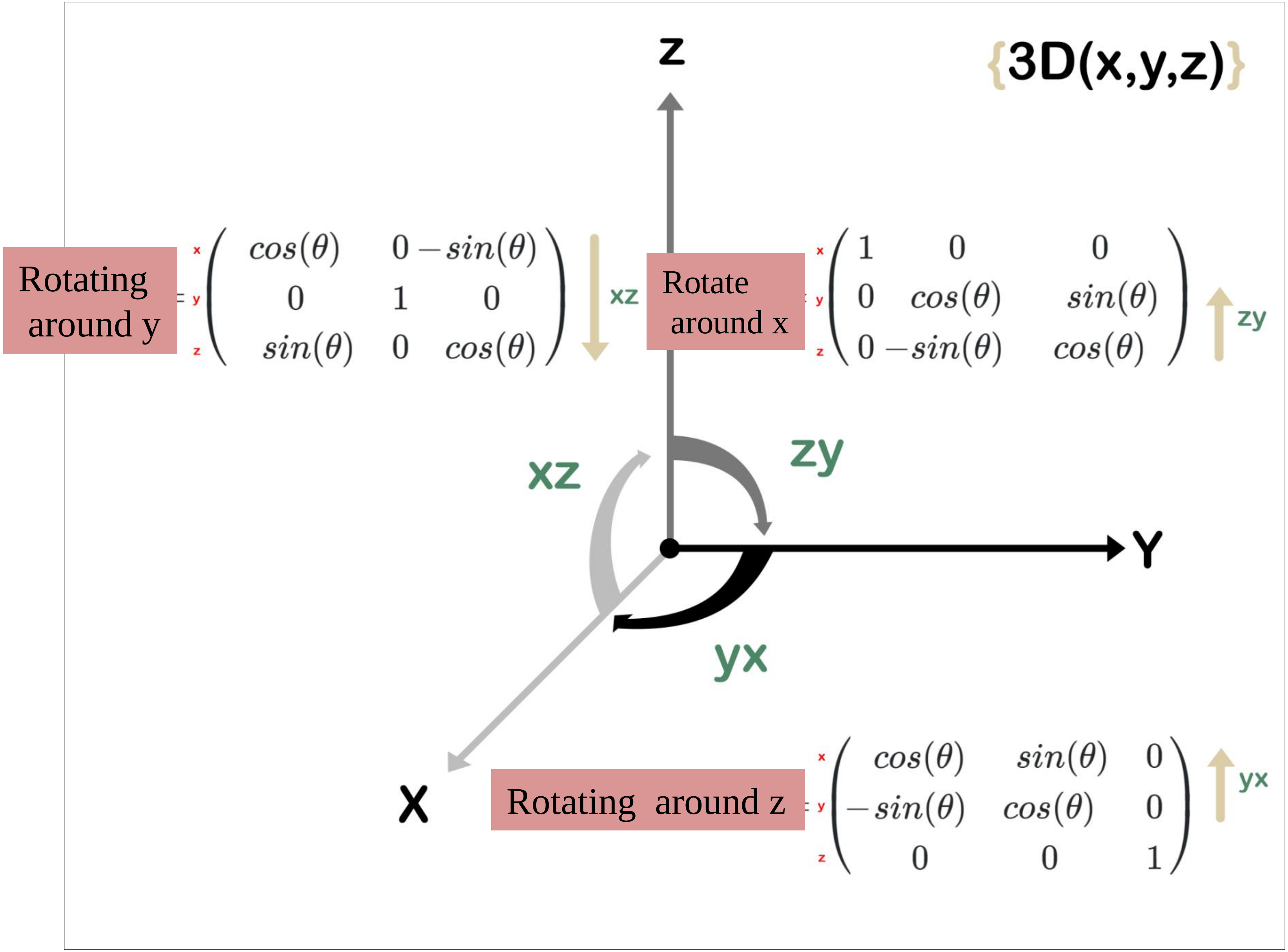
$$\hat{k}\cdot\hat{i}' = \cos(90+60) = -\sin 60 = -\frac{\sqrt{3}}{2}$$

The rest are zero

Therefore the transformation matrix and the equation of transformation becomes:

$$\begin{pmatrix} A'_x \\ A'_y \\ A'_z \end{pmatrix} = \begin{pmatrix} \hat{i}\cdot\hat{i}' & \hat{j}\cdot\hat{i}' & \hat{k}\cdot\hat{i}' \\ \hat{i}\cdot\hat{j}' & \hat{j}\cdot\hat{j}' & \hat{k}\cdot\hat{j}' \\ \hat{i}\cdot\hat{k}' & \hat{j}\cdot\hat{k}' & \hat{k}\cdot\hat{k}' \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} \Rightarrow \begin{pmatrix} A'_x \\ A'_y \\ A'_z \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 & -\frac{\sqrt{3}}{2} \\ 0 & 1 & 0 \\ \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$\vec{A} = \frac{1}{2}\hat{i}' + \hat{j}' + \frac{\sqrt{3}}{2}\hat{k}'$$



Rotating around y:

$$\begin{pmatrix} cos(\theta) & 0 & -sin(\theta) \\ 0 & 1 & 0 \\ sin(\theta) & 0 & cos(\theta) \end{pmatrix}$$

Rotate around x:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & cos(\theta) & sin(\theta) \\ 0 & -sin(\theta) & cos(\theta) \end{pmatrix}$$

Rotating around z:

$$\begin{pmatrix} cos(\theta) & sin(\theta) & 0 \\ -sin(\theta) & cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Some important trigonometric functions in transform operations

$$\cos\left(\frac{\pi}{2} + x\right) = -\sin x$$
$$\cos\left(\frac{\pi}{2} - x\right) = \sin x$$
$$\cos(\pi \pm x) = -\cos x$$
$$\cos\left(\frac{3\pi}{2} + x\right) = \sin x$$
$$\cos\left(\frac{3\pi}{2} - x\right) = -\sin x$$

$$\tan\left(\frac{\pi}{2} + x\right) = -\cot x$$
$$\tan\left(\frac{\pi}{2} - x\right) = \cot x$$
$$\tan(\pi + x) = \tan x$$
$$\tan(\pi - x) = -\tan x$$
$$\tan\left(\frac{3\pi}{2} + x\right) = -\cot x$$
$$\tan\left(\frac{3\pi}{2} - x\right) = \cot x$$

$$\sin\left(\frac{\pi}{2} \pm x\right) = \cos x$$
$$\sin(\pi + x) = -\sin x$$
$$\sin(\pi - x) = \sin x$$
$$\sin\left(\frac{3\pi}{2} \pm x\right) = -\cos x$$

$$\cos^2 x + \sin^2 x = 1$$

$$\cos(2\alpha) = 2\cos^2\alpha - 1$$
$$\cos(2\alpha) = \cos^2\alpha - \sin^2\alpha$$
$$\cos(2\alpha) = 1 - 2\sin^2\alpha$$

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$
$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$
$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

Chapter One
The Vectors (5)

Solved Problems

Analytical mechanics
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1.1: Given the two vectors **A = i +j** and **B =j + k**, find the following:

(a) **A+B** and **|A + B|** (b) 3**A**-2**B** (c) **A. B** (d) **A×B** and **|A**x**B|**

Solution:

(a) $\vec{A}+\vec{B}=(\hat{i}+\hat{j})+(\hat{j}+\hat{k})=\hat{i}+2\hat{j}+\hat{k}$

$|\vec{A}+\vec{B}|=(1+4+1)^{\frac{1}{2}}=\sqrt{6}$

(b) $3\vec{A}-2\vec{B}=3(\hat{i}+\hat{j})-2(\hat{j}+\hat{k})=3\hat{i}+\hat{j}-2\hat{k}$

(c) $\vec{A}\cdot\vec{B}=(1)(0)+(1)(1)+(0)(1)=1$

(d) $\vec{A}\times\vec{B}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 1 & 0\\ 0 & 1 & 1\end{vmatrix}=\hat{i}(1-0)+\hat{j}(0-1)+\hat{k}(1-0)=\hat{i}-\hat{j}+\hat{k}$

$|\vec{A}\times\vec{B}|=(1+1+1)^{\frac{1}{2}}=\sqrt{3}$

1.2 Given the three vectors **A = 2i +j**, **B = i + k**, and **C = 4j**, find the following: **(a) A. (B + C)** and **(A + B) .C**

(b) A.(B×C) and **(A×B).C** **(c) A×(B×C)** and **(A×B)×C**

Solution:

(a) $\vec{A}\cdot\left(\vec{B}+\vec{C}\right)=\left(2\hat{i}+\hat{j}\right)\cdot\left(\hat{i}+4\hat{j}+\hat{k}\right)=(2)(1)+(1)(4)+(0)(1)=6$

$\left(\vec{A}+\vec{B}\right)\cdot\vec{C}=\left(3\hat{i}+\hat{j}+\hat{k}\right)\cdot 4\hat{j}=(3)(0)+(1)(4)+(1)(0)=4$

(b) $\vec{A}\cdot\left(\vec{B}\times\vec{C}\right)=\begin{vmatrix}2 & 1 & 0\\ 1 & 0 & 1\\ 0 & 4 & 0\end{vmatrix}=-8$

$\left(\vec{A}\times\vec{B}\right)\cdot\vec{C}=\vec{A}\cdot\left(\vec{B}\times\vec{C}\right)=-8$

(c) $\vec{A}\times\left(\vec{B}\times\vec{C}\right)=\left(\vec{A}\cdot\vec{C}\right)\vec{B}-\left(\vec{A}\cdot\vec{B}\right)\vec{C}=4\left(\hat{i}+\hat{k}\right)-2\left(4\hat{j}\right)=4\hat{i}-8\hat{j}+4\hat{k}$

$\left(\vec{A}\times\vec{B}\right)\times\vec{C}=-\vec{C}\times\left(\vec{A}\times\vec{B}\right)=-\left[\left(\vec{C}\cdot\vec{B}\right)\vec{A}-\left(\vec{C}\cdot\vec{A}\right)\vec{B}\right]$

1.3: Find the angle between the vectors

$\mathbf{A} = a\mathbf{i} + 2a\mathbf{j}$ and $\mathbf{B} = a\mathbf{i} + 2a\mathbf{j} + 3a\mathbf{k}$.

(Note: These two vectors define a face diagonal and a body diagonal of a rectangular block of sides a, 2a, and 3a.)

Solution:

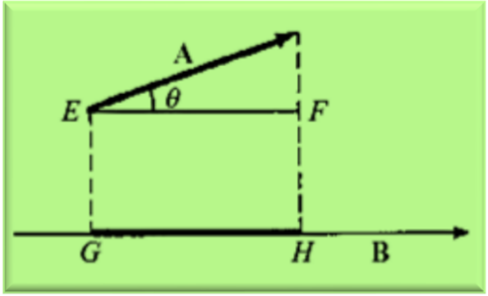
$$\cos\theta = \frac{\vec{A}\cdot\vec{B}}{AB} = \frac{(a)(a)+(2a)(2a)+(0)(3a)}{\sqrt{5a^2}\sqrt{14a^2}} = \frac{5a^2}{a^2\sqrt{5}\sqrt{14}}$$

$$\theta = \cos^{-1}\sqrt{\frac{5}{14}} \approx 53^\circ$$

1.4 Prove that the projection of **A** on B is equal to **A .b**, where **b** is a unit vector in the direction of **B**.

Solution:

Through the initial and terminal points of **A** pass planes perpendicular to **B** at G and **H** respectively, as in the adjacent Figure: then



Projection of **A** on **B** = GH = EF = $A\cos\theta$ = **A .b**

1.5: Find the volume of a parallelepiped with sides

A= 3**i** –**j** , **B**= **j** +5**k** and **C** =**i** +5**j**+4**k**

Solution:

$$\text{volume of parallelepiped} = |\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})| = \left|\begin{vmatrix} 3 & -1 & 0 \\ 0 & 1 & 2 \\ 1 & 5 & 4 \end{vmatrix}\right|$$

$$= |-20| = 20.$$

1.6: A= i +j , B= 2i -3j +k and C = 4j −3k , find

(a) (A×B) × C (b) A × (B× C)

Solution:

(A×B)×C=(A . C)B – (B . C) A

= **(4)(2i -3j +k) − (-15) (i +j)**

= 8**i**-12**j** +4**k** +15**i** +15**j**

= 23**i** +3**j** + 4**k**

A.C = 4

B.C = -12-3=-15

$$(a)\quad \mathbf{A}\times\mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 2 & -3 & 1 \end{vmatrix} = \mathbf{i}-\mathbf{j}-5\mathbf{k}. \quad \text{Then} \quad (\mathbf{A}\times\mathbf{B})\times\mathbf{C} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & -5 \\ 0 & 4 & -3 \end{vmatrix} = 23\mathbf{i}+3\mathbf{j}+4\mathbf{k}.$$

A×(B×C)= (A.C) B – (A.B) C

A.C = 4

A.B = -1

= (4)(2**i** -3**j** +**k**) − (−1) 4**j** −3**k**

= 8**i** − 12**j** +4**k** + 4**j** −3**k**

= 8**i** − 12**j** + **k**

(*b*) $B \times C = \begin{vmatrix} i & j & k \\ 2 & -3 & 1 \\ 0 & 4 & -3 \end{vmatrix} = 5i + 6j + 8k.$ Then $A \times (B \times C) = \begin{vmatrix} i & j & k \\ 1 & 1 & 0 \\ 5 & 6 & 8 \end{vmatrix} = 8i - 8j + k.$

It can be proved that, in general, $(A \times B) \times C \neq A \times (B \times C)$.

1.6:Find a vector perpendicular to **i** +2**k** and **i** + **j** + **k**, and find the area of the triangle with these two vectors as adjacent sides

$$(\hat{i}+2\hat{k})\times(\hat{i}+\hat{j}-\hat{k})=\begin{vmatrix}\hat{i} & -\hat{j} & \hat{k}\\ 1 & 0 & 2\\ 1 & 1 & -1\end{vmatrix}=-2\hat{i}+3\hat{j}+\hat{k}$$

This vector is perpendicular to the given vectors.

The area of the triangle is half the area of the parallelogram and hence the desired area is :

$$|-2\boldsymbol{i}+3\boldsymbol{j}+\boldsymbol{k}| = \sqrt{4+9+1} = \sqrt{14}$$

$$\frac{1}{2}\left|-2\hat{i}+3\hat{j}+\hat{k}\right| = \frac{\sqrt{14}}{2}$$

1.7: Find a unit vector normal to the plane containing the two vectors **A** = 2i +j− **k and B = i −j +2k**

Solution:

Unit vector = $\frac{\textit{the vector}}{\textit{Its magnitude}} = \frac{\mathbf{A\times B}}{|\mathbf{A\times B}|}$

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & -\hat{j} & \hat{k} \\ 2 & 1 & -1 \\ 1 & -1 & 2 \end{vmatrix} = \hat{i} - 5\hat{j} - 3\hat{k}$$

$$\left|\vec{A}\times\vec{B}\right| = \sqrt{1^2 + (-5)^2 + (-3)^2} = \sqrt{35}$$

$$\therefore \quad \hat{n} = \frac{1}{\sqrt{35}}(\hat{i} - 5\hat{j} - 3\hat{k})$$

1.8: Find the value of λ for which the vectors a and b are perpendicular, where **a**= 2**i** + λ **j** + **k** and **b** = **i** -2**j** +3**k**

Solution:

Given $\vec{a} = 2\hat{i} + \lambda\hat{j} + \hat{k}$

and $\vec{b} = \hat{i} - 2\hat{j} + 3\hat{k}$

Since the vectors are perpendicular, $\vec{a} \cdot \vec{b} = 0$

$$\Rightarrow \quad (2\hat{i} + \lambda\hat{j} + \hat{k}) \cdot (\hat{i} - 2\hat{j} + 3\hat{k}) = 0$$

$$2(1) + \lambda(-2) + 1(3) = 0$$

$$2 - 2\lambda + 3 = 0$$

$$5 - 2\lambda = 0 \Rightarrow 2\lambda = 5$$

$$\lambda = \frac{5}{2}$$

1.9:Find the projection of the vector **i** +3**j** + 7**k** on the vector 2**i**+ 6**j**+ 3**k**

Solution:

Let $\vec{a} = \hat{i}+3\hat{j}+7\hat{k}$ and $\vec{b} = 2\hat{i}+6\hat{j}+3\hat{k}$

$$\vec{a}\cdot\vec{b} = (\hat{i}+3\hat{j}+7\hat{k})\cdot(2\hat{i}+6\hat{j}+3\hat{k})$$
$$= 1(2) + 3(6) + 7(3)$$
$$= 2 + 18 + 21 = 41$$

$$|\vec{b}| = \sqrt{2^2+6^2+3^2} = \sqrt{4+36+9}$$
$$= \sqrt{49} = 7$$

Now, projection of $\vec{a}$ on $\vec{b}$ $= \dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|} = \dfrac{41}{7}$

Chapter One
The Vectors (6)

1-10: Derivative and Integration of a Vector
1-11: Position Vector of a Particle

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1-10: Derivative and Integration of a Vector

Consider a vector **A**, whose components are functions of a single variable u. The vector may represent position, velocity, and so on. The parameter u is usually the time t, but it can be any quantity that determines the components of **A**:

$$\mathbf{A}(u) = \mathbf{i}A_x(u) + \mathbf{j}A_y(u) + \mathbf{k}A_z(u)$$

The derivative of **A** with respect to u is defined, quite analogously to the ordinary derivative of a scalar function, by the limit

$$\frac{d\mathbf{A}}{du} = \lim_{\Delta u \to 0} \frac{\Delta \mathbf{A}}{\Delta u} = \lim_{\Delta u \to 0} \left(\mathbf{i}\frac{\Delta A_x}{\Delta u} + \mathbf{j}\frac{\Delta A_y}{\Delta u} + \mathbf{k}\frac{\Delta A_z}{\Delta u} \right)$$

Where $\Delta A_x = A_x$ (u + Δu) − A_x (u) and so on. Hence:

$$\frac{d\mathbf{A}}{du} = \mathbf{i}\frac{dA_x}{du} + \mathbf{j}\frac{dA_y}{du} + \mathbf{k}\frac{dA_z}{du}$$

The derivative of a vector is a vector whose Cartesian components are ordinary derivatives.

Derivative of the sum of two vectors is equal to the sum of the derivatives, namely,

$$\frac{d}{du}(\mathbf{A}+\mathbf{B}) = \frac{d\mathbf{A}}{du} + \frac{d\mathbf{B}}{du}$$

The rules for differentiating vector products obey similar rules of vector calculus. For example,

- $$\frac{d(n\mathbf{A})}{du} = \frac{dn}{du}\mathbf{A} + n\frac{d\mathbf{A}}{du}$$
- $$\frac{d(\mathbf{A}\cdot\mathbf{B})}{du} = \frac{d\mathbf{A}}{du}\cdot\mathbf{B} + \mathbf{A}\cdot\frac{d\mathbf{B}}{du}$$
- $$\frac{d(\mathbf{A}\times\mathbf{B})}{du} = \frac{d\mathbf{A}}{du}\times\mathbf{B} + \mathbf{A}\times\frac{d\mathbf{B}}{du}$$

Notice that it is necessary to preserve the order of the terms in the derivative of the cross product.

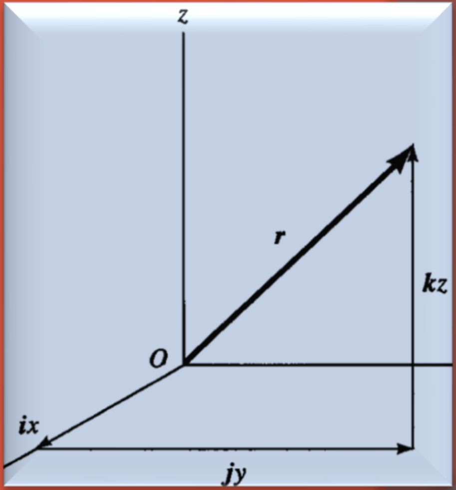
The integration of a vector as a function of any variable say time (t) can be written as:

$$\int \vec{A}\,dt = \hat{i}\int A_x(t)\,dt + \hat{j}\int A_y(t)\,dt + \hat{k}\int A_z(t)\,dt$$

1.11: Position Vector of a Particle: Velocity and Acceleration in Rectangular Coordinates

The position of a particle can be specified by a single vector, namely, the **displacement** of the particle relative to the origin of the coordinate system. This vector is called the **position vector** of the particle. In rectangular coordinates (as shown in figure), the position vector is simply

$$\vec{r} = \hat{i}\,x + \hat{j}\,y + \hat{k}\,z$$



The components of the position vector of a moving particle are functions of the time, namely,

$$x = x(t) \qquad y = y(t) \qquad z = z(t)$$

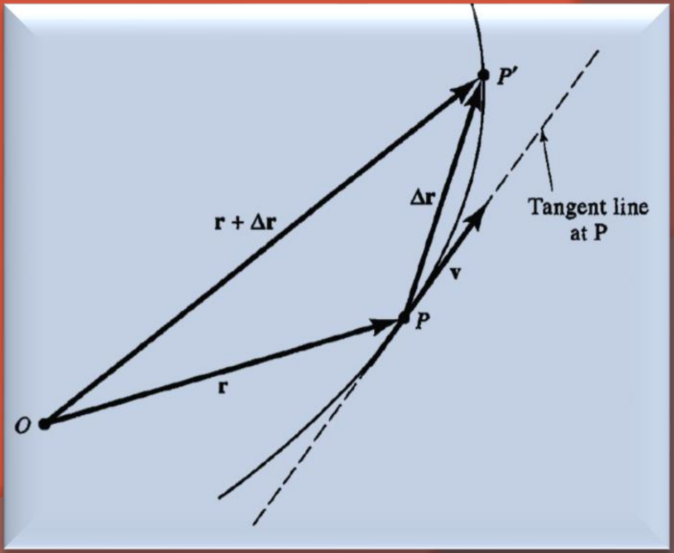
If the vector is the position vector **r** of a moving particle and the parameter is the time t, the derivative of **r** with respect to t is called the **velocity**, which we shall denote by **v**.

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \mathbf{i}\dot{x} + \mathbf{j}\dot{y} + \mathbf{k}\dot{z}$$

where the dots indicate differentiation with respect to t.

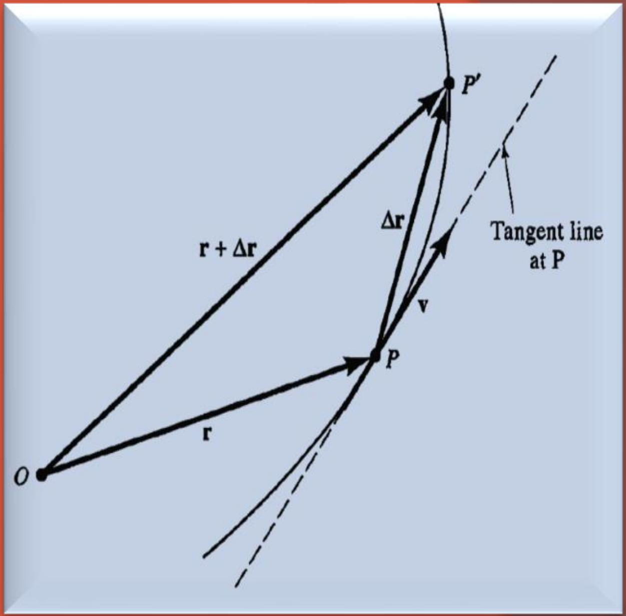
The geometric significance of the velocity vector

Suppose a particle is at a certain position at time t. At a time Δt later, the particle will have moved from the position r(t) to the position r(t + Δt). The vector displacement during the time interval Δt is:



$$\Delta r = r\,(\mathrm{t} + \Delta \mathrm{t}) - r\,(\mathrm{t})$$

So the quotient Δ**r** /Δt is a vector that is parallel to the displacement. As we consider smaller and smaller time intervals, the quotient Δ**r** / Δt approaches a limit dr/dt, which we call the **velocity**. The vector dr/dt expresses both the direction of motion and the rate. This is shown graphically in Figure



In the time interval Δt, the particle moves along the path from P to P'. As Δt approaches zero, the point P' approaches P, and the direction of the vector Δ **r**/ Δt approaches the direction of the tangent to the path at P. The **velocity vector**, therefore, is always tangent to the path of motion.

The magnitude of the velocity is called the speed. In rectangular components the speed is just:

$$v = |\mathbf{v}| = (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)^{1/2}$$

The time derivative of the velocity is called the acceleration. Denoting the acceleration with **a**, we have

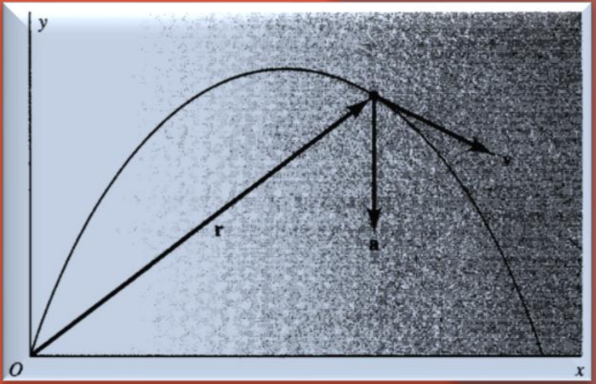
$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2}$$

In rectangular components, $\mathbf{a} = \mathbf{i}\ddot{x} + \mathbf{j}\ddot{y} + \mathbf{k}\ddot{z}$

Thus, **acceleration** is a vector quantity whose components, in rectangular coordinates, are the second derivatives of the positional coordinates of a moving particle.

Example7: Examine the motion represented by the equation:

$$\mathbf{r}(t) = \mathbf{i}bt + \mathbf{j}\left(ct - \frac{gt^2}{2}\right) + \mathbf{k}0$$



This represents motion in the xy plane, because the z component is constant and equal to zero. The velocity **v** is obtained by differentiating with respect to t, namely

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \mathbf{i}b + \mathbf{j}(c - gt)$$

The speed v varies with t according to the equation:

$$v = [b^2 + (c - gt)^2]^{1/2}$$

The acceleration, likewise, is given by

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = -\mathbf{j}g$$

Thus, **a** is in the negative y direction and has the constant magnitude g. The path of motion is a parabola, as shown in Figure.

Example 8: Circular Motion

Let us **analyze** the circular motion. Suppose the position vector of a particle is given by:

$$\mathbf{r} = \mathbf{i}b \sin \omega t + \mathbf{j}b \cos \omega t$$

where ω, is a constant and *b* is the radius.

The distance from the origin remains constant:

$$|\mathbf{r}| = r = (b^2 \sin^2 \omega t + b^2 \cos^2 \omega t)^{1/2} = b$$

So the path is a circle of radius b centered at the origin. Differentiating r, we find the velocity vector

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \mathbf{i}b\omega \cos \omega t - \mathbf{j}b\omega \sin \omega t$$

The particle traverses its path with constant speed:

$$v = |\mathbf{v}| = (b^2\omega^2 \cos^2 \omega t + b^2\omega^2 \sin^2 \omega t)^{1/2} = b\omega$$

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \mathbf{i}b\omega \cos\omega t - \mathbf{j}b\omega \sin\omega t$$

The acceleration is

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = -\mathbf{i}b\omega^2 \sin\omega t - \mathbf{j}b\omega^2 \cos\omega t$$

In this case the acceleration is perpendicular to the velocity, because the dot product of **v** and **a** vanishes:

$$\mathbf{v}\cdot\mathbf{a} = (b\omega\cos\omega t)(-b\omega^2\sin\omega t) + (-b\omega\sin\omega t)(-b\omega^2\cos\omega t) = 0$$

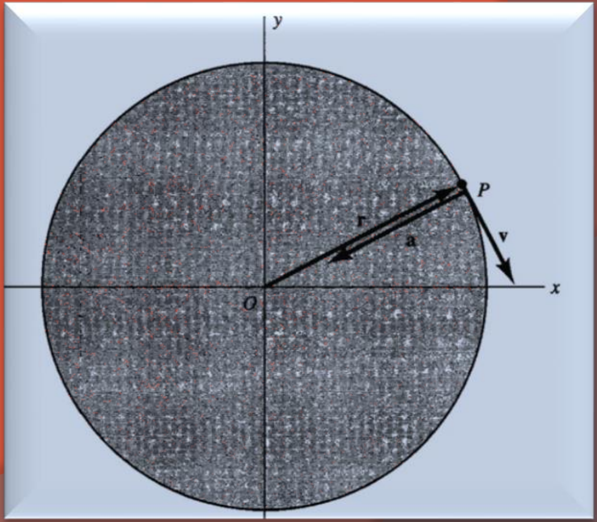
$$\vec{v}\cdot\vec{a} = -\omega^3 b^2 \sin\omega t\cos\omega t + \omega^3 b^2 \sin\omega t\cos\omega t = 0$$

Comparing the two expressions for **a** and **r**, we find:

$$\vec{a} = -\hat{i}\,\omega^2 b\sin\omega t - \hat{j}\,\omega^2 b\cos\omega t$$

$$\vec{r} = \hat{i}\,b\sin\omega t + \hat{j}\,b\cos\omega t$$

$$\mathbf{a} = -\omega^2\mathbf{r}$$



so **a** and **r** are oppositely directed: that is. a always points toward the center of the circular path (as shown in Figure).

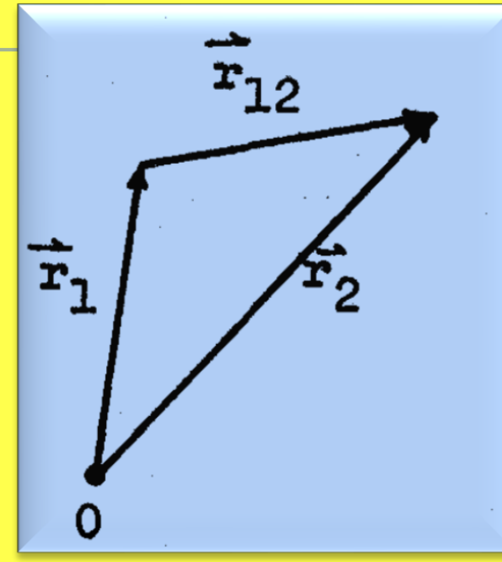
Chapter One
Lecture (7)

1-12:Relative velocity:
1-13:Ttangential and Normal Components of Acceleration:

Analytical mechanics
Dr. Ali A. Mohammed Saleh

1-12:Relative velocity:

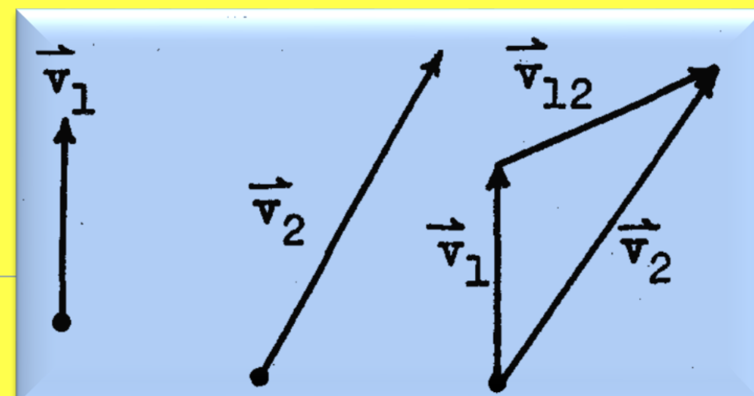
Consider two particles whose position vectors are $\mathbf{r}_1$ and $\mathbf{r}_2$ respectively. The displacement of the second particle with respect to the first is the difference $\mathbf{r}_2 - \mathbf{r}_1$ which we shall call $\mathbf{r}_{12}$. The velocity of the second particle relative to the first which we shall call the **relative velocity** is :



$$\vec{v}_{12} = \frac{d\vec{r}_{12}}{dt} = \frac{d(\vec{r}_2 - \vec{r}_1)}{dt} = \vec{v}_2 - \vec{v}_1$$

By transposing $\mathbf{v}_1$, for the actual velocity of particle 2 in term of the particle 1 and the relative velocity of the two particles we have:

$$\mathbf{v}_2 = \mathbf{v}_1 - \mathbf{v}_{12}$$



Example 9:

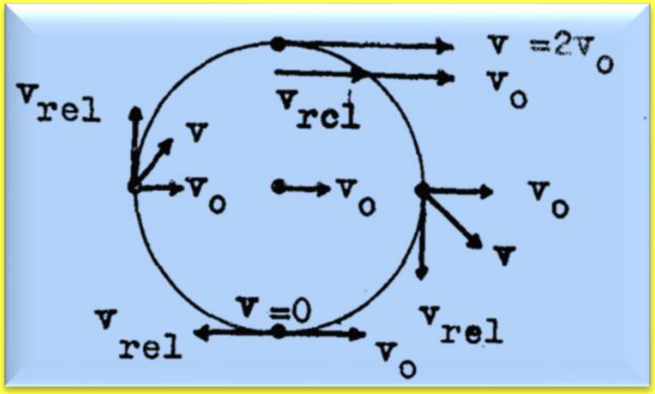
A wheel of radius b rolls along the ground with a forward speed v_o, find the velocity of any point P on the rim relative to the ground.

Solution:

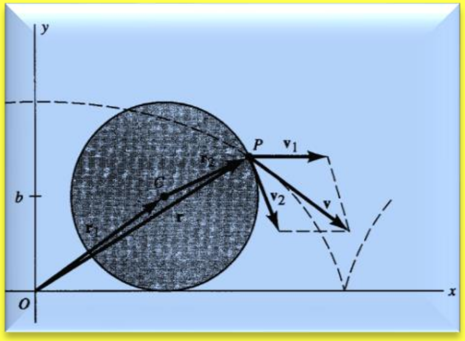
The motion of the point P is circular and suppose that the motion is clockwise about the origin center of wheel, in this case. Then, we first consider the position vector is given be expression:

$$\vec{r}_{op} = \hat{i} b \cos\theta - \hat{j} b \sin\theta$$

where $\theta = \omega t$



The time derivative gives the velocity of P relative to the center of the wheel as:



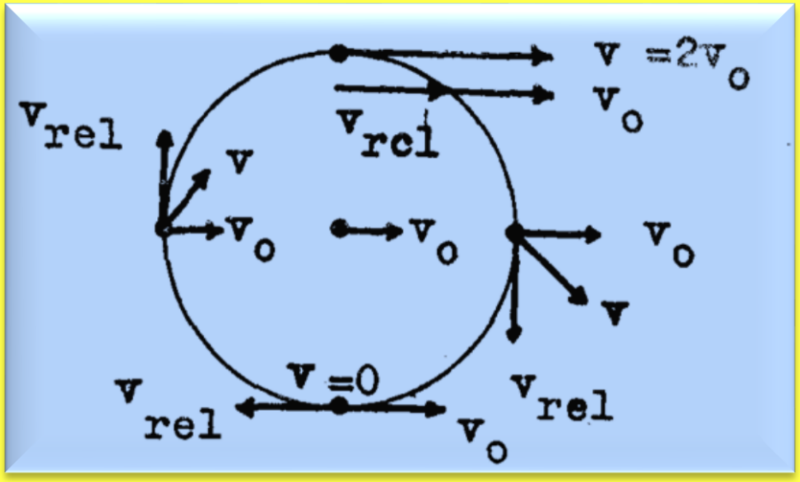
$$\mathbf{v}_{rel} = -\mathbf{i} b\omega \sin\theta - \mathbf{j} b\omega \cos\theta$$

Since the angular velocity $\omega = v_o/b$, and since the velocity of the center of the wheel relative to the ground is $\mathbf{i}v_o$,then the true velocity of P relative to the ground is

$$\mathbf{v} = \mathbf{i}v_0 - \mathbf{i}b\omega \sin\theta - \mathbf{j}b\omega\cos\theta$$
$$= \mathbf{i}v_0(1 - \sin\theta) - \mathbf{j}v_0\cos\theta$$

In particular, for we find that $\mathbf{v} = \mathbf{i}2v_o$, which is just twice the velocity of the center C. At these points the particle is at the uppermost part of its path.

Furthermore we obtain $\mathbf{v} = 0$. At these points the particle is at its lowest point and is instantaneously in contact with the ground.



Example 10:

The following relations represent the position of two particles move in the same circular path:

$$\mathbf{r}_1 = \mathbf{i}\, b\sin\omega t + \mathbf{j}\, b\cos\omega t \quad \text{and} \quad \mathbf{r}_2 = \mathbf{i}\, b\cos\omega t - \mathbf{j}\, b\sin\omega t$$

Find **1-** relative velocity **2-** magnitude of the relative velocity, **3-** the time rate of the change of the displacement between the two particles, all as a function of time.

Solution:

since $\mathbf{v}_{12} = \mathbf{v}_2 - \mathbf{v}_1$ Then

$$\mathbf{v}_1 = \mathbf{i}\, b\omega\cos\omega t - \mathbf{j}\, b\omega\sin\omega t$$

$$\mathbf{v}_2 = -\mathbf{i}\, b\omega\sin\omega t - \mathbf{j}\, b\omega\cos\omega t$$

Therefor the relative velocity is:

1- $\mathbf{v}_{12} = -\mathbf{i}\, b\omega(\sin\omega t + \cos\omega t) - \mathbf{j}\, b\omega(\cos\omega t - \sin\omega t)$

2- $|\mathbf{v}_{12}| = b\omega\,(\sin^2\omega t + \cos^2\omega t + 2\sin\omega t\cos\omega t + \cos^2\omega t + \sin^2\omega t - 2\sin\omega t\cos\omega t)^{1/2}$

$= b\omega(2)^{1/2}$ this is the magnitude of the relative velocity

3- The time rate of the change of the displacement between the two particles is:

$$\frac{d}{dt}\left|\vec{r_2}-\vec{r_1}\right|$$

and since

$$\mathbf{r}_1 = \mathbf{i}\ b\sin\omega t + \mathbf{j}\ b\cos\omega t \quad \text{and} \quad \mathbf{r}_2 = \mathbf{i}\ b\cos\omega t - \mathbf{j}\ b\sin\omega t$$

Then $\mathbf{r}_2 - \mathbf{r}_1 = \mathbf{i}\ b\ (\cos\omega t - \sin\omega t) - \mathbf{j}\ b\ (\sin\omega t + \cos\omega t)$

$$|\mathbf{r}_2 - \mathbf{r}_1| = b\ (\cos^2\omega t + \sin^2\omega t - 2\sin\omega t\cos\omega t + \sin^2\omega t + \cos^2\omega t + 2\sin\omega t\cos\omega t)^{1/2}$$

$$= b(2)^{1/2}$$

Then

$$\frac{d}{dt}\left|\vec{r_2}-\vec{r_1}\right| = 0$$

Note the time rate of the change of the displacement between the two particles is not the same as the magnitude of the relative velocity.

1-13: Ttangential and Normal Components of Acceleration:

Any vector can be expressed as the product of its magnitude and a unit vector giving its direction. The velocity vector **v** can be written as the product of the particle's speed *v* and a unit vector τ that give the direction of the particle's motion, thus:

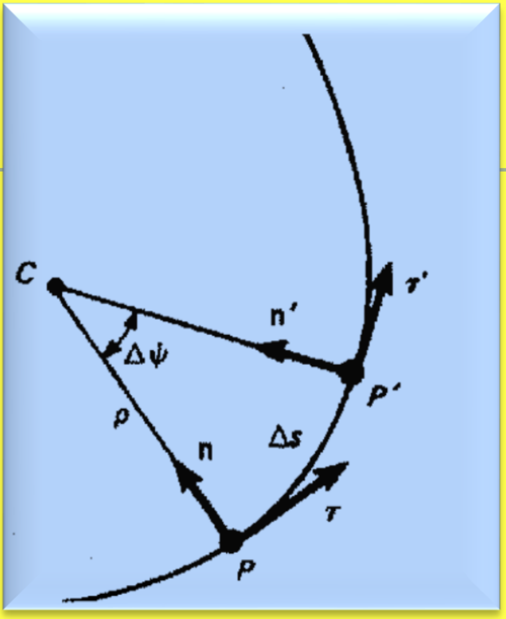
$$\mathbf{v} = v\boldsymbol{\tau}$$

The vector τ is called **the unit tangent vector**. As the particle moves the speed *v* may change and the direction of τ may change. By using the differentiation of product of scalar and vector to obtain acceleration vector the result is:

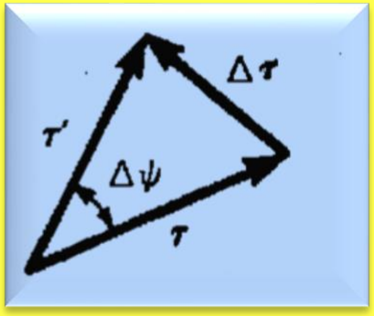
$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d(v\boldsymbol{\tau})}{dt} = \dot{v}\boldsymbol{\tau} + v\,\frac{d\boldsymbol{\tau}}{dt}$$

The unit vector being of constant magnitude, has a derivative $\frac{d\tau}{dt}$ express the change in direction of τ with respect to time

The particle move a distance Δs along the path from the initial point P to another point $\acute{P}$ at time interval Δt. let denote the unit tangent vector at P and $\acute{P}$ by τ and τ́ , respectively as shown in figure.



The directions of these two unit vectors differ by angle Δψ as shown in the figure, for small value of Δψ, the difference Δτ approaches Δψ in magnitude $(\Delta\tau \rightarrow \Delta\psi)$ and the direction of Δτ become perpendicular to the direction of τ $(\Delta\tau \perp \tau)$ when $\Delta\psi$ and Δs approach to zero. The derivative $\frac{d\tau}{d\psi}$ is of magnitude unity $\left(\left|\frac{d\tau}{d\psi}\right| = 1\right)$and is perpendicular to τ $\left(\frac{d\tau}{d\psi} \perp \tau\right)$

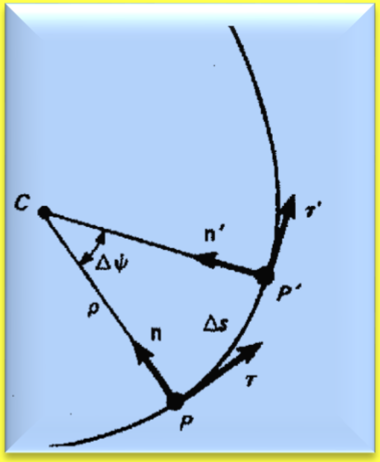


So That we call it the **unit normal vector** and denote it by **n**

$$\frac{d\boldsymbol{\tau}}{d\psi} = \mathbf{n}$$

To find the time derivative unit vector $\frac{d\tau}{dt}$, we use the chain rule as follows:

$$\frac{d\boldsymbol{\tau}}{dt} = \frac{d\boldsymbol{\tau}}{d\psi}\frac{d\psi}{dt} = \mathbf{n}\,\frac{d\psi}{ds}\frac{ds}{dt} = \mathbf{n}\,\frac{v}{\rho}$$



In which $\rho = \frac{ds}{d\psi}$ is the radius of curvature of the path of moving particle at P. Substitute the value of $\frac{d\tau}{dt}$ into equation:

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d(v\boldsymbol{\tau})}{dt} = \dot{v}\boldsymbol{\tau} + v\,\frac{d\boldsymbol{\tau}}{dt}$$

To find

$$\mathbf{a} = \dot{v}\boldsymbol{\tau} + \frac{v^2}{\rho}\,\mathbf{n}$$

Thus we can note that the acceleration of a moving particle has two components:

$$\mathbf{a} = \dot{v}\boldsymbol{\tau} + \frac{v^2}{\rho}\mathbf{n}$$

1- **Tangential component** (in the direction of motion) $a_T = \dot{v} = \ddot{s}$

2- **Normal component**, also called centripetal acceleration (directed toward the center of curvature on the concave side of the path motion) $a_N = \frac{v^2}{\rho}$

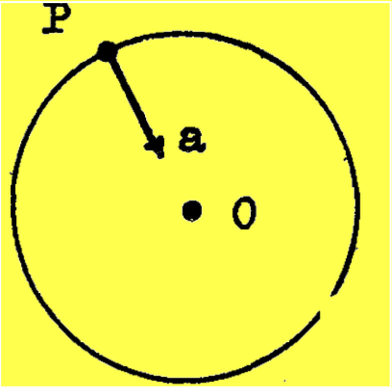
The magnitude of the total acceleration is given by:

$$|\mathbf{a}| = \left|\frac{d\mathbf{v}}{dt}\right| = \left(\dot{v}^2 + \frac{v^4}{\rho^2}\right)^{1/2}$$

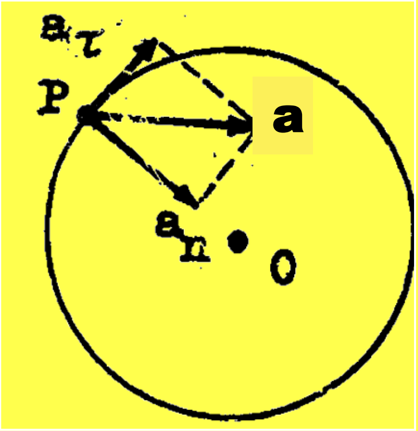
$$\mathbf{a} = \dot{v}\boldsymbol{\tau} + \frac{v^2}{\rho}\mathbf{n}$$

$$|\mathbf{a}| = \left|\frac{d\mathbf{v}}{dt}\right| = \left(\dot{v}^2 + \frac{v^4}{\rho^2}\right)^{1/2}$$

If the particle moves on a circle with constant speed Then the acceleration vector is of magnitude $\frac{v^2}{R_o}$, where R_o is the radius of the circle. The acceleration vector is always points to the center in this case as shown in figure.



However if the speed is not constant but increase at a certain rate $\dot{v}$ then the acceleration has a forward component and is slanted away from the center of circle towards the direction as shown in figure. **What happen if the speed decrease at a certain rate $\dot{v}$?**



Chapter One
Lecture (8)

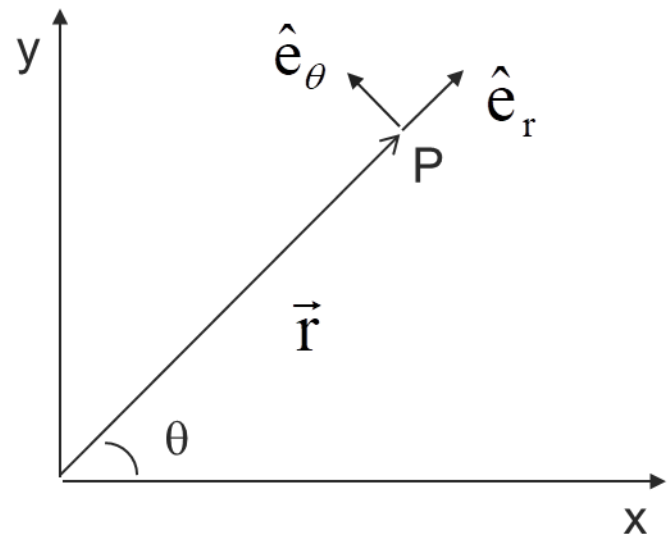
1-14: Velocity and acceleration in plane polar coordinate

Analytical mechanics
Dr. Ali A. Mohammed Saleh

1-14: Velocity and acceleration in plane polar coordinate

It is often convenient to employ polar coordinates r, θ to express the position of a particle moving in a plane. The position vector of the particle can be written as the product of the radial distance r by a unit radial vector $\mathbf{e}_r$:

$$\mathbf{r} = r\mathbf{e}_r$$



The coordinates of a point P are described by the radial distance from the origin “r” and the angle with respect to the x- axis.

$$x = r\cos\theta$$

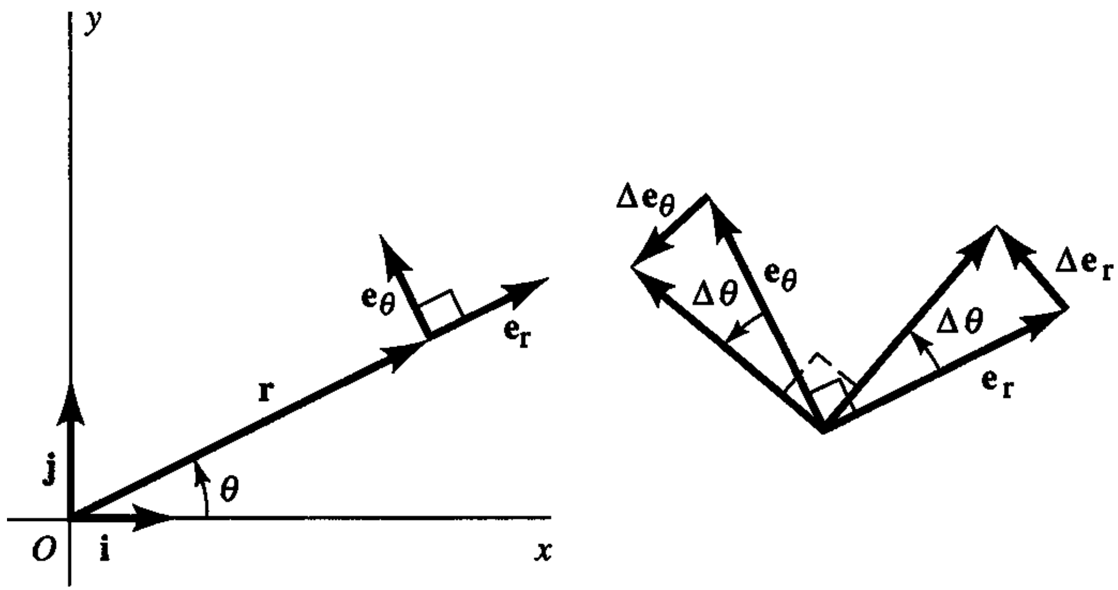
$$y = r\sin\theta$$

Along r the unit vector is $\hat{e}_r$ and along θ , the unit vector is $\hat{e}_\theta$

As the particle moves, both r and $\hat{e}_r$ vary; thus, they are both functions of the time. Hence, if we differentiate with respect to t, we have

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \dot{r}\mathbf{e}_r + r\frac{d\mathbf{e}_r}{dt} \qquad \text{--------} \ 1$$

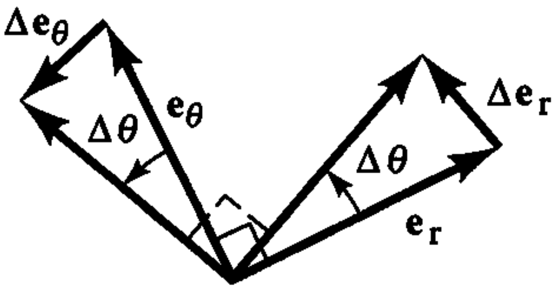
To calculate the derivative d $\hat{e}_r$ /dt, let us consider the vector diagram shown in figure:



When the direction of r changes by an amount the Δθ corresponding change $\Delta\hat{e}_r$ of the unit radial vector is as follows:

The magnitude $|\Delta\hat{e}_r|$ is approximately equal to Δθ ($|\Delta\hat{e}_r| \approx \Delta\theta$) and the direction of is very nearly perpendicular to $\hat{e}_r$ ($\Delta\hat{e}_r \perp \hat{e}_r$). Let us introduce another unit vector, $\hat{e}_\theta$, whose direction is perpendicular to $\hat{e}_r$. Then we have

$$\Delta\mathbf{e}_r \simeq \mathbf{e}_\theta \Delta\theta$$



If we divide by Δt and take the limit, we get:

$$\frac{d\mathbf{e}_r}{dt} = \mathbf{e}_\theta \frac{d\theta}{dt} \qquad \text{-------- 2}$$

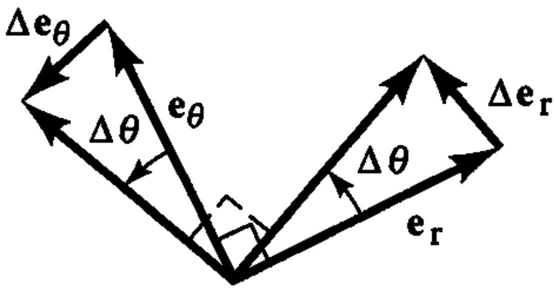
$$|\Delta\hat{e}_r| \cong \Delta\theta \quad \mathit{and} \quad \Delta\hat{e}_r \perp \hat{e}_r$$

And since

$$\hat{e}_\theta \perp \hat{e}_r \;\Rightarrow\; \Delta\hat{e}_r = \Delta\theta\,\hat{e}_\theta$$

For the time derivative of the unit radial vector. In similar way, we can argue that the change in the unit vector $\hat{e}_\theta$ is given by the approximation:

$$\Delta \mathbf{e}_r \simeq \mathbf{e}_\theta \Delta\theta$$



Here the minus sign is inserted to indicate that the direction of the change is opposite to the direction of $\hat{e}_r$, as shown in the figure. Consequently, the time derivative is given by

$$\frac{d\mathbf{e}_\theta}{dt} = -\mathbf{e}_r \frac{d\theta}{dt} \quad = -\,\mathbf{e}_r\dot{\theta} \qquad \text{--------} \; 3$$

We can finally write the equation for the velocity by substituting equation (2) for the derivative of the unit radial vector in equation (1) to find:

$$\mathbf{v} = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta \qquad \text{-------- } 4$$

Where, $\dot{r}$ is the radial component of the velocity vector, and $r\dot{\theta}$ is the transverse component.

To find the acceleration vector, we take the derivative of the velocity with respect to time for equation (4). This gives

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \ddot{r}\mathbf{e}_r + \dot{r}\frac{d\mathbf{e}_r}{dt} + (\dot{r}\dot{\theta} + r\ddot{\theta})\mathbf{e}_\theta + r\dot{\theta}\frac{d\mathbf{e}_\theta}{dt}$$

Substituting the value of $\frac{d\hat{e}_r}{dt}$ and $\frac{d\hat{e}_\theta}{dt}$ in the equation of acceleration we obtain:

$$\vec{a} = \ddot{r}\,\hat{e}_r + \dot{r}\dot{\theta}\,\hat{e}_\theta + (\dot{r}\dot{\theta} + r\ddot{\theta})\hat{e}_\theta - r\dot{\theta}^2\,\hat{e}_r$$

$$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{e}_\theta \qquad \text{-------- } 5$$

Thus, the radial component of the acceleration vector is:

$$a_r = \ddot{r} - r\dot{\theta}^2$$

and the transverse component is

$$a_\theta = 2\dot{r}\dot{\theta} + r\ddot{\theta}$$

$$= \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})$$

We study here two special cases

For a particle moves **on a circle** of constant radius b, so that $\dot{r} = 0$, then the radial component of the acceleration is of magnitude $b\dot{\theta}^2$ and is directed toward the center of the circular path. The transverse component in this case is $b\ddot{\theta}$.

While if the particle moves along a fixed **radial line**, that is if θ is constant then the radial component is $\ddot{r}$ and the transverse component is zero.

If **r and both vary**, then the general expression (6) gives the acceleration.

$$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{e}_\theta$$

Example 11:

A particle moves in a spiral path in such away that the radial distance decreases at a constant rate, r=b −ct , while the angular speed increases at a constant rate, $\dot{\theta} = kt$. **Find** the speed as a function of time.

Solution:

$$we\ have\ \ \dot{r} = -c\ \ and\ \ \ddot{r} = 0.$$

$$and\ \ the\ velocity\ \ is\quad \vec{v} = \dot{r}\,\hat{e}_r + r\dot{\theta}\,\hat{e}_\theta$$

$$\Rightarrow \vec{v} = -c\,\hat{e}_r + (b - ct)kt\,\hat{e}_\theta$$

$$the\ speed\ \ v = \sqrt{c^2 + (b - ct)^2 k^2 t^2}$$

Which is valid for $t \leq b/c$. **Note** that v = c for both t=0, r=b and for t=b/c, r=0.

Example 12:

On a horizontal turntable that is rotating at constant angular speed a bug is crawling outward on a radial line such that its distance from the center increases quadratically with time $\vec{r} = bt^2$, θ=ωt where b and ω are constants. **Find** the acceleration of the bug.

Solution:

$$we\ have\ \dot{r} = 2bt,\ \ddot{r} = 2b,\ \dot{\theta} = \omega,\ \ddot{\theta} = 0.$$

$$\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\mathbf{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\mathbf{e}_\theta$$

$$a = (2b - bt^2\omega^2)\hat{e}_r + (0 + 2(2bt)\omega)\hat{e}_\theta$$

$$= b(2 - t^2\omega^2)\hat{e}_r + (4b\omega t)\hat{e}_\theta$$

Note that the radial component of the acceleration becomes negative for large t in this example , although the radius is always increasing monotonically with time.

Chapter two
Lecture (3)

***The force as a function of time**
***The Forces as a function of Velocity**

Analytical mechanics
Dr. Ali A. Mohammed Saleh

2-2-3: The force as a function of time: The concept of impulse

The equation of motion is :

$$F(t) = m\frac{dv}{dt}$$

The linear momentum (and velocity) can be obtained as a function of time by integrating the equation of motion to get is :

$$\int F(t)\,dt = mv(t) + C$$

Where C is the constant of integration. The integral $\int f(t)dt$ is called **impulse**. The position of the particle as a function of time can be found by a second integration as follow:

$$x = \int v(t)\,dt = \int\left[\int \frac{F(t')}{m}\,dt'\right]dt$$

Example 3:

A block is initially at rest on a smooth horizontal surface. At time t=0 a constantly increasing horizontal force is applied F= ct. **Find** the displacement as a function of time

Solution:

$$ct = m\frac{dv}{dt}$$

$$v = \frac{1}{m}\int_0^t ct\,dt = \frac{ct^2}{2m}$$

$$x = \int v(t)\,dt = \int\left[\int \frac{F(t')}{m}\,dt'\right]dt$$

$$x = \int_0^t \frac{ct^2}{2m}\,dt = \frac{ct^3}{6m}$$

2-2-4:The Forces as a function of Velocity

Forces like viscous resistance exerted on a body moving through a fluid or like Air resistance on falling (or rising) body are called viscous forces. If the force can be expressed as a function of v only, the differential equation of motion $F = m\ddot{x}$ may be written in the form:

$$F(v) = m\frac{dv}{dt} \quad \text{------------ 1}$$

$$F(v) = m\frac{dv}{dx}\frac{dx}{dx} = mv\frac{dv}{dx} \quad \text{------------2}$$

from eq. (1) we found:

$$t = t(v) = \int \frac{mdv}{F(v)} \quad \text{------------3}$$

from eq. (2) we found:

$$x = x(v) = \int \frac{mvdv}{F(v)} \quad \text{------------4}$$

Solving eq. (3) gives v as a function of time, v =v(t) and we can solve it for x as given:

$$v = \frac{dx}{dt}$$

$$x = \int v(t)dt = x(t)$$

Solving eq. (4) gives v as a function of position we get:

v =v(x)

$$v(x) = \frac{dx}{dt}$$

$$t = \int \frac{dx}{v(x)} = t(x)$$

Example 4:

Suppose a block is projected with initial velocity v_o on a smooth horizontal surface and that there is air resistance such that F(v) = -cv the linear case dominates. Calculate v and x as a function of time

Solution:

The differential equation of motion is:

$$F(v) = -cv = m\frac{dv}{dt}$$

Which gives upon integrating,

$$t = \int_{v_0}^{v} \frac{-m\,dv}{cv} = -\frac{m}{c}\ln\left(\frac{v}{v_0}\right)$$

We can easily solve for (v) as a function of (t) multiplying by $-{}^{c}/_{m}$ and this give :

$$-\frac{c}{m}t = \ln\left(\frac{v}{v_0}\right)$$

Taking the exponential of both side. The result is:

$$v = v_0 \, e^{-\frac{c}{m}t}$$

Thus the velocity decreases exponentially with time. A second integration gives the position (x):

$$x = \int_0^t \left(v_0 \, e^{-\frac{c}{m}t} \right) dt$$

$$x = \frac{mv_0}{c}\left(1 - e^{-\frac{c}{m}t}\right)$$

Showing that the block approaches a limiting position (i.e. the block never goes beyond the limiting position) given by:

$$x_{lim} = \frac{mv_0}{c}$$

Chapter two
Lecture (4)

Vertical fall through a Fluid and Terminal Velocity

Analytical mechanics
Dr. Ali A. Mohammed Saleh

2-3: Vertical fall through a Fluid and Terminal Velocity:

The viscous force that a fluid exerts on a particle depends on velocity, F = F(v). in the case of viscous resistance exerted on a body moving through a fluid. If the force can be expressed as a function of **v** only, the differential equation of motion may be written in either of the two forms:

$$F_0 + F(v) = m\frac{dv}{dt}$$

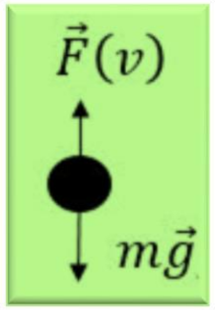
$$F_0 + F(v) = mv\frac{dv}{dx}$$

F_o is any constant force that does not depend on **v**. Upon separating variables, integration yields either t or x as a function of v. A second integration can then yield a functional relationship between x and t.

2-3-1:Linear case (F=−cv):

For an object falling vertically in a resisting fluid, the force F_o in equations above is the weight of the object − mg for the x-axis positive in the upward direction. The resistance force is proportional to the first power of v. we can express the force as − cv regardless of the sign of v because the resistance is always opposite to the direction of motion. Then the differential equation of motion is given by:

$$-mg - cv = m\frac{dv}{dt} \qquad \text{-------------- } 1$$



The constant of proportionality c depends on the size and shape of the object and the viscosity of the fluid. Separating variables and integrating, we find:

$$t = \int \frac{m\,dv}{F(v)} = \int_{v_0}^{v} \frac{m\,dv}{-mg - cv}$$

$$= -\frac{m}{c}\ln\frac{mg + cv}{mg + cv_0} \qquad \text{-------------- } 2$$

in which v_o is the initial velocity at t = 0. Upon multiplying by − c/m and taking the exponential, we can solve for v: $(mg + c\, v_0)e^{-\frac{c}{m}t} = (mg + c\, v)$

$$v = -\frac{mg}{c} + \left(\frac{mg}{c} + v_0\right)e^{-ct/m} \qquad \text{-----------} \; 3$$

After a sufficient time (t >> m/c), the exponential term drops to a negligible value and the velocity approaches the limiting value −mg/c. The limiting velocity of a falling body is called the **terminal velocity**; it is that velocity at which the force of resistance is just equal and opposite to the weight of the body so that the total force is zero, and so the acceleration is zero.

The magnitude of the terminal velocity is the ***terminal speed*** **mg/c** which is designate by v_t and let us write the ***characteristic time*** τ by **m/c**. Equation 3 can then be written as

$$v = -v_t(1 - e^{-t/\tau}) + v_0 e^{-t/\tau} \qquad \text{----------} \; 4$$

In particular, for an object dropped from rest at time t =0, v_o =0, we find:

$$v = -v_t(1 - e^{-t/\tau})$$

Integrating equation 3 with give x as a function of t, then we get:

$$x - x_0 = \int_0^t v(t)\, dt$$

$$= -\frac{mg}{c}t + \left(\frac{m^2 g}{c^2} + \frac{mv_0}{c}\right)(1 - e^{-ct/m}) \qquad \text{--------5}$$

We can write equation 5 in term of the terminal speed v_t and characteristic time τ :

$$x = x_o - v_t t + x_1(1 - e^{-t/\tau})$$

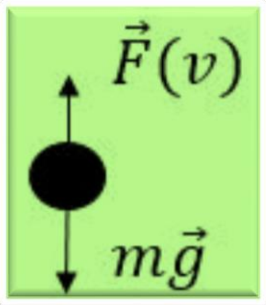
Where:

$$x_1 = \frac{m^2 g}{c^2} + \frac{mv_o}{c} = g\,\tau^2 + v_o\tau$$

The ratio between the terminal speed and the characteristic time is:

$$ratio = \frac{v_t}{\tau} = \frac{{mg}/{c}}{{m}/{c}} = g$$

2-3-2: non linear case (Quadratic case, $F(v) = \mp c\, v^2$:



In this case, the magnitude of F(v) is proportional to v^2. To ensure that the force remains resistive, we must remember that the **sign** preceding the F(V) term **depends** on whether or not the motion of the object is upward or downward. This is the case for any resistive force proportional to an even power of velocity.

A general solution involves treating the upward and downward motions separately. Here, we things somewhat by considering only the situation in which the body is either dropped from rest or projected downward with an initial velocity v_o.

Then the equation of motion is :

$$-mg \pm cv^2 = m\frac{dv}{dt}$$

For rising bodies F_r is negative (downward)
For falling bodies F_r is positive (upward)

For rising body we have:

The differential equation of motion can be integrated to give t as a function of v

$$-mg - cv^2 = m\frac{dv}{dt}$$

$$t = \int_{t_o}^{t} dt = -\frac{1}{g}\int \frac{dv}{\left(1 + \frac{c}{mg}v^2\right)}$$

Let

$$u = \sqrt{\frac{c}{mg}}\ \mathrm{v}, \quad v = \sqrt{\frac{mg}{c}}\ \mathrm{u} \quad \Rightarrow \quad dv = \sqrt{\frac{mg}{c}}\ \mathrm{du}$$

$$t - t_o = -\frac{1}{g}\int \frac{\sqrt{\frac{mg}{c}}\ \mathrm{du}}{(1+u^2)}$$

$$t = -\sqrt{\frac{m}{gc}}\ tan^{-1}u\ + t_o \quad \Rightarrow \quad t = -\sqrt{\frac{m}{gc}}\ tan^{-1}\sqrt{\frac{c}{mg}}\ \mathrm{v} + t_o$$

Where terminal speed v_t and characteristic time τ are equal:

$$v_t = \sqrt{\frac{mg}{c}} \qquad \tau = \sqrt{\frac{m}{gc}}$$

$$t = -\tau\ tan^{-1}\frac{v}{v_t} + t_o$$

Now we find v as a function of t from equation above :

$$v_t\ tan\ \frac{t - t_o}{-\tau} = v \quad \Rightarrow \quad v = v_t\ tan\ \frac{t_o - t}{\tau}$$

for rising

For falling body we have:

In the same way we can find the equation of motion for falling body and is given by :

$$-mg + cv^2 = m\,\frac{dv}{dt}$$

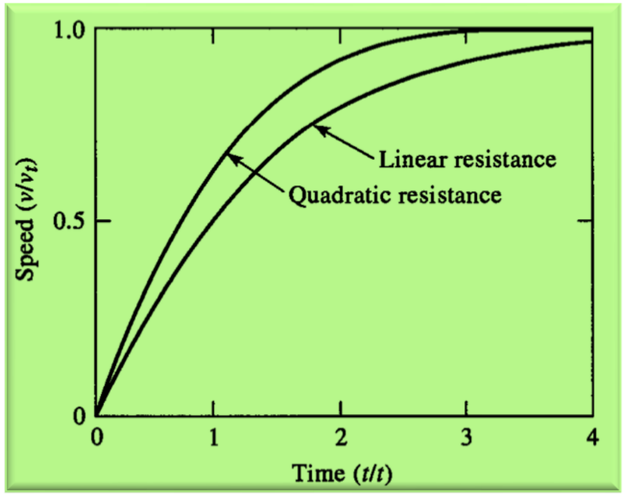
$$t = \int_{\acute{t}_o}^{t} dt = -\frac{1}{g}\int \frac{m\,dv}{(-mg + cv^2)}$$

$$t = -\tau\, tanh^{-1}\,\frac{v}{v_t} + \acute{t}_o$$

Now we find v as a function of t :

for rising

$$v\; = -\, v_t\, tanh\;\frac{t - \acute{t}_o}{\tau}$$



Graphs of speed versus time for a falling body subject to linear and quadratic air resistance

Chapter two
Lecture (4)

Vertical fall through a Fluid and Terminal Velocity

Analytical mechanics
Dr. Ali A. Mohammed Saleh

2-3: Vertical fall through a Fluid and Terminal Velocity:

The viscous force that a fluid exerts on a particle depends on velocity, F = F(v). in the case of viscous resistance exerted on a body moving through a fluid. If the force can be expressed as a function of **v** only, the differential equation of motion may be written in either of the two forms:

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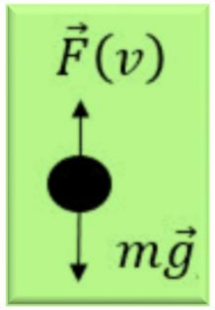

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In particular, for an object dropped from rest at time t =0, v_o =0, we find:

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Where:

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The ratio between the terminal speed and the characteristic time is:

$$ratio = \frac{v_t}{\tau} = \frac{mg/_c}{m/_c} = g$$

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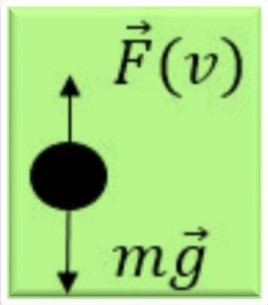

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$$dv = \sqrt{\frac{mg}{c}}\ \mathrm{du}$$

$$t - t_o = -\frac{1}{g}\int \frac{\sqrt{\frac{mg}{c}}\ \mathrm{du}}{(1+u^2)}$$

$$t = -\sqrt{\frac{m}{gc}}\ tan^{-1}u\ + t_o$$

$$t = -\sqrt{\frac{m}{gc}}\ tan^{-1}\sqrt{\frac{c}{mg}}\ \mathrm{v} + t_o$$

Where terminal speed v_t and characteristic time τ are equal:

$$v_t = \sqrt{\frac{mg}{c}}$$

$$\tau = \sqrt{\frac{m}{gc}}$$

$$t = -\tau\, tan^{-1}\frac{v}{v_t} + t_o$$

Now we find v as a function of t from equation above :

$$v_t\ tan\ \frac{t - t_o}{-\tau} = v$$

$$v\ =\ v_t\ tan\ \frac{t_o - t}{\tau}$$

for rising

For falling body we have:

In the same way we can find the equation of motion for falling body and is given by :

$$-mg + cv^2 = m\,\frac{dv}{dt}$$

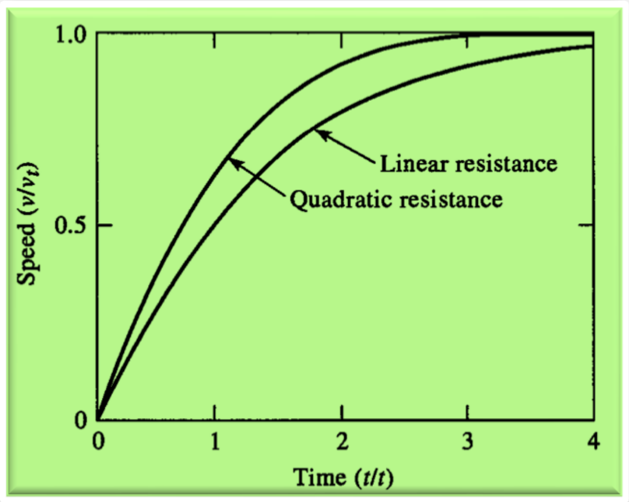
$$t = \int_{\acute{t}_o}^{t} dt = -\frac{1}{g}\int \frac{m\,dv}{(-mg + cv^2)}$$

$$t = -\tau\, tanh^{-1}\,\frac{v}{v_t} + \acute{t}_o$$

Now we find v as a function of t :

for rising

$$v \;=\; -\,v_t\, tanh\;\frac{t - \acute{t}_o}{\tau}$$



Graphs of speed versus time for a falling body subject to linear and quadratic air resistance

Chapter two
Lecture (5)

Linear Restoring Force: Harmonic Motion

Analytical mechanics
Dr. Ali A. Mohammed Saleh

2-4:Linear Restoring Force: Harmonic Motion

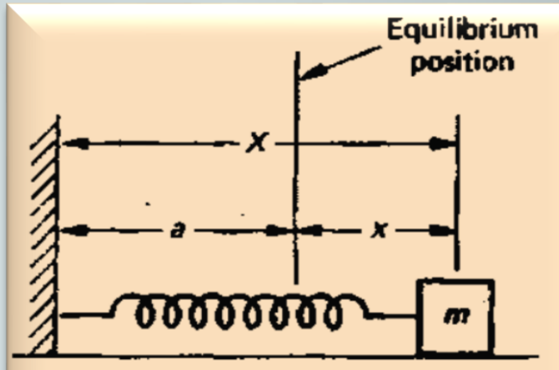
One of the most important cases of rectilinear motion is that produced by **linear restoring force**. This is a force whose magnitude, is proportional to the displacement of a particle from some equilibrium position and whose direction is always opposite to that of the displacement. Such force is exerted by an elastic cord or by spring obeying Hooke's law.

$$F = -k(X - a) = -kx \qquad \text{---------} \ 1$$

Where X is the total length and a is the unstretched (zero load) length of the spring. The variable x= (X-a) is the displacement of the spring from it is equilibrium. The proportionality constant k is called **stiffness**

Let a particle of mass m be attached to the spring as shown in figure. The force acting on particle is given by equation (1).

$$F = -kx$$



a: horizontal motion

Let the same spring be hold vertically as shown in figure. The total force now on particle is;

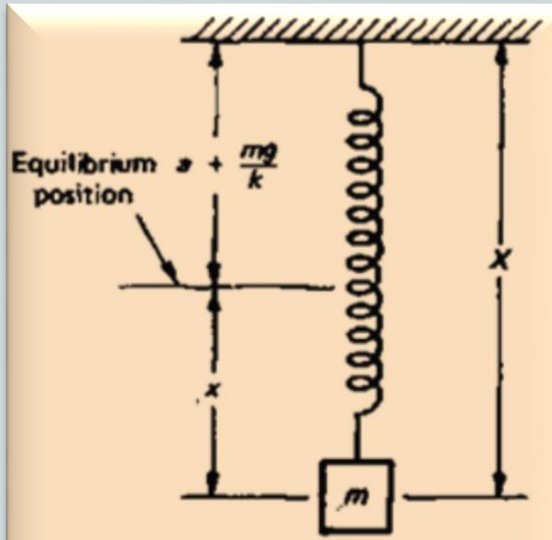
$$F = -k(X - a) + mg \qquad \text{---------- } 2$$

Where the positive is downward. Let measure x relative to new equilibrium position, that is the:

$$x = X - a - mg/k$$

This given again:

$$F = -kx$$



b: vertical motion

So the differential equation of motion is again:

$$m\ddot{x} + kx = 0 \qquad \text{------------- } 3$$

Equation (3) is linear differential equation of motion with constant coefficient. We try $A\,e^{qt}$, where q is a constant can be determined.

If $x = A\,e^{qt}$ a solution for all values of time. Apply this solution to eq. 3 we have:

$$m\frac{d^2}{dt^2}(Ae^{qt}) + k(Ae^{qt}) = 0 \qquad \text{---------- } 4$$

Which reduces to the equation:

$$mq^2 + k = 0$$

That is:

$$q = \pm i\sqrt{\frac{k}{m}} = \pm i\omega_0$$

$$i = \sqrt{-1}$$

$$w_o = \sqrt{\frac{k}{m}}$$

If f_1 and f_2 are solution then the sum of $f_1 + f_2$ is a solution too. The general con of equation (3) is then:

$$x = A_+ e^{i\omega_0 t} + A_- e^{-i\omega_0 t} \quad \text{------- } 5$$

Since $e^{iu} = \cos u + i \sin u$ the alternate forms of the solution are:

$$x = a \sin \omega_0 t + b \cos \omega_0 t \quad \text{------ } 6$$

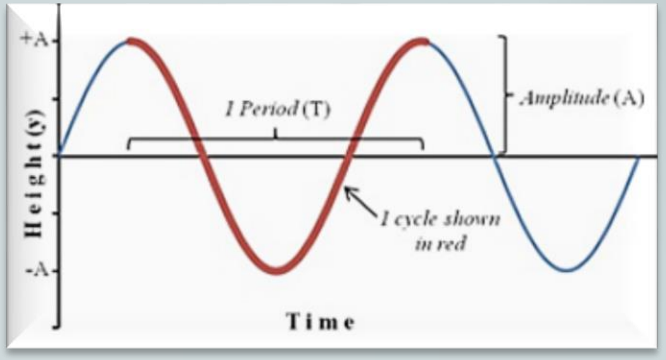
$$x = A \cos(\omega_0 t + \theta_0) \quad \text{-----------}7$$

The constant A of integration in the above equations are determined From the initial conditions. Equations. 5,6and 7 are solution of eq.(3). The Motion of sinusoidal oscillation of the displacement x, so that the eq.(3) is the differential equation of the harmonic oscillator. The coefficient ω_o is called **angular frequency** ($\omega_o = 2\pi f_o$ where f_o is the linear frequency). The maximum value of c is called the **amplitude** of the oscillation, that is a constant in equation (7).

The period T_o of the oscillation is the time required for complete cycle:

$$T_0 = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{m}{k}}$$

$$f_0 = \frac{1}{T_0} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$



Example:

A light spring is found to stretch an amount *b* when it supports A block of mass *m*. if the block is pulled downward a distance *l* From its equilibrium position and released at time $t = 0$, find the resulting motion as a function of time

Solution:

First to found the spring stiffness, we note that in the static equilibrium condition

$$F = -kb = -mg \quad \Rightarrow \quad k = \frac{mg}{b}$$

Hence, the angular frequency of oscillation is:

$$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{g}{b}}$$

In order to find constants for the equation of motion

$$x = A\cos(\omega_0 t + \theta_0)$$

We have at t = 0:

$$x = l \quad \text{and} \quad \dot{x} = 0$$

But

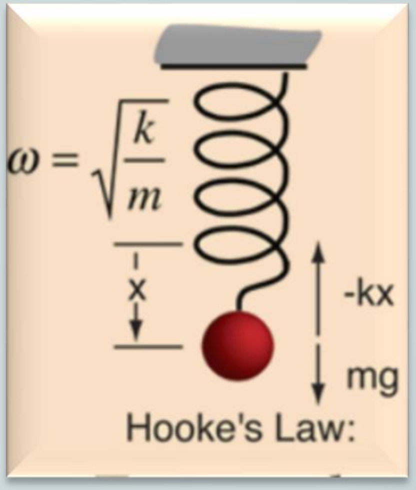
$$\dot{x} = -A\omega_0 \sin(\omega_0 t + \theta_0)$$

Thus

$$A = l \qquad \theta_0 = 0$$

So

$$x = l\cos\left(\sqrt{\frac{g}{b}}\, t\right)$$



2-5: Energy Considerations in Harmonic Motion

Consider a particle under the action of a linear restoring force $F_x = -kx$. Let us calculate the work done by an external force in moving the particle from the equilibrium position (x=0) to some position x. Assume that we move the particle very slowly so that it does not gain any kinetic energy; that is, the applied external force is barely greater in magnitude than the restoring force $-kx$; hence, $F_{ext} = -F_x = kx$ so:

$$W = \int_0^x F_{ext}\, dx = \int_0^x kx\, dx = \frac{k}{2}x^2 \qquad \text{---------} \; 1$$

In the case of a spring obeying Hooke's law, the work is stored in the spring as potential energy: W = V(x), where

$$V(x) = \frac{1}{2}kx^2 \qquad \text{------------} \; 2$$

Thus, $F_x = -\,dV/dx = -\,kx$, as required by the definition of V. The total when the particle is undergoing harmonic motion, is given by the sum of the kinetic and potential energies, namely

$$E = \frac{1}{2} m\dot{x}^2 + \frac{1}{2} kx^2 \qquad \text{-------- } 3$$

The kinetic energy is quadratic in the velocity variable, and the potential energy is quadratic in the displacement variable. The total energy is constant if there are no other forces except the restoring force acting on the particle.

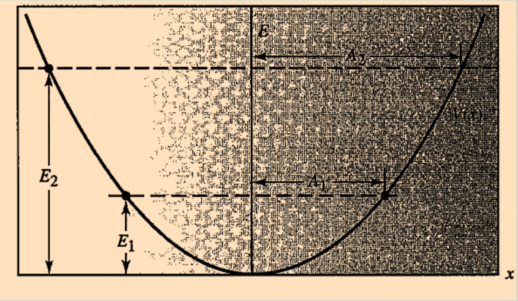
The motion of the particle can be found by starting with the energy equation (3). Solving for the velocity gives:

$$\dot{x} = \pm\left(\frac{2E}{m} - \frac{kx^2}{m}\right)^{1/2} \qquad \text{-------- } 4$$

which can be integrated to give t as a function of x as follows:

$$t = \int \frac{dx}{\pm[(2E/m)-(k/m)x^2]^{1/2}} = \mp(m/k)^{1/2}\cos^{-1}(x/A)+C \quad \text{-------- } 5$$

in which C is a constant of integration and A is the amplitude given by



$$A = \left(\frac{2E}{k}\right)^{1/2} \quad \text{-------- } 6$$

We see from the energy equation (eq.3) that the maximum value of the speed, which is call v_{max} occur at x = 0. we can write:

$$E = \frac{1}{2}mv_{max}^2 = \frac{1}{2}kA^2 \quad \text{--------}7$$

As the particle oscillates, the kinetic and potential energies continually change. The constant total energy is entirely in the form of kinetic energy at the center, where x = 0 and $\dot{x} = \mp v_{max}$ and it is all potential energy at extrema where $\dot{x} = 0$ and x= ∓ A.

1-10: Derivative and Integration of a Vector:

Consider a vector **A**, whose components are functions of a single variable u. The vector may represent position, velocity, and so on. The parameter u is usually the time t, but it can be any quantity that determines the components of **A**. Let:

$$\vec{A}(u) = \hat{i}\,A_x(u) + \hat{j}\,A_y(u) + \hat{k}\,A_z(u)$$

Then

$$\frac{d}{du}\vec{A} = \hat{i}\frac{d}{du}A_x + \hat{j}\frac{d}{du}A_y + \hat{k}\frac{d}{du}A_z$$

Derivative of a Vector is a vector whose components are ordinary derivatives.

The derivative of the sum of two vectors is equal to the sum of the derivatives, namely,

$$\frac{d}{du}(\vec{A}+\vec{B}) = \frac{d}{du}\vec{A} + \frac{d}{du}\vec{B}$$

The rules for differentiating vector products obey similar rules of vector calculus. If n is a scalar function of u then,

$$\frac{d}{du}n\vec{A} = \frac{dn}{du}\vec{A} + n\frac{d\vec{A}}{du}$$

$$\frac{d}{du}(\vec{A}\cdot\vec{B}) = \frac{d\vec{A}}{du}\cdot\vec{B} + \vec{A}\cdot\frac{d\vec{B}}{du}$$

$$\frac{d}{du}(\vec{A}\times\vec{B}) = \frac{d\vec{A}}{du}\times\vec{B} + \vec{A}\times\frac{d\vec{B}}{du}$$

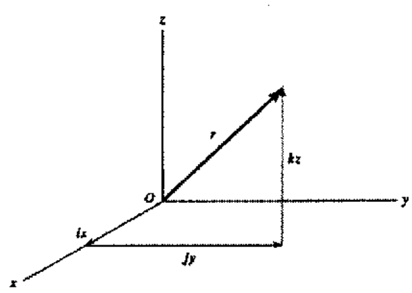
Notice that it is necessary to preserve يحافظ the order of the terms in the derivative of the cross product.

The integration of a vector A as a function of any variable say time (t) can be written as:

$$\int \vec{A}\,dt = \hat{i}\int A_x(t)\,dt + \hat{j}\int A_y(t)\,dt + \hat{k}\int A_z(t)\,dt$$

1-11: Position Vector of a Particle: Velocity and Acceleration in Rectangular Coordinates

The position of a particle can be specified by a single vector, namely, the displacement الازاحة of the particle relative to the origin of the coordinate system. This vector is called the position vector of the particle. In rectangular coordinates (as shown in the Figure), the position vector is simply:



$$\vec{r} = \hat{i}\,x + \hat{j}\,y + \hat{k}\,z$$

The components of the position vector of a moving particle are functions of the time, i.e.

$$x=x(t)\,, \quad y= y(t) \quad , z = z(t)$$

The derivative of **r** with respect to *t* is called the velocity, which we shall denote by *v*:

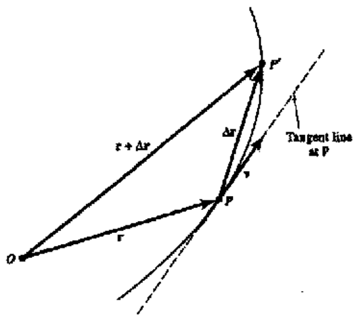
$$\vec{v} = \frac{d\vec{r}}{dt} = \hat{i}\,\dot{x} + \hat{j}\,\dot{y} + \hat{k}\,\dot{z}$$

where the dots indicate تشير الى differentiation with respect to *t*.

Let us examine the **geometric significance** of the velocity vector. Suppose a particle is at a certain position at time t. At a time Δt later, the particle will have moved from the position r(t) to the position r(t + Δt). The vector displacement during the lime interval Δt is:

$$\Delta\mathbf{r} = r(t + \Delta t) - r(t)$$

so the quotient حاصل قسمة Δr/Δt is a vector that is parallel to the displacement. As we consider smaller and smaller time intervals, the quotient Δr/Δt approaches a limit *d***r**/*d*t, which we call the *velocity*. The vector **dr**/dt expresses both the direction of motion and the rate. This is shown graphically in figure. In the time interval Δt, the particle moves along the path from P to P'. As Δt approaches zero, the point P' approaches P, and the



direction of the vector $\Delta\mathbf{r}/\Delta t$ approaches the direction of the tangent to the path at P. The velocity vector is always tangent to the path of motion.

The magnitude of the velocity is called the speed. In rectangular components the speed is :

$$v = |\vec{v}| = (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)^{1/2}$$

The time derivative of the velocity is called the *acceleration*. Denoting the acceleration with **a**, we have

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}$$

In rectangular components,

$$\vec{a} = \hat{i}\,\ddot{x} + \hat{j}\,\ddot{y} + \hat{k}\,\ddot{z}$$

Thus, acceleration is a vector quantity whose components, in rectangular coordinates, are the second derivatives of the positional coordinates of a moving particle.

Example7: Let us examine نختبر the projectile قذيفة motion represented by the equation:

$$\vec{r}(t) = \hat{i}\,bt + \hat{j}\left(ct - \frac{gt^2}{2}\right)$$

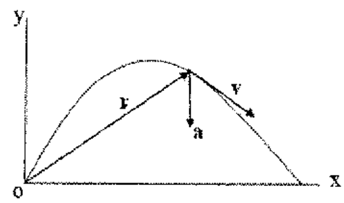
This represents motion in the xy plane, because the z component is constant and equal to zero. The velocity v is obtained by differentiating with respect to *t*, hence,

$$\vec{v} = \frac{d\vec{r}}{dt} = \hat{i}\,b + \hat{j}\,(c - gt)$$

The acceleration is given by

$$\vec{a} = \frac{d\vec{v}}{dt} = -g\,\hat{j}$$

Thus, **a** is in the negative y direction and has the constant magnitude g. The path of motion is a parabola قطع مكافئ.



Example 8: (Circular Motion)

Suppose the position vector of a particle is given by:

$$\vec{r} = \hat{\imath} b \sin\omega t + \hat{\jmath} b \cos\omega t$$

The distance from the origin remains constant:

$$r = |\vec{r}| = (b^2 \sin^2 \omega t + b^2 \cos^2 \omega t)^{1/2} = b$$

So the path is a circle of radius b centered at the origin. Differentiating **r**, we find the velocity vector

$$\vec{v} = \frac{d\vec{r}}{dt} = \hat{\imath}\, \omega b \cos\omega t - \hat{\jmath}\, \omega b \sin\omega t$$

The particle traverses its path with constant speed:

$$v = |\vec{v}| = (\omega^2 b^2 \cos^2 \omega t + \omega^2 b^2 \sin^2 \omega t)^{1/2} = b\omega$$

The acceleration is:

$$\vec{a} = \frac{d\vec{v}}{dt} = -\hat{\imath}\, \omega^2 b \sin\omega t - \hat{\jmath}\, \omega^2 b \cos\omega t$$

In this case the acceleration is perpendicular to the velocity $\vec{v} \perp \vec{a}$, because the dot product of **v** and **a** vanishes:

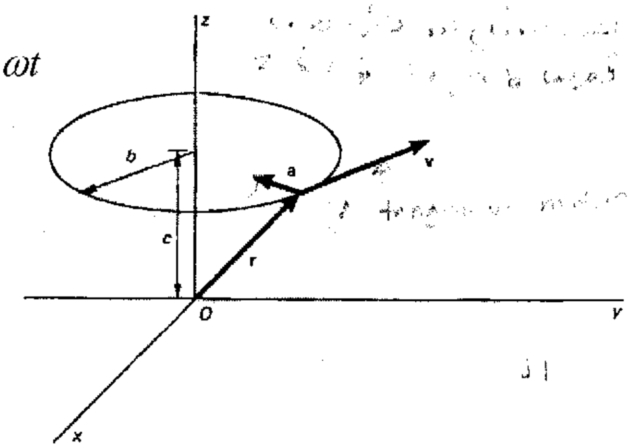
$$\vec{v} \cdot \vec{a} = -\omega^3 b^2 \sin\omega t \cos\omega t + \hat{\jmath}\omega^3 b^2 \sin\omega t \cos\omega t = 0$$

Comparing the two expressions for **a** and **r**, we find

$$\vec{a} = -\hat{\imath}\, \omega^2 b \sin\omega t - \hat{\jmath}\, \omega^2 b \cos\omega t$$

$$\vec{r} = \hat{\imath} b \sin\omega t + \hat{\jmath} b \cos\omega t$$

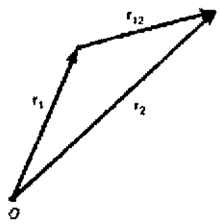
$$\vec{a} = -\omega^2 \vec{r}$$



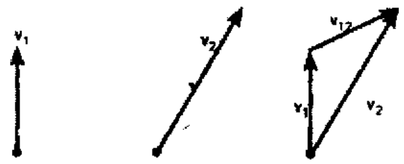
The path is a circle with radius b

1-12: Relative velocity:

Consider two particles whose position are $\vec{r}_1$ and $\vec{r}_2$ respectively. The displacement of the second particle with respect to the first is the difference $\vec{r}_2 - \vec{r}_1$ which we shall call r_{12}.



The velocity of the second particle relative to the first which we shall call the relative velocity is:



$$\mathbf{v}_{12} = \frac{d\mathbf{r}_{12}}{dt} = \frac{d(\mathbf{r}_2 - \mathbf{r}_1)}{dt} = \mathbf{v}_2 - \mathbf{v}_1$$

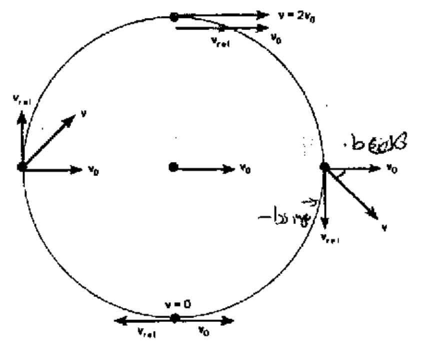
By transposing v1, for the actual velocity of particle 2 in terms of the velocity of particle 1 and the relative velocity of the two particles we have:

$$V_2 = V_1 + V_{12}$$

Example 9:

A wheel of radius b rolls along the ground with a forward speed, find the velocity of any point P on the rim relative to the ground.

Solution: The motion of point P is circular and suppose that the motion is clockwise about the origin center of wheel, in this case. Then, we First consider the position vector is given by expression:



$$\mathbf{r}_{OP} = \mathbf{i}b\cos\theta - \mathbf{j}b\sin\theta$$

where $\theta = \omega t$

the time derivative gives the velocity of P relative to the center of the wheel as:

$$\mathbf{v}_{rel} = -\mathbf{i}b\omega\sin\theta - \mathbf{j}b\omega\cos\theta$$

Since the angular velocity $\omega = v_o/b$, and since the velocity of the center of the wheel relative to the ground is $\mathbf{i}v_o$, then the true velocity of P relative to the ground is:

$$\begin{aligned}\mathbf{v} &= \mathbf{i}v_0 - \mathbf{i}b\omega\sin\theta - \mathbf{j}b\omega\cos\theta \\ &= \mathbf{i}v_0(1 - \sin\theta) - \mathbf{j}v_0\cos\theta\end{aligned}$$

Example 10: The following relations represent the position of two particles move in the same circular path.

$$\mathbf{r}_1 = \mathbf{i}\, b\sin\omega t + \mathbf{j}\, b\cos\omega t \quad \text{and} \quad \mathbf{r}_2 = \mathbf{i}\, b\cos\omega t - \mathbf{j}\, b\sin\omega t$$

Find the relative velocity, the magnitude of the relative velocity, and the time rate of the change of the displacement between the two particles, all as a function of time?

solution:

since $\mathbf{v}_{21} = \mathbf{v}_2 - \mathbf{v}_1$

$$\mathbf{v}_1 = \mathbf{i}\, b\omega\cos\omega t - \mathbf{j}\, b\omega\sin\omega t$$

$$\mathbf{v}_2 = -\mathbf{i}\, b\omega\sin\omega t - \mathbf{j}\, b\omega\cos\omega t$$

therefore the relative velocity is

$$\mathbf{v}_{21} = -\mathbf{i}\, b\omega(\sin\omega t + \cos\omega t) - \mathbf{j}\, b\omega(\cos\omega t - \sin\omega t)$$

$$|\mathbf{v}_{21}| = b\omega\,(\sin^2\omega t + \cos^2\omega t + 2\sin\omega t\cos\omega t + \cos^2\omega t + \sin^2\omega t - 2\sin\omega t\cos\omega t)^{1/2}$$

$= b\omega(2)^{1/2}$ this is the magnitude of the relative velocity

The time rate of the change of the displacement between the two particles is :

$$\frac{d}{dt}|\vec{r_2} - \vec{r_1}|$$

and since

$$\mathbf{r}_2 - \mathbf{r}_1 = \mathbf{i}\, b\,(\cos\omega t - \sin\omega t) - \mathbf{j}\, b\,(\sin\omega t + \cos\omega t)$$

$$|\boldsymbol{r}_2 - \boldsymbol{r}_1| = b\,(\cos^2\omega t + \sin^2\omega t - 2\sin\omega t\cos\omega t + \sin^2\omega t + \cos^2\omega t + 2\sin\omega t\cos\omega t)^{1/2}$$

$$= b(2)^{1/2}$$

then $$\frac{d}{dt}|\vec{r_2} - \vec{r_1}| = 0$$

Note: That the time rate of the change of the displacement between the two particles is not the same as the magnitude of the relative velocity.

1-13: Tangential and normal componenet of acceleration:

Any vector can be expressed as the product of its magnitude and a unit vector giving its direction. the velocity vector **v** can be written as the product of the particle's speed v and a unit vector $\boldsymbol{\tau}$ that give the direction of particle's motion, thus

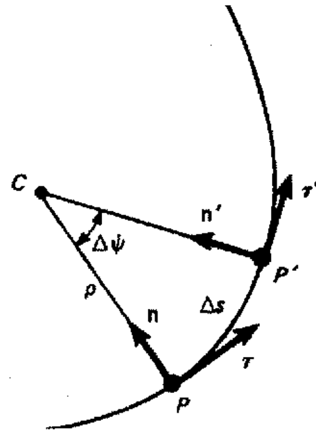
$$\mathbf{v} = v\boldsymbol{\tau}$$

the vector $\boldsymbol{\tau}$ is called the **unit tangent vector**. As the particle moves the speed v may change and the direction of $\boldsymbol{\tau}$ may change. By using the differentiation of product of scalar and vector to obtain acceleration vector the result is:

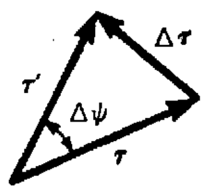
$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d(v\boldsymbol{\tau})}{dt} = \dot{v}\boldsymbol{\tau} + v\,\frac{d\boldsymbol{\tau}}{dt} \quad \ldots\ldots\ldots.(1)$$

The unit vector $\boldsymbol{\tau}$ being of constant magnitude, has a derivative $\frac{d\boldsymbol{\tau}}{dt}$ express the change in direction of $\boldsymbol{\tau}$ with respect to time.

The particle moves a distance Δs along the path from the initial point P to another point P' at time interval Δt. Let denote the unit tangent vectors at P and P' by $\boldsymbol{\tau}$ and $\boldsymbol{\tau}'$, respectively as shown in the figure.



The directions of these two units vectors differ by angle $\Delta\psi$ as shown in figure, for small value of $\Delta\psi$, the difference $\Delta\boldsymbol{\tau}$ approaches $\Delta\psi$ in magnitude ($\Delta\tau \rightarrow \Delta\psi$) and the direction of $\Delta\boldsymbol{\tau}$ become perpendicular to the direction of $\boldsymbol{\tau}$ ($\Delta\boldsymbol{\tau} \perp \boldsymbol{\tau}$) when $\Delta\psi$ and $\Delta s \rightarrow$ zero. The derivative $\frac{d\boldsymbol{\tau}}{d\psi}$ is of magnitude unity ($\left|\frac{d\boldsymbol{\tau}}{d\psi}\right| = 1$) and is perpendicular to $\boldsymbol{\tau}$ ($\frac{d\boldsymbol{\tau}}{d\psi} \perp \boldsymbol{\tau}$)



So that we call it the unit normal vector and denote it by **n**:

$$\frac{d\boldsymbol{\tau}}{d\psi} = \mathbf{n}$$

To find the time derivative $\frac{d\tau}{dt}$, we use the chain rule as follows:

$$\frac{d\tau}{dt} = \frac{d\tau}{d\psi}\frac{d\psi}{dt} = \mathbf{n}\ \frac{d\psi}{ds}\frac{ds}{dt} = \mathbf{n}\ \frac{v}{\rho}$$

in which $\rho = \frac{ds}{d\psi}$ is the radius of curvature of the path of moving particle at P.

Substitute the value of $\frac{d\tau}{dt}$ into equation (1) to find:

$$\mathbf{a} = \dot{v}\boldsymbol{\tau} + \frac{v^2}{\rho}\,\mathbf{n}$$

Thus we can note that the acceleration of a moving paricle has two components:

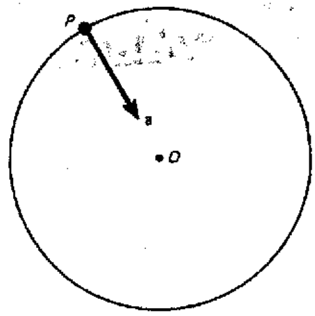
1- Tangential component (in the direction of motion) $a_\tau = \dot{v} = \ddot{s}$

2- Normal component, also called centripetal acceleration (directed toword the center of curvature on the concave side of the path motion $a_n = \frac{v^2}{\rho}$

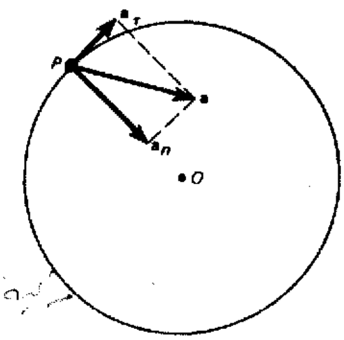
The magnitude of the total acceleration is given by:

$$|\mathbf{a}| = \left|\frac{d\mathbf{v}}{dt}\right| = \left(\dot{v}^2 + \frac{v^4}{\rho^2}\right)^{1/2}$$

If the particle moves on a circle with constant speed then the acceleration vector is of magnitude v^2/R_0, where R_0 is the raduis of the circle.The acceleration vector is always points to the center in this case as shown in figure.



However if the speed is not constant but increase at a certain rate $\dot{v}$ then the acceleration has a forward component and is slanted تنحرف away مبتعدة from the center of circle towards the direction as shown in figure. What happen if the speed decrease at a certain rate $\dot{v}$?



1-14: Velocity and Acceleration in Plane Polar Coordinates:

It is often convenient to employ polar coordinates r, θ to express the position of a particle moving in a plane. The position of the particle can be written as the product of the radial distance r by a unit radial vector $\mathbf{e}_r$:

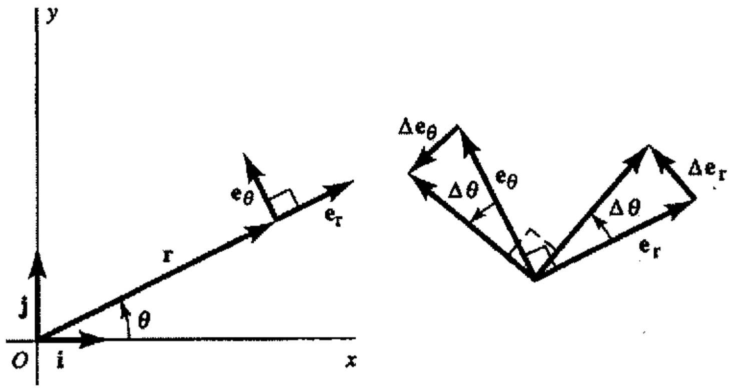
$$\mathbf{r} = r\mathbf{e}_r$$

As the particle moves, both r and $\mathbf{e}_r$ vary; thus, they are both functions of the time.

Hence, if we differentiate with respect to t, we have

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \dot{r}\mathbf{e}_r + r\frac{d\mathbf{e}_r}{dt} \qquad \text{---------- } 1$$

To calculate the derivative $d\mathbf{e}_r/dt$, let us consider the vector diagram shown in Figure.



When the direction of **r** changes by an amount Δθ the corresponding change $\Delta\mathbf{e}_r$ of the unit radial vector is as follows:

The magnitude $|\Delta\mathbf{e}_r|$ is approximately equal to Δθ ($|\Delta\mathbf{e}_r| \approx \Delta\theta$) and the direction of $\Delta\mathbf{e}_r$ is very nearly perpendicular to $\mathbf{e}_r$ ($\Delta\mathbf{e}_r \perp \mathbf{e}_r$). Let us introduce another unit vector, e_θ, whose direction is perpendicular to $\mathbf{e}_r$. Then we have

$$\Delta\mathbf{e}_r \approx \mathbf{e}_\theta\Delta\theta$$

If we divide by Δt and take the limit as $\Delta t \to 0$, we get

$$\frac{d\mathbf{e}_r}{dt} = \mathbf{e}_\theta\frac{d\theta}{dt} = \dot{\theta}\mathbf{e}_\theta \qquad \text{-------- } 2$$

For the time derivative of the unit radial vector, we can argue in similar way that the change in the unit vector $\mathbf{e}_\theta$ is given by:

$$\Delta \mathbf{e}_\theta \approx -\mathbf{e}_r \Delta\theta$$

Here the minus sign is inserted to indicate that the direction of the change is opposite to the direction of $\mathbf{e}_r$, as shown in Figure. Consequently, the time derivative is given by

$$\frac{d\mathbf{e}_\theta}{dt} = -\mathbf{e}_r \frac{d\theta}{dt} = -\mathbf{e}_r\dot{\theta} \qquad \text{-------- 3}$$

By using Equation 2 for the derivative of the unit radial vector, we can finally write the equation for the velocity as:

$$\mathbf{v} = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta \qquad \text{--------- 4}$$

Thus, $\dot{r}$ is the radial القطبية component المركبة of the velocity vector, and $r\dot{\theta}$ is the transverse المستعرضة component.

To find the acceleration vector, we take the derivative of the velocity with respect to time. This gives

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \ddot{r}\mathbf{e}_r + \dot{r}\frac{d\mathbf{e}_r}{dt} + (\dot{r}\dot{\theta} + r\ddot{\theta})\mathbf{e}_\theta + r\dot{\theta}\frac{d\mathbf{e}_\theta}{dt}$$

The values of $d\mathbf{e}_r/dt$ and $d\mathbf{e}_\theta/dt$ are given by Equations 2 and 3 and yield the following equation for the acceleration vector in plane polar coordinates:

$$\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\mathbf{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\mathbf{e}_\theta \qquad \text{----------5}$$

Thus, the radial component of the acceleration vector is

$$a_r = \ddot{r} - r\dot{\theta}^2$$

and the transverse component is:

$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} = \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})$$

The above results show, for instance, that if a particle moves on a circle of constant radius b, so that $\dot{r} = 0$, then the radial component of the acceleration is of magnitude $b\dot{\theta}^2$ and is directed inward الداخل toward the center of the circular path.

The transverse component in this case is $b\varepsilon$. On the other hand, if the particle moves along a fixed radial line, that is, if θ is constant, then the radial component is just $\ddot{r}$ and the transverse component is zero. If **r** and θ both vary, then the general expression (6) gives the acceleration.

Example11:

A honey bee^(نحل العسل) hones^(بيوت) in on its hive^(خلية) in a spiral^(لولبي) path^(مسار) in such a way that the radial distance decreases^(يتناقص) at a constant rate, r = b — Ct, while the angular speed increases^(يتزايد) at a constant rate, $\dot{\theta}$ = kt, Find the speed as a function of time.

Solution:

We have $\dot{r}$ = -c and $\ddot{r}$ =0. Thus, from Equation 5,

$$\mathbf{v} = -c\mathbf{e}_r + (b - ct)kt\mathbf{e}_\theta$$

So

$$v = [c^2 + (b-ct)^2k^2t^2]^{1/2}$$

which is valid for t ≤ b/c. Note that v= c both for t = 0, r = b and for t=b/c ,r=0.

Example12

On a horizontal turntable^(قرص دوار) that is rotating at constant angular speed, a bug^(بق) is crawling^(الزحف) outward^(نحو الخارج) on a radial line such that its distance from the center increases quadratically^(تربيعيا) with time: **r** = bt², θ= ωt, where b and ω are constants. Find the acceleration of the bug.

Solution:

We have $\dot{r}$= 2bt, $\ddot{r}$ = 2b, $\dot{\theta}$ = ω, $\ddot{\theta}$ = 0. Substituting into Equation 5, we find

$$\begin{aligned}\mathbf{a} &= \mathbf{e}_r(2b - bt^2\omega^2) + \mathbf{e}_\theta[0 + 2(2bt)\omega] \\ &= b(2 - t^2\omega^2)\mathbf{e}_r + 4b\omega t\mathbf{e}_\theta\end{aligned}$$

Note that the radial component of the acceleration becomes negative for large t in this example, although the radius is always increasing monotonically with time.

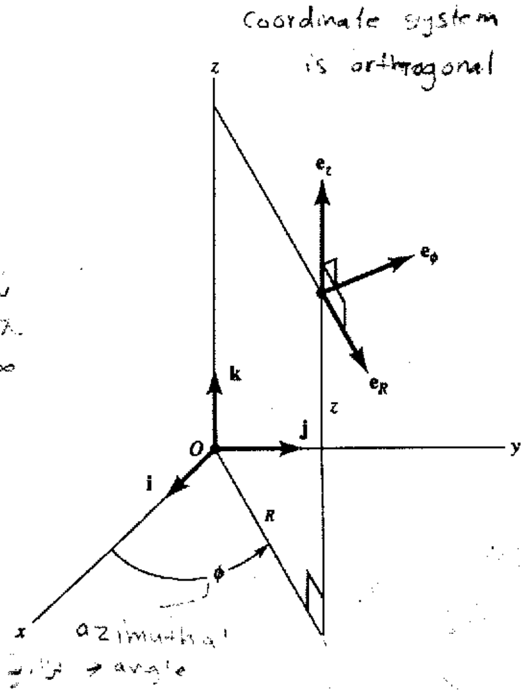
1-15:Velocity and Acceleration in Cylindrical Coordinates:

In the case of three-dimensional motion, the position of a particle can be described in cylindrical coordinates R, ϕ, z. The position vector is then written as:

$$\mathbf{r} = R\mathbf{e}_R + Z\mathbf{e}_z \quad \text{---------------- } 1$$

where e_R is a unit radial (نصف قطري) vector in the xy plane and $\mathbf{e}_z$ is the unit vector in the z direction. A third unit vector e_ϕ is needed so that the three vectors $\mathbf{e}_R\ \mathbf{e}_z\ \mathbf{e}_\phi$ constitute (انشاء) a right-handed triad, as illustrated in Figure. We note that $k = \mathbf{e}_z$.

The velocity and acceleration vectors are found by differentiating, as before. This again involves derivatives of the unit vectors. An argument similar to that used for the plane case shows that



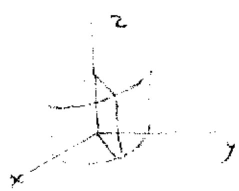
$$\frac{d\hat{e}_R}{dt} = \dot{\phi}\,\hat{e}_\phi \quad and \quad \frac{d\hat{e}_\phi}{dt} = -\dot{\phi}\,\hat{e}_R$$

And the unit vector $\hat{e}_z$ does not change in direction so

$$\frac{d\hat{e}_z}{dt} = 0$$

$$\vec{v} = \frac{d\vec{r}}{dt} = \dot{R}\hat{e}_R + R\frac{d\hat{e}_R}{dt} + \dot{z}\hat{e}_z + z\frac{d\hat{e}_z}{dt}$$

Subsitituting for the time derivative, we get



$$\vec{v} = \dot{R}\hat{e}_R + R\dot{\phi}\hat{e}_\phi + \dot{z}\hat{e}_z \quad \text{------------- } 2$$

$$\vec{a} = \ddot{R}\hat{e}_R + \dot{R}\frac{d\hat{e}_R}{dt} + (\dot{R}\dot{\phi} + R\ddot{\phi})\hat{e}_\phi + R\dot{\phi}\frac{d\hat{e}_\phi}{dt} + \ddot{z}\,\hat{e}_z$$

Subsitituting for the time derivative, we get

$$\vec{a} = \ddot{R}\hat{e}_R + \dot{R}\,\dot{\phi}\hat{e}_\phi + \dot{R}\dot{\phi}\hat{e}_\phi + R\ddot{\phi}\hat{e}_\phi - R\dot{\phi}^2\hat{e}_R + \ddot{z}\,\hat{e}_z$$

$$\vec{a} = (\ddot{R} - R\dot{\phi}^2)\hat{e}_R + (2\dot{R}\dot{\phi} + R\ddot{\phi})\hat{e}_\phi + \ddot{z}\,\hat{e}_z \quad \text{-------------- } 3$$

The derivatives of the unit vectors is to differentiate the following equations, which are the relationships between the fixed unit triad i ,j ,k and the rotated triad $\hat{e}_R, \hat{e}_\phi, \hat{e}_z$:

$$\hat{e}_R = \hat{i}\cos\phi + \hat{j}\sin\phi$$

$$\hat{e}_\phi = -\hat{i}\sin\phi + \hat{j}\cos\phi$$

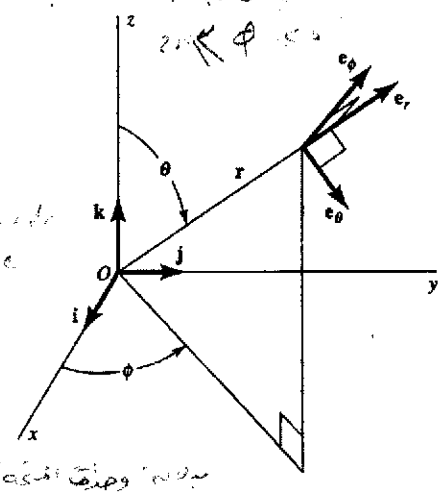
$$\hat{e}_z = \hat{k}$$

The steps are left as an exercise. The result can also be found by use of the rotation matrix.

1-16: Velocity and Acceleration in Spherical Coordinates:

When spherical coordinates r, θ, ϕ are employed to describe the position of a particle, the position vector is written as the product of the radial distance *r* and the unit radial vector $\mathbf{e}_r$, as with plane polar coordinates. Thus,

$$\vec{r} = r\,\hat{e}_r \text{ ---------------- } 1$$



The direction of $\mathbf{e}_r$ is now specified by the two angles θ and ϕ. We introduce two more unit vectors, $\mathbf{e}_\theta$ and $\mathbf{e}_\phi$, as shown in Figure.

The velocity is:

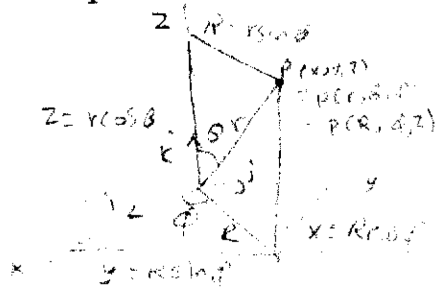
$$\vec{v} = \frac{d\vec{r}}{dt} = \dot{r}\hat{e}_r + r\,\frac{d\hat{e}_r}{dt} \text{ ------------} 2$$

Our next problem is how to express the derivative $\frac{d\hat{e}_r}{dt}$ in term of the unit vectors in the rotated triad. We can derive relationships between $\hat{\imath}, \hat{\jmath}, \hat{k}$ and $\hat{e}_r, \hat{e}_\theta, \hat{e}_\phi$. By expressing any unit vector in terms of its projections on the x,y,z coordinate axes. as an example

$$\hat{e}_r = \hat{i}(\hat{e}_r\cdot\hat{i}) + \hat{j}(\hat{e}_r\cdot\hat{j}) + \hat{k}(\hat{e}_r\cdot\hat{k})$$

$\hat{e}_r \cdot \hat{\imath}$ is the projection of the unit vector $\hat{e}_r$ directly onto the unit vector $\hat{\imath}$ and is equal to cos α, (see page 8) the cosine of the angle between those two unit vectors.

We need to express these dot products in terms of θ and ϕ not α. We can obtain desired relation by making two successive projections to get to the x-axis. First project $\hat{e}_r$ onto the xy plane, and then project from there onto the x-axis. The first projection gives us a factor of sin θ, while the second yields a factor of cos ϕ. The magnitude of the projection obtained in this way is the desired dot product:



$$\hat{e}_r \cdot \hat{i} = \sin\theta\cos\phi, \quad \hat{e}_r \cdot \hat{j} = \sin\theta\sin\phi$$

$$and \qquad \hat{e}_r \cdot \hat{k} = \cos\theta$$

In a similar way the relationship for $\hat{e}_\theta$ and $\hat{e}_\phi$ can be opbtained yailding the fellowing relation:

$$\left.\begin{aligned} \hat{e}_r &= \hat{i}\sin\theta\cos\phi + \hat{j}\sin\theta\sin\phi + \hat{k}\cos\theta \\ \hat{e}_\theta &= \hat{i}\cos\theta\cos\phi + \hat{j}\cos\theta\sin\phi - \hat{k}\sin\theta \\ \hat{e}_\phi &= -\hat{i}\sin\phi + \hat{j}\cos\phi \end{aligned}\right| \quad \text{--------- 3}$$

which express the unit vectors of the rotated triad in terms of the fixed triad **i,j,k**.

Let us differentiate the first equation with respect to time. The result is:

$$\frac{d\hat{e}_r}{dt} = \hat{i}(\dot{\theta}\cos\theta\cos\phi - \dot{\phi}\sin\theta\sin\phi) + \hat{j}(\dot{\theta}\cos\theta\sin\phi + \dot{\phi}\sin\theta\cos\phi) - \hat{k}\dot{\theta}\sin\theta$$

We find that the above equation reduces to:

$$\frac{d\hat{e}_r}{dt} = \hat{e}_\phi\dot{\phi}\sin\theta + \hat{e}_\theta\dot{\theta}$$

The other two derivatives are found through a similar procedure. The results are

$$\frac{d\hat{e}_\theta}{dt} = -\hat{e}_r\dot{\theta} + \hat{e}_\phi\dot{\phi}\cos\theta \qquad \text{and} \qquad \frac{d\hat{e}_\phi}{dt} = -\hat{e}_r\dot{\phi}\sin\theta - \hat{e}_\theta\dot{\phi}\cos\theta \qquad \text{(H.w)}$$

Finally substituting eqs. ($\frac{d\hat{e}_r}{dt}$, $\frac{d\hat{e}_\theta}{dt}$, $\frac{d\hat{e}_\phi}{dt}$) in eq.(2) we obtain:

$$\vec{v} = \hat{e}_r \dot{r} + \hat{e}_\phi r\dot{\phi}\sin\theta + \hat{e}_\theta r\dot{\theta} \ ) \qquad \text{------------- 4}$$

To find the acceleration, we differentiate eq. 4 with respect to time. This gives:

$$\vec{a} = \frac{dv}{dt}$$

$$\vec{a} = \hat{e}_r\ddot{r} + \dot{r}\frac{d\hat{e}_r}{dt} + \hat{e}_\phi \frac{d(r\dot{\phi}\sin\theta)}{dt} + r\dot{\phi}\sin\theta\frac{d\hat{e}_\phi}{dt} + \hat{e}_\theta\frac{d(r\dot{\theta})}{dt} + r\dot{\theta}\frac{d\hat{e}_\theta}{dt}$$

Substituting for the unit vectors derivatives ($\frac{d\hat{e}_r}{dt}$, $\frac{d\hat{e}_\theta}{dt}$, $\frac{d\hat{e}_\phi}{dt}$) we obtain:

$$\begin{aligned}\vec{a} &= (\ddot{r} - r\dot{\phi}^2\sin^2\theta - r\dot{\theta}^2)\hat{e}_r \\ &+ (r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2\sin\theta\cos\theta)\hat{e}_\theta \\ &+ (r\ddot{\phi}\sin\theta + 2\dot{r}\dot{\phi}\sin\theta + 2r\dot{\theta}\dot{\phi}\cos\theta)\hat{e}_\phi\end{aligned} \qquad \text{----------- 5}$$

giving the acceleration vector in terms of its components in the triad $\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi$

Example13:

A bead[خرزة] slides[تنزلق] on a wire bent[ينحني] into the form[شكل] of a helix[حلزوني], the motion of the bead being given in cylindrical coordinates by R = b, ϕ= ωt, z =ct. Find the velocity and acceleration vectors as functions of time.

Solution:

$\vec{v} = \dot{R}\,\hat{e}_R + R\dot{\phi}\,\hat{e}_\phi + \dot{z}\,\hat{e}_z$

$\vec{a} = (\ddot{R} - R\dot{\phi}^2)\hat{e}_R + (2\dot{R}\dot{\phi} + R\ddot{\phi})\hat{e}_\phi + \ddot{z}\,\hat{e}_z$

Differentiating, we find $\dot{R} = \ddot{R} = 0, \dot{\phi} = \omega, \ddot{\phi} = 0, \dot{z} = c, \ddot{z} = 0$. So, from equation for velocity and acceleration in cylindrical coordinates , we have:

$$\vec{v} = b\omega\hat{e}_\phi + c\hat{e}_z$$

$$\vec{a} = -b\omega^2\hat{e}_R$$

1-17: Gradient , Divergence, and Curl :

The del operater (∇) in rectangular coordinate is given by:

$$\vec{\nabla} = \hat{\imath}\frac{d}{dx} + \hat{\jmath}\frac{d}{dy} + \hat{k}\frac{d}{dz}$$

The del operator is a vector that has no physical meaning or vector direction by itself.

1-17-1: The gradient الانحدار :

Let $\phi(x,y,z)$ be a differentiable at any point, therefore the gradient of ϕ is

$$\vec{\nabla}\phi = \hat{\imath}\frac{d\phi}{dx} + \hat{\jmath}\frac{d\phi}{dy} + \hat{k}\frac{d\phi}{dz}$$

The following computation formulas on gradient, which are easily proved, should be noted:

1- $\nabla(A+B)=\nabla A + \nabla B$

2- $\nabla(AB)=A\nabla B + B\nabla A$

3- $\nabla\left[\frac{A}{B}\right] = \frac{B\,\nabla A - A\nabla\,B}{B^2}$

4- $\nabla\, V^n = n\, V^{n-1}\,\nabla\, V$

1-17-2: The divergence التباعد :

Let

$$\vec{v}(x,y.z) = \hat{\imath}\, v_x + \hat{\jmath}\, v_y + \hat{k}\, v_z$$

Then the divergence of $\vec{v}$ (written div $\vec{v}$ or $\vec{\nabla}.\vec{v}$) is

$$\vec{\nabla}.\vec{v} = \frac{dv_x}{dx} + \frac{dv_y}{dy} + \frac{dv_z}{dz}$$

Note that $\vec{\nabla}.\vec{v} \neq \vec{v}.\vec{\nabla}$

Note the following properties of the divergence of a vector:

1- Its produces a scalar.

2- The divergence of a scalar V, div V make no sense.

3- $\nabla.(\mathbf{A}+\mathbf{B}) = \nabla.\mathbf{A} + \nabla.\mathbf{B}$

4- $\nabla.(V\mathbf{A}) = V\nabla.\mathbf{A} + \mathbf{A} . V\nabla$

1-17-2: The curl الالتفاف**:**

The curl of the vector field $\vec{v}(x, y.z) = \hat{\imath}\, v_x + \hat{\jmath}\, v_y + \hat{k}\, v_z$ is denoted by $\overrightarrow{\text{curl v}}$ or $(\vec{\nabla} \times \vec{v})$ is:

$$\overrightarrow{\text{curl v}} = \vec{\nabla} \times \vec{v} = \begin{vmatrix} i & j & k \\ \frac{d}{d_x} & \frac{d}{d_y} & \frac{d}{d_z} \\ v_x & v_y & v_z \end{vmatrix}$$

Note the following properties of the curl:

1. The curl of a vector is another vector.
2. The curl of a scalar V, $\nabla\times V$, makes no sense.
3. $\nabla \times (\mathbf{A}+\mathbf{B}) = \nabla \times \mathbf{A} + \nabla \times \mathbf{B}$

$(\vec{B}\cdot\vec{\nabla})A - (\vec{\nabla}\cdot\vec{A})B - (A\cdot\nabla)B + A(\nabla\cdot B)$

4. $\nabla \times (\mathbf{A}\times\mathbf{B}) = \mathbf{A}(\nabla . \mathbf{B}) - \mathbf{B}(\nabla . \mathbf{A}) + (\mathbf{B}.\nabla)\mathbf{A} - (\mathbf{A} . \nabla)\mathbf{B}$

6- The divergence of the curl of a vector vanishes,

that is, $\nabla . (\nabla \times \mathbf{A}) = \mathbf{0}$

7. The curl of the gradient of a scalar is vanishes,

that is, $\nabla \times \nabla V = \mathbf{0}$

$$\nabla\cdot(\nabla u) \equiv \nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \Rightarrow \text{laplacian of } u$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \quad \text{laplacian operator}$$

Notice

We can defined $d\vec{r}$ and $\vec{\nabla}$ in the:

1- Cartesian Coordinate:

$$d\vec{r} = \hat{\imath}dx + \hat{\jmath}dy + \hat{k}dz$$

$$\vec{\nabla} = \hat{\imath}\frac{d}{dx} + \hat{\jmath}\frac{d}{dy} + \hat{k}\frac{d}{dz}$$

2- Cylindrical Coordinate:

$$d\vec{r} = \hat{e}_R dR + \hat{e}_\phi R d\phi + \hat{e}_Z dz$$

$$\vec{\nabla} = \hat{e}_R\frac{d}{dR} + \hat{e}_\phi\frac{1}{R}\frac{d}{d\phi} + \hat{e}_Z\frac{d}{dz}$$

3- Spherical coordinate:

$$d\vec{r} = \hat{e}_r dr + \hat{e}_\theta r d\theta + \hat{e}_\phi r\sin\theta\, d\phi$$

$$\vec{\nabla} = \hat{e}_r\frac{d}{dr} + \hat{e}_\theta\frac{1}{r}\frac{d}{d\theta} + \hat{e}_\phi\frac{1}{r\sin\theta}\frac{d}{d\phi}$$

Example 14:

The potential that represents an inverse square force is $V(r) = \frac{k}{r}$, where $r = (x^2 + y^2 + z^2)^{1/2}$. Using the definition $\vec{F} = -\vec{\nabla}v$, calculate the component of this force.

Solution:

$$\vec{F} = -\vec{\nabla}v = -\left(\hat{\imath}\frac{dv_x}{dx} + \hat{\jmath}\frac{dv_y}{dy} + \hat{k}\frac{dv_z}{dz}\right)$$

Since $V(r) = \frac{k}{r}$ and $\frac{dV}{dx} = -\frac{kx}{r^3}$, $\frac{dV}{dy} - \frac{ky}{r^3}$, and $\frac{dV}{dz} = -\frac{kz}{r^3}$ then

$$\vec{F} = \frac{k}{r^3}\left(\hat{\imath}x + \hat{\jmath}y + \hat{k}z\right) = \frac{k}{r^3}\vec{r}$$

Another solution by using spherical coordinate

$$\vec{\nabla}= \hat{e}_r\frac{d}{dr} + \hat{e}_\theta\frac{1}{r}\frac{d}{d\theta} + \hat{e}_\phi\frac{1}{r\sin\theta}\frac{d}{d\phi}$$

$$\vec{F} = -\vec{\nabla}V = -k\hat{e}_r\frac{d}{dr}\left(\frac{1}{r}\right) + 0 + 0$$

$$\vec{F} = -k\hat{e}_r\left(-\frac{1}{r^2}\right)$$

$$\vec{F} = \frac{k}{r^2}\,\hat{e}_r$$

Example 15: If $\varphi = \frac{1}{r}$ where $r = (x^2 + y^2 + z^2)^{1/2}$, show that $\nabla\varphi = \frac{-\boldsymbol{r}}{r^3}$

Solution:

$$\nabla\Phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2 + y^2 + z^2)^{-1/2}$$

$$= \left(-\frac{1}{2}.2x\hat{i} - \frac{1}{2}.2y\hat{j} - \frac{1}{2}.2z\hat{k}\right)(x^2 + y^2 + z^2)^{-3/2}$$

$$= -(x\hat{i} + y\hat{j} + z\hat{k})(x^2 + y^2 + z^2)^{-\frac{3}{2}} = -\frac{\vec{r}}{r^3}$$

Example 16: Find a unit vector normal to the surface $xy^2+xz=1$ at the point $(-1,1,1)$.

Solution:

$$\nabla(xy^2 + xz) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(xy^2 + xz)$$

$$= (y^2 + z)\hat{i} + (2xy)\hat{j} + x\hat{k}$$

$$= 2\hat{i} - 2\hat{j} - \hat{k}, at(-1, 1, 1)$$

A unit vector normal to the surface is obtained by dividing the above vector by its magnitude. Hence the unit vector is

$$(2\hat{i} - 2\hat{j} - \hat{k})[(2)^2 + (-2)^2 + (-1)^2]^{-1/2} = \frac{2}{3}\hat{i} - \frac{2}{3}\hat{j} - \frac{1}{3}\hat{k}$$

Example 17:

(a) Show that F = $(2xy + z^2)\ \hat{\imath} + x^2\ \hat{\jmath} +$ ~~xyz~~ 2xz $\hat{k}$, is a conservative force field.

(b) Find the scalar potential.

(c) Find the work done in moving a unit mass in this field from the point (1, 0, 1) to (2, 1, −1).

Solution: (a) It is sufficient to show that Curl $\boldsymbol{F} = 0$

$$\nabla \times \boldsymbol{F} = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2xy + z^2 & x^2 & 2xz \end{vmatrix} = \hat{\imath}.0 - \hat{\jmath}(2z - 2z) + \hat{k}(2x - 2x) = 0$$

(b) $\mathrm{d}\Phi = F.\,\mathrm{d}\boldsymbol{r} = ((2xy + z^2)\hat{\imath} + x^2\hat{\jmath} + 2xz\hat{k}).(\hat{\imath}\mathrm{d}x + \hat{\jmath}\mathrm{d}y + \hat{k}\mathrm{d}z)$

$= (2xy + z^2)\ \mathrm{d}x + x^2\ \mathrm{d}y + 2xz\mathrm{d}z$

$= (2xy\mathrm{d}x + x^2\ \mathrm{d}y) + (z^2\ \mathrm{d}x + 2xz\mathrm{d}z)$

$= \mathrm{d}(x^2 y) + \mathrm{d}(z^2 x) = \mathrm{d}(x^2 y + xz^2)$

Therefore $\Phi = x^2 y + xz^2 + \text{constant}$

(c) Work done $= \Phi_2 - \Phi_1 = 5.0$

Example 18: If $\phi = x^2yz^3$ and A= $xz\mathbf{i} - y^2\mathbf{j} + 2x^2y\mathbf{k}$, find:

a- $\nabla\phi$ **b-** ∇. **A** **c-** $\nabla\times$ **A** **d-** div (ϕ**A**) **e-** curl (ϕ**A**)

Solution:

(a) $\nabla\phi = \left(\mathbf{i}\dfrac{\partial}{\partial x} + \mathbf{j}\dfrac{\partial}{\partial y} + \mathbf{k}\dfrac{\partial}{\partial z}\right)\phi = \dfrac{\partial\phi}{\partial x}\mathbf{i} + \dfrac{\partial\phi}{\partial y}\mathbf{j} + \dfrac{\partial\phi}{\partial z}\mathbf{k} = \dfrac{\partial}{\partial x}(x^2yz^3)\mathbf{i} + \dfrac{\partial}{\partial y}(x^2yz^3)\mathbf{j} + \dfrac{\partial}{\partial z}(x^2yz^3)\mathbf{k}$

$= 2xyz^3\mathbf{i} + x^2z^3\mathbf{j} + 3x^2yz^2\mathbf{k}$

(b) $\nabla\cdot\mathbf{A} = \left(\mathbf{i}\dfrac{\partial}{\partial x} + \mathbf{j}\dfrac{\partial}{\partial y} + \mathbf{k}\dfrac{\partial}{\partial z}\right)\cdot(xz\mathbf{i} - y^2\mathbf{j} + 2x^2y\mathbf{k})$

$= \dfrac{\partial}{\partial x}(xz) + \dfrac{\partial}{\partial y}(-y^2) + \dfrac{\partial}{\partial z}(2x^2y) = z - 2y$

(c) $\nabla\times\mathbf{A} = \left(\mathbf{i}\dfrac{\partial}{\partial x} + \mathbf{j}\dfrac{\partial}{\partial y} + \mathbf{k}\dfrac{\partial}{\partial z}\right)\times(xz\mathbf{i} - y^2\mathbf{j} + 2x^2y\mathbf{k})$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xz & -y^2 & 2x^2y \end{vmatrix}$$

$= \left(\dfrac{\partial}{\partial y}(2x^2y) - \dfrac{\partial}{\partial z}(-y^2)\right)\mathbf{i} + \left(\dfrac{\partial}{\partial z}(xz) - \dfrac{\partial}{\partial x}(2x^2y)\right)\mathbf{j} + \left(\dfrac{\partial}{\partial x}(-y^2) - \dfrac{\partial}{\partial y}(xz)\right)\mathbf{k}$

$= 2x^2\mathbf{i} + (x - 4xy)\mathbf{j}$

(*d*) $\text{div}\,(\phi\mathbf{A}) = \nabla\cdot(\phi\mathbf{A}) = \nabla\cdot(x^3yz^4\mathbf{i} - x^2y^3z^3\mathbf{j} + 2x^4y^2z^3\mathbf{k})$

$$= \frac{\partial}{\partial x}(x^3yz^4) + \frac{\partial}{\partial y}(-x^2y^3z^3) + \frac{\partial}{\partial z}(2x^4y^2z^3)$$

$$= 3x^2yz^4 - 3x^2y^2z^3 + 6x^4y^2z^2$$

(*e*) $\text{curl}\,(\phi\mathbf{A}) = \nabla\times(\phi\mathbf{A}) = \nabla\times(x^3yz^4\mathbf{i} - x^2y^3z^3\mathbf{j} + 2x^4y^2z^3\mathbf{k})$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^3yz^4 & -x^2y^3z^3 & 2x^4y^2z^3 \end{vmatrix}$$

$$= (4x^4yz^3 - 3x^2y^3z^2)\mathbf{i} + (4x^3yz^3 - 8x^3y^2z^3)\mathbf{j} - (2xy^3z^3 + x^3z^4)\mathbf{k}$$

Example 19: Prove $\nabla\cdot(\phi\mathbf{A}) = (\nabla\phi)\cdot\mathbf{A} + \phi(\nabla\cdot\mathbf{A})$.

Solution:

$$\nabla\cdot(\phi\mathbf{A}) = \nabla\cdot(\phi A_1\mathbf{i} + \phi A_2\mathbf{j} + \phi A_3\mathbf{k})$$

$$= \frac{\partial}{\partial x}(\phi A_1) + \frac{\partial}{\partial y}(\phi A_2) + \frac{\partial}{\partial z}(\phi A_3)$$

$$= \frac{\partial\phi}{\partial x}A_1 + \frac{\partial\phi}{\partial y}A_2 + \frac{\partial\phi}{\partial z}A_3 + \phi\left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right)$$

$$= \left(\frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}\right)\cdot(A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k})$$

$$+ \phi\left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)\cdot(A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k})$$

$$= (\nabla\phi)\cdot\mathbf{A} + \phi(\nabla\cdot\mathbf{A})$$

Example 20: prove div curl **A** = 0

Solution:

$$\text{div curl}\,\mathbf{A} = \nabla\cdot(\nabla\times\mathbf{A}) = \nabla\cdot\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_1 & A_2 & A_3 \end{vmatrix}$$

$$= \nabla\cdot\left[\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)\mathbf{k}\right]$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)$$

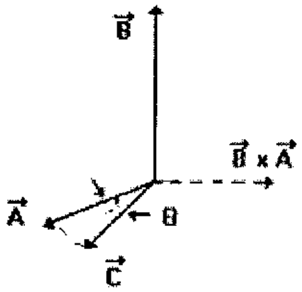
$$= \frac{\partial^2 A_3}{\partial x\,\partial y} - \frac{\partial^2 A_2}{\partial x\,\partial z} + \frac{\partial^2 A_1}{\partial y\,\partial z} - \frac{\partial^2 A_3}{\partial y\,\partial x} + \frac{\partial^2 A_2}{\partial z\,\partial x} - \frac{\partial^2 A_1}{\partial z\,\partial y}$$

$$= 0$$

Problems of chapter one

1- **Assume that two vectors A and B are known. Let C be an unknown vector such that A · C = *u* is a known quantity and A × C = B. Find C in terms of A, B, *u*, and the magnitude of A.**

Solution:



$$B=|\vec{B}|=|\vec{A}\times\vec{C}|=AC\sin\theta \qquad \therefore C_y=C\sin\theta=\frac{B}{A}$$

$$\vec{A}\cdot\vec{C}=AC\cos\theta=u \qquad \therefore C_x=C\cos\theta=\frac{u}{A}$$

$$\vec{C}=\frac{\vec{A}}{A}C_x+\frac{\vec{B}\times\vec{A}}{|\vec{B}\times\vec{A}|}C_y=\frac{u}{A^2}\vec{A}+\frac{\vec{B}\times\vec{A}}{AB}\left(\frac{B}{A}\right)$$

$$=\frac{u}{A^2}\vec{A}+\frac{1}{A^2}\vec{B}\times\vec{A}$$

2- Find a unit vector normal to the plane containing the two vectors

$$\vec{A}=2\hat{i}+\hat{j}-\hat{k} \qquad and \qquad \vec{B}=\hat{i}-\hat{j}+2\hat{k}$$

Solution:

$$\vec{A}\times\vec{B}=\begin{vmatrix}\hat{i} & -\hat{j} & \hat{k}\\ 2 & 1 & -1\\ 1 & -1 & 2\end{vmatrix}=\hat{i}-5\hat{j}-3\hat{k} \text{ and } |\vec{A}\times\vec{B}|=\sqrt{1^2+(-5)^2+(-3)^2}=\sqrt{35}$$

$$\because \hat{n}=\frac{\vec{A}\times\vec{B}}{|\vec{A}\times\vec{B}|} \qquad \left(unit\ vector=\frac{the\ vector}{its\ magnitude}\right)$$

$$\therefore \hat{n}=\frac{1}{\sqrt{35}}(\hat{i}-5\hat{j}-3\hat{k})$$

3- A racing car moves on a circle of constant radius b. If the speed of the car varies with time *t* according to the equation $v=ct$ where c is a positive constant, show that the angle between the velocity vector and the acceleration vector is 45° at time t = $\sqrt{b/c}$ (Hint: *At this time the **tangential** and **normal** components of the acceleration **are equal in magnitude**.*)

Solution:

$$\mathbf{v}=v\boldsymbol{\tau}=ct\boldsymbol{\tau}$$

$$\mathbf{a}=\dot{v}\boldsymbol{\tau}+\frac{v^2}{\rho}\mathbf{n}=c\,\boldsymbol{\tau}+\frac{c^2t^2}{b}\mathbf{n}$$

at $t=\sqrt{\frac{b}{c}}$. $\vec{v}=\hat{\tau}\sqrt{bc}$ and $\vec{a}=c\hat{\tau}+c\hat{n}$

$$\cos\theta=\frac{\vec{v}\cdot\vec{a}}{va}=\frac{c\sqrt{bc}}{\sqrt{bc}\sqrt{2c^2}}=\frac{1}{\sqrt{2}}$$

$$\theta=45^\circ$$

4- A small ball is fastened to a long rubber band and twirled around in such away that the ball moves in an elliptical path given by the equation:

$$\mathbf{r}(t)=\mathbf{i}b\cos\omega t+\mathbf{j}2b\sin\omega t$$

where b and ω are constants. Find the speed of the ball as a function of *t*. In particular, find v at t =0 and at t = π /2ω, at which times the ball is, respectively, at its minimum and maximum distances from the origin.

Solution: $\vec{v}(t)=-\hat{i}b\omega\sin(\omega t)+\hat{j}2b\omega\cos(\omega t)$

$$|\vec{v}|=\left(b^2\omega^2\sin^2\omega t+4b^2\omega^2\cos^2\omega t\right)^{\frac{1}{2}}=b\omega\left(1+3\cos^2\omega t\right)^{\frac{1}{2}}$$

$$\vec{a}(t)=-\hat{i}b\omega^2\cos\omega t-\hat{j}2b\omega^2\sin\omega t$$

$$|\vec{a}|=b\omega^2\left(1+3\sin^2\omega t\right)^{\frac{1}{2}}$$

at $t=0$, $|\vec{v}|=2b\omega$; at $t=\frac{\pi}{2\omega}$, $|\vec{v}|=b\omega$

5- A bee goes out from its hive in a spiral path given in plane polar coordinates by $r=be^{kt}$, $\theta=ct$ where b, k, and c are positive constants. Show that the angle between the velocity vector and the acceleration vector remains constant as the bee moves outward. (Hint: Find **v. a**/va)

Solution: $\vec{v}=\dot{r}\hat{e}_r+r\dot{\theta}\hat{e}_\theta=bke^{kt}\hat{e}_r+bce^{kt}\hat{e}_\theta$

$$\vec{a}=\left(\ddot{r}-r\dot{\theta}^2\right)\hat{e}_r+\left(r\ddot{\theta}+2\dot{r}\dot{\theta}\right)\hat{e}_\theta=b\left(k^2-c^2\right)e^{kt}\hat{e}_r+2bcke^{kt}\hat{e}_\theta$$

$$\cos\phi=\frac{\vec{v}\cdot\vec{a}}{va}=\frac{b^2k\left(k^2-c^2\right)e^{2kt}+2b^2c^2ke^{2kt}}{be^{kt}\left(k^2+c^2\right)^{\frac{1}{2}}be^{kt}\left[\left(k^2-c^2\right)^2+4c^2k^2\right]^{\frac{1}{2}}}$$

$$\cos\phi=\frac{k\left(k^2+c^2\right)}{\left(k^2+c^2\right)^{\frac{1}{2}}\left(k^2+c^2\right)}=\frac{k}{\left(k^2+c^2\right)^{\frac{1}{2}}},\quad \text{a constant}$$

6- An ant crawls on the surface of a ball of radius b in such a manner that the ant's motion is given in spherical coordinates by the equations

$$r = b \qquad \phi = \omega t \qquad \theta = \frac{\pi}{2}\left[1 + \frac{1}{4}\cos(4\omega t)\right]$$

Find the speed of the ant as a function of the lime t. What sort of path is represented by the above equations?

Solution:

$$\vec{v} = \hat{e}_r \dot{r} + \hat{e}_\phi r\dot{\phi}\sin\theta + \hat{e}_\theta r\dot{\theta}$$

$$\vec{v} = \hat{e}_\phi b\omega \sin\left\{\frac{\pi}{2}\left[1 + \frac{1}{4}\cos(4\omega t)\right]\right\} - \hat{e}_\theta b\frac{\pi}{2}\omega\sin(4\omega t)$$

$$\vec{v} = \hat{e}_\phi b\omega \cos\left[\frac{\pi}{8}\cos(4\omega t)\right] - \hat{e}_\theta b\omega\frac{\pi}{2}\sin(4\omega t)$$

$$|\vec{v}| = b\omega\left[\cos^2\left(\frac{\pi}{8}\cos 4\omega t\right) + \frac{\pi^2}{4}\sin^2 4\omega t\right]^{\frac{1}{2}}$$

Path is sinusoidal oscillation about the equator.

7- Prove that **v. a** = $v\dot{v}$ and, hence, that for a moving particle **v** and **a** are perpendicular to each other if the speed v is constant. (Hint: *Differentiate both sides of the equation* $v \bullet v = v^2$ *with respect to t. Note,* $\dot{v}$ *is not the same as* $|a|$ *I. It is the magnitude of the acceleration of the partied along its instantaneous direction of motion.*)

Solution:

$$\vec{v}\cdot\vec{v} = v^2$$

$$\frac{d\vec{v}}{dt}\cdot\vec{v} + \vec{v}\cdot\frac{d\vec{v}}{dt} = 2v\dot{v}$$

$$2\vec{v}\cdot\vec{a} = 2v\dot{v}$$

$$\vec{v}\cdot\vec{a} = v\dot{v}$$

8- Show that the tangential component of The acceleration of a moving particle is given by the expression:

$$a_\tau = \frac{\mathbf{v} \cdot \mathbf{a}}{v}$$

and the normal component is therefore $a_n = \left(a^2 - a_\tau^2\right)^{1/2} = \left[a^2 - \frac{(\mathbf{v} \cdot \mathbf{a})^2}{v^2}\right]^{1/2}$

solution:

$$\vec{v} = v\hat{\tau} \text{ and } \vec{a} = a_\tau \hat{\tau} + a_n \hat{n}$$

$$\vec{v} \cdot \vec{a} = v a_\tau, \text{ so } a_\tau = \frac{\vec{v} \cdot \vec{a}}{v}$$

$$a^2 = a_\tau^2 + a_n^2, \text{ so } a_n = \left(a^2 - a_\tau^2\right)^{\frac{1}{2}}$$

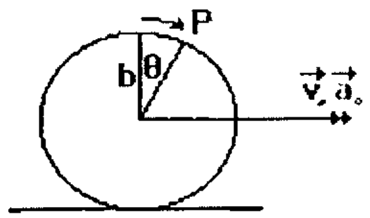
9- A wheel of radius b rolls along the ground with constant forward acceleration a_o. Show that, at any given instant, the magnitude of the acceleration of any point on the wheel is $(a_o^2 + v^4/b^2)^{1/2}$ relative to the center of the wheel and is also $a_o[2 + 2cos\theta + v^4/a_o^2 b^2 - (2v^2/a_o b)sin\theta]^{1/2}$ relative to the ground. Here v is the instantaneous forward speed, and θ defines the location of the point on the wheel, measured forward from the highest point. Which point has the greatest acceleration relative to the ground?

Solution

$$\vec{r}_{oP} = \hat{i}b\sin\theta + \hat{j}b\cos\theta$$

$$\vec{v}_{rel} = \hat{i}b\dot{\theta}\cos\theta - \hat{j}b\dot{\theta}\sin\theta$$

$$\vec{a}_{rel} = \hat{i}b\left(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta\right) - \hat{j}b\left(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta\right)$$



at the point $\theta = \frac{\pi}{2}$, $\vec{v}_{rel} = -\vec{v}$

So, $|\vec{v}_{rel}| = b\dot{\theta} = v$

$$\dot{\theta} = \frac{v}{b} \qquad \ddot{\theta} = \frac{\dot{v}}{b} = \frac{a_o}{b}$$

Now, $\vec{a}_{rel} = \dot{v}_{rel}\hat{\tau} + \frac{v_{rel}^2}{\rho}\hat{n} = a_o\hat{\tau} + \frac{v^2}{b}\hat{n}$

$$|\vec{a}_{rel}| = \left(a_o^2 + \frac{v^4}{b^2}\right)^{\frac{1}{2}}$$

$\vec{v}_P = \vec{v} + \vec{v}_{rel}$ and $\vec{a}_P = \vec{a}_o + \vec{a}_{rel}$

$$\vec{a}_P = \hat{i}\left[a_o + b\left(\frac{a_o}{b}\cos\theta - \frac{v^2}{b^2}\sin\theta\right)\right] - \hat{j}b\left(\frac{a_o}{b}\sin\theta + \frac{v^2}{b^2}\cos\theta\right)$$

$$|\vec{a}_P| = a_o\left(2 + 2\cos\theta + \frac{v^4}{a_o^2 b^2} - \frac{2v^2}{a_o b}\sin\theta\right)^{\frac{1}{2}}$$

$\vec{a}_P$ is a maximum at $\theta = 0$, i.e., at the top of the wheel.

$$-2\sin\theta - \frac{2v^2}{a_o b}\cos\theta = 0$$

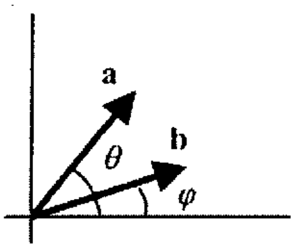
$$\theta = \tan^{-1}\left(-\frac{v^2}{a_o b}\right)$$

10- Use vector algebra to derive the following trigonometric identities

(a) $\cos(\theta - \phi) = \cos\theta\cos\phi + \sin\theta\sin\phi$

(b) $\sin(\theta - \phi) = \sin\theta\cos\phi - \cos\theta\sin\phi$

Solution:



(a) $a = \hat{i}\cos\theta + \hat{j}\sin\theta$

$b = \hat{i}\cos\varphi + \hat{j}\sin\varphi$

$a \cdot b = \cos(\theta - \varphi) = \left(\hat{i}\cos\theta + \hat{j}\sin\theta\right)\cdot\left(\hat{i}\cos\varphi + \hat{j}\sin\varphi\right)$

$\cos(\theta - \varphi) = \cos\theta\cos\varphi + \sin\theta\sin\varphi$

(b) $b \times a = |\hat{k}|\sin(\theta - \varphi) = \left|\left(\hat{i}\cos\theta + \hat{j}\sin\theta\right)\times\left(\hat{i}\cos\varphi + \hat{j}\sin\varphi\right)\right|$

$\sin(\theta - \varphi) = \sin\theta\cos\varphi - \cos\theta\sin\varphi$

Problems (homework)

1- Consider a cube whose edges are each of unit length. One corner coincides with the origin of an xyz Cartesian coordinate system. Three of the cube's edges extend from the origin along the positive direction of each coordinate axis. Find the vector that begins at the origin and extends
   (a) along a major diagonal of the cube;
   (b) along the diagonal of the lower face of the cube.
   (c) Calling these vectors **A** and **B**, find $\mathbf{C} = \mathbf{A} \times \mathbf{B}$.
   (d) Find the angle between **A** and **B**.

2- Given the time-varying vector

$$\mathbf{A} = \mathbf{i}\alpha t + \mathbf{j}\beta t^2 + \mathbf{k}\gamma t^3$$

where α, β, and γ are constants, find the first and second time derivatives $d\mathbf{A}/dt$ and $d^2\mathbf{A}/dt^2$.

For what value (or values) of q is the vector $\mathbf{A} = \mathbf{i}q + 3\mathbf{j} + \mathbf{k}$ perpendicular to the vector $\mathbf{B} = \mathbf{i}q - q\mathbf{j} + 2\mathbf{k}$?

3- A buzzing fly moves in a helical path given by the equation

$$\mathbf{r}(t) = \mathbf{i}b \sin \omega t + \mathbf{j}b \cos \omega t + \mathbf{k}ct^2$$

Show that the magnitude of the acceleration of the fly is constant, provided b, ω, and c are constant.

4- A particle moves in a helical path such that its position in cylindrical coordinate given by

$$R=b \qquad \phi=\omega t \qquad z=ct^2$$

Find the speed and the magnitude of the acceleration as a function of t.

5- Express the vector $2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$ in the primed triad $\mathbf{i}'\mathbf{j}'\mathbf{k}'$ in which the $x'y'$-axes are rotated about the z-axis (which coincides with the z'-axis) through an angle of 30°.

1-10: Derivative and Integration of a Vector:

Consider a vector **A**, whose components are functions of a single variable u. The vector may represent position, velocity, and so on. The parameter u is usually the time t, but it can be any quantity that determines the components of **A**. Let:

$$\vec{A}(u) = \hat{i}\, A_x(u) + \hat{j}\, A_y(u) + \hat{k}\, A_z(u)$$

Then

$$\frac{d}{du}\vec{A} = \hat{i}\frac{d}{du}A_x + \hat{j}\frac{d}{du}A_y + \hat{k}\frac{d}{du}A_z$$

Derivative of a Vector is a vector whose components are ordinary derivatives.

The derivative of the sum of two vectors is equal to the sum of the derivatives, namely,

$$\frac{d}{du}(\vec{A}+\vec{B}) = \frac{d}{du}\vec{A} + \frac{d}{du}\vec{B}$$

The rules for differentiating vector products obey similar rules of vector calculus. If n is a scalar function of u then,

$$\frac{d}{du}n\vec{A} = \frac{dn}{du}\vec{A} + n\frac{d\vec{A}}{du}$$

$$\frac{d}{du}(\vec{A}\cdot\vec{B}) = \frac{d\vec{A}}{du}\cdot\vec{B} + \vec{A}\cdot\frac{d\vec{B}}{du}$$

$$\frac{d}{du}(\vec{A}\times\vec{B}) = \frac{d\vec{A}}{du}\times\vec{B} + \vec{A}\times\frac{d\vec{B}}{du}$$

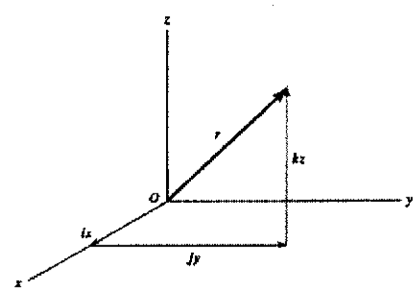
Notice that it is necessary to preserve يحافظ the order of the terms in the derivative of the cross product.

The integration of a vector A as a function of any variable say time (t) can be written as:

$$\int \vec{A}\, dt = \hat{i}\int A_x(t)\, dt + \hat{j}\int A_y(t)\, dt + \hat{k}\int A_z(t)\, dt$$

1-11: Position Vector of a Particle: Velocity and Acceleration in Rectangular Coordinates

The position of a particle can be specified by a single vector, namely, the displacement الازاحة of the particle relative to the origin of the coordinate system. This vector is called the position vector of the particle. In rectangular coordinates (as shown in the Figure), the position vector is simply:



$$\vec{r} = \hat{i}\,x + \hat{j}\,y + \hat{k}\,z$$

The components of the position vector of a moving particle are functions of the time, i.e.

$$x = x(t), \quad y = y(t) \quad, z = z(t)$$

The derivative of **r** with respect to *t* is called the velocity, which we shall denote by *v*:

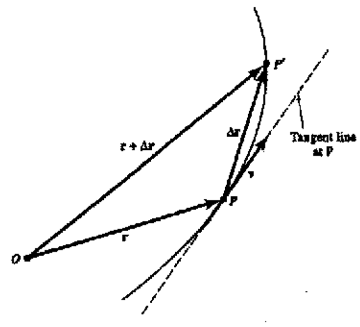
$$\vec{v} = \frac{d\vec{r}}{dt} = \hat{i}\,\dot{x} + \hat{j}\,\dot{y} + \hat{k}\,\dot{z}$$

where the dots indicate تشير الى differentiation with respect to *t*.

Let us examine the **geometric significance** of the velocity vector. Suppose a particle is at a certain position at time t. At a time Δt later, the particle will have moved from the position r(t) to the position $r(t + \Delta t)$. The vector displacement during the lime interval Δt is:

$$\Delta \mathbf{r} = r(t + \Delta t) - r(t)$$

so the quotient حاصل قسمة $\Delta r/\Delta t$ is a vector that is parallel to the displacement. As we consider smaller and smaller time intervals, the quotient $\Delta r/\Delta t$ approaches a limit $d\mathbf{r}/dt$, which we call the *velocity*. The vector **dr**/dt expresses both the direction of motion and the rate. This is shown graphically in figure. In the time interval Δt, the particle moves along the path from P to P'. As Δt approaches zero, the point P' approaches P, and the



direction of the vector $\Delta \mathbf{r}/\Delta t$ approaches the direction of the tangent to the path at P. The velocity vector is always tangent to the path of motion.

The magnitude of the velocity is called the speed. In rectangular components the speed is :

$$v = |\vec{v}| = (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)^{1/2}$$

The time derivative of the velocity is called the *acceleration*. Denoting the acceleration with **a**, we have

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}$$

In rectangular components,

$$\vec{a} = \hat{i}\,\ddot{x} + \hat{j}\,\ddot{y} + \hat{k}\,\ddot{z}$$

Thus, acceleration is a vector quantity whose components, in rectangular coordinates, are the second derivatives of the positional coordinates of a moving particle.

Example7: Let us examine (نختبر) the projectile (قذيفة) motion represented by the equation:

$$\vec{r}(t) = \hat{i}\,bt + \hat{j}\left(ct - \frac{gt^2}{2}\right)$$

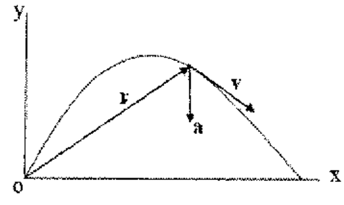
This represents motion in the xy plane, because the z component is constant and equal to zero. The velocity v is obtained by differentiating with respect to *t*, hence,

$$\vec{v} = \frac{d\vec{r}}{dt} = \hat{i}\,b + \hat{j}\,(c - gt)$$

The acceleration is given by

$$\vec{a} = \frac{d\vec{v}}{dt} = -g\,\hat{j}$$

Thus, **a** is in the negative y direction and has the constant magnitude g. The path of motion is a parabola (قطع مكافئ).



Example 8: (Circular Motion)

Suppose the position vector of a particle is given by:

$$\vec{r} = \hat{i}b\sin\omega t + \hat{j}b\cos\omega t$$

The distance from the origin remains constant:

$$r = |\vec{r}| = (b^2\sin^2\omega t + b^2\cos^2\omega t)^{1/2} = b$$

So the path is a circle of radius b centered at the origin. Differentiating **r**, we find the velocity vector

$$\vec{v} = \frac{d\vec{r}}{dt} = \hat{i}\,\omega b\cos\omega t - \hat{j}\,\omega b\sin\omega t$$

The particle traverses(يجتاز) its path with constant speed:

$$v = |\vec{v}| = (\omega^2 b^2\cos^2\omega t + \omega^2 b^2\sin^2\omega t)^{1/2} = b\omega$$

The acceleration is:

$$\vec{a} = \frac{d\vec{v}}{dt} = -\hat{i}\,\omega^2 b\sin\omega t - \hat{j}\,\omega^2 b\cos\omega t$$

In this case the acceleration is perpendicular to the velocity $\vec{v}\perp\vec{a}$, because the dot product of **v** and **a** vanishes:

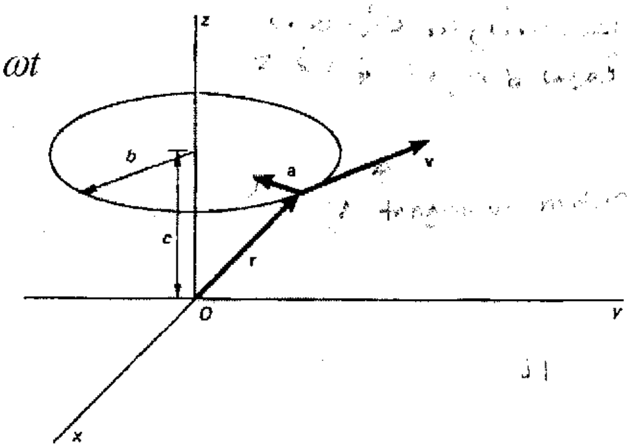
$$\vec{v}\cdot\vec{a} = -\omega^3 b^2\sin\omega t\cos\omega t + \hat{j}\omega^3 b^2\sin\omega t\cos\omega t = 0$$

Comparing the two expressions for **a** and **r**, we find

$$\vec{a} = -\hat{i}\,\omega^2 b\sin\omega t - \hat{j}\,\omega^2 b\cos\omega t$$

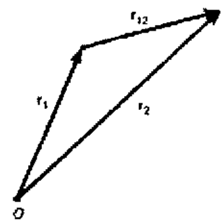
$$\vec{r} = \hat{i}b\sin\omega t + \hat{j}b\cos\omega t$$

$$\vec{a} = -\omega^2\vec{r}$$



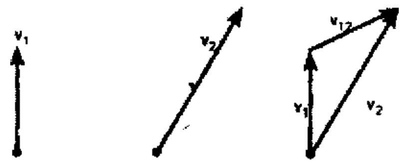
The path is a circle with radius b

1-12: Relative velocity:



Consider two particles whose position are $\vec{r}_1$ and $\vec{r}_2$ respectively. The displacement of the second particle with respect to the first is the difference $\vec{r}_2-\vec{r}_1$ which we shall call r_{12}.

The velocity of the second particle relative to the first which we shall call the relative velocity is:



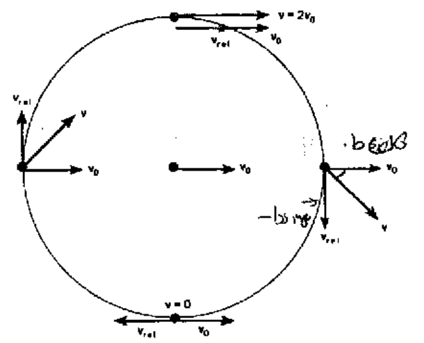
$$\mathbf{v}_{12} = \frac{d\mathbf{r}_{12}}{dt} = \frac{d(\mathbf{r}_2 - \mathbf{r}_1)}{dt} = \mathbf{v}_2 - \mathbf{v}_1$$

By transposing v1, for the actual velocity of particle 2 in terms of the velocity of particle 1 and the relative velocity of the two particles we have:

$$V_2 = V_1 + V_{12}$$

Example 9:

A wheel of radius b rolls along the ground with a forward speed, find the velocity of any point P on the rim relative to the ground.



Solution: The motion of point P is circular and suppose that the motion is clockwise about the origin center of wheel, in this case. Then, we First consider the position vector is given by expression:

$$\mathbf{r}_{OP} = \mathbf{i}b\cos\theta - \mathbf{j}b\sin\theta$$

where $\theta = \omega t$

the time derivative gives the velocity of P relative to the center of the wheel as:

$$\mathbf{v}_{rel} = -\mathbf{i}b\omega\sin\theta - \mathbf{j}b\omega\cos\theta$$

Since the angular velocity $\omega = v_o/b$, and since the velocity of the center of the wheel relative to the ground is $\mathbf{i}v_o$, then the true velocity of P relative to the ground is:

$$\mathbf{v} = \mathbf{i}v_0 - \mathbf{i}b\omega\sin\theta - \mathbf{j}b\omega\cos\theta$$
$$= \mathbf{i}v_0(1 - \sin\theta) - \mathbf{j}v_0\cos\theta$$

Example 10: The following relations represent the position of two particles move in the same circular path.

$$\mathbf{r}_1 = \mathbf{i}\, b\sin\omega t + \mathbf{j}\, b\cos\omega t \quad \text{and} \quad \mathbf{r}_2 = \mathbf{i}\, b\cos\omega t - \mathbf{j}\, b\sin\omega t$$

Find the relative velocity, the magnitude of the relative velocity, and the time rate of the change of the displacement between the two particles, all as a function of time?

solution:

since $\mathbf{v}_{21} = \mathbf{v}_2 - \mathbf{v}_1$

$$\mathbf{v}_1 = \mathbf{i}\, b\omega\cos\omega t - \mathbf{j}\, b\omega\sin\omega t$$

$$\mathbf{v}_2 = -\mathbf{i}\, b\omega\sin\omega t - \mathbf{j}\, b\omega\cos\omega t$$

therefore the relative velocity is

$$\mathbf{v}_{21} = -\mathbf{i}\, b\omega(\sin\omega t + \cos\omega t) - \mathbf{j}\, b\omega(\cos\omega t - \sin\omega t)$$

$$|\mathbf{v}_{21}| = b\omega\,(\sin^2\omega t + \cos^2\omega t + 2\sin\omega t\cos\omega t + \cos^2\omega t + \sin^2\omega t - 2\sin\omega t\cos\omega t)^{1/2}$$

$= b\omega(2)^{1/2}$ this is the magnitude of the relative velocity

The time rate of the change of the displacement between the two particles is :

$$\frac{d}{dt}|\overrightarrow{r_2} - \overrightarrow{r_1}|$$

and since

$$\mathbf{r}_2 - \mathbf{r}_1 = \mathbf{i}\, b\,(\cos\omega t - \sin\omega t) - \mathbf{j}\, b\,(\sin\omega t + \cos\omega t)$$

$$|\boldsymbol{r}_2 - \boldsymbol{r}_1| = b\,(\cos^2\omega t + \sin^2\omega t - 2\sin\omega t\cos\omega t + \sin^2\omega t + \cos^2\omega t + 2\sin\omega t\cos\omega t)^{1/2}$$

$$= b(2)^{1/2}$$

then $\qquad \frac{d}{dt}|\overrightarrow{r_2} - \overrightarrow{r_1}| = 0$

Note: That the time rate of the change of the displacement between the two particles is not the same as the magnitude of the relative velocity.

1-13: Tangential and normal componenet of acceleration:

Any vector can be expressed as the product of its magnitude and a unit vector giving its direction. the velocity vector **v** can be written as the product of the particle's speed v and a unit vector $\boldsymbol{\tau}$ that give the direction of particle's motion, thus

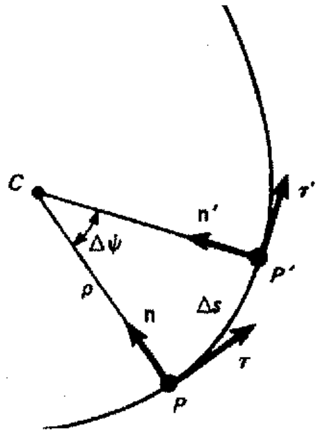
$$\mathbf{v} = v\boldsymbol{\tau}$$

the vector $\boldsymbol{\tau}$ is called the **unit tangent vector**. As the particle moves the speed v may change and the direction of $\boldsymbol{\tau}$ may change. By using the differentiation of product of scalar and vector to obtain acceleration vector the result is:

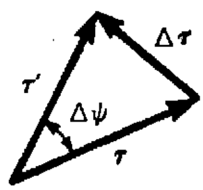
$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d(v\boldsymbol{\tau})}{dt} = \dot{v}\boldsymbol{\tau} + v\,\frac{d\boldsymbol{\tau}}{dt} \quad \ldots\ldots\ldots.(1)$$

The unit vector $\boldsymbol{\tau}$ being of constant magnitude, has a derivative $\frac{d\boldsymbol{\tau}}{dt}$ express the change in direction of $\boldsymbol{\tau}$ with respect to time.

The particle moves a distance Δs along the path from the initial point P to another point P' at time interval Δt. Let denote the unit tangent vectors at P and P' by $\boldsymbol{\tau}$ and $\boldsymbol{\tau}'$, respectively as shown in the figure.



The directions of these two units vectors differ by angle $\Delta\psi$ as shown in figure, for small value of $\Delta\psi$, the difference $\Delta\boldsymbol{\tau}$ approaches $\Delta\psi$ in magnitude ($\Delta\tau \rightarrow \Delta\psi$) and the direction of $\Delta\boldsymbol{\tau}$ become perpendicular to the direction of $\boldsymbol{\tau}$ ($\Delta\boldsymbol{\tau} \perp \boldsymbol{\tau}$) when $\Delta\psi$ and $\Delta s \rightarrow$ zero. The derivative $\frac{d\boldsymbol{\tau}}{d\psi}$ is of magnitude unity ($\left|\frac{d\boldsymbol{\tau}}{d\psi}\right| = 1$) and is perpendicular to $\boldsymbol{\tau}$ ($\frac{d\boldsymbol{\tau}}{d\psi} \perp \boldsymbol{\tau}$)



So that we call it the unit normal vector and denote it by **n**:

$$\frac{d\boldsymbol{\tau}}{d\psi} = \mathbf{n}$$

To find the time derivative $\frac{d\tau}{dt}$, we use the chain rule as follows:

$$\frac{d\tau}{dt} = \frac{d\tau}{d\psi}\frac{d\psi}{dt} = \mathbf{n}\,\frac{d\psi}{ds}\frac{ds}{dt} = \mathbf{n}\,\frac{v}{\rho}$$

in which $\rho = \frac{ds}{d\psi}$ is the radius of curvature of the path of moving particle at P.

Substitute the value of $\frac{d\tau}{dt}$ into equation (1) to find:

$$\mathbf{a} = \dot{v}\boldsymbol{\tau} + \frac{v^2}{\rho}\,\mathbf{n}$$

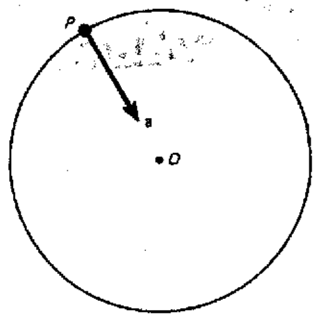
Thus we can note that the acceleration of a moving paricle has two components:

1- Tangential component (in the direction of motion) $a_\tau = \dot{v} = \ddot{s}$

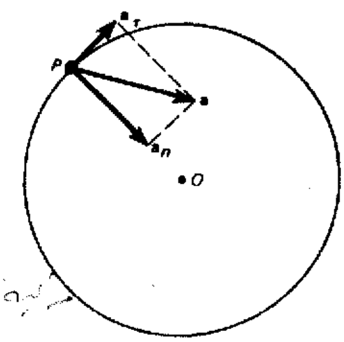
2- Normal component, also called centripetal acceleration (directed toword the center of curvature on the concave side of the path motion $a_n = \frac{v^2}{\rho}$

The magnitude of the total acceleration is given by:

$$|\mathbf{a}| = \left|\frac{d\mathbf{v}}{dt}\right| = \left(\dot{v}^2 + \frac{v^4}{\rho^2}\right)^{1/2}$$



If the particle moves on a circle with constant speed then the acceleration vector is of magnitude v^2/R_0, where R_0 is the raduis of the circle.The acceleration vector is always points to the center in this case as shown in figure.



However if the speed is not constant but increase at a certain rate $\dot{v}$ then the acceleration has a forward component and is slanted تنحرف away مبتعدة from the center of circle towards the direction as shown in figure. What happen if the speed decrease at a certain rate $\dot{v}$?

1-14: Velocity and Acceleration in Plane Polar Coordinates:

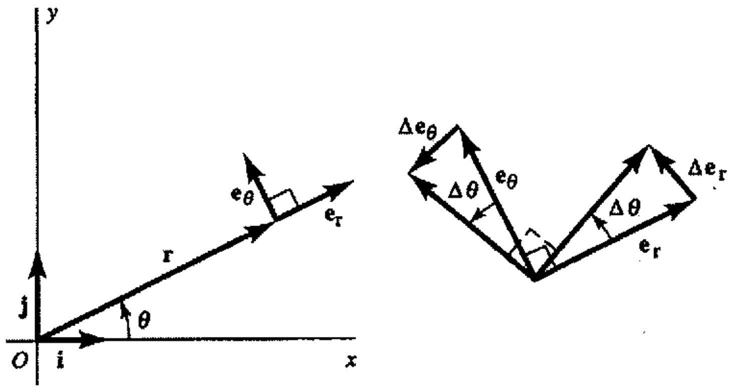
It is often convenient to employ polar coordinates r, θ to express the position of a particle moving in a plane. The position of the particle can be written as the product of the radial distance *r* by a unit radial vector $\mathbf{e}_r$:

$$\mathbf{r} = r\mathbf{e}_r$$

As the particle moves, both r and $\mathbf{e}_r$ vary; thus, they are both functions of the time. Hence, if we differentiate with respect to t, we have

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \dot{r}\mathbf{e}_r + r\frac{d\mathbf{e}_r}{dt} \qquad \text{--------- } 1$$

To calculate the derivative $d\mathbf{e}_r/dt$, let us consider the vector diagram shown in Figure.



When the direction of **r** changes by an amount Δθ the corresponding change $\Delta\mathbf{e}_r$ of the unit radial vector is as follows:

The magnitude $|\Delta\mathbf{e}_r|$ is approximately equal to Δθ ($|\Delta\mathbf{e}_r| \approx \Delta\theta$) and the direction of $\Delta\mathbf{e}_r$ is very nearly perpendicular to $\mathbf{e}_r$ ($\Delta\mathbf{e}_r \perp \mathbf{e}_r$). Let us introduce another unit vector, e_θ, whose direction is perpendicular to $\mathbf{e}_r$. Then we have

$$\Delta\mathbf{e}_r \approx \mathbf{e}_\theta\Delta\theta$$

If we divide by Δt and take the limit as $\Delta t \to 0$, we get

$$\frac{d\mathbf{e}_r}{dt} = \mathbf{e}_\theta\frac{d\theta}{dt} = \dot{\theta}\mathbf{e}_\theta \qquad \text{-------- } 2$$

For the time derivative of the unit radial vector, we can argue in similar way that the change in the unit vector $\mathbf{e}_\theta$ is given by:

$$\Delta\dot{\mathbf{e}}_\theta \approx -\mathbf{e}_r\Delta\theta$$

Here the minus sign is inserted to indicate that the direction of the change is opposite to the direction of $\mathbf{e}_r$, as shown in Figure. Consequently, the time derivative is given by

$$\frac{d\mathbf{e}_\theta}{dt} = -\mathbf{e}_r\frac{d\theta}{dt} \qquad = -\,\mathbf{e}_r\dot{\theta} \qquad \text{-------- } 3$$

By using Equation 2 for the derivative of the unit radial vector, we can finally write the equation for the velocity as:

$$\mathbf{v} = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta \qquad \text{--------- } 4$$

Thus, $\dot{r}$ is the radial القطبية component المركبة of the velocity vector, and $r\dot{\theta}$ is the transverse المستعرضة component.

To find the acceleration vector, we take the derivative of the velocity with respect to time. This gives

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \ddot{r}\mathbf{e}_r + \dot{r}\frac{d\mathbf{e}_r}{dt} + (\dot{r}\dot{\theta} + r\ddot{\theta})\mathbf{e}_\theta + r\dot{\theta}\frac{d\mathbf{e}_\theta}{dt}$$

The values of $d\mathbf{e}_r/dt$ and $d\mathbf{e}_\theta/dt$ are given by Equations 2 and 3 and yield the following equation for the acceleration vector in plane polar coordinates:

$$\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\mathbf{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\mathbf{e}_\theta \qquad \text{----------}5$$

Thus, the radial component of the acceleration vector is

$$a_r = \ddot{r} - r\dot{\theta}^2$$

and the transverse component is:

$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} = \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})$$

The above results show, for instance, that if a particle moves on a circle of constant radius b, so that $\dot{r} = 0$, then the radial component of the acceleration is of magnitude $b\dot{\theta}^2$ and is directed inward الداخل toward the center of the circular path.

The transverse component in this case is $b\dot{\theta}$. On the other hand, if the particle moves along a fixed radial line, that is, if θ is constant, then the radial component is just $\ddot{r}$ and the transverse component is zero. If **r** and θ both vary, then the general expression (6) gives the acceleration.

Example11:

A honey bee (نحل العسل) hones (بيوت) in on its hive (خلية) in a spiral (لولبي) path (مسار) in such a way that the radial distance decreases (يتناقص) at a constant rate, r = b — Ct, while the angular speed increases (يتزايد) at a constant rate, $\dot{\theta}$ = kt, Find the speed as a function of time.

Solution:

We have $\dot{r}$ = -c and $\ddot{r}$ =0. Thus, from Equation 5,

$$\mathbf{v} = -c\mathbf{e}_r + (b - ct)kt\mathbf{e}_\theta$$

So

$$v = [c^2 + (b-ct)^2 k^2 t^2]^{1/2}$$

which is valid for t ≤ *b/c*. Note that *v*= c both for *t* = 0, r = b and for t=b/c ,r=0.

Example12

On a horizontal turntable (قرص دوار) that is rotating at constant angular speed, a bug (بق) is crawling (الزحف) outward (نحو الخارج) on a radial line such that its distance from the center increases quadratically (تربيعيا) with time: **r** = bt^2, θ= ωt, where b and ω are constants. Find the acceleration of the bug.

Solution:

We have $\dot{r}$= 2bt, $\ddot{r}$ = 2b, $\dot{\theta}$ = ω, $\ddot{\theta}$ = 0. Substituting into Equation 5, we find

$$\begin{aligned}\mathbf{a} &= \mathbf{e}_r(2b - bt^2\omega^2) + \mathbf{e}_\theta[0 + 2(2bt)\omega] \\ &= b(2 - t^2\omega^2)\mathbf{e}_r + 4b\omega t\mathbf{e}_\theta\end{aligned}$$

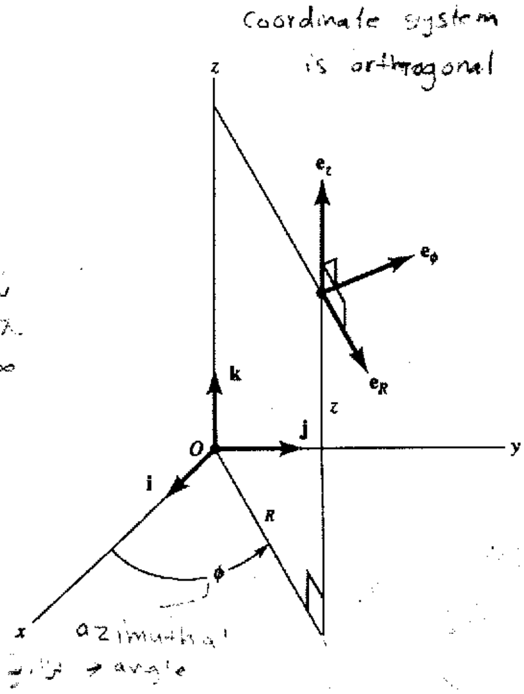
Note that the radial component of the acceleration becomes negative for large t in this example, although the radius is always increasing monotonically with time.

1-15:Velocity and Acceleration in Cylindrical Coordinates:

In the case of three-dimensional motion, the position of a particle can be described in cylindrical coordinates R, ϕ, z. The position vector is then written as:

$$\mathbf{r} = R\mathbf{e}_R + Z\mathbf{e}_z \quad \text{---------------- } 1$$

where e_R is a unit radial vector in the xy plane and $\mathbf{e}_z$ is the unit vector in the z direction. A third unit vector e_ϕ is needed so that the three vectors $\mathbf{e}_R\ \mathbf{e}_z\ \mathbf{e}_\phi$ constitute a right-handed triad, as illustrated in Figure. We note that $k = \mathbf{e}_z$.



The velocity and acceleration vectors are found by differentiating, as before. This again involves derivatives of the unit vectors. An argument similar to that used for the plane case shows that

$$\frac{d\hat{e}_R}{dt} = \dot{\phi}\,\hat{e}_\phi \quad and \quad \frac{d\hat{e}_\phi}{dt} = -\dot{\phi}\,\hat{e}_R$$

And the unit vector $\hat{e}_z$ does not change in direction so

$$\frac{d\hat{e}_z}{dt} = 0$$

$$\vec{v} = \frac{d\vec{r}}{dt} = \dot{R}\hat{e}_R + R\frac{d\hat{e}_R}{dt} + \dot{z}\hat{e}_z + z\frac{d\hat{e}_z}{dt}$$

Subsitituting for the time derivative, we get

$$\vec{v} = \dot{R}\hat{e}_R + R\dot{\phi}\hat{e}_\phi + \dot{z}\hat{e}_z \quad \text{------------- } 2$$

$$\vec{a} = \ddot{R}\hat{e}_R + \dot{R}\frac{d\hat{e}_R}{dt} + \left(\dot{R}\dot{\phi} + R\ddot{\phi}\right)\hat{e}_\phi + R\dot{\phi}\frac{d\hat{e}_\phi}{dt} + \ddot{z}\,\hat{e}_z$$

Subsitituting for the time derivative, we get

$$\vec{a} = \ddot{R}\hat{e}_R + \dot{R}\,\dot{\phi}\hat{e}_\phi + \dot{R}\dot{\phi}\hat{e}_\phi + R\ddot{\phi}\hat{e}_\phi - R\dot{\phi}^2\hat{e}_R + \ddot{z}\,\hat{e}_z$$

$$\vec{a} = \left(\ddot{R} - R\dot{\phi}^2\right)\hat{e}_R + \left(2\dot{R}\dot{\phi} + R\ddot{\phi}\right)\hat{e}_\phi + \ddot{z}\,\hat{e}_z \quad \text{-------------- } 3$$

The derivatives of the unit vectors is to differentiate the following equations, which are the relationships between the fixed unit triad i ,j ,k and the rotated triad $\hat{e}_R, \hat{e}_\emptyset, \hat{e}_Z$:

$$\hat{e}_R = \hat{i}\cos\phi + \hat{j}\sin\phi$$

$$\hat{e}_\phi = -\hat{i}\sin\phi + \hat{j}\cos\phi$$

$$\hat{e}_z = \hat{k}$$

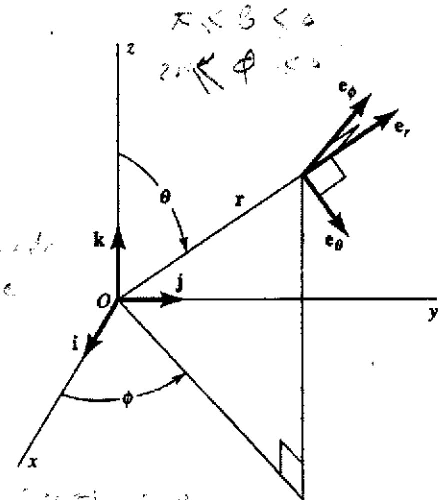
The steps are left as an exercise. The result can also be found by use of the rotation matrix.

1-16: Velocity and Acceleration in Sphericalكروية Coordinatesمحاور:

When spherical coordinates r, θ, ϕ are employed to describe the position of a particle, the position vector is written as the product of the radial distance *r* and the unit radial vector $\mathbf{e}_r$, as with plane polar coordinates. Thus,

$$\vec{r} = r\,\hat{e}_r \text{ ---------------- } 1$$

The direction of $\mathbf{e}_r$ is now specified by the two angles θ and ϕ. We introduce two more unit vectors, $\mathbf{e}_\theta$ and $\mathbf{e}_\phi$, as shown in Figure.



The velocity is:

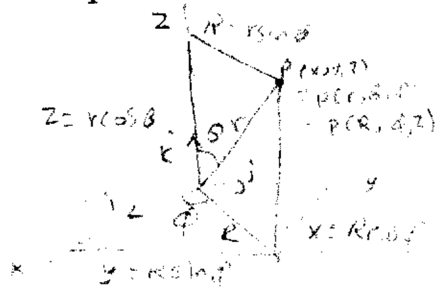
$$\vec{v} = \frac{d\vec{r}}{dt} = \dot{r}\hat{e}_r + r\,\frac{d\hat{e}_r}{dt} \text{ ------------}2$$

Our next problem is how to express the derivative $\frac{d\hat{e}_r}{dt}$ in term of the unit vectors in the rotated triad. We can derive relationships between $\hat{\imath}, \hat{\jmath}, \hat{k}$ and $\hat{e}_r, \hat{e}_\theta, \hat{e}_\phi$. By expressing any unit vector in terms of its projections on the x,y,z coordinate axes. as an example

$$\hat{e}_r = \hat{i}(\hat{e}_r \cdot \hat{i}) + \hat{j}(\hat{e}_r \cdot \hat{j}) + \hat{k}(\hat{e}_r \cdot \hat{k})$$

$\hat{e}_r \cdot \hat{\imath}$ is the projection of the unit vector $\hat{e}_r$ directly onto the unit vector $\hat{\imath}$ and is equal to cos α, (see page 8) the cosine of the angle between those two unit vectors.

We need to express these dot products in terms of θ and ϕ not α. We can obtain desired relation by making two successive (متعاقبة) projections to get to the x-axis. First project $\hat{e}_r$ onto the xy plane, and then project from there onto the x-axis. The first projection gives us a factor of sin θ, while the second yields a factor of cos ϕ. The magnitude of the projection obtained in this way is the desired dot product:



$$\hat{e}_r \cdot \hat{i} = \sin\theta\cos\phi, \quad \hat{e}_r \cdot \hat{j} = \sin\theta\sin\phi$$

$$\text{and} \qquad \hat{e}_r \cdot \hat{k} = \cos\theta$$

In a similar way the relationship for $\hat{e}_\theta$ and $\hat{e}_\phi$ can be opbtained yailding the fellowing relation:

$$\left.\begin{aligned} \hat{e}_r &= \hat{i}\sin\theta\cos\phi + \hat{j}\sin\theta\sin\phi + \hat{k}\cos\theta \\ \hat{e}_\theta &= \hat{i}\cos\theta\cos\phi + \hat{j}\cos\theta\sin\phi - \hat{k}\sin\theta \\ \hat{e}_\phi &= -\hat{i}\sin\phi + \hat{j}\cos\phi \end{aligned}\right\} \quad \text{--------- 3}$$

which express the unit vectors of the rotated triad in terms of the fixed triad **i,j,k**.

Let us differentiate the first equation with respect to time. The result is:

$$\frac{d\hat{e}_r}{dt} = \hat{i}(\dot{\theta}\cos\theta\cos\phi - \dot{\phi}\sin\theta\sin\phi) + \hat{j}(\dot{\theta}\cos\theta\sin\phi + \dot{\phi}\sin\theta\cos\phi) - \hat{k}\dot{\theta}\sin\theta$$

We find that the above equation reduces to:

$$\frac{d\hat{e}_r}{dt} = \hat{e}_\phi\dot{\phi}\sin\theta + \hat{e}_\theta\dot{\theta}$$

The other two derivatives are found through a similar procedure. The results are

$$\frac{d\hat{e}_\theta}{dt} = -\hat{e}_r\dot{\theta} + \hat{e}_\phi\dot{\phi}\cos\theta \qquad \text{and} \qquad \frac{d\hat{e}_\phi}{dt} = -\hat{e}_r\dot{\phi}\sin\theta - \hat{e}_\theta\dot{\phi}\cos\theta \qquad \text{(H.w)}$$

Finally substituting eqs. ($\frac{d\hat{e}_r}{dt}$, $\frac{d\hat{e}_\theta}{dt}$, $\frac{d\hat{e}_\phi}{dt}$) in eq.(2) we obtain:

$$\vec{v} = \hat{e}_r \dot{r} + \hat{e}_\phi r\dot{\phi}\sin\theta + \hat{e}_\theta r\dot{\theta}\ ) \quad \text{------------- 4}$$

To find the acceleration, we differentiate eq. 4 with respect to time. This gives:

$$\vec{a} = \frac{dv}{dt}$$

$$\vec{a} = \hat{e}_r\ddot{r} + \dot{r}\,\frac{d\hat{e}_r}{dt} + \hat{e}_\phi\,\frac{d(r\dot{\phi}\sin\theta)}{dt} + r\dot{\phi}\sin\theta\,\frac{d\hat{e}_\phi}{dt} + \hat{e}_\theta\frac{d(r\dot{\theta})}{dt} + r\dot{\theta}\,\frac{d\hat{e}_\theta}{dt}$$

Substituting for the unit vectors derivatives ($\frac{d\hat{e}_r}{dt}$, $\frac{d\hat{e}_\theta}{dt}$, $\frac{d\hat{e}_\phi}{dt}$) we obtain:

$$\begin{aligned}\vec{a} &= (\ddot{r} - r\dot{\phi}^2\sin^2\theta - r\dot{\theta}^2)\hat{e}_r \qquad \text{----------- 5}\\ &+ (r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2\sin\theta\cos\theta)\hat{e}_\theta\\ &+ (r\ddot{\phi}\sin\theta + 2\dot{r}\dot{\phi}\sin\theta + 2r\dot{\theta}\dot{\phi}\cos\theta)\hat{e}_\phi\end{aligned}$$

giving the acceleration vector in terms of its components in the triad $\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi$

Example13:

A bead[خرزة] slides[تنزلق] on a wire bent[ينحني] into the form[شكل] of a helix[حلزوني], the motion of the bead being given in cylindrical coordinates by R = b, $\phi = \omega t$, z =ct. Find the velocity and acceleration vectors as functions of time.

Solution:

$\vec{v} = \dot{R}\,\hat{e}_R + R\dot{\phi}\,\hat{e}_\phi + \dot{z}\,\hat{e}_z$

$\vec{a} = (\ddot{R} - R\dot{\phi}^2)\hat{e}_R + (2\dot{R}\dot{\phi} + R\ddot{\phi})\hat{e}_\phi + \ddot{z}\,\hat{e}_z$

Differentiating, we find $\dot{R} = \ddot{R} = 0, \phi = \omega, \ddot{\phi} = 0, \dot{z} = c, \ddot{z} = 0$. So, from equation for velocity and acceleration in cylindrical coordinates , we have:

$$\vec{v} = b\omega\hat{e}_\phi + c\hat{e}_z$$

$$\vec{a} = -b\omega^2\hat{e}_R$$

1-17: Gradient , Divergence, and Curl :

The del operater (∇) in rectangular coordinate is given by:

$$\vec{\nabla} = \hat{\imath}\frac{d}{dx} + \hat{\jmath}\frac{d}{dy} + \hat{k}\frac{d}{dz}$$

The del operator is a vector that has no physical meaning or vector direction by itself.

1-17-1: The gradient :

Let $\phi(x,y,z)$ be a differentiable at any point, therefore the gradient of ϕ is

$$\vec{\nabla}\phi = \hat{\imath}\frac{d\phi}{dx} + \hat{\jmath}\frac{d\phi}{dy} + \hat{k}\frac{d\phi}{dz}$$

The following computation formulas on gradient, which are easily proved, should be noted:

1- $\nabla(A+B) = \nabla A + \nabla B$

2- $\nabla(AB) = A\nabla B + B\nabla A$

3- $\nabla\left[\frac{A}{B}\right] = \frac{B\,\nabla A - A\nabla\, B}{B^2}$

4- $\nabla\, V^n = n\, V^{n-1}\, \nabla\, V$

1-17-2: The divergence :

Let

$$\vec{v}(x,y.z) = \hat{\imath}\, v_x + \hat{\jmath}\, v_y + \hat{k}\, v_z$$

Then the divergence of $\vec{v}$ (written div $\vec{v}$ or $\vec{\nabla}.\vec{v}$) is

$$\vec{\nabla}.\vec{v} = \frac{dv_x}{dx} + \frac{dv_y}{dy} + \frac{dv_z}{dz}$$

Note that $\vec{\nabla}.\vec{v} \neq \vec{v}.\vec{\nabla}$

Note the following properties of the divergence of a vector:

1- Its produces a scalar.

2- The divergence of a scalar V, div V make no sense.

3- $\nabla.(\mathbf{A}+\mathbf{B}) = \nabla.\mathbf{A} + \nabla.\mathbf{B}$

4- $\nabla.(V\mathbf{A}) = V\nabla.\mathbf{A} + \mathbf{A} . V\nabla$

1-17-2: The curl الالتفاف:

The curl of the vector field $\vec{v}(x, y.z) = \hat{\imath}\, v_x + \hat{\jmath}\, v_y + \hat{k}\, v_z$ is denoted by $\overrightarrow{\text{curl v}}$ or $(\vec{\nabla} \times \vec{v})$ is:

$$\overrightarrow{\text{curl v}} = \vec{\nabla} \times \vec{v} = \begin{vmatrix} i & j & k \\ \dfrac{d}{d_x} & \dfrac{d}{d_y} & \dfrac{d}{d_z} \\ v_x & v_y & v_z \end{vmatrix}$$

Note the following properties of the curl:

1. The curl of a vector is another vector.
2. The curl of a scalar V, $\nabla\times V$, makes no sense.
3. $\nabla \times (\mathbf{A}+\mathbf{B}) = \nabla \times \mathbf{A} + \nabla \times \mathbf{B}$ curl $\vec{A}$ + curl $\vec{B}$

$(\vec{B}.\vec{\nabla})A - (\vec{\nabla}.\vec{A})B - (A.\nabla)B + A(\nabla.B)$

4. $\nabla \times (\mathbf{A}\times\mathbf{B}) = \mathbf{A}(\nabla . \mathbf{B}) - \mathbf{B}(\nabla . \mathbf{A}) + (\mathbf{B}.\nabla\mathbf{A}) - (\mathbf{A} . \nabla\mathbf{B})$

6- The divergence of the curl of a vector vanishes,

that is, $\nabla . (\nabla \times \mathbf{A}) = \mathbf{0}$ = div curl $\vec{F}$ = 0

7. The curl of the gradient of a scalar is vanishes,

that is, $\nabla \times \nabla V = \mathbf{0}$ curl $\vec{\nabla} V = 0$

$$\nabla\cdot(\nabla u) \equiv \nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \Rightarrow \text{laplacian of } u$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \quad \text{laplacian operator}$$

Notice

We can defined $d\vec{r}$ and $\vec{\nabla}$ in the:

1- Cartesian Coordinate:

$$d\vec{r} = \hat{\imath}dx + \hat{\jmath}dy + \hat{k}dz$$

$$\vec{\nabla} = \hat{\imath}\frac{d}{dx} + \hat{\jmath}\frac{d}{dy} + \hat{k}\frac{d}{dz}$$

2- Cylindrical Coordinate:

$$d\vec{r} = \hat{e}_R dR + \hat{e}_\phi R d\phi + \hat{e}_Z dz$$

$$\vec{\nabla} = \hat{e}_R\frac{d}{dR} + \hat{e}_\phi\frac{1}{R}\frac{d}{d\phi} + \hat{e}_Z\frac{d}{dz}$$

3- Spherical coordinate:

$$d\vec{r} = \hat{e}_r dr + \hat{e}_\theta r d\theta + \hat{e}_\phi r sin\theta\, d\phi$$

$$\vec{\nabla} = \hat{e}_r\frac{d}{dr} + \hat{e}_\theta\frac{1}{r}\frac{d}{d\theta} + \hat{e}_\phi\frac{1}{r sin\theta}\frac{d}{d\phi}$$

Example 14:

The potential that represents an inverse square force is $V(r) = \frac{k}{r}$, where $r = (x^2 + y^2 + z^2)^{1/2}$. Using the definition $\vec{F} = -\vec{\nabla}v$, calculate the component of this force.

Solution:

$$\vec{F} = -\vec{\nabla}v = -\left(\hat{\imath}\frac{dv_x}{dx} + \hat{\jmath}\frac{dv_y}{dy} + \hat{k}\frac{dv_z}{dz}\right)$$

Since $V(r) = \frac{k}{r}$ and $\frac{dV}{dx} = -\frac{kx}{r^3}$, $\frac{dV}{dy} - \frac{ky}{r^3}$, and $\frac{dV}{dz} = -\frac{kz}{r^3}$ then

$$\vec{F} = \frac{k}{r^3}\left(\hat{\imath}x + \hat{\jmath}y + \hat{k}z\right) = \frac{k}{r^3}\vec{r}$$

Another solution by using spherical coordinate

$$\vec{\nabla} = \hat{e}_r \frac{d}{dr} + \hat{e}_\theta \frac{1}{r}\frac{d}{d\theta} + \hat{e}_\phi \frac{1}{r\sin\theta}\frac{d}{d\phi}$$

$$\vec{F} = -\vec{\nabla}V = -k\hat{e}_r \frac{d}{dr}\left(\frac{1}{r}\right) + 0 + 0$$

$$\vec{F} = -k\hat{e}_r\left(-\frac{1}{r^2}\right)$$

$$\vec{F} = \frac{k}{r^2}\,\hat{e}_r$$

Example 15: If $\varphi = \frac{1}{r}$ where $r = (x^2 + y^2 + z^2)^{1/2}$, show that $\nabla\varphi = \frac{-\boldsymbol{r}}{r^3}$

Solution:

$$\nabla\Phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2 + y^2 + z^2)^{-1/2}$$

$$= \left(-\frac{1}{2}.2x\hat{i} - \frac{1}{2}.2y\hat{j} - \frac{1}{2}.2z\hat{k}\right)(x^2 + y^2 + z^2)^{-3/2}$$

$$= -(x\hat{i} + y\hat{j} + z\hat{k})(x^2 + y^2 + z^2)^{-\frac{3}{2}} = -\frac{\vec{r}}{r^3}$$

Example 16: Find a unit vector normal to the surface $xy^2+xz=1$ at the point $(-1,1,1)$.

Solution:

$$\nabla(xy^2 + xz) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(xy^2 + xz)$$

$$= (y^2 + z)\hat{i} + (2xy)\hat{j} + x\hat{k}$$

$$= 2\hat{i} - 2\hat{j} - \hat{k},\ at(-1, 1, 1)$$

A unit vector normal to the surface is obtained by dividing the above vector by its magnitude. Hence the unit vector is

$$(2\hat{i} - 2\hat{j} - \hat{k})[(2)^2 + (-2)^2 + (-1)^2]^{-1/2} = \frac{2}{3}\hat{i} - \frac{2}{3}\hat{j} - \frac{1}{3}\hat{k}$$

Example 17:

(a) Show that F = $(2xy + z^2)\,\hat{\imath} + x^2\,\hat{\jmath} +$ ~~xyz~~ 2xz $\hat{k}$, is a conservative force field.

(b) Find the scalar potential.

(c) Find the work done in moving a unit mass in this field from the point (1, 0, 1) to (2, 1, −1).

Solution: (a) It is sufficient to show that Curl $\boldsymbol{F} = 0$

$$\nabla \times \boldsymbol{F} = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2xy + z^2 & x^2 & 2xz \end{vmatrix} = \hat{\imath}.0 - \hat{\jmath}(2z - 2z) + \hat{k}(2x - 2x) = 0$$

(b) $d\Phi = F.\,d\boldsymbol{r} = ((2xy + z^2)\hat{\imath} + x^2\hat{\jmath} + 2xz\hat{k}).(\hat{\imath}dx + \hat{\jmath}dy + \hat{k}dz)$

$= (2xy + z^2)\,dx + x^2\,dy + 2xz dz$

$= (2xydx + x^2\,dy) + (z^2\,dx + 2xzdz)$

$= d(x^2y) + d(z^2x) = d(x^2y + xz^2)$

Therefore $\Phi = x^2y + xz^2 + \text{constant}$

(c) Work done $= \Phi_2 - \Phi_1 = 5.0$

Example 18: If $\phi = x^2yz^3$ and A= $xz\mathbf{i} - y^2\mathbf{j} + 2x^2y\mathbf{k}$, find:

a- $\nabla\phi$ **b-** $\nabla.$ **A** **c-** $\nabla\times$ **A** **d-** div (ϕ**A**) **e-** curl (ϕ**A**)

Solution:

(a) $$\nabla\phi = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k} = \frac{\partial}{\partial x}(x^2yz^3)\mathbf{i} + \frac{\partial}{\partial y}(x^2yz^3)\mathbf{j} + \frac{\partial}{\partial z}(x^2yz^3)\mathbf{k}$$
$$= 2xyz^3\mathbf{i} + x^2z^3\mathbf{j} + 3x^2yz^2\mathbf{k}$$

(b) $$\nabla\cdot\mathbf{A} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\cdot(xz\mathbf{i} - y^2\mathbf{j} + 2x^2y\mathbf{k})$$
$$= \frac{\partial}{\partial x}(xz) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(2x^2y) = z - 2y$$

(c) $$\nabla\times\mathbf{A} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\times(xz\mathbf{i} - y^2\mathbf{j} + 2x^2y\mathbf{k})$$
$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xz & -y^2 & 2x^2y \end{vmatrix}$$
$$= \left(\frac{\partial}{\partial y}(2x^2y) - \frac{\partial}{\partial z}(-y^2)\right)\mathbf{i} + \left(\frac{\partial}{\partial z}(xz) - \frac{\partial}{\partial x}(2x^2y)\right)\mathbf{j} + \left(\frac{\partial}{\partial x}(-y^2) - \frac{\partial}{\partial y}(xz)\right)\mathbf{k}$$
$$= 2x^2\mathbf{i} + (x - 4xy)\mathbf{j}$$

$$(d)\quad \text{div}\,(\phi\mathbf{A}) = \nabla\cdot(\phi\mathbf{A}) = \nabla\cdot(x^3yz^4\mathbf{i} - x^2y^3z^3\mathbf{j} + 2x^4y^2z^3\mathbf{k})$$
$$= \frac{\partial}{\partial x}(x^3yz^4) + \frac{\partial}{\partial y}(-x^2y^3z^3) + \frac{\partial}{\partial z}(2x^4y^2z^3)$$
$$= 3x^2yz^4 - 3x^2y^2z^3 + 6x^4y^2z^2$$

$$(e)\quad \text{curl}\,(\phi\mathbf{A}) = \nabla\times(\phi\mathbf{A}) = \nabla\times(x^3yz^4\mathbf{i} - x^2y^3z^3\mathbf{j} + 2x^4y^2z^3\mathbf{k})$$
$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^3yz^4 & -x^2y^3z^3 & 2x^4y^2z^3 \end{vmatrix}$$
$$= (4x^4yz^3 - 3x^2y^3z^2)\mathbf{i} + (4x^3yz^3 - 8x^3y^2z^3)\mathbf{j} - (2xy^3z^3 + x^3z^4)\mathbf{k}$$

Example 19: Prove $\nabla\cdot(\phi\mathbf{A}) = (\nabla\phi)\cdot\mathbf{A} + \phi(\nabla\cdot\mathbf{A})$.

Solution:

$$\nabla\cdot(\phi\mathbf{A}) = \nabla\cdot(\phi A_1\mathbf{i} + \phi A_2\mathbf{j} + \phi A_3\mathbf{k})$$
$$= \frac{\partial}{\partial x}(\phi A_1) + \frac{\partial}{\partial y}(\phi A_2) + \frac{\partial}{\partial z}(\phi A_3)$$
$$= \frac{\partial\phi}{\partial x}A_1 + \frac{\partial\phi}{\partial y}A_2 + \frac{\partial\phi}{\partial z}A_3 + \phi\left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right)$$
$$= \left(\frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}\right)\cdot(A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k})$$
$$+ \phi\left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)\cdot(A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k})$$
$$= (\nabla\phi)\cdot\mathbf{A} + \phi(\nabla\cdot\mathbf{A})$$

Example 20: prove div curl $\mathbf{A} = 0$

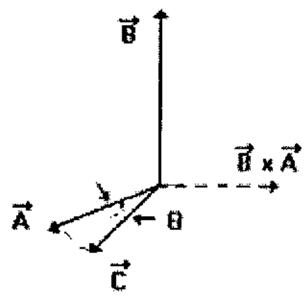
Solution:

$$\text{div curl}\,\mathbf{A} = \nabla\cdot(\nabla\times\mathbf{A}) = \nabla\cdot\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_1 & A_2 & A_3 \end{vmatrix}$$
$$= \nabla\cdot\left[\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)\mathbf{k}\right]$$
$$= \frac{\partial}{\partial x}\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)$$
$$= \frac{\partial^2 A_3}{\partial x\,\partial y} - \frac{\partial^2 A_2}{\partial x\,\partial z} + \frac{\partial^2 A_1}{\partial y\,\partial z} - \frac{\partial^2 A_3}{\partial y\,\partial x} + \frac{\partial^2 A_2}{\partial z\,\partial x} - \frac{\partial^2 A_1}{\partial z\,\partial y}$$
$$= 0$$

Problems of chapter one

1- **Assume that two vectors A and B are known. Let C be an unknown vector such that A · C = *u* is a known quantity and A × C = B. Find C in terms of A, B, *u*, and the magnitude of A.**

Solution:



$$B = |\vec{B}| = |\vec{A}\times\vec{C}| = AC\sin\theta \qquad \therefore C_y = C\sin\theta = \frac{B}{A}$$

$$\vec{A}\cdot\vec{C} = AC\cos\theta = u \qquad \therefore C_x = C\cos\theta = \frac{u}{A}$$

$$\vec{C} = \frac{\vec{A}}{A}C_x + \frac{\vec{B}\times\vec{A}}{|\vec{B}\times\vec{A}|}C_y = \frac{u}{A^2}\vec{A} + \frac{\vec{B}\times\vec{A}}{AB}\left(\frac{B}{A}\right)$$

$$= \frac{u}{A^2}\vec{A} + \frac{1}{A^2}\vec{B}\times\vec{A}$$

2- Find a unit vector normal to the plane containing the two vectors

$$\vec{A} = 2\hat{i} + \hat{j} - \hat{k} \qquad and \qquad \vec{B} = \hat{i} - \hat{j} + 2\hat{k}$$

Solution:

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & -\hat{j} & \hat{k} \\ 2 & 1 & -1 \\ 1 & -1 & 2 \end{vmatrix} = \hat{i} - 5\hat{j} - 3\hat{k} \text{ and } |\vec{A}\times\vec{B}| = \sqrt{1^2 + (-5)^2 + (-3)^2} = \sqrt{35}$$

$$\because \hat{n} = \frac{\vec{A}\times\vec{B}}{|\vec{A}\times\vec{B}|} \qquad (unit\ vector = \frac{the\ vector}{its\ magnitude})$$

$$\therefore \hat{n} = \frac{1}{\sqrt{35}}(\hat{i} - 5\hat{j} - 3\hat{k})$$

3- A racing car moves on a circle of constant radius b. If the speed of the car varies with time t according to the equation $v = ct$ where c is a positive constant, show that the angle between the velocity vector and the acceleration vector is 45° at time t = $\sqrt{b/c}$ (Hint: *At this time the* ***tangential*** *and* ***normal*** *components of the acceleration* ***are equal in magnitude.***)

Solution:

$$\mathbf{v} = v\boldsymbol{\tau} = ct\boldsymbol{\tau}$$

$$\mathbf{a} = \dot{v}\boldsymbol{\tau} + \frac{v^2}{\rho}\mathbf{n} = c\,\boldsymbol{\tau} + \frac{c^2t^2}{b}\mathbf{n}$$

$$\text{at } t=\sqrt{\frac{b}{c}}.\quad \vec{v}=\hat{\tau}\sqrt{bc} \quad\text{and}\quad \vec{a}=c\hat{\tau}+c\hat{n}$$

$$\cos\theta=\frac{\vec{v}\cdot\vec{a}}{va}=\frac{c\sqrt{bc}}{\sqrt{bc}\sqrt{2c^2}}=\frac{1}{\sqrt{2}}$$

$$\theta=45^\circ$$

4- A small ball is fastened to a long rubber band and twirled around in such away that the ball moves in an elliptical path given by the equation:

$$\mathbf{r}(t)=\mathbf{i}b\cos\omega t+\mathbf{j}2b\sin\omega t$$

where b and ω are constants. Find the speed of the ball as a function of t. In particular, find v at t =0 and at t = $\pi/2\omega$, at which times the ball is, respectively, at its minimum and maximum distances from the origin.

Solution: $\vec{v}(t)=-\hat{i}b\omega\sin(\omega t)+\hat{j}2b\omega\cos(\omega t)$

$$|\vec{v}|=\left(b^2\omega^2\sin^2\omega t+4b^2\omega^2\cos^2\omega t\right)^{\frac{1}{2}}=b\omega\left(1+3\cos^2\omega t\right)^{\frac{1}{2}}$$

$$\vec{a}(t)=-\hat{i}b\omega^2\cos\omega t-\hat{j}2b\omega^2\sin\omega t$$

$$|\vec{a}|=b\omega^2\left(1+3\sin^2\omega t\right)^{\frac{1}{2}}$$

$$\text{at}\quad t=0,\quad |\vec{v}|=2b\omega;\quad \text{at}\quad t=\frac{\pi}{2\omega},\quad |\vec{v}|=b\omega$$

5- A bee goes out from its hive in a spiral path given in plane polar coordinates by $r=be^{kt}$, $\theta=ct$ where b, k, and c are positive constants. Show that the angle between the velocity vector and the acceleration vector remains constant as the bee moves outward. (Hint: Find **v**. **a**/va)

Solution: $\vec{v}=\dot{r}\hat{e}_r+r\dot{\theta}\hat{e}_\theta=bke^{kt}\hat{e}_r+bce^{kt}\hat{e}_\theta$

$$\vec{a}=\left(\ddot{r}-r\dot{\theta}^2\right)\hat{e}_r+\left(r\ddot{\theta}+2\dot{r}\dot{\theta}\right)\hat{e}_\theta=b\left(k^2-c^2\right)e^{kt}\hat{e}_r+2bcke^{kt}\hat{e}_\theta$$

$$\cos\phi=\frac{\vec{v}\cdot\vec{a}}{va}=\frac{b^2k\left(k^2-c^2\right)e^{2kt}+2b^2c^2ke^{2kt}}{be^{kt}\left(k^2+c^2\right)^{\frac{1}{2}}be^{kt}\left[\left(k^2-c^2\right)^2+4c^2k^2\right]^{\frac{1}{2}}}$$

$$\cos\phi=\frac{k\left(k^2+c^2\right)}{\left(k^2+c^2\right)^{\frac{1}{2}}\left(k^2+c^2\right)}=\frac{k}{\left(k^2+c^2\right)^{\frac{1}{2}}},\quad\text{a constant}$$

6- An ant crawls on the surface of a ball of radius b in such a manner that the ant's motion is given in spherical coordinates by the equations

$$r = b \qquad \phi = \omega t \qquad \theta = \frac{\pi}{2}\left[1 + \frac{1}{4}\cos(4\omega t)\right]$$

Find the speed of the ant as a function of the lime t. What sort of path is represented by the above equations?

Solution:

$$\vec{v} = \hat{e}_r \dot{r} + \hat{e}_\phi r\dot{\phi}\sin\theta + \hat{e}_\theta r\dot{\theta}$$

$$\vec{v} = \hat{e}_\phi b\omega \sin\left\{\frac{\pi}{2}\left[1 + \frac{1}{4}\cos(4\omega t)\right]\right\} - \hat{e}_\theta b\frac{\pi}{2}\omega \sin(4\omega t)$$

$$\vec{v} = \hat{e}_\phi b\omega \cos\left[\frac{\pi}{8}\cos(4\omega t)\right] - \hat{e}_\theta b\omega \frac{\pi}{2}\sin(4\omega t)$$

$$|\vec{v}| = b\omega\left[\cos^2\left(\frac{\pi}{8}\cos 4\omega t\right) + \frac{\pi^2}{4}\sin^2 4\omega t\right]^{\frac{1}{2}}$$

Path is sinusoidal oscillation about the equator.

7- Prove that **v. a** = $v\dot{v}$ and, hence, that for a moving particle **v** and **a** are perpendicular to each other if the speed v is constant. (Hint: *Differentiate both sides of the equation* $v \bullet v = v^2$ *with respect to t. Note,* $\dot{v}$ *is not the same as* $|a|$ *I. It is the magnitude of the acceleration of the partied along its instantaneous direction of motion.*)

Solution:

$$\vec{v} \cdot \vec{v} = v^2$$

$$\frac{d\vec{v}}{dt} \cdot \vec{v} + \vec{v} \cdot \frac{d\vec{v}}{dt} = 2v\dot{v}$$

$$2\vec{v} \cdot \vec{a} = 2v\dot{v}$$

$$\vec{v} \cdot \vec{a} = v\dot{v}$$

8- Show that the tangential component of The acceleration of a moving particle is given by the expression:

$$a_\tau = \frac{\mathbf{v}\cdot\mathbf{a}}{v}$$

and the normal component is therefore $a_n = \left(a^2 - a_\tau^2\right)^{1/2} = \left[a^2 - \frac{(\mathbf{v}\cdot\mathbf{a})^2}{v^2}\right]^{1/2}$

solution:

$$\vec{v} = v\hat{\tau} \text{ and } \vec{a} = a_\tau\hat{\tau} + a_n\hat{n}$$

$$\vec{v}\cdot\vec{a} = va_\tau, \text{ so } a_\tau = \frac{\vec{v}\cdot\vec{a}}{v}$$

$$a^2 = a_\tau^2 + a_n^2, \text{ so } a_n = \left(a^2 - a_\tau^2\right)^{\frac{1}{2}}$$

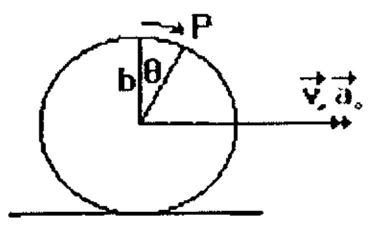
9- A wheel of radius b rolls along the ground with constant forward acceleration a_o. Show that, at any given instant, the magnitude of the acceleration of any point on the wheel is $(a_o^2 + v^4/b^2)^{1/2}$ relative to the center of the wheel and is also $a_o[2 + 2cos\theta + v^4/a_o^2b^2 - (2v^2/a_ob)sin\theta]^{1/2}$ relative to the ground. Here v is the instantaneous forward speed, and θ defines the location of the point on the wheel, measured forward from the highest point. Which point has the greatest acceleration relative to the ground?

Solution

$$\vec{r}_{oP} = \hat{i}b\sin\theta + \hat{j}b\cos\theta$$

$$\vec{v}_{rel} = \hat{i}b\dot{\theta}\cos\theta - \hat{j}b\dot{\theta}\sin\theta$$

$$\vec{a}_{rel} = \hat{i}b\left(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta\right) - \hat{j}b\left(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta\right)$$



at the point $\theta = \frac{\pi}{2}$, $\vec{v}_{rel} = -\vec{v}$

So, $|\vec{v}_{rel}| = b\dot{\theta} = v$

$$\dot{\theta} = \frac{v}{b} \qquad \ddot{\theta} = \frac{\dot{v}}{b} = \frac{a_o}{b}$$

Now, $\vec{a}_{rel} = \dot{v}_{rel}\hat{\tau} + \frac{v_{rel}^2}{\rho}\hat{n} = a_o\hat{\tau} + \frac{v^2}{b}\hat{n}$

$$|\vec{a}_{rel}| = \left(a_o^2 + \frac{v^4}{b^2}\right)^{\frac{1}{2}}$$

$\vec{v}_P = \vec{v} + \vec{v}_{rel}$ and $\vec{a}_P = \vec{a}_o + \vec{a}_{rel}$

$$\vec{a}_P = \hat{i}\left[a_o + b\left(\frac{a_o}{b}\cos\theta - \frac{v^2}{b^2}\sin\theta\right)\right] - \hat{j}b\left(\frac{a_o}{b}\sin\theta + \frac{v^2}{b^2}\cos\theta\right)$$

$$|\vec{a}_P| = a_o\left(2 + 2\cos\theta + \frac{v^4}{a_o^2 b^2} - \frac{2v^2}{a_o b}\sin\theta\right)^{\frac{1}{2}}$$

$\vec{a}_P$ is a maximum at $\theta = 0$, i.e., at the top of the wheel.

$$-2\sin\theta - \frac{2v^2}{a_o b}\cos\theta = 0$$

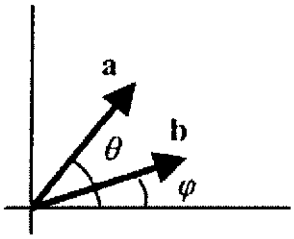
$$\theta = \tan^{-1}\left(-\frac{v^2}{a_o b}\right)$$

10- Use vector algebra to derive the following trigonometric identities

(a) $\cos(\theta - \phi) = \cos\theta\cos\phi + \sin\theta\sin\phi$

(b) $\sin(\theta - \phi) = \sin\theta\cos\phi - \cos\theta\sin\phi$

Solution:



(a) $a = \hat{i}\cos\theta + \hat{j}\sin\theta$

$b = \hat{i}\cos\varphi + \hat{j}\sin\varphi$

$a \cdot b = \cos(\theta - \varphi) = \left(\hat{i}\cos\theta + \hat{j}\sin\theta\right)\cdot\left(\hat{i}\cos\varphi + \hat{j}\sin\varphi\right)$

$\cos(\theta - \varphi) = \cos\theta\cos\varphi + \sin\theta\sin\varphi$

(b) $b \times a = |\hat{k}|\sin(\theta - \varphi) = \left|\left(\hat{i}\cos\theta + \hat{j}\sin\theta\right)\times\left(\hat{i}\cos\varphi + \hat{j}\sin\varphi\right)\right|$

$\sin(\theta - \varphi) = \sin\theta\cos\varphi - \cos\theta\sin\varphi$

Problems (homework)

1- Consider a cube whose edges are each of unit length. One corner coincides with the origin of an xyz Cartesian coordinate system. Three of the cube's edges extend from the origin along the positive direction of each coordinate axis. Find the vector that begins at the origin and extends

(a) along a major diagonal of the cube;

(b) along the diagonal of the lower face of the cube.

(c) Calling these vectors **A** and **B**, find $\mathbf{C} = \mathbf{A} \times \mathbf{B}$.

(d) Find the angle between **A** and **B**.

2- Given the time-varying vector

$$\mathbf{A} = \mathbf{i}\alpha t + \mathbf{j}\beta t^2 + \mathbf{k}\gamma t^3$$

where α, β, and γ are constants, find the first and second time derivatives $d\mathbf{A}/dt$ and $d^2\mathbf{A}/dt^2$.

For what value (or values) of q is the vector $\mathbf{A} = \mathbf{i}q + 3\mathbf{j} + \mathbf{k}$ perpendicular to the vector $\mathbf{B} = \mathbf{i}q - q\mathbf{j} + 2\mathbf{k}$?

3- A buzzing fly moves in a helical path given by the equation

$$\mathbf{r}(t) = \mathbf{i}b \sin \omega t + \mathbf{j}b \cos \omega t + \mathbf{k}ct^2$$

Show that the magnitude of the acceleration of the fly is constant, provided b, ω, and c are constant.

4- A particle moves in a helical path such that its position in cylindrical coordinate given by

$$R=b \qquad \phi=\omega t \qquad z=ct^2$$

Find the speed and the magnitude of the acceleration as a function of t.

5- Express the vector $2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$ in the primed triad $\mathbf{i}'\mathbf{j}'\mathbf{k}'$ in which the $x'y'$-axes are rotated about the z-axis (which coincides with the z'-axis) through an angle of 30°.

Analytical Mechanics

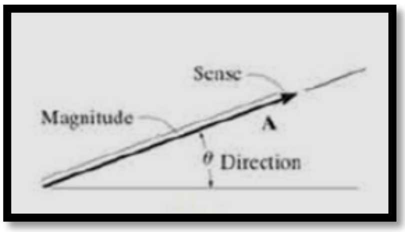
Chapter One

1-1: Scalar and vectors:

A **scalar** is any **positive** or **negative physical quantity** that can be completely specified by its **magnitude**. such as mass, length, density, volume and temperature.

A **vector** is any physical quantity that requires both a **magnitude** and **direction** for its complete description. A vector is shown **graphically** by an **arrow**. The **length** of the arrow represents the **magnitude** of the vector, and a fixed axis defines the **direction** of its line of action .The **head** of the arrow indicates the **sense** of **direction of the vector** (Fig 1- 1).

Fig. (1-1)



For **handwritten** work, it is often **convenient** to denote a vector quantity by simply drawing an arrow on top it A, ($\vec{A}$) .In **print**, vector quantities are represented by **bold** face letters such as **A**, and its **magnitude** of the vector is **italicized**, *A*.

1-2:Geometric Properties :

1. Two vectors A and B are equal if they have the same magnitude and direction regardless of their initial points. Thus **A** = **B**.

2. A vector having direction opposite to that of vector A but with the same magnitude is denoted by -A .

3. The sum or resultant of vectors **A** and **B** of Fig. 1-2(a) below is a vector C formed by placing the initial point of B on the terminal point of A and joining the initial point of A to the terminal point of B [see Fig. 1-2(b) below].

The sum C is written C = A + B. The definition here is equivalent to the parallelogram law for vector addition as indicated in Fig.1-2(c) below.

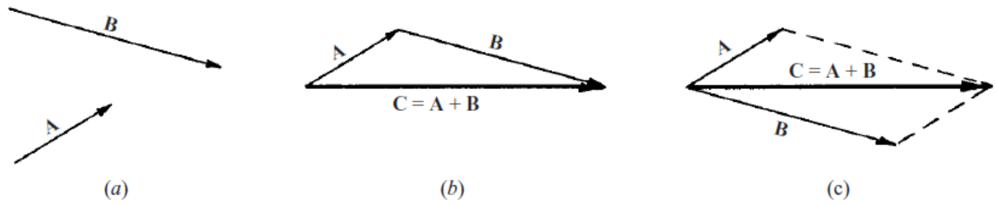


Fig.(1-2)

4. The difference of vectors **A** and **B**, represented by A - B, is that vector C which added to B gives A. Equivalently, A - B may be defined as A + (-B). If A = B, then A - B is defined as the null or zero vector and is represented by the symbol **0**. This has a magnitude of zero but its direction is not defined.

5. Multiplication of a vector **A** by a scalar m produces a vector m**A** with magnitude $|m|$ times the magnitude of **A** and direction the same as or opposite to that of **A** according as m is positive or negative. If m = 0, mA = 0, the null vector.

1-3: Algebraic Properties of Vectors:

The following algebraic properties are consequences of the geometric definition of a vector. If A, B and C are vectors, and m and n are scalars, then:

1. A + B = B + A Commutative Law for Addition

2. A +(B + C) = (A + B) + C Associative Law for Addition

3. m(n**A**)= **(**mn**)A =** n(m**A**) Associative Law for Multiplication

4. (m + n)**A =** m**A +** n**A** Distributive Law

5. m**(A + B)** = m**A** + m**B** Distributive Law

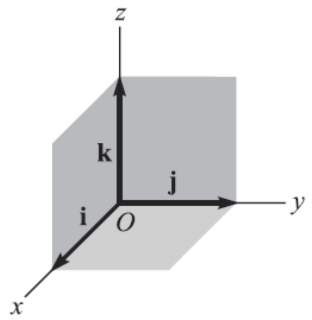
6. A+0=A **7. A+(-A)= 0**

Note that in these laws only multiplication of a vector by one or more scalars is defined.

1-4: Unit Vectors:

Unit vectors are vectors having unit length. If **A** is any vector with length $A > 0$, then A/A is a unit vector, denoted by **a**, having the same direction as A. Then **A** = Aa. The rectangular unit vectors i, j, and k are unit vectors having the direction of the positive x, y, and z axes of a rectangular coordinate system [see Fig. 1-3].

Fig.(1-3)



1-5 Components of A Vectors:

Any vector **A** in 3 dimensions can be represented with initial point at the origin O of a rectangular coordinate system [see Fig. 1-4]. Let (A_1;A_2;A_3) be the rectangular coordinates of the terminal point of vector A with initial point at O. The vectors A_1i;A_2j; and A_3k are called the rectangular component vectors, or simply **component vectors**, of A in the x, y; and z directions respectively. The sum or resultant of A_1i;A_2j; and A_3k is the vector **A,** so that we can write:

$$A = A_1 i + A_2 j + A_3 k$$

The magnitude of **A** is:

$$A = |\mathbf{A}| = \sqrt{A_1^2 + A_2^2 + A_3^2}$$

In particular, the position vector or radius vector r from O to the point (x; y; z) is written:

$$r = xi + yj + zk$$

and has magnitude:

$$r = |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}.$$

1-6: Dot OR Scalar Product:

The dot or scalar product of two vectors **A** and **B**, denoted by A.B (read **A** dot **B**) is defined as the product of the magnitudes of *A* and *B* and the of the angle between them(projection of **A** on **B**). In symbols,

$$\mathbf{A} \cdot \mathbf{B} = AB \cos \theta; \quad 0 < \theta < \pi$$

Assuming that neither **A** nor **B** is the zero vector, an immediate consequence of the definition is that **A . B** = 0 if and only if **A** and **B** are perpendicular. Note that A . B is a scalar and not a vector. The following laws are valid:

1. **A . B = B . A** Commutative Law for Dot Products
2. **A . (B + C)** = **A . B** + **A . C** Distributive Law
3. m(**A . B**) = (m**A**) . **B** = **A** . (m**B**) = (**A . B**)m, where m is a scalar.
4. i . i = j . j = k . k = 1; i . j = j . k = k . i = 0
5. If **A** = A_1i + A_2j + A_3k and **B** = B_1i + B_2j + B_3k, then

$$\mathbf{A} \cdot \mathbf{B} = A_1B_1 + A_2B_2 + A_3B_3$$

In particular, we can write:

$$\mathbf{A} \cdot \mathbf{B} = A_xB_x + A_yB_y + A_zB_z$$

6. The square of the magnitude of a vector **A** is given by the dot product of A with itself,

$$\mathbf{A}^2 = |\mathbf{A}|^2 = \mathbf{A}.\mathbf{A}$$

As an example of the dot product, suppose that an object under the action of a constant force undergoes a linear displacement **Δs**, (as shown in Figure 1-4) .By definition, the work ΔW done by the force is given by the product of the component of the force **F** in the direction of **Δs**, multiplied by the magnitude Δ*s* of the displacement; that is,

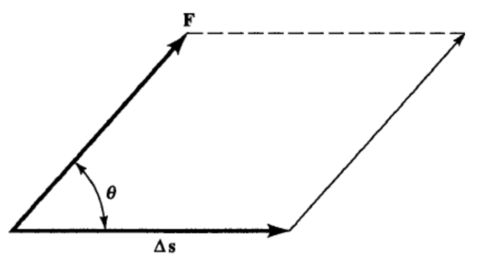


Fig.1-4

$$\Delta W=(F\cos\theta)\Delta s$$

where θ is the angle between **F** and **Δs**. But the expression on the right is just the dot product of **F** and **Δs**, that is,

$$\mathbf{\Delta W=F.\Delta s}$$

1-7: CROSS OR VECTOR PRODUCT:

The cross or vector product of **A** and **B** is a vector **C** = **A** × **B** (read **A** cross **B**). The magnitude of **A** × **B** is defined as the product of the magnitudes of **A** and **B** and the sine of the angle between them. The direction of the vector **C = A × B** is perpendicular to the plane of **A** and **B** and such that **A, B,** and **C** form a right-handed system. In symbols,

$$\mathbf{A} \times \mathbf{B} = AB \sin \theta \mathbf{u}; \qquad 0 \leq \theta \leq \pi$$

where **u** is a unit vector indicating the direction of **A** × **B**. IfA = B or if A is parallel to B, then sin θ = 0 and A × B = 0.

The following laws are valid:

1. **A × B = -B × A** (Commutative Law for Cross Products Fails)

2. **A × (B + C) = A× B + A × C** Distributive Law

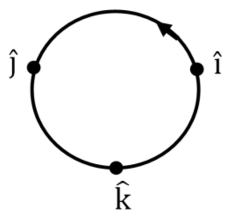
3. m(**A × B**) = (m**A**) × **B = A** × (m**B**) = (**A × B**)m, where m is a scalar.

Also the following consequences of the definition are important:

4. i × i = j × j = k × k = 0,

i × j = k = - j × i; j × k = i = -k × j ; k × i = j= -i × k

This cyclic nature of the cross product can be emphasized by diagramming the multiplication table as shown in Figure



5. |**A** × **B** |= the area of a parallelogram with sides **A** and **B.**

6. If **A** × **B** = 0 and neither **A** nor **B** is a null vector, then **A** and **B** are parallel.

7. If **A** =A_xi + A_yj + A_zk and **B** = B_xi + B_yj + B_zk, then

$$\mathbf{A}\times\mathbf{B} = \begin{vmatrix} i & j & k \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$\mathbf{A}\times\mathbf{B} = \mathbf{i}\begin{vmatrix} A_y & A_z \\ B_y & B_z \end{vmatrix} + \mathbf{j}\begin{vmatrix} A_z & A_x \\ B_z & B_x \end{vmatrix} + \mathbf{k}\begin{vmatrix} A_x & A_y \\ B_x & B_y \end{vmatrix}$$

The cross product expressed in i, j ,k form is:

$$\mathbf{A}\times\mathbf{B} = \mathbf{i}(A_yB_z - A_z\mathrm{B}_y) + \mathbf{j}(A_zB_x - A_xB_z) + \mathbf{k}(A_xB_y - \mathrm{A}_yB_x)$$

Let us calculate the magnitude of the cross product. We have

$$|\mathbf{A}\times\mathbf{B}|^2 = (A_yB_z - A_zB_y)^2 + (A_zB_x - A_xB_z)^2 + (A_xB_y - A_yB_x)^2$$

This can be reduced to:

$$|\mathbf{A}\times\mathbf{B}|^2 = \left(A_x^2 + A_y^2 + A_z^2\right)\left(B_x^2 + B_y^2 + B_z^2\right) - (A_xB_x + A_yB_y + A_zB_z)^2$$

from the definition of the dot product, the above equation may be written in the form

$$|\mathbf{A}\times\mathbf{B}|^2 = A^2B^2 - (\mathbf{A}\cdot\mathbf{B})^2$$

Taking the square root of both sides of Equation above we can express the magnitude of the cross product as

$$|\mathbf{A}\times\mathbf{B}| = AB(1-\cos^2\theta)^{1/2} = AB\sin\theta$$

where θ is the angle between **A** and **B**.

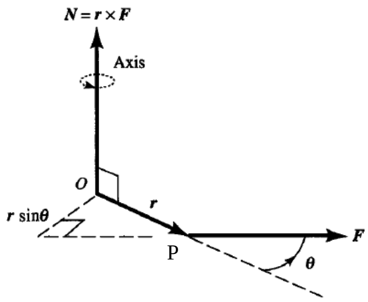
physical example of the cross product is the rotational moment (torque)

$$\boldsymbol{\tau} = \mathbf{r}\times\mathbf{F}$$

Let a force **F** act at a point P(x, y, z), as shown in Figure 1-5, and let the vector **OP** be designated by **r**; that is,

OP=r=ix+jy+kz

Figure 1-5: Illustration of the moment of a force about a point 0.



The moment **N** of force, or the torque **N**, about a given point 0 is defined as the cross product

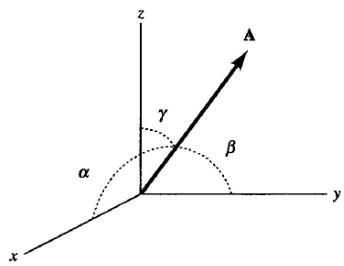
$$\mathbf{N} = \mathbf{r} \times \mathbf{F}$$

Thus, the moment of a force about a point is a having a magnitude and a direction. If a single force is applied at a point P on a body that is initially at rest and is free to turn about a fixed point 0 as a pivotمحور, then the body tends to rotate. The axis of this rotation is perpendicular to the force F, and it is also perpendicular to the line OP; therefore, the direction of the torque vector **N** is along the axis of rotation. The magnitude of the torque is given by:

$$|N| = |\mathbf{r} \times \mathbf{F}| = rF\sin\theta$$

in which θ is the angle between **r** and **F**. Thus, |N| can be regarded as the product of the magnitude of the force and the quantity $r \sin \theta$, which is just the perpendicular distance from the line of action of the force to the point 0.

1-8: Representation of a given vector as the product of a scalar and a single unit vector:



Consider the equation:

$$\mathbf{A} = \mathbf{i}A_x + \mathbf{j}A_x + \mathbf{k}A_x$$

Multiply and divide on the right by the magnitude of **A**

$$\mathbf{A} = A\left(\mathbf{i}\frac{A_x}{A} + \mathbf{j}\frac{A_y}{A} + \mathbf{k}\frac{A_z}{A}\right)$$

Now $A_x/A = \cos\alpha$, $A_y/A = \cos\beta$, and = $A_z/A \cos\gamma$ are the direction cosines of the vector A, and α, β, and γ are the direction angles. Thus, we can write are the direction cosines of vector A, and are the direction angles.

Thus we can write

$$A = A(i \cos\alpha + j \cos\beta + k \cos\gamma) = A(\cos\alpha, \cos\beta, \cos\gamma)$$

or

$$\mathbf{A} = \mathbf{n}A$$

where **n** is a unit vector whose components are cosα ,cos β, and cos γ. Consider any other vector B. Clearly, the projection of B on A is just

$$B\cos\theta = \frac{\mathbf{B}\cdot\mathbf{A}}{A} = \mathbf{B}\cdot\mathbf{n}$$

where θ is the angle between A and B.

Example 1:

1- Find the angle between the vectors

2- Find the vector product of the vectors

$$\vec{A} = 2\hat{i} + 3\hat{j} - \hat{k} \qquad and \qquad \vec{B} = -\hat{i} + \hat{j} + 2\hat{k}$$

solution:

1- $\vec{A}.\vec{B} = A_x B_x + A_y B_y + A_z B_z = 2\times(-1) + 3\times 1 + (-1)\times 2 = -1$

$$A = \left|\vec{A}\right| = \sqrt{2^2 + 3^2 + (-1)^2} = 3.74 \ units$$

$$B = \left|\vec{B}\right| = \sqrt{(-1)^2 + 1^2 + 2^2} = 2.45 \ units$$

and since $\vec{A}.\vec{B} = AB\cos\theta \quad \Rightarrow \quad \theta = \cos^{-1}\left(\frac{\vec{A}.\vec{B}}{AB}\right)$

$$\theta = \cos^{-1}\left(\frac{-1}{3.74\times 2.45}\right) = \cos^{-1}(-0.109) = 96.3^0$$

2-

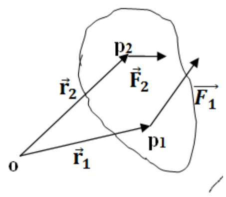
$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & -1 \\ -1 & 1 & 2 \end{vmatrix} = \hat{i}\,(6+1) - \hat{j}\,(4-1) + \hat{k}\,(2+3) = 7\hat{i} - 3\hat{j} + 5\hat{k}$$

Example 2:

A force $\vec{F}_1 = \hat{i} + \hat{j}$ **was applied on a body at a point p_1 such that** $o\vec{p}_1 = \vec{r}_1 = 2\hat{i} + \hat{j}$ **and a second force** $\vec{F}_2 = \hat{j} - \hat{k}$ **was applied at a point** $o\vec{p}_2 = \vec{r}_2 = \hat{i} + \hat{j} + \hat{k}$**. Find: (a) the total moment** $\vec{N}$ **(b) the magnitude of** $\vec{N}$ **(c) the direction cosines of** $\vec{N}$**.**

solution:

(a) $\vec{N}_1 = \vec{r}_1 \times \vec{F}_1$ **and** $\vec{N}_2 = \vec{r}_2 \times \vec{F}_2$



$$\vec{N}_1 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & 0 \\ 1 & 1 & 0 \end{vmatrix} = \hat{k} \qquad \vec{N}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 0 & 1 & -1 \end{vmatrix} = -2\hat{i} + \hat{j} + \hat{k}$$

The total moment is

$$\vec{N} = \vec{N}_1 + \vec{N}_2 = \hat{k} + (-2\,\hat{i} + \hat{j} + \hat{k}) = -2\,\hat{i} + \hat{j} + 2\,\hat{k}$$

(b) $\vec{N} = \sqrt{4+1+4} = 3$

(c)

$$\cos\ \alpha = \frac{N_x}{N} = \frac{-2}{3}, \quad \cos\ \beta = \frac{N_y}{N} = \frac{1}{3} \quad \text{and} \quad \cos\ \gamma = \frac{N_z}{N} = \frac{2}{3}$$

1-9:Triple Products:

The expression

$$\mathbf{A.\,(B \times C)}$$

is called the **scalar triple product** of **A**, **B**, and **C**. It is a scalar because it is the dot product of two vectors. Referring to the determinant expressions for the cross product, we see that the scalar triple product may be written:

$$\mathbf{A}.\,(\mathbf{B} \times \mathbf{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

Because the exchange of the terms of two rows or of two columns of a determinant changes its sign but not its absolute value, we can derive the following useful equation:

1- (A . B)C ≠ A(B . C) in general

2- A . (B × C) =B. (C × A) = C . (A× B) = volume of a parallelepiped having A, B, and C as edges, or the negative of this volume according as A, B, and C do or do not form a right handed system. Thus, the dot and the cross may be interchanged in the scalar triple product.

The expression

$$\mathbf{A\times (B \times C)}$$

is called the **vector triple product**. The following equation holds for the vector triple product:

1- **A× (B × C) ≠ (A× B) × C** (Associative Law for Cross Products Fails)

2- **A×(B×C)= (A.C) B – (A.B) C**

A×(B×C)=(A . C)B – (B . C) A

Vector triple products are particularly useful in the study of rotating coordinate systems and rotations of rigid bodies.

Example 3:

Given the three vectors **A = 1, B = i—j,** and **C = k**, find

1- **A. (B × C)**

2- A× (B × C)

Solution:

1-

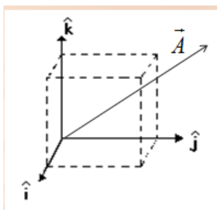
$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \begin{vmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1(-1+0) = -1$$

2-

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B}) = (\mathbf{i}-\mathbf{j})0 - \mathbf{k}(1-0) = -\mathbf{k}$$

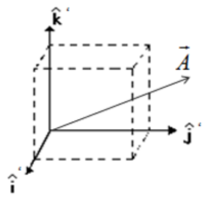
1-10:Change of Coordinate System :The Transformation Matrix:

The rotation of a Cartesian coordinate system is an example of an orthogonal transformation. Consider the vector A expressed relative to the triad ijk:



$$\mathbf{A} = \mathbf{i}A_x + \mathbf{j}A_y + \mathbf{k}A_z$$

Relative to a new triad **i'j'k'** having a different orientation from that of **ijk**, the same vector **A** is expressed as



$$\mathbf{A} = \mathbf{i}'A_{x'} + \mathbf{j}'A_{y'} + \mathbf{k}'A_{z'}$$

Now the dot product **A. i'** is just $A_{x'}$, that is, the projection of **A** on the unit vector **i'**. Thus, we may write

$$A_{x'} = \mathbf{A}\cdot\mathbf{i}' = (\mathbf{i}\cdot\mathbf{i}')A_x + (\mathbf{j}\cdot\mathbf{i}')A_y + (\mathbf{k}\cdot\mathbf{i}')A_z$$
$$A_{y'} = \mathbf{A}\cdot\mathbf{j}' = (\mathbf{i}\cdot\mathbf{j}')A_x + (\mathbf{j}\cdot\mathbf{j}')A_y + (\mathbf{k}\cdot\mathbf{j}')A_z$$
$$A_{z'} = \mathbf{A}\cdot\mathbf{k}' = (\mathbf{i}\cdot\mathbf{k}')A_x + (\mathbf{j}\cdot\mathbf{k}')A_y + (\mathbf{k}\cdot\mathbf{k}')A_z$$

The scalar products (**i . i'**), (**i . j'**), and so on are called the **coefficients of transformation**. They are equal to the direction cosines of the axes of the primed coordinate system relative to the unprimed system. The unprimed components are similarly expressed as

$$A_x = \mathbf{A}\cdot\mathbf{i} = (\mathbf{i}'\cdot\mathbf{i})A_{x'} + (\mathbf{j}'\cdot\mathbf{i})A_{y'} + (\mathbf{k}'\cdot\mathbf{i})A_{z'}$$
$$A_y = \mathbf{A}\cdot\mathbf{j} = (\mathbf{i}'\cdot\mathbf{j})A_{x'} + (\mathbf{j}'\cdot\mathbf{j})A_{y'} + (\mathbf{k}'\cdot\mathbf{j})A_{z'}$$
$$A_z = \mathbf{A}\cdot\mathbf{k} = (\mathbf{i}'\cdot\mathbf{k})A_{x'} + (\mathbf{j}'\cdot\mathbf{k})A_{y'} + (\mathbf{k}'\cdot\mathbf{k})A_{z'}$$

The equations of transformation are conveniently expressed in matrix notation. Thus,

$$\begin{pmatrix} A_{x'} \\ A_{y'} \\ A_{z'} \end{pmatrix} = \begin{pmatrix} \mathbf{i}\cdot\mathbf{i}' & \mathbf{j}\cdot\mathbf{i}' & \mathbf{k}\cdot\mathbf{i}' \\ \mathbf{i}\cdot\mathbf{j}' & \mathbf{j}\cdot\mathbf{j}' & \mathbf{k}\cdot\mathbf{j}' \\ \mathbf{i}\cdot\mathbf{k}' & \mathbf{j}\cdot\mathbf{k}' & \mathbf{k}\cdot\mathbf{k}' \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$$

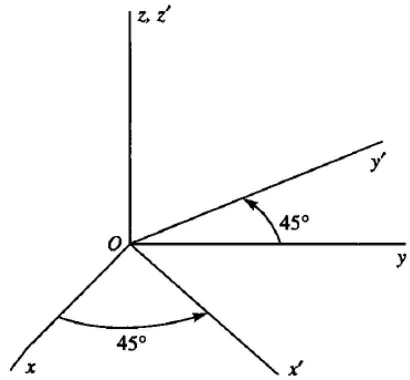
The 3-by-3 matrix in Equation above is called the **transformation matrix**.

Example 4:

Express the vector **A = 3i + 2j + k** in terms of the triad **i'j'k'**, where the x'y'—axes are rotated 45° around the z-axis, with the z- and z'-axes coinciding, as shown in Figure. Referring to the figure, we have for the coefficients of transformation **i . i'** = cos 45° and so on,

we have from the figure:

$$\mathbf{i}\cdot\mathbf{i}' = 1/\sqrt{2} \qquad \mathbf{j}\cdot\mathbf{i}' = 1/\sqrt{2} \qquad \mathbf{k}\cdot\mathbf{i}' = 0$$
$$\mathbf{i}\cdot\mathbf{j}' = -1/\sqrt{2} \qquad \mathbf{j}\cdot\mathbf{j}' = 1/\sqrt{2} \qquad \mathbf{k}\cdot\mathbf{j}' = 0$$
$$\mathbf{i}\cdot\mathbf{k}' = 0 \qquad \mathbf{j}\cdot\mathbf{k}' = 0 \qquad \mathbf{k}\cdot\mathbf{k}' = 1$$



$$\begin{pmatrix} A'_x \\ A'_y \\ A'_z \end{pmatrix} = \begin{pmatrix} \hat{i}\cdot\hat{i}' & \hat{j}\cdot\hat{i}' & \hat{k}\cdot\hat{i}' \\ \hat{i}\cdot\hat{j}' & \hat{j}\cdot\hat{j}' & \hat{k}\cdot\hat{j}' \\ \hat{i}\cdot\hat{k}' & \hat{j}\cdot\hat{k}' & \hat{k}\cdot\hat{k}' \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$$

$$\begin{pmatrix} A'_x \\ A'_y \\ A'_z \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$$

$$\begin{pmatrix} A'_x \\ A'_y \\ A'_z \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$$

These give:

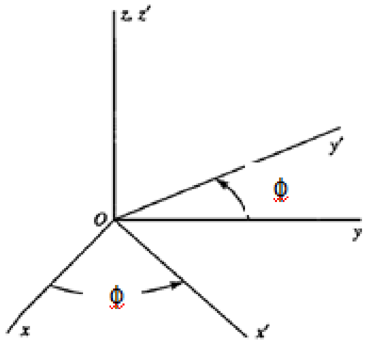
$$A_{x'} = \frac{3}{\sqrt{2}} + \frac{2}{\sqrt{2}} = \frac{5}{\sqrt{2}} \qquad A_{y'} = \frac{-3}{\sqrt{2}} + \frac{2}{\sqrt{2}} = \frac{-1}{\sqrt{2}} \qquad A_{z'} = 1$$

so that, in the primed system, the vector A is given by

$$\vec{A} = \frac{5}{\sqrt{2}}\hat{i}' - \frac{1}{\sqrt{2}}\hat{j}' + \hat{k}'$$

Example 5:

Find the transformation matrix for a rotation of the primed coordinate system through an angle ϕ about the z-axis. We have



$$\mathbf{i}\cdot\mathbf{i}'=\mathbf{j}\cdot\mathbf{j}'=\cos\phi$$
$$\mathbf{j}\cdot\mathbf{i}'=-\mathbf{i}\cdot\mathbf{j}'=\sin\phi$$
$$\mathbf{k}\cdot\mathbf{k}'=1$$

and all other dot products are zero; hence, the transformation matrix is:

$$\begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

the transformation matrix for a rotation about a different coordinate axis-say, the y-axis through an angle (θ) given by the matrix

$$\begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix}$$

Consequently, the matrix for the combination of two rotations, the first being about the z-axis (angle ϕ) and the second being about the new y' -axis (angle (θ), is given by the matrix product

$$\begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}=\begin{pmatrix} \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \\ \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \end{pmatrix}$$

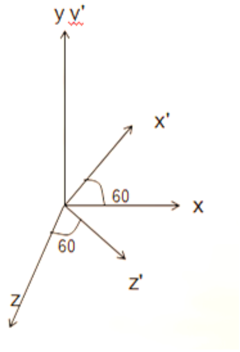
Now matrix multiplication is, in general, noncommutative; therefore, we might expect that the result would be different if the order of the rotations, and, therefore, the order of the matrix multiplication, were reversed.

Example6:

Express the vector $\hat{i}+\hat{j}$ in terms of the triad $\hat{i}'\ \hat{j}'\ \hat{k}'$ where the x' z' axes are rotated 60 degree around the y axis, and the y' axes coinciding.

Solution:

From the figure . we can see that



$$\hat{i}\cdot\hat{i}' = \hat{k}\cdot\hat{k}' = \cos 60^0 = \frac{1}{2}$$

$$\hat{i}\cdot\hat{k}' = \sin 60^0 = \frac{\sqrt{3}}{2} \qquad \hat{i}\cdot\hat{j}' = \cos 90 = 0$$

$$\hat{k}\cdot\hat{i}' = \cos(90+60) = -\sin 60 = -\frac{\sqrt{3}}{2}$$

the rest are zero

Therefore the transformation matrix and the equation of transformation becomes:

$$\begin{pmatrix} A'_x \\ A'_y \\ A'_z \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 & -\frac{\sqrt{3}}{2} \\ 0 & 1 & 0 \\ \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$\vec{A} = \frac{1}{2}\hat{i}' + \hat{j}' + \frac{\sqrt{3}}{2}\hat{k}'$$

$$\cos\left(\frac{\pi}{2}+x\right) = -\sin x$$
$$\cos\left(\frac{\pi}{2}-x\right) = \sin x$$
$$\cos(\pi \pm x) = -\cos x$$
$$\cos\left(\frac{3\pi}{2}+x\right) = \sin x$$
$$\cos\left(\frac{3\pi}{2}-x\right) = -\sin x$$

$$\tan\left(\frac{\pi}{2}+x\right) = -\cot x$$
$$\tan\left(\frac{\pi}{2}-x\right) = \cot x$$
$$\tan(\pi + x) = \tan x$$
$$\tan(\pi - x) = -\tan x$$
$$\tan\left(\frac{3\pi}{2}+x\right) = -\cot x$$
$$\tan\left(\frac{3\pi}{2}-x\right) = \cot x$$

$$\sin\left(\frac{\pi}{2}\pm x\right) = \cos x$$
$$\sin(\pi + x) = -\sin x$$
$$\sin(\pi - x) = \sin x$$
$$\sin\left(\frac{3\pi}{2}\pm x\right) = -\cos x$$

$$\cos^2 x + \sin^2 x = 1$$

$$\cos(2\alpha) = 2\cos^2\alpha - 1 \qquad \cos(2\alpha) = \cos^2\alpha - \sin^2\alpha \qquad \cos(2\alpha) = 1 - 2\sin^2\alpha$$

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \qquad \sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta \qquad \cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

Solved Problems (page 44):

1.1: Given the two vectors **A = i +j** and **B =j + k**, find the following:

(a) **A+B** and **|A + B|** (b) 3**A**-2**B** (c) **A. B** (d) **A×B** and **|A + B|**

Solution:

(a) $\vec{A}+\vec{B}=(\hat{i}+\hat{j})+(\hat{j}+\hat{k})=\hat{i}+2\hat{j}+\hat{k}$

$$\left|\vec{A}+\vec{B}\right|=(1+4+1)^{\frac{1}{2}}=\sqrt{6}$$

(b) $3\vec{A}-2\vec{B}=3(\hat{i}+\hat{j})-2(\hat{j}+\hat{k})=3\hat{i}+\hat{j}-2\hat{k}$

(c) $\vec{A}\cdot\vec{B}=(1)(0)+(1)(1)+(0)(1)=1$

(d) $\vec{A}\times\vec{B}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 1 & 0\\ 0 & 1 & 1\end{vmatrix}=\hat{i}(1-0)+\hat{j}(0-1)+\hat{k}(1-0)=\hat{i}-\hat{j}+\hat{k}$

$$\left|\vec{A}\times\vec{B}\right|=(1+1+1)^{\frac{1}{2}}=\sqrt{3}$$

1.2 Given the three vectors **A = 2i +j**, **B = i + k**, and **C = 4j**, find the following:

(a) **A. (B + C)** and **(A + B) .C** (b**) A.(B×C)** and **(A×B).C**

(c) **A×(B×C)** and **(A×B)×C**

Solution:

(a) $\vec{A}\cdot\left(\vec{B}+\vec{C}\right)=\left(2\hat{i}+\hat{j}\right)\cdot\left(\hat{i}+4\hat{j}+\hat{k}\right)=(2)(1)+(1)(4)+(0)(1)=6$

$$\left(\vec{A}+\vec{B}\right)\cdot\vec{C}=\left(3\hat{i}+\hat{j}+\hat{k}\right)\cdot4\hat{j}=(3)(0)+(1)(4)+(1)(0)=4$$

(b) $\vec{A}\cdot\left(\vec{B}\times\vec{C}\right)=\begin{vmatrix}2 & 1 & 0\\ 1 & 0 & 1\\ 0 & 4 & 0\end{vmatrix}=-8$

(c) $\vec{A}\times\left(\vec{B}\times\vec{C}\right)=\left(\vec{A}\cdot\vec{C}\right)\vec{B}-\left(\vec{A}\cdot\vec{B}\right)\vec{C}=4\left(\hat{i}+\hat{k}\right)-2\left(4\hat{j}\right)=4\hat{i}-8\hat{j}+4\hat{k}$

$$\left(\vec{A}\times\vec{B}\right)\times\vec{C}=-\vec{C}\times\left(\vec{A}\times\vec{B}\right)=-\left[\left(\vec{C}\cdot\vec{B}\right)\vec{A}-\left(\vec{C}\cdot\vec{A}\right)\vec{B}\right]$$

1.3: Find the angle between the vectors **A** = a**i** + 2a**j** and **B** = a**i** + 2a**j** + 3a**k**. (Note: These two vectors define a face diagonal and a body diagonal of a rectangular block of sides a, 2a, and 3a.)

Solution:

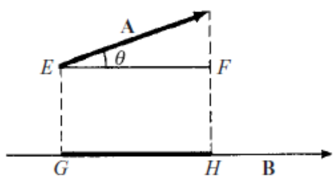
$$\cos\theta = \frac{\vec{A}\cdot\vec{B}}{AB} = \frac{(a)(a)+(2a)(2a)+(0)(3a)}{\sqrt{5a^2}\sqrt{14a^2}} = \frac{5a^2}{a^2\sqrt{5}\sqrt{14}}$$

$$\theta = \cos^{-1}\sqrt{\frac{5}{14}} \approx 53°$$

1.4 Prove that the projection of **A** on B is equal to **A** .**b**, where **b** is a unit vector in the direction of **B**.

Solution:

Through the initial and terminal points of **A** pass planes perpendicular to **B** at G and **H** respectively, as in the adjacent Figure: then



Projection of **A** on **B** = $\overline{GH}$ = $\overline{EF}$ = $A\cos\theta$ = **A** .**b**

1.5 Find the volume of a parallelepiped with sides $\mathbf{A} = 3\mathbf{i} - \mathbf{j}$, $\mathbf{B} = \mathbf{j} + 2\mathbf{k}$, $\mathbf{C} = \mathbf{i} + 5\mathbf{j} + 4\mathbf{k}$.

Solution:

$$\text{volume of parallelepiped} = |\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})| = \left|\begin{vmatrix} 3 & -1 & 0 \\ 0 & 1 & 2 \\ 1 & 5 & 4 \end{vmatrix}\right|$$

$$= |-20| = 20.$$

1.6 If $\mathbf{A} = \mathbf{i} + \mathbf{j}$, $\mathbf{B} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$, $\mathbf{C} = 4\mathbf{j} - 3\mathbf{k}$, find (*a*) $(\mathbf{A} \times \mathbf{B}) \times \mathbf{C}$, (*b*) $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$.

Solution:

(*a*) $\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 2 & -3 & 1 \end{vmatrix} = \mathbf{i} - \mathbf{j} - 5\mathbf{k}$. Then $(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & -5 \\ 0 & 4 & -3 \end{vmatrix} = 23\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$.

(*b*) $\mathbf{B} \times \mathbf{C} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -3 & 1 \\ 0 & 4 & -3 \end{vmatrix} = 5\mathbf{i} + 6\mathbf{j} + 8\mathbf{k}$. Then $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 5 & 6 & 8 \end{vmatrix} = 8\mathbf{i} - 8\mathbf{j} + \mathbf{k}$.

It can be proved that, in general, $(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} \neq \mathbf{A} \times (\mathbf{B} \times \mathbf{C})$.

Problems (homework)

Answer to the following problems.

1- If $\mathbf{A} = 5\mathbf{i} - \mathbf{j} - 2\mathbf{k}$ and $\mathbf{B} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$, find

(a) $\mathbf{A} \times \mathbf{B}$ and $\mathbf{B} \times \mathbf{A}$ (b) $|\mathbf{A} \times \mathbf{B}|$

(c) sin ϕ and ϕ where ϕ is the smaller angle between **A** and **B**.

(d) $(\mathbf{A} \times \mathbf{B}).\mathbf{B}$, and $(\mathbf{A} \times \mathbf{B}).\mathbf{A}$.

2- If $\mathbf{A} = 3\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$, $\mathbf{B} = 2\mathbf{i} - 4\mathbf{j} + 5\mathbf{k}$, and $\mathbf{C} = \mathbf{i} + \mathbf{j} - 2\mathbf{k}$, find

(a) $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$ (b) $(\mathbf{A} \times \mathbf{B}) \times \mathbf{C}$

3- Evaluate

(a) $2\mathbf{i} \times (3\mathbf{j} - 4\mathbf{k})$ (b) $(\mathbf{i} + 2\mathbf{j}) \times \mathbf{k}$ (c) $(2\mathbf{i} - 4\mathbf{j}) \times (\mathbf{i} + \mathbf{k})$

4- prove that the transformation matrix for a rotation about y-axis through an angle (θ) is given by the matrix :

$$\begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix}$$

Analytical Mechanics

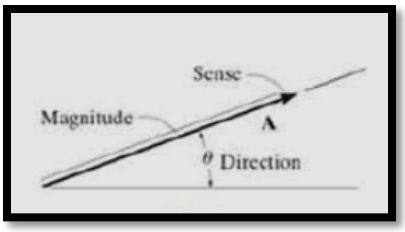
Chapter One

1-1: Scalar and vectors:

A **scalar** is any **positive** or **negative physical quantity** that can be completely specified by its **magnitude**. such as mass, length, density, volume and temperature.

A **vector** is any physical quantity that requires both a **magnitude** and **direction** for its complete description. A vector is shown **graphically** by an **arrow**. The **length** of the arrow represents the **magnitude** of the vector, and a fixed axis defines the **direction** of its line of action .The **head** of the arrow indicates the **sense** of **direction of the vector** (Fig 1- 1).

Fig. (1-1)



For **handwritten** work, it is often **convenient** to denote a vector quantity by simply drawing an arrow on top it A, ($\vec{A}$) .In **print**, vector quantities are represented by **bold** face letters such as **A**, and its **magnitude** of the vector is **italicized**, *A*.

1-2:Geometric Properties :

1. Two vectors A and B are equal if they have the same magnitude and direction regardless of their initial points. Thus **A** = **B**.

2. A vector having direction opposite to that of vector A but with the same magnitude is denoted by -A .

3. The sum or resultant of vectors **A** and **B** of Fig. 1-2(a) below is a vector C formed by placing the initial point of B on the terminal point of A and joining the initial point of A to the terminal point of B [see Fig. 1-2(b) below].

The sum C is written C = A + B. The definition here is equivalent to the parallelogram law for vector addition as indicated in Fig.1-2(c) below.

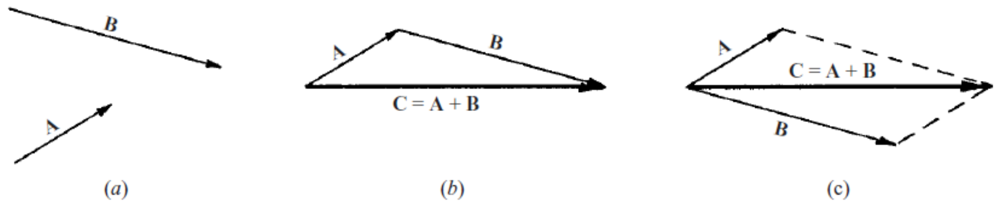


Fig.(1-2)

4. The difference of vectors **A** and **B**, represented by A - B, is that vector C which added to B gives A. Equivalently, A - B may be defined as A + (-B). If A = B, then A - B is defined as the null or zero vector and is represented by the symbol **0**. This has a magnitude of zero but its direction is not defined.

5. Multiplication of a vector **A** by a scalar m produces a vector m**A** with magnitude $|m|$ times the magnitude of **A** and direction the same as or opposite to that of **A** according as m is positive or negative. If m = 0, mA = 0, the null vector.

1-3: Algebraic Properties of Vectors:

The following algebraic properties are consequences of the geometric definition of a vector. If A, B and C are vectors, and m and n are scalars, then:

1. A + B = B + A Commutative Law for Addition

2. A +(B + C) = (A + B) + C Associative Law for Addition

3. m(n**A**)= **(**mn**)A =** n(m**A**) Associative Law for Multiplication

4. (m + n)**A =** m**A +** n**A** Distributive Law

5. m**(A + B)** = m**A** + m**B** Distributive Law

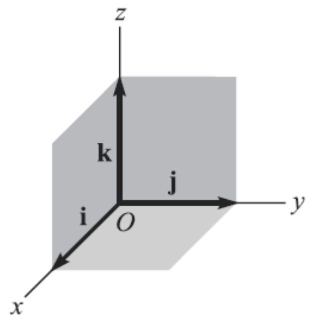
6. A+0=A **7. A+(-A)= 0**

Note that in these laws only multiplication of a vector by one or more scalars is defined.

1-4: Unit Vectors:

Unit vectors are vectors having unit length. If **A** is any vector with length $A > 0$, then A/A is a unit vector, denoted by **a**, having the same direction as A. Then **A** = *A*a. The rectangular unit vectors i, j, and k are unit vectors having the direction of the positive x, y, and z axes of a rectangular coordinate system [see Fig. 1-3].

Fig.(1-3)



1-5 Components of A Vectors:

Any vector **A** in 3 dimensions can be represented with initial point at the origin O of a rectangular coordinate system [see Fig. 1-4]. Let $(A_1;A_2;A_3)$ be the rectangular coordinates of the terminal point of vector A with initial point at O. The vectors A_1i;A_2j; and A_3k are called the rectangular component vectors, or simply **component vectors**, of A in the x, y; and z directions respectively. The sum or resultant of A_1i;A_2j; and A_3k is the vector **A,** so that we can write:

$$A = A_1 i + A_2 j + A_3 k$$

The magnitude of **A** is:

$$A = |\mathbf{A}| = \sqrt{A_1^2 + A_2^2 + A_3^2}$$

In particular, the position vector or radius vector r from O to the point (x; y; z) is written:

$$r = xi + yj + zk$$

and has magnitude:

$$r = |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}.$$

1-6: Dot OR Scalar Product:

The dot or scalar product of two vectors **A** and **B**, denoted by A.B (read **A** dot **B**) is defined as the product of the magnitudes of *A* and *B* and the of the angle between them(projection of **A** on **B**). In symbols,

$$\mathbf{A} \cdot \mathbf{B} = AB \cos\theta; \quad 0 < \theta < \pi$$

Assuming that neither **A** nor **B** is the zero vector, an immediate consequence of the definition is that **A . B** = 0 if and only if **A** and **B** are perpendicular. Note that A . B is a scalar and not a vector. The following laws are valid:

1. **A . B = B . A** Commutative Law for Dot Products
2. **A . (B + C)** = **A . B** + **A . C** Distributive Law
3. m(**A . B**) = (m**A**) . **B** = **A** . (m**B**) = (**A . B**)m, where m is a scalar.
4. i . i = j . j = k . k = 1; i . j = j . k = k . i = 0
5. If **A** = A_1i + A_2j + A_3k and **B** = B_1i + B_2j + B_3k, then

$$\mathbf{A} \cdot \mathbf{B} = A_1B_1 + A_2B_2 + A_3B_3$$

In particular, we can write:

$$\mathbf{A} \cdot \mathbf{B} = A_xB_x + A_yB_y + A_zB_z$$

6. The square of the magnitude of a vector **A** is given by the dot product of A with itself,

$$\mathbf{A}^2 = |\mathbf{A}|^2 = \mathbf{A}.\mathbf{A}$$

As an example of the dot product, suppose that an object under the action of a constant force undergoes a linear displacement **Δs**, (as shown in Figure 1-4) .By definition, the work ΔW done by the force is given by the product of the component of the force **F** in the direction of **Δs**, multiplied by the magnitude Δ*s* of the displacement; that is,

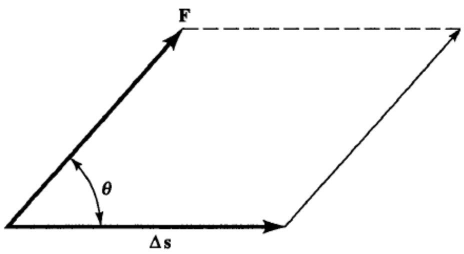


Fig.1-4

$$\Delta W=(F\cos\theta)\Delta s$$

where θ is the angle between **F** and **Δs**. But the expression on the right is just the dot product of **F** and **Δs**, that is,

$$\Delta \mathbf{W}=\mathbf{F}.\Delta \mathbf{s}$$

1-7: CROSS OR VECTOR PRODUCT:

The cross or vector product of **A** and **B** is a vector **C** = **A** × **B** (read **A** cross **B**). The magnitude of **A** × **B** is defined as the product of the magnitudes of **A** and **B** and the sine of the angle between them. The direction of the vector **C = A** × **B** is perpendicular to the plane of **A** and **B** and such that **A, B,** and **C** form a right-handed system. In symbols,

$$\mathbf{A} \times \mathbf{B} = AB \sin \theta \mathbf{u}; \qquad 0 \le \theta \le \pi$$

where **u** is a unit vector indicating the direction of **A** × **B**. IfA = B or if A is parallel to B, then sin θ = 0 and A × B = 0.

The following laws are valid:

1. **A × B = -B × A** (Commutative Law for Cross Products Fails)

2. **A × (B + C) = A× B + A × C** Distributive Law

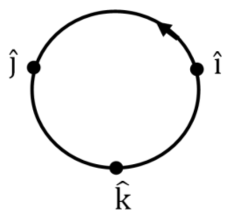
3. m(**A × B**) = (m**A**) × **B = A** × (m**B**) = (**A × B**)m, where m is a scalar.

Also the following consequences of the definition are important:

4. i × i = j × j = k × k = 0,

i × j = k = - j × i; j × k = i = -k × j ; k × i = j= -i × k

This cyclic nature of the cross product can be emphasized by diagramming the multiplication table as shown in Figure



5. |**A** × **B** |= the area of a parallelogram with sides **A** and **B.**

6. If **A** × **B** = 0 and neither **A** nor **B** is a null vector, then **A** and **B** are parallel.

7. If **A** $= A_x i + A_y j + A_z k$ and **B** $= B_x i + B_y j + B_z k$, then

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} i & j & k \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$\mathbf{A} \times \mathbf{B} = \mathbf{i}\begin{vmatrix} A_y & A_z \\ B_y & B_z \end{vmatrix} + \mathbf{j}\begin{vmatrix} A_z & A_x \\ B_z & B_x \end{vmatrix} + \mathbf{k}\begin{vmatrix} A_x & A_y \\ B_x & B_y \end{vmatrix}$$

The cross product expressed in i, j ,k form is:

$$\mathbf{A} \times \mathbf{B} = \mathbf{i}(A_y B_z - A_z B_y) + \mathbf{j}(A_z B_x - A_x B_z) + \mathbf{k}(A_x B_y - A_y B_x)$$

Let us calculate the magnitude of the cross product. We have

$$|\mathbf{A} \times \mathbf{B}|^2 = (A_y B_z - A_z B_y)^2 + (A_z B_x - A_x B_z)^2 + (A_x B_y - A_y B_x)^2$$

This can be reduced to:

$$|\mathbf{A} \times \mathbf{B}|^2 = \left(A_x^2 + A_y^2 + A_z^2\right)\left(B_x^2 + B_y^2 + B_z^2\right) - (A_x B_x + A_y B_y + A_z B_z)^2$$

from the definition of the dot product, the above equation may be written in the form

$$|\mathbf{A} \times \mathbf{B}|^2 = A^2 B^2 - (\mathbf{A} \cdot \mathbf{B})^2$$

Taking the square root of both sides of Equation above we can express the magnitude of the cross product as

$$|\mathbf{A} \times \mathbf{B}| = AB(1 - \cos^2\theta)^{1/2} = AB \sin\theta$$

where θ is the angle between **A** and **B**.

physical example of the cross product is the rotational moment (torque)

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$$

Let a force **F** act at a point P(x, y, z), as shown in Figure 1-5, and let the vector **OP** be designated by **r**; that is,

$$\mathbf{OP}=\mathbf{r}=\mathbf{i}x+\mathbf{j}y+\mathbf{k}z$$

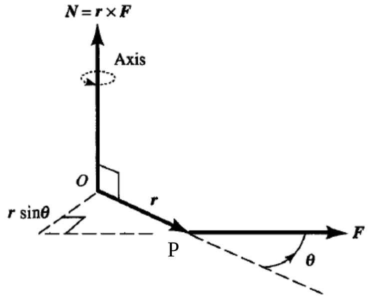


Figure 1-5: Illustration of the moment of a force about a point 0.

The moment **N** of force, or the torque **N**, about a given point 0 is defined as the cross product

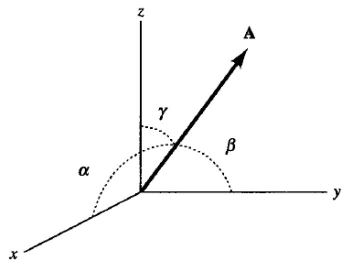
$$\mathbf{N}=\mathbf{r}\times\mathbf{F}$$

Thus, the moment of a force about a point is a having a magnitude and a direction. If a single force is applied at a point P on a body that is initially at rest and is free to turn about a fixed point 0 as a pivotمحور, then the body tends to rotate. The axis of this rotation is perpendicular to the force F, and it is also perpendicular to the line OP; therefore, the direction of the torque vector **N** is along the axis of rotation. The magnitude of the torque is given by:

$$|N|=|\mathbf{r}\times\mathbf{F}|=rF\sin\theta$$

in which θ is the angle between **r** and **F**. Thus, |N| can be regarded as the product of the magnitude of the force and the quantity $r \sin \theta$, which is just the perpendicular distance from the line of action of the force to the point 0.

1-8: Representation of a given vector as the product of a scalar and a single unit vector:



Consider the equation:

$$\mathbf{A}=\mathbf{i}A_x+\mathbf{j}A_x+\mathbf{k}A_x$$

Multiply and divide on the right by the magnitude of **A**

$$\mathbf{A}=A\left(\mathbf{i}\frac{A_x}{A}+\mathbf{j}\frac{A_y}{A}+\mathbf{k}\frac{A_z}{A}\right)$$

Now $A_x/A=\cos\alpha$, $A_y/A=\cos\beta$, and $= A_z/A \cos\gamma$ are the direction cosines of the vector A, and α, β, and γ are the direction angles. Thus, we can write are the direction cosines of vector A, and are the direction angles.

Thus we can write

$$A = A(i \cos\alpha + j \cos\beta + k \cos\gamma) = A(\cos\alpha, \cos\beta, \cos\gamma)$$

or

$$\mathbf{A} = \mathbf{n}A$$

where **n** is a unit vector whose components are cosα ,cos β, and cos γ. Consider any other vector B. Clearly, the projection of B on A is just

$$B\cos\theta = \frac{\mathbf{B}\cdot\mathbf{A}}{A} = \mathbf{B}\cdot\mathbf{n}$$

where θ is the angle between A and B.

Example 1:

1- Find the angle between the vectors

2- Find the vector product of the vectors

$$\vec{A} = 2\hat{i} + 3\hat{j} - \hat{k} \qquad and \qquad \vec{B} = -\hat{i} + \hat{j} + 2\hat{k}$$

solution:

1- $\vec{A}.\vec{B} = A_x B_x + A_y B_y + A_z B_z = 2\times(-1) + 3\times 1 + (-1)\times 2 = -1$

$$A = \left|\vec{A}\right| = \sqrt{2^2 + 3^2 + (-1)^2} = 3.74 \ units$$

$$B = \left|\vec{B}\right| = \sqrt{(-1)^2 + 1^2 + 2^2} = 2.45 \ units$$

and since $\vec{A}.\vec{B} = AB\cos\theta \quad \Rightarrow \quad \theta = \cos^{-1}\left(\frac{\vec{A}.\vec{B}}{AB}\right)$

$$\theta = \cos^{-1}\left(\frac{-1}{3.74\times 2.45}\right) = \cos^{-1}(-0.109) = 96.3^0$$

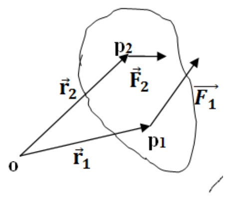
2-

$$\vec{A}\times\vec{B}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 3 & -1\\ -1 & 1 & 2\end{vmatrix}=\hat{i}\,(6+1)-\hat{j}\,(4-1)+\hat{k}\,(2+3)=7\hat{i}-3\hat{j}+5\hat{k}$$

Example 2:

A force $\vec{F}_1=\hat{i}+\hat{j}$ **was applied on a body at a point** p_1 **such that** $o\vec{p}_1=\vec{r}_1=2\hat{i}+\hat{j}$ **and a second force** $\vec{F}_2=\hat{j}-\hat{k}$ **was applied at a point** $o\vec{p}_2=\vec{r}_2=\hat{i}+\hat{j}+\hat{k}$**. Find: (a) the total moment** $\vec{N}$ **(b) the magnitude of** $\vec{N}$ **(c) the direction cosines of** $\vec{N}$**.**

solution:



(a) $\vec{N}_1=\vec{r}_1\times\vec{F}_1$ **and** $\vec{N}_2=\vec{r}_2\times\vec{F}_2$

$$\vec{N}_1=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 1 & 0\\ 1 & 1 & 0\end{vmatrix}=\hat{k} \qquad \vec{N}_2=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 1 & 1\\ 0 & 1 & -1\end{vmatrix}=-2\hat{i}+\hat{j}+\hat{k}$$

The total moment is

$$\vec{N}=\vec{N}_1+\vec{N}_2=\hat{k}+(-2\,\hat{i}+\hat{j}+\hat{k})=-2\,\hat{i}+\hat{j}+2\,\hat{k}$$

(b) $\vec{N}=\sqrt{4+1+4}=3$

(c)

$$\cos\alpha=\frac{N_x}{N}=\frac{-2}{3}\,,\quad \cos\beta=\frac{N_y}{N}=\frac{1}{3}\quad \textbf{and}\quad \cos\gamma=\frac{N_z}{N}=\frac{2}{3}$$

1-9:Triple Products:

The expression

$$\mathbf{A}.\,(\mathbf{B} \times \mathbf{C})$$

is called the **scalar triple product** of **A**, **B**, and **C**. It is a scalar because it is the dot product of two vectors. Referring to the determinant expressions for the cross product, we see that the scalar triple product may be written:

$$\mathbf{A}.\,(\mathbf{B} \times \mathbf{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

Because the exchange of the terms of two rows or of two columns of a determinant changes its sign but not its absolute value, we can derive the following useful equation:

1- (A . B)C ≠ A(B . C) in general

2- A . (B × C) =B. (C × A) = C . (A× B) = volume of a parallelepiped having A, B, and C as edges, or the negative of this volume according as A, B, and C do or do not form a right handed system. Thus, the dot and the cross may be interchanged in the scalar triple product.

The expression

$$\mathbf{A}\times (\mathbf{B} \times \mathbf{C})$$

is called the **vector triple product**. The following equation holds for the vector triple product:

1- **A× (B × C) ≠ (A× B) × C** (Associative Law for Cross Products Fails)

2- **A×(B×C)= (A.C) B – (A.B) C**

A×(B×C)=(A . C)B – (B . C) A

Vector triple products are particularly useful in the study of rotating coordinate systems and rotations of rigid bodies.

Example 3:

Given the three vectors **A = 1, B = i—j,** and **C = k**, find

1- **A. (B × C)**

2- A× (B × C)

Solution:

1-

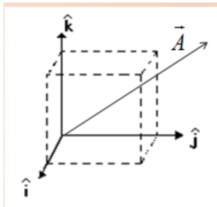
$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})=\begin{vmatrix}1 & 0 & 0\\ 1 & -1 & 0\\ 0 & 0 & 1\end{vmatrix}=1(-1+0)=-1$$

2-

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C})=\mathbf{B}(\mathbf{A}\cdot\mathbf{C})-\mathbf{C}(\mathbf{A}\cdot\mathbf{B})=(\mathbf{i}-\mathbf{j})0-\mathbf{k}(1-0)=-\mathbf{k}$$

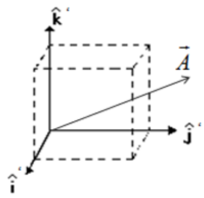
1-10:Change of Coordinate System :The Transformation Matrix:

The rotation of a Cartesian coordinate system is an example of an orthogonal transformation. Consider the vector A expressed relative to the triad ijk:



$$\mathbf{A}=\mathbf{i}A_x+\mathbf{j}A_y+\mathbf{k}A_z$$

Relative to a new triad **i'j'k'** having a different orientation from that of **ijk**, the same vector **A** is expressed as



$$\mathbf{A}=\mathbf{i}'A_{x'}+\mathbf{j}'A_{y'}+\mathbf{k}'A_{z'}$$

Now the dot product **A. i'** is just $A_{x'}$, that is, the projection of **A** on the unit vector **i'**. Thus, we may write

$$\begin{aligned}A_{x'}&=\mathbf{A}\cdot\mathbf{i}'=(\mathbf{i}\cdot\mathbf{i}')A_x+(\mathbf{j}\cdot\mathbf{i}')A_y+(\mathbf{k}\cdot\mathbf{i}')A_z\\ A_{y'}&=\mathbf{A}\cdot\mathbf{j}'=(\mathbf{i}\cdot\mathbf{j}')A_x+(\mathbf{j}\cdot\mathbf{j}')A_y+(\mathbf{k}\cdot\mathbf{j}')A_z\\ A_{z'}&=\mathbf{A}\cdot\mathbf{k}'=(\mathbf{i}\cdot\mathbf{k}')A_x+(\mathbf{j}\cdot\mathbf{k}')A_y+(\mathbf{k}\cdot\mathbf{k}')A_z\end{aligned}$$

The scalar products (**i . i'**), (**i . j'**), and so on are called the **coefficients of transformation**. They are equal to the direction cosines of the axes of the primed coordinate system relative to the unprimed system. The unprimed components are similarly expressed as

$$
\begin{aligned}
A_x &= \mathbf{A}\cdot\mathbf{i} = (\mathbf{i}'\cdot\mathbf{i})A_{x'} + (\mathbf{j}'\cdot\mathbf{i})A_{y'} + (\mathbf{k}'\cdot\mathbf{i})A_{z'} \\
A_y &= \mathbf{A}\cdot\mathbf{j} = (\mathbf{i}'\cdot\mathbf{j})A_{x'} + (\mathbf{j}'\cdot\mathbf{j})A_{y'} + (\mathbf{k}'\cdot\mathbf{j})A_{z'} \\
A_z &= \mathbf{A}\cdot\mathbf{k} = (\mathbf{i}'\cdot\mathbf{k})A_{x'} + (\mathbf{j}'\cdot\mathbf{k})A_{y'} + (\mathbf{k}'\cdot\mathbf{k})A_{z'}
\end{aligned}
$$

The equations of transformation are conveniently expressed in matrix notation. Thus,

$$
\begin{pmatrix} A_{x'} \\ A_{y'} \\ A_{z'} \end{pmatrix} = \begin{pmatrix} \mathbf{i}\cdot\mathbf{i}' & \mathbf{j}\cdot\mathbf{i}' & \mathbf{k}\cdot\mathbf{i}' \\ \mathbf{i}\cdot\mathbf{j}' & \mathbf{j}\cdot\mathbf{j}' & \mathbf{k}\cdot\mathbf{j}' \\ \mathbf{i}\cdot\mathbf{k}' & \mathbf{j}\cdot\mathbf{k}' & \mathbf{k}\cdot\mathbf{k}' \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}
$$

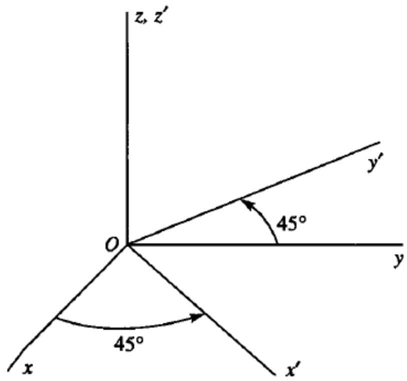
The 3-by-3 matrix in Equation above is called the **transformation matrix**.

Example 4:

Express the vector **A = 3i + 2j + k** in terms of the triad **i'j'k'**, where the x'y'—axes are rotated 45° around the z-axis, with the z- and z'-axes coinciding, as shown in Figure. Referring to the figure, we have for the coefficients of transformation **i . i'** = cos 45° and so on,

we have from the figure:

$$
\begin{aligned}
&\mathbf{i}\cdot\mathbf{i}' = 1/\sqrt{2} \qquad && \mathbf{j}\cdot\mathbf{i}' = 1/\sqrt{2} \qquad && \mathbf{k}\cdot\mathbf{i}' = 0 \\
&\mathbf{i}\cdot\mathbf{j}' = -1/\sqrt{2} && \mathbf{j}\cdot\mathbf{j}' = 1/\sqrt{2} && \mathbf{k}\cdot\mathbf{j}' = 0 \\
&\mathbf{i}\cdot\mathbf{k}' = 0 && \mathbf{j}\cdot\mathbf{k}' = 0 && \mathbf{k}\cdot\mathbf{k}' = 1
\end{aligned}
$$



$$\begin{pmatrix} A'_x \\ A'_y \\ A'_z \end{pmatrix} = \begin{pmatrix} \hat{i}\cdot\hat{i}' & \hat{j}\cdot\hat{i}' & \hat{k}\cdot\hat{i}' \\ \hat{i}\cdot\hat{j}' & \hat{j}\cdot\hat{j}' & \hat{k}\cdot\hat{j}' \\ \hat{i}\cdot\hat{k}' & \hat{j}\cdot\hat{k}' & \hat{k}\cdot\hat{k}' \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$$

$$\begin{pmatrix} A'_x \\ A'_y \\ A'_z \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$$

$$\begin{pmatrix} A'_x \\ A'_y \\ A'_z \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$$

These give:

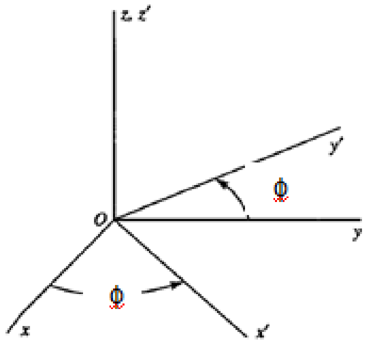
$$A_{x'} = \frac{3}{\sqrt{2}} + \frac{2}{\sqrt{2}} = \frac{5}{\sqrt{2}} \qquad A_{y'} = \frac{-3}{\sqrt{2}} + \frac{2}{\sqrt{2}} = \frac{-1}{\sqrt{2}} \qquad A_{z'} = 1$$

so that, in the primed system, the vector A is given by

$$\vec{A} = \frac{5}{\sqrt{2}}\hat{i}' - \frac{1}{\sqrt{2}}\hat{j}' + \hat{k}'$$

Example 5:

Find the transformation matrix for a rotation of the primed coordinate system through an angle ϕ about the z-axis. We have



$$\mathbf{i}\cdot\mathbf{i}'=\mathbf{j}\cdot\mathbf{j}'=\cos\phi$$
$$\mathbf{j}\cdot\mathbf{i}'=-\mathbf{i}\cdot\mathbf{j}'=\sin\phi$$
$$\mathbf{k}\cdot\mathbf{k}'=1$$

and all other dot products are zero; hence, the transformation matrix is:

$$\begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

the transformation matrix for a rotation about a different coordinate axis-say, the y-axis through an angle (θ) given by the matrix

$$\begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix}$$

Consequently, the matrix for the combination of two rotations, the first being about the z-axis (angle ϕ) and the second being about the new y' -axis (angle (θ), is given by the matrix product

$$\begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}=\begin{pmatrix} \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \\ \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \end{pmatrix}$$

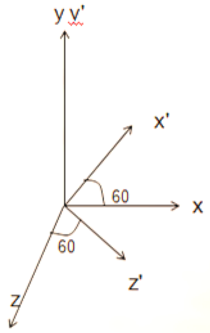
Now matrix multiplication is, in general, noncommutative; therefore, we might expect that the result would be different if the order of the rotations, and, therefore, the order of the matrix multiplication, were reversed.

Example6:

Express the vector $\hat{i}+\hat{j}$ in terms of the triad $\hat{i}'\ \hat{j}'\ \hat{k}'$ where the x' z' axes are rotated 60 degree around the y axis, and the y' axes coinciding.

Solution:

From the figure . we can see that



$$\hat{i}\cdot\hat{i}' = \hat{k}\cdot\hat{k}' = \cos 60^0 = \frac{1}{2}$$

$$\hat{i}\cdot\hat{k}' = \sin 60^0 = \frac{\sqrt{3}}{2} \qquad \hat{i}\cdot\hat{j}' = \cos 90 = 0$$

$$\hat{k}\cdot\hat{i}' = \cos(90+60) = -\sin 60 = -\frac{\sqrt{3}}{2}$$

the rest are zero

Therefore the transformation matrix and the equation of transformation becomes:

$$\begin{pmatrix} A'_x \\ A'_y \\ A'_z \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 & -\frac{\sqrt{3}}{2} \\ 0 & 1 & 0 \\ \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$\vec{A} = \frac{1}{2}\hat{i}' + \hat{j}' + \frac{\sqrt{3}}{2}\hat{k}'$$

$$\cos\left(\frac{\pi}{2}+x\right) = -\sin x$$
$$\cos\left(\frac{\pi}{2}-x\right) = \sin x$$
$$\cos(\pi \pm x) = -\cos x$$
$$\cos\left(\frac{3\pi}{2}+x\right) = \sin x$$
$$\cos\left(\frac{3\pi}{2}-x\right) = -\sin x$$

$$\tan\left(\frac{\pi}{2}+x\right) = -\cot x$$
$$\tan\left(\frac{\pi}{2}-x\right) = \cot x$$
$$\tan(\pi + x) = \tan x$$
$$\tan(\pi - x) = -\tan x$$
$$\tan\left(\frac{3\pi}{2}+x\right) = -\cot x$$
$$\tan\left(\frac{3\pi}{2}-x\right) = \cot x$$

$$\sin\left(\frac{\pi}{2}\pm x\right) = \cos x$$
$$\sin(\pi + x) = -\sin x$$
$$\sin(\pi - x) = \sin x$$
$$\sin\left(\frac{3\pi}{2}\pm x\right) = -\cos x$$

$$\cos^2 x + \sin^2 x = 1$$

$$\cos(2\alpha) = 2\cos^2\alpha - 1 \qquad \cos(2\alpha) = \cos^2\alpha - \sin^2\alpha \qquad \cos(2\alpha) = 1 - 2\sin^2\alpha$$

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \qquad \sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta \qquad \cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

Solved Problems (page 44):

1.1: Given the two vectors **A = i +j** and **B =j + k**, find the following:

(a) **A+B** and **|A + B|** (b) 3**A**-2**B** (c) **A. B** (d) **A×B** and **|A + B|**

Solution:

(a) $\vec{A}+\vec{B}=(\hat{i}+\hat{j})+(\hat{j}+\hat{k})=\hat{i}+2\hat{j}+\hat{k}$

$|\vec{A}+\vec{B}|=(1+4+1)^{\frac{1}{2}}=\sqrt{6}$

(b) $3\vec{A}-2\vec{B}=3(\hat{i}+\hat{j})-2(\hat{j}+\hat{k})=3\hat{i}+\hat{j}-2\hat{k}$

(c) $\vec{A}\cdot\vec{B}=(1)(0)+(1)(1)+(0)(1)=1$

(d) $\vec{A}\times\vec{B}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 1 & 0\\ 0 & 1 & 1\end{vmatrix}=\hat{i}(1-0)+\hat{j}(0-1)+\hat{k}(1-0)=\hat{i}-\hat{j}+\hat{k}$

$|\vec{A}\times\vec{B}|=(1+1+1)^{\frac{1}{2}}=\sqrt{3}$

1.2 Given the three vectors **A = 2i +j**, **B = i + k**, and **C = 4j**, find the following:

(a) **A. (B + C)** and **(A + B) .C** (b**) A.(B×C)** and **(A×B).C**

(c) **A×(B×C)** and **(A×B)×C**

Solution:

(a) $\vec{A}\cdot\left(\vec{B}+\vec{C}\right)=\left(2\hat{i}+\hat{j}\right)\cdot\left(\hat{i}+4\hat{j}+\hat{k}\right)=(2)(1)+(1)(4)+(0)(1)=6$

$\left(\vec{A}+\vec{B}\right)\cdot\vec{C}=\left(3\hat{i}+\hat{j}+\hat{k}\right)\cdot4\hat{j}=(3)(0)+(1)(4)+(1)(0)=4$

(b) $\vec{A}\cdot\left(\vec{B}\times\vec{C}\right)=\begin{vmatrix}2 & 1 & 0\\ 1 & 0 & 1\\ 0 & 4 & 0\end{vmatrix}=-8$

(c) $\vec{A}\times\left(\vec{B}\times\vec{C}\right)=\left(\vec{A}\cdot\vec{C}\right)\vec{B}-\left(\vec{A}\cdot\vec{B}\right)\vec{C}=4\left(\hat{i}+\hat{k}\right)-2\left(4\hat{j}\right)=4\hat{i}-8\hat{j}+4\hat{k}$

$\left(\vec{A}\times\vec{B}\right)\times\vec{C}=-\vec{C}\times\left(\vec{A}\times\vec{B}\right)=-\left[\left(\vec{C}\cdot\vec{B}\right)\vec{A}-\left(\vec{C}\cdot\vec{A}\right)\vec{B}\right]$

1.3: Find the angle between the vectors **A** = a**i** + 2a**j** and **B** = a**i** + 2a**j** + 3a**k**. (Note: These two vectors define a face diagonal and a body diagonal of a rectangular block of sides a, 2a, and 3a.)

Solution:

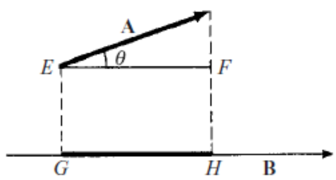
$$\cos\theta = \frac{\vec{A}\cdot\vec{B}}{AB} = \frac{(a)(a)+(2a)(2a)+(0)(3a)}{\sqrt{5a^2}\sqrt{14a^2}} = \frac{5a^2}{a^2\sqrt{5}\sqrt{14}}$$

$$\theta = \cos^{-1}\sqrt{\frac{5}{14}} \approx 53°$$

1.4 Prove that the projection of **A** on B is equal to **A** .**b**, where **b** is a unit vector in the direction of **B**.

Solution:

Through the initial and terminal points of **A** pass planes perpendicular to **B** at G and **H** respectively, as in the adjacent Figure: then



Projection of **A** on **B** = $\overline{GH}$ = $\overline{EF}$ = $A\cos\theta$ = **A** .**b**

1.5 Find the volume of a parallelepiped with sides **A** = 3**i** − **j**, **B** = **j** + 2**k**, **C** = **i** + 5**j** + 4**k**.

Solution:

$$\text{volume of parallelepiped} = |\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})| = \left|\begin{vmatrix} 3 & -1 & 0 \\ 0 & 1 & 2 \\ 1 & 5 & 4 \end{vmatrix}\right|$$

$$= |-20| = 20.$$

1.6 If $\mathbf{A} = \mathbf{i} + \mathbf{j}$, $\mathbf{B} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$, $\mathbf{C} = 4\mathbf{j} - 3\mathbf{k}$, find (*a*) $(\mathbf{A} \times \mathbf{B}) \times \mathbf{C}$, (*b*) $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$.

Solution:

$$(a)\quad \mathbf{A}\times\mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 2 & -3 & 1 \end{vmatrix} = \mathbf{i} - \mathbf{j} - 5\mathbf{k}. \quad \text{Then} \quad (\mathbf{A}\times\mathbf{B})\times\mathbf{C} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & -5 \\ 0 & 4 & -3 \end{vmatrix} = 23\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}.$$

$$(b)\quad \mathbf{B}\times\mathbf{C} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -3 & 1 \\ 0 & 4 & -3 \end{vmatrix} = 5\mathbf{i} + 6\mathbf{j} + 8\mathbf{k}. \quad \text{Then} \quad \mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 5 & 6 & 8 \end{vmatrix} = 8\mathbf{i} - 8\mathbf{j} + \mathbf{k}.$$

It can be proved that, in general, $(\mathbf{A}\times\mathbf{B})\times\mathbf{C} \neq \mathbf{A}\times(\mathbf{B}\times\mathbf{C})$.

Problems (homework)

Answer to the following problems.

1- If $\mathbf{A} = 5\mathbf{i} - \mathbf{j} - 2\mathbf{k}$ and $\mathbf{B} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$, find

(a) $\mathbf{A}\times\mathbf{B}$ and $\mathbf{B}\times\mathbf{A}$ (b) $|\mathbf{A}\times\mathbf{B}|$

(c) sin ϕ and ϕ where ϕ is the smaller angle between **A** and **B**.

(d) $(\mathbf{A}\times\mathbf{B}).\mathbf{B}$, and $(\mathbf{A}\times\mathbf{B}).\mathbf{A}$.

2- If $\mathbf{A} = 3\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$, $\mathbf{B} = 2\mathbf{i} - 4\mathbf{j} + 5\mathbf{k}$, and $\mathbf{C} = \mathbf{i} + \mathbf{j} - 2\mathbf{k}$, find

(a) $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})$ (b) $(\mathbf{A}\times\mathbf{B})\times\mathbf{C}$

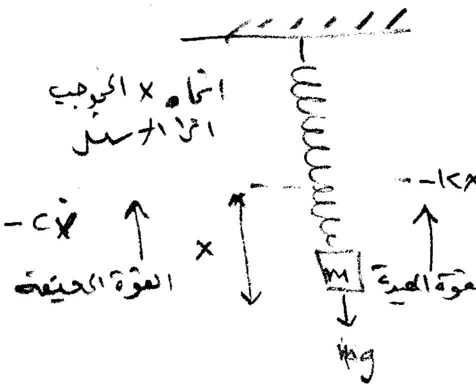
3- Evaluate

(a) $2\mathbf{i}\times(3\mathbf{j} - 4\mathbf{k})$ (b) $(\mathbf{i} + 2\mathbf{j})\times\mathbf{k}$ (c) $(2\mathbf{i} - 4\mathbf{j})\times(\mathbf{i} + \mathbf{k})$

4- prove that the transformation matrix for a rotation about y-axis through an angle (θ) is given by the matrix :

$$\begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix}$$

$$\sum F_x = m\ddot{x} = -c\dot{x} - kx$$



$$\therefore\ m\ddot{x} + c\dot{x} + kx = 0$$ ⟸ المعادلة التفاضلية للحركة

Differential eq. of motion

نجرب الحل الاتي

$$x = Ae^{qt}$$

$$\therefore\ \dot{x} = Aqe^{qt}$$

$$\ddot{x} = Aq^2e^{qt}$$

نعوض الحلول السابقة في معادلة الحركة التفاضلية

$$mAq^2e^{qt} + cAqe^{qt} + kAe^{qt} = 0$$

$$mq^2 + cq + k = 0$$ ⟸ معادلة من الدرجة الثانية نحلها بطريقة الدستور، وتعطي جذور المعادلة كالاتي

$$q = \frac{-c \pm \sqrt{c^2 - 4km}}{2m}$$

$$q_1 = \frac{-c + \sqrt{c^2 - 4mk}}{2m}, \quad q_2 = \frac{-c - \sqrt{c^2 - 4mk}}{2m}$$

وهناك ثلاث حالات حسب العامل المميز (تحت الجذر)

1- $q > 0$ ⟹ over damping (فوق التضاؤل)

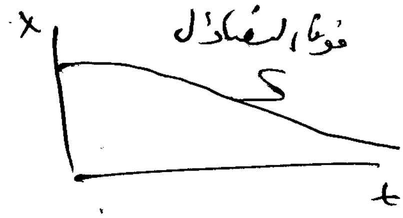
2- $q = 0$ ⟹ critical damping (التضاؤل الحرج)

} قيمة q تكون حقيقية وسالبة

3- $q < 0$ ⟹ under damping (دون/تحت التضاؤل)

(1) Over damping

لنفرض ان $q_1=-\gamma_1$ ، $q_2=-\gamma_2$ (حيث يمثل القيم الحقيقية لـ q)



$$x = x_1 + x_2$$

$$\therefore x_1 = A_1 e^{-\gamma_1 t} \quad , \quad x_2 = A e^{-\gamma_2 t}$$

$$x = A_1 e^{-\gamma_1 t} + A_2 e^{-\gamma_2 t}$$

نلاحظ ان الحركة تكون غير تذبذبية في هذه الحالة . اذ ان الازاحة x تقل مع الزمن بصورة تدريجية .

2- Critical damping

في هذه الحالة $q=0$ اي ان

$$q = q_1 = q_2 = 0$$

لايجاد الحل العام للجذور المتساوية نبدأ من المعادلة التفاضلية العامة للحركة

$m\ddot{x} + c\dot{x} + kx = 0$ نقسم على معامل $\ddot{x}$

$$\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = 0$$

$$\therefore \gamma = \frac{c}{2m} \qquad \therefore 2\gamma = \frac{c}{m}$$

$$\ddot{x} + 2\gamma\dot{x} + \gamma^2 x = 0$$

$$\frac{d^2}{dt^2}x + 2\gamma\frac{d}{dt}x + \gamma^2 x = 0$$

$$\left[\frac{d^2}{dt^2} + 2\gamma\frac{d}{dt} + \gamma^2\right]x = 0$$

$$\left(\frac{d}{dt}+\gamma\right)\left(\frac{d}{dt}+\gamma\right)x = 0$$

نفرض ان

$$\left[\frac{d}{dt}+\gamma\right]x = u$$

$$\therefore \left(\frac{d}{dt}+\gamma\right)u = 0$$

$$\frac{du}{dt} + \gamma u = 0 \quad \Rightarrow \quad \frac{du}{dt} = -\gamma u$$

$$= \int \frac{du}{u} = -\gamma \int dt$$

$$= \ln u = -\gamma t$$

بأخذ ln الطرفين ينتج

$$\Rightarrow u = e^{-\gamma t}$$

$$\Rightarrow u = A_1 e^{-\gamma t}$$

ثابت [illegible]

$$u = \left[\frac{d}{dt} + \gamma\right] x = A_1 e^{-\gamma t}$$

$$\left[\frac{dx}{dt} + \gamma x\right] e^{\gamma t} = A_1$$

مشتقة حاصل ضرب دالتين

$$\frac{d}{dt}\left(x e^{\gamma t}\right) = A_1$$

وبالتكامل مرة أخرى نجد

$$\int d\left(x e^{\gamma t}\right) = A_1 \int dt$$

ثابت التكامل

$$x e^{\gamma t} = A_1 t + A_2$$

$$\therefore x = (A_1 t + A_2) e^{-\gamma t}$$

تبين المعادلة (من اكل العام) ان الدالة x دالة اسية تتناقص مع الزمن وبذلك تكون الحركة غير تذبذبية

وعليه فإنه في حالة التضاؤل الثقيل وحالة التضاؤل الحرج للاهتزازات الازاحة x تتناقص مع الزمن الى الصفر وبذلك تكون الحركة غير تذبذبية

* under damping

$$x = A_1 e^{q_1 t} + A_2 e^{q_2 t} \qquad \text{—(1)}$$

$$\therefore q = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}$$

لإيجاد q_1 و q_2

$$q_1 = \frac{-c + i\sqrt{4mk - c^2}}{2m} = -\frac{c}{2m} + i\sqrt{\frac{4mk}{4m^2} - \frac{c^2}{4m^2}}$$

$$= -\frac{c}{2m} + \sqrt{\frac{k}{m} + \frac{c^2}{4m^2}} = -\frac{c}{2m} + \sqrt{\omega_0^2 - \gamma^2}$$

$$= -\gamma + i\omega_1$$

$$\omega_1 = \sqrt{\omega_0^2 - \gamma^2}$$

حيث أن ω_1 ثابت

حيث أن ω_0 ، ω_1 تمثل التردد الزاوي للمتذبذب المثير مضمحل والتردد الزاوي للمتذبذب التوافقي (تحت المحصول) على التوالي الاضمحلال الكلي

$$\therefore q_1 = -\gamma + i\omega_1$$

وبنفس الطريقة نجد أن

$$q_2 = -\gamma - i\omega_1$$

نعوض عن q_1 و q_2 في المعادلة (1) ينتج

$$x = A_+ e^{(-\gamma + i\omega_1)t} + A_- e^{(-\gamma - i\omega_1)t}$$

$$= e^{-\gamma t}\left[A_+ e^{i\omega_1 t} + A_- e^{-i\omega_1 t}\right]$$

بالتعويض من العلاقات التالية

$$e^{iu} = \cos u + i\sin u$$

$$e^{-iu} = \cos u - i\sin u$$

$$\therefore x = e^{-\gamma t}\left[A_+(\cos\omega_1 t + i\sin\omega_1 t) + A_-(\cos\omega_1 t - i\sin\omega_1 t)\right]$$

$$= e^{-\gamma t}\left[(A_+ + A_-)\cos\omega_1 t + (A_+ - A_-)\, i\sin\omega_1 t)\right]$$

$$x = e^{-\gamma t}(b\cos\omega_1 t + a\sin\omega_1 t)$$

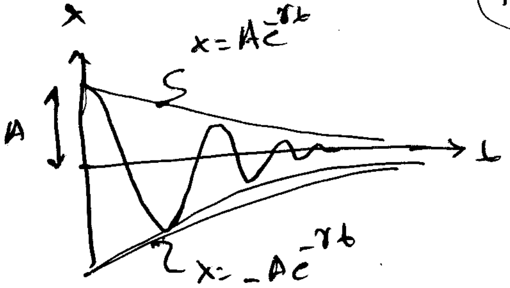
$a = i(A_+ - A_-$

$b = (A_+ - A_-$

(السعة تكون غير ثابتة (تكون دالة $e^{-\gamma t}$))

يمكن كتابة حل اخر بالشكل الاتي

الحركة تكون تذبذبية نوع ← $x = Ae^{-\gamma t}\cos(\omega_1 t + \theta_0)$



حيث ان

$$A = (a^2+b^2)^{1/2}$$

$$\theta_0 = -\tan^{-1}(b/a)$$

فيزيائيا الجزء الحقيقي يمثل ان الحركة تذبذبية. والسعة $Ae^{-\gamma t}$ تضمحل اسيا مع الزمن. وكذلك نلاحظ ان التردد الزاوي للتذبذب ω_1 هو اقل من التردد الزاوي للتذبذب غير المضمحل (المتضائل) ω_0 ويدعى التردد ω_1 بالتردد الطبيعي

$$\omega_1=\sqrt{\frac{k}{m}-\frac{c^2}{4m^2}} \Rightarrow \omega_1=\sqrt{\omega_0^2-\gamma^2}$$

$$=\omega_0\sqrt{1-\frac{\gamma^2}{\omega_0^2}} \Rightarrow \omega_1=\omega_0\left(1-\frac{\gamma^2}{2\omega_0^2}\right)$$

$$\omega_1=\omega_0-\frac{\gamma^2}{2\omega_0}$$

اعتبارات الطاقة

الطاقة الكلية للمتذبذب التوافقي المضمحل تعطى على انه حاصل جمع الطاقة الحركية + الطاقة الكامنة

$$E=\frac{1}{2}m\dot{x}^2+\frac{1}{2}kx^2$$

وهذه الطاقة تكون ثابتة في المتذبذب غير المضمحل. بتفاضل هذه المعادلة بالنسبة للزمن ينتج

$$\frac{dE}{dt}=m\dot{x}\ddot{x}+kx\dot{x}=(m\ddot{x}+kx)\dot{x}$$

من معادلة التفاضلية للحركة

$$m\ddot{x}+c\dot{x}+kx=0$$

$$m\ddot{x}+kx=-c\dot{x}$$

بالتعويض عنه

$$\therefore \frac{dE}{dt}=-c\dot{x}^2$$

المعدل الزمني لتغير الطاقة يساوي حاصل ضرب القوة المعيقة مع السرعة. ولان المعدل الزمني دائما سالب فان الطاقة الكلية تقل بصورة مستمرة وهذه الطاقة تتبدد كحرارة .

68

ص

Exampel

Ans. : 1

$$\omega_1 = \sqrt{\frac{k}{m} - \frac{c^2}{4m^2}} = \sqrt{\omega_0^2 - \gamma^2}$$

نعوض عن $\gamma = \frac{\omega_0}{4}$

$$= \sqrt{\omega_0^2 - \frac{\omega_0^2}{16}} = \omega_0 \sqrt{1 - \frac{1}{16}}$$

$$\boxed{\omega_1 = \omega_0 \sqrt{\frac{15}{16}} = \sqrt{\frac{k}{m}} \sqrt{\frac{15}{16}}}$$

at time $t = 0$ $\quad A(t) = A$

$t = T_1$ $\quad A(t) = A e^{-\gamma t}$

The ratio: (النسبة) $= \frac{A e^{-\gamma T_1}}{A} = e^{-\gamma T_1}$

نحسب حساب γT_1

$$T_1 = \frac{1}{f_1} = \frac{2\pi}{\omega_1} \leftarrow \text{نعوض من العلاقة (1)}$$

$$= \frac{2\pi}{\omega_0} \sqrt{\frac{16}{15}} = \frac{2\pi}{4\gamma} \sqrt{\frac{16}{15}}$$

$\omega_0 = 4\gamma$

$$\therefore \gamma T_1 = \frac{\pi}{2} \sqrt{\frac{16}{15}} = 1.56$$

وعليه فان النسبة بين تذبذبين متعاقبين هو

$$e^{-\gamma T_1} = e^{-1.56} = 0.21$$

Chapter Two

Dynamic of a Particle Rectilinear Motion

2-1: Newton's Laws of Motion:

- Three laws of motion are now known collectively as Newton's laws of motion or as Newton's laws:

I. **The first law** describes a common property shared by all matter, named **inertia.** The law state that everybody continues in its state of rest, or of uniform motion in a straight line, unless it is compelled to change that state by forces impressed upon it.

II. **Mass and Force: Newton's Second law**: The quantitative measure of inertia is called **mass.** The change of motion is proportional to the motive force impressed and is made in the direction of the line in which that force is impressed. Thus, the second law can be written as:

$$\vec{F} = \frac{d\,(mv)}{dt}$$

Considering the mass to be a constant, independent of velocity (which is not true of objects moving at "relativistic" speeds or speeds approaching the speed of light, 3×10^8 m/s) we can write:

$$\vec{F} = m\vec{a}$$

Where **a** is the resultant acceleration of a mass m subjected to a force **F**.

III. **Newton's third law**: To every action there is always imposed an equal reaction; or, the mutual actions of two bodies upon each other are always equal and directed to contrary parts.

$$\mathbf{F_1 = -F_2}$$

Newton's third law, namely, that two interacting bodies exert equal and opposite forces upon one another.

2-2: Linear Momentum:

Linear momentum proves to be such a useful notion that it is given its own symbol:

$$p=mv$$

So the Newton's second law can be rephrased as follows: The time rate of change of an object's linear momentum is proportional to the impressed force and may be written as:

$$\vec{F} = \frac{d\vec{p}}{dt}$$

Newton's third law, can be expressed in terms of linear momentum. Thus for two mutually interacting bodies A and B, we have

$$\frac{d\overrightarrow{p_A}}{dt} = -\frac{d\overrightarrow{p_B}}{dt}$$

$$\frac{d}{dt}(P_A + P_B) = 0$$

$$P_A + P_B = \text{constant}$$

Thus the third law implies يتطلب that the total linear momentum زخم of two interacting bodies always remain constant. This constancy is a special case of the more general situation in which the total linear momentum of an isolated معزول system نظام (a system subject يتعرض to no net externally خارجية applied forces قوى) is a **conserved** محفوظة **quantity** كمية. The law of linear momentum conservation is one of the most fundamental أساسي laws of physics and is valid even in situations in which Newtonian mechanics fails.

2-3: Motion of a Particle

The fundamental equation of motion for a particle subject to the influence of a net force, F, is given by Newton's second law:

$$\vec{F} = m\vec{a}$$

We can write **F** as $\mathbf{F}_{net}$, the vector sum of all the forces acting on the particle is given by:

$$\mathbf{F}_{net} = \sum \mathbf{F}_i = m\frac{d^2\vec{\mathbf{r}}}{dt^2} = m\vec{\mathbf{a}} \qquad \text{-------1}$$

The usual problem of dynamics can be expressed in the following way: Given a knowledge of the forces acting on a particle (or system of particles), calculate the acceleration of the particle.Knowing the acceleration, calculate the velocity and position as functions of time. This process involves solving the second - order differential equation of motion represented by equation (1).

In most problems $\vec{F}$ is Known as a function of coordinate (including time), and we have to find the position of the particle as a function of time. To do so, we have to solve a set of differential equations

$$\frac{d\vec{v}}{dt} = \frac{\vec{F}}{m}$$

The solution of these equations depends upon $\vec{F}$

2-2-1: Rectilinear Motion: Uniform Acceleration Under a Constant Force:

When a moving particle remains on a single straight line, the motion is said to be **rectilinear**. We can choose the x-axis as the line of motion. The general equation of motion is then:

$$F(x, \dot{x}, t) = m\ddot{x}\ ) \qquad \text{---------2}$$

The simplest situation is that in which the **force is constant**. In this case we have constant acceleration. and the solution is readily obtained by direct integration with respect to time:

$$F = \text{constant}$$

$$\frac{dv}{dt} = \frac{F}{m} = \text{constant} = a$$

$$\int_{v_0}^{v} dv = a\int_{0}^{t} dt \qquad \Rightarrow v = v_0 + at$$

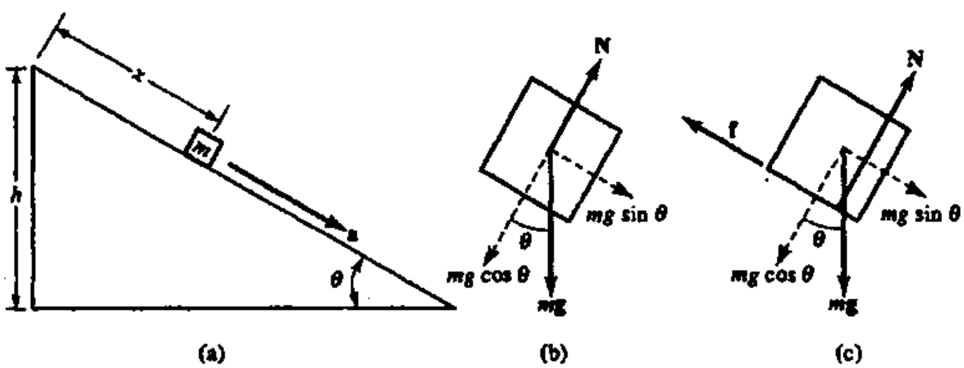
Since $v = \frac{dx}{dt}$ Then $dx = (v_0 + at)dt$

The solution is readily obtained by direct integration with respect to time:

$$x = x_0 + v_0 t + \frac{1}{2}at^2$$

Where v_0 is the velocity and x_0 is the position at t =0.

Example 1: Consider a block that is free to slide ينزلق down a smooth, frictionless plane that is inclined يميل at an angle θ to the horizontal الأفق, as shown in Figure 1(a). If the height of the plane is h and the block is released حرر from rest at the top, what will be its speed when it reaches the bottom?



Solution:

The only force along the x direction is the component of gravitational force, mg sinθ, as shown in Figure 1(b). It is constant. Thus,

$$\vec{F} = m\vec{a}$$

$$a = \frac{dv}{dt} = \frac{F}{m} = g\sin\theta$$

$$x - x_0 = \frac{h}{\sin\theta}$$

Thus $$v^2 = 2(g\sin\theta)\left(\frac{h}{\sin\theta}\right) = 2gh$$

Suppose that, instead of being smooth ناعم, the plane is rough خشن; that is, it exerts a frictional احتكاك force f on the particle. Then the net force in the x-direction, (see Figure 1(c)), is equal to mg sinθ $-f$. Now, for sliding contact it is found that the magnitude of the frictional force is proportional to the magnitude of the normal force N; that is,

$$f = \mu_k N$$

where the constant of proportionality μ_k is known as the *coefficient of sliding*. The normal force, as shown in the figure, is equal to mg cosθ; hence,

$$f = \mu_k mg \cos\theta$$

The net force in the x direction is equal to

$$F = mg\sin\theta - \mu_k mg\cos\theta$$

$$a = \frac{F}{m} = g(\sin\theta - \mu_k \cos\theta)$$

The speed of the particle increases if the expression in parentheses is positive that is **a** to be positive $\sin\theta > \mu_k \cos\theta$ or $\tan\theta > \mu_k$

$$\text{i.e. } \theta \geq \tan^{-1} \mu_k \qquad \theta > \varepsilon$$

$\tan^{-1}\mu_k$ is called the ***angle of friction*** and denoted by **ε**

If $\theta = \varepsilon$ then a=0 and the particle slides down the plane with constant speed (v = constant)

If $\theta < \varepsilon$ then a is negative (a< 0), and so the particle eventually comes to rest the body eventually come to rest.

For motion up the plane, the direction of the frictional force is reversed; that is, it is in the positive x-direction. The acceleration (actually deceleration تباطؤ) is then $a = g(\sin\theta + \mu_k \cos\theta)$.

2-2-2: Forces that Depend on Position: The Concepts [مبدأ] of Kinetic [حركية] and Potential [كامنة] Energy [طاقة]:

It is true that the force a particle experiences depends on the particle's position with respect to other bodies. This is the case, for example, with electrostatic and gravitational forces. If the force is independent of velocity or time, then the differential equation for rectilinear motion is simply:

$$F(x) = m\ddot{x} \qquad \text{-----------} \; 1$$

By using the chain rule to solve such differential equation to write the acceleration in the following way:

$$\ddot{x} = \frac{d\dot{x}}{dt} = \frac{dx}{dt}\frac{d\dot{x}}{dx} = v\frac{dv}{dx}$$

So the differential equation of motion may be written:

$$F(x) = mv\frac{dv}{dx} = \frac{m}{2}\frac{dv^2}{dx} = \frac{dT}{dx} \qquad \text{-------------} \; 2$$

Where $T = \frac{1}{2}mv^2$ is the **Kinetic energy** of the particle .We can now express equation 2 in integral form:

$$\int F(x)dx = \int dT = \frac{1}{2}m\dot{x}^2 + \text{constant} \qquad \text{--------} \; 3$$

Where $\int F(x)dx$ represent the work done on the particle by the force F(x). The work is equal to the change in the kinetic energy of the particle.

$$W = \int_{x_0}^{x} F(x)\,dx = T - T_0 \qquad \text{-------------} \; 4$$

Let us define a function V(x) such that:

$$-\frac{dV(x)}{dx} = F(x) \qquad \text{------------} 5$$

The function V(x) is called the **Potential energy**.

In terms of V(x), the work integral is:

$$W = \int_{x_0}^{x} F(x)\,dx = -\int_{x_0}^{x} dV = -V(x) + V(x_0) = T - T_0 \qquad \text{-------------- } 6$$

We now transpose terms and write Equation 6 in the following form:

$$T_o + V(x_o) = \text{constant} = T + V(x) = E$$

$$T + V(x) = \frac{1}{2}m\dot{x}^2 + V(x) = \text{constant} = E \qquad \text{-------------}7$$

This is the energy equation. E is defined to be the **total energy** of the particle. It is equal to the sum of the kinetic and potential energies and is constant throughout the motion of the particle. For one-dimensional motion, if the impressed force is a function of position only, then the sum of the kinetic and potential energies remain constant throughout the motion, such for is called conservative .Nonconservative forces such as friction which is no potential exists.

The motion of the particle can be obtained by solving the energy equation:

$$\frac{1}{2}m\dot{x}^2 + V(x) = E \quad \text{and} \quad v = \frac{dx}{dt}$$

$$v = \frac{dx}{dt} = \pm\sqrt{\frac{2}{m}[E - V(x)]}$$

$$\int_{x_0}^{x} \frac{dx}{\pm\sqrt{\frac{2}{m}[E - V(x)]}} = \int_{0}^{t} dt$$

This given t as a function of x.

Example 2: An example of conservative motion is the motion of a freely falling body. If we choose the x- direction to be positive upward, then the gravitational force is equal to –mg. Therefore, $-dV/dx = -mg$ and then V=mgx +C. the constant of integration C is arbitrary constant. We can choice C=0 which mean V=0 when x=0. The energy equation is then:

$$E = \frac{1}{2}m\dot{x}^2 + mgx$$

Let the body be projected upward with initial speed v_o from the origin x =0. These values give:

$E = \frac{1}{2} m v_o^2$ عند قذف الجسم لأعلى بسرعة ابتدائية v_o ومن السكون $x = 0$

$$E = \frac{1}{2}mv_0^2 = \frac{1}{2}mv^2 + mgx$$

$$v^2 = v_0^2 - 2gx$$

The turning point of the motion, which is in this case the maximum height, is given by setting v =0.

$$0 = v_0^2 - 2gx$$

$$E = V = mgx_{max} \Rightarrow h = x_{max} = \frac{v_0^2}{2g}$$

2-2-3:The Force as a function of time: The Concepts of impulse الدفع : F=F(t)

The equation of motion is:

$$F(t) = m\frac{dv}{dt}$$

The linear momentum (and velocity) can be obtained as a function of time by integrating the equation of motion to get:

$$\int F(t)\,dt = mv(t) + C$$

C is the constant of integration. The integral $\int F(t)$ is called the impulse.

The position of the particle as a function of time can be found by a second integration as follow:

$$x = \int v(t)\,dt = \int\left[\int \frac{F(t')}{m}\,dt'\right]dt$$

Example 3: A block is initially at rest on a smooth horizontal surface. At time t=0 a constantly increasing horizontal force is applied: F=ct. find the velocity and the displacement as a function of time.

Solution:

$$ct = m\frac{dv}{dt}$$

$$v = \frac{1}{m}\int_0^t ct\,dt = \frac{ct^2}{2m}$$

$$x = \int_0^t \frac{ct^2}{2m}\,dt = \frac{ct^3}{6m}$$

2-2-4:The force as a function of velocity:

Forces like viscous[لزوجة] resistance exerted[مسلطة] on a body moving through a fluid[مائع] or like air resistance[مقاومة] on falling (or rising) body are called **viscous forces**. If the force can be expressed as a function of v only, the differential equation of motion $F = m\ddot{x}$ may be written in this case in the form

$$F(v) = m\frac{dv}{dt} \qquad \text{------------ 1}$$

Or $$F(v) = m\frac{dv}{dx}\frac{dx}{dx} = mv\frac{dv}{dx} \qquad \text{------------2}$$

From eq. 1 we found $$t = t(v) = \int \frac{mdv}{F(v)} \qquad \text{-------------3}$$

From eq. 2 we found $$x = x(t) = \int \frac{mvdv}{F(v)} \qquad \text{-------------4}$$

Solving eq. 3 gives v as a function of t, that is v=v(t)

We can solve for x

$$v = \frac{dx}{dt}$$

$$dx = v(t)dt$$

By integration to get

$$x = x(t) = \int v(t)dt \qquad \text{-------------5}$$

Example 4: Suppose a block is projected قذف with initial ابتدائية velocity v_o on a smooth horizontal surface and that there is air resistance such that $F(v) = -cv$ the linear case dominates. Calculate v and x as a function of t

Solution: The differential equation of motion is:

$$F(v) = -cv = m\frac{dv}{dt}$$

Which gives, upon integrating,

$$t = \int_{v_0}^{v} \frac{-m dv}{cv} = -\frac{m}{c}\ln\left(\frac{v}{v_0}\right)$$

We can easily solve for v as a function of t by multiplying by -c/m and

$$-\frac{c}{m}t = \ln\left(\frac{v}{v_0}\right)$$

Taking the exponential of both sides. The result is:

$$v = v_0 e^{-\frac{c}{m}t}$$

Thus, the velocity decreases exponentially with time. A second integration gives:

$$x = \int_0^t \left(v_0 e^{-\frac{c}{m}t}\right)dt$$

That is :

$$x = \frac{mv_0}{c}\left(1 - e^{-\frac{c}{m}t}\right)$$

Showing that the block approaches a limiting position (in other words the block never goes beyond the limiting position) given by:

$$x_{lim} = \frac{mv_0}{c}$$

Problems (homework)

1- Find the velocity $\dot{x}$ and the position x as functions of the time t for a particle of mass m, which starts from rest at x =0 and t =0, subject to the following force functions:

(a) $F_x = F_o + ct$

(b) $F_x = F_o \sin ct$

(c) $F_x = F_o e^{ct}$

Where F_o and c are positive constants.

2- A: Find the velocity $\dot{x}$ as a function of the displacement x for a particle of mass m, which starts from rest at x =0, subject to the following force functions:

(a) $F_x = F_o + cx$

(b) $F_x = F_o \cos cx$

(c) $F_x = F_o e^{cx}$

Where F_o and c are positive constants.

B: Find the potential energy function V(x) for each of the forces

3- A particle of mass m moves along a frictionless, horizontal plane with speed given by v(x)=k/x, where x is its distance from the origin and k is a positive constant. Find the force F(x) to which the particle is subject.

4- Given that the velocity of a particle varies with the displacement x according to the equation $\dot{x} = bx^{-3}$ where b is positive constant, find the force acting on the particle as a function of x.

5- A particle moves with an acceleration which inversely proportional to its velocity (a=k/v) and start from the origin with speed u cm s^{-1}. After 3 second its speed is 2u. find the distance covered in the 3 seconds

6- if a=2-x and v =0 when x=1, describe the motion.

7- A particle at t=0 has the positive x_o, the velocity v_o is being acted on it by sinusoidal force **F=Fo sinωt,** calculate its position as a function of time.

2-3: Vertical Fall through a Fluid: Terminal Velocity

It often happens that the force that acts on a body is a function of the velocity of the body, the case of viscous resistance exerted on a body moving through a fluid. If the force can be expressed as a function of v only, the differential equation of motion may be written in either of the two forms:

$$F_0 + F(v) = m\frac{dv}{dt}$$

$$F_0 + F(v) = mv\frac{dv}{dx}$$

Here F_o is any constant force that does not depend on v.

2-3-1: Linear case (F= -cv) :-

For an object falling vertically in a resisting fluid, the force F_o in equations above is the weight of the object (-mg), and the resistance is proportional to the first power of v, we can express this force as $-cv$ regardless of the sign of v because the resistance is always opposite to the direction of motion. Let us take x-axis to be positive in the upward direction. Then the differential equation of motion is:

$$-mg - cv = m\frac{dv}{dt} \qquad \text{-------------- } 1$$

The constant of proportionality c depends on the size and shape of the object and the viscosity of the fluid. Separating variables and integrating, we find

$$\begin{aligned} t &= \int \frac{m\,dv}{F(v)} = \int_{v_0}^{v} \frac{m\,dv}{-mg - cv} \qquad \text{-------------- } 2 \\ &= -\frac{m}{c}\ln\frac{mg + cv}{mg + cv_0} \end{aligned}$$

In which v_o is the initial velocity at t = 0. Upon multiplying by -c/m and taking the exponential, we can solve for v:

$$v = -\frac{mg}{c} + \left(\frac{mg}{c} + v_0\right)e^{-ct/m} \qquad \text{------------ } 3$$

After a sufficient time (t >> m/c) the exponential الاسي term الحد drops يهبط to a negligible مهملة value قيمة and the velocity approaches the limiting value -mg/c, which is called the **terminal** المنتهي **velocity;**

It is that velocity at which the force of resistance is just equal and opposite to the weight of the body so that the total force is zero, and so the acceleration is zero. The magnitude of the terminal velocity is the terminal speed انطلاق. Let us designate the terminal speed mg/c by v_t and let us write τ (which we may call the characteristic النوعي time) for m/c, then equ. 3 can be write as:

$$v = -v_t + (v_t + v_0)e^{-t/\tau} \qquad \text{----------- 4}$$

In particular, for an object dropped from rest at time t =0, v0 =0, we find

$$v = -v_t(1 - e^{-t/\tau})$$

Integration of equation 3 will give x as a function of t, then we get:

$$x - x_0 = \int_0^t v(t)\, dt$$

$$= -\frac{mg}{c}t + \left(\frac{m^2g}{c^2} + \frac{mv_0}{c}\right)(1 - e^{-ct/m}) \qquad \text{---------5}$$

We can write equation 5 in the form:

$$x = x_o - v_t t + x_1(1 - e^{-t/\tau})$$

Where

$$x_1 = \frac{m^2g}{c^2} + \frac{mv_o}{c} = g\,\tau^2 + v_o\,\tau$$

The ratio between the terminal speed and τ is:

$$ratio = \frac{v_t}{\tau} = \frac{mg/c}{m/c} = g$$

2-3-2: non-Linear case (Quadratic case, $F(v)=\pm cv^2$):-

In this case, the magnitude of F(v) is proportional to v^2. we must remember that the sign preceding the F(v) term depends on whether or not the motion of the object is upward or downward.

$$-mg \pm cv^2 = m\frac{dv}{dt}$$

For rising bodies F_r is negative (downward)

For falling bodies F_r is positive (upward)

The differential equation of motion can be integrated to give t as a function of v:

$$t = \int \frac{m\,dv}{-mg - cv^2} = -\tau \tan^{-1}\frac{v}{v_t} + t_0 \qquad (rising)$$

$$t = \int \frac{m\,dv}{-mg + cv^2} = -\tau \tanh^{-1}\frac{v}{v_t} + t_0' \qquad (falling)$$

Where

$$\sqrt{\frac{m}{cg}} = \tau \text{ (the characteristic time)}$$

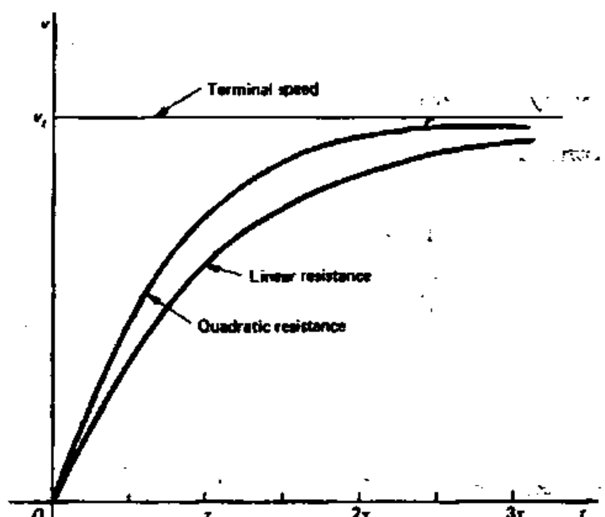
and

$$\sqrt{\frac{mg}{c}} = v_t \text{ (the terminal speed)}$$

Solving for v,

$$v = v_t \tan\frac{t_0 - t}{\tau} \qquad (rising)$$

$$v = -v_t \tanh\frac{t - t_0'}{\tau} \qquad (falling)$$



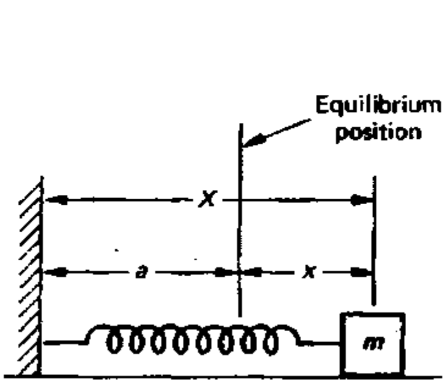
Graphs of speed versus time for a falling body subject to linear and quadratic air resistance.

2-4: Linear Restoring Force: Harmonic Motion:

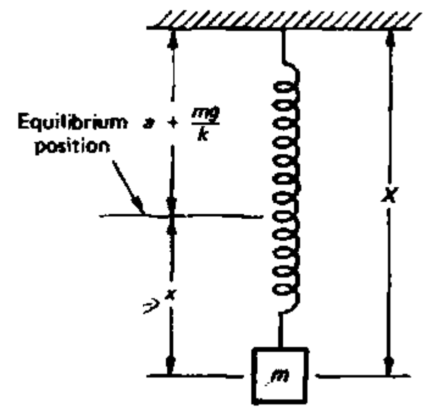
One of the most important cases of rectilinear motion is that produced by **linear restoring force**. This is a force whose magnitude is proportional to displacement of a particle from some equilibrium position and whose direction s always opposite to that of displacement. Such force is exerted by an elastic مرن cord وتر or by spring نابض obeying يخضعان Hooke's law.

$$F = -k(X - a) = -kx \qquad \text{--------- 1}$$

Where X is the total length and a is unstretched غير منبسط (zero load) length of the spring. The variable x=(X-a) is the displacement of the spring from its equilibrium. The proportionality constant k is called **stiffness** المرونة. Let a particle of mass m be attached to the spring as shown in figure (a) below. The force acting on particle is giving by equation 1.



a: horizontal motion



b: vertical motion

Let the same spring be hold vertically as shown in figure (b). The total force now on particle is:

$$F = -k(X - a) + mg \qquad \text{--------- 2}$$

Where the positive direction is downward الاسفل. For this case, let us measure x relative to new equilibrium position, that is :

$$x = X - a - mg/k$$

This given again:

$$F = -kx$$

In both cases then (a and b) the equation of motion, becomes:

$$m\ddot{x} + kx = 0 \qquad \text{------------ } 3$$

Equation 3 is the linear differential equation of motion with constant coefficient. We try the function Ae^{qt} where q is a constant to be determined يمكن ايجاده.

If $x = Ae^{qt}$ a solution for all values of time (applying this solution in equ.3), then we have:

$$m\frac{d^2}{dt^2}(Ae^{qt}) + k(Ae^{qt}) = 0 \qquad \text{---------- } 4$$

which reduces تختزل to the equation:

$$mq^2 + k = 0$$

that is:

$$q = \pm i\sqrt{\frac{k}{m}} = \pm i\omega_0$$

where $i = \sqrt{-1}$ and $w_o = \sqrt{\frac{k}{m}}$, if f_1 and f_2 are solution then the sum of $f_1 + f_2$ is a solution too. The general solution of equation 3 is then:

$$x = A_+e^{i\omega_0 t} + A_-e^{-i\omega_0 t} \qquad \text{------- } 5$$

Since $e^{iu} = \cos u + i \sin u$, alternate forms of the solution are:

$$x = a \sin \omega_0 t + b \cos \omega_0 t \qquad \text{------ } 6$$

Or

$$x = A \cos(\omega_0 t + \theta_0) \qquad \text{-----------} 7$$

The constant A of integration in the above solutions and are determined يتحدد from the initial conditions. Equations 5,6, and 7 are solution of equation 3. The motion is sinusoidal الجيبي oscillation التذبذب of the displacement الازحة x, so that equation 3 is the differential equation of the harmonic التوافقي oscillator المتذبذب. The coefficient w_o is called the angular الزاوي frequency التردد ($w_o = 2\pi f_o$ where f_o is the linear frequancy). The maximum value of x is called the amplitude of the oscillation, it's the constant A in equation 7.

The period T_o of the oscillation is the time required for complete كاملة cycle دورة .

$$T_0 = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{m}{k}}$$

$$f_0 = \frac{1}{T_0} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

Example: A light spring is found to stretch an amount b when it supports a block of mass m. if the block is pulled downward a distance l from its equilibrium position and released at time $t = 0$, find the resulting motion as a function of t. $x(t)$

Solution: $x = A\cos(\omega_0 t + \theta_0)$

First, to find the spring stiffness, we note that in the static equilibrium condition

$$F = -kb = -mg$$

so that

$$k = \frac{mg}{b}$$

Hence the angular frequency of oscillation is

$$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{g}{b}}$$

In order to find the constants for the equation of motion

$$x = A\cos(\omega_0 t + \theta_0)$$

we have

$$x = l \qquad \text{and} \qquad \dot{x} = 0$$

at time $t = 0$. But

$$\dot{x} = -A\omega_0 \sin(\omega_0 t + \theta_0)$$

Thus

$$A = l \qquad \theta_0 = 0$$

so

$$x = l\cos\left(\sqrt{\frac{g}{b}}\,t\right)$$

is the required expression.

2-5:Energy Considerations in Harmonic Motion:

Consider a particle under the action of a linear restoring force $F_x = -kx$. Let us calculate the work done by an external force F_{ext} in moving the particle from the equilibrium position (x = 0) to some position x. Assume that we move the particle very slowly so that it does not gain any kinetic energy; that is, the applied external force is barely greater in magnitude than the restoring force -kx; hence, $F_{ext} = - F_x = kx$, so:

$$W = \int_0^x F_{ext}\, dx = \int_0^x kx\, dx = \frac{k}{2}x^2 \qquad \text{--------- 1}$$

In the case of a spring obeying Hooke's law, the work is stored in the spring as potential energy: W = V(x), where:

$$V(x) = \tfrac{1}{2}kx^2 \qquad \text{------------ 2}$$

Thus, $F_x = - dV/dx = - kx$, as required by the definition of V. The total energy when the particle is undergoing harmonic motion, is given by the sum of the kinetic and potential energies, namely,

$$E = \tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}kx^2 \qquad \text{------- 3}$$

The kinetic energy is quadratic in the velocity variable, and the potential energy is quadratic in the displacement variable. The total energy is constant if there are no other forces except the restoring force acting on the particle.

The motion of the particle can be found by starting with the energy equation (3). Solving for the velocity gives:

$$\dot{x} = \pm\left(\frac{2E}{m} - \frac{kx^2}{m}\right)^{1/2} \qquad \text{-------- 4}$$

Which can be integrated to give **t** as a function of **x** as follows :

$$t = \int \frac{dx}{\pm[(2E/m) - (k/m)x^2]^{1/2}} = \mp(m/k)^{1/2} \cos^{-1}(x/A) + C \qquad \text{------- 5}$$

In which C is a constant of integration and A is the amplitude given by:

$$A = \left(\frac{2E}{k}\right)^{1/2} \qquad \text{--------- 6}$$

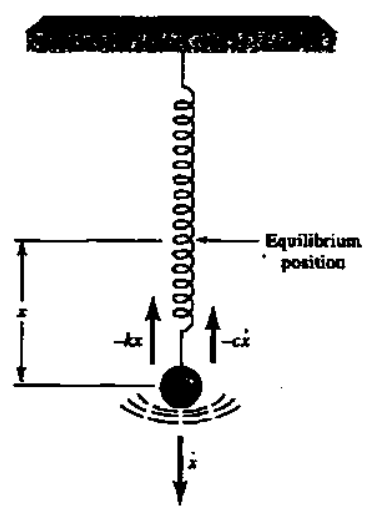
The energy equation(3) that the maximum value of the speed, which is call v_{max} occurs at x =0. Accordingly, we can write:

$$E = \frac{1}{2} m v_{max}^2 = \frac{1}{2} k A^2 \qquad \text{--------7}$$

As the particle oscillates, the kinetic and potential energies continually change. The constant total energy is entirely in the form of kinetic energy at the center, where x = 0 and $\dot{x} = \pm v_{max}$ and it is all potential energy $\dot{x} = 0$ and x = ±A.

2-6: Damped Harmonic Motion:

Let us consider an object of mass m that is supported by a light spring of stiffness k. We assume that there is a viscous retarding[المعيقة] force that is a linear function of the velocity, such as is produced by air drag[مقاومة] at low speeds. The forces are indicated in Figure below.



If x is the displacement from equilibrium, then the restoring force is $-kx$ and the retarding force is $-c\dot{x}$, where c is a constant of proportionality. The differential equation of motion is, therefore

$$m\ddot{x} + c\dot{x} + kx = 0$$

We trying the solutions:

$x = Ae^{qt}$, $\dot{x} = Aqe^{qt}$ and $\ddot{x} = Aq^2e^{qt}$

Substituting in equation of motion we obtain:

$$mq^2 + cq + k = 0$$

The roots are given by the well known quadratic formula:

$$q = \frac{-c \pm (c^2 - 4mk)^{1/2}}{2m}$$

There are three possible physically distinct مميزة cases حالات:

1- q real > 0 , $c^2 > 4mk$ over damping فوق التضاؤل

2- q real = 0 , $c^2 = 4mk$ critical damping التضاؤل الحرج

3- q imaginary , $c^2 < 4mk$ under damping دون التضاؤل

1-Overdamping: let call $-\gamma_1$ and $-\gamma_2$ the two real values of q. the general solution may then be written:

$$x = A_1 e^{-\gamma_1 t} + A_2 e^{-\gamma_2 t}$$

We see that the motion is non-oscillatory, the displacement x decaying to zero in an exponential manner as a time go on.

2- Critical damping: (here q =0). The two root are equal. To find the general solution, for equal roots, from the original differential equation of motion:

$$m\ddot{x} + c\dot{x} + kx = 0 \qquad \text{------- 1}$$

we can written:

$$\left(\frac{d}{dt} + \gamma\right)\left(\frac{d}{dt} + \gamma\right)x = 0 \qquad \text{------- 2}$$

Where

$$\gamma = \frac{c}{2m}$$

$$\left(\frac{d}{dt} + \gamma\right)u = 0$$

We make the substitution $u = \gamma x + dx/dt$ which then gives:

$$u = \left(\frac{d}{dt} + \gamma\right)x$$

This is integrated to give $u = A_1 e^{-\gamma t}$. Substitute back for u we obtain:

$$u = \left(\frac{d}{dt} + \gamma\right)x = A_1 e^{-\gamma t}$$

Which can also be written:

$$A_1 = \left(\frac{dx}{dt} + \gamma x\right)e^{\gamma t} \qquad \Rightarrow \qquad A_1 = \frac{d}{dt}\left(xe^{\gamma t}\right)$$

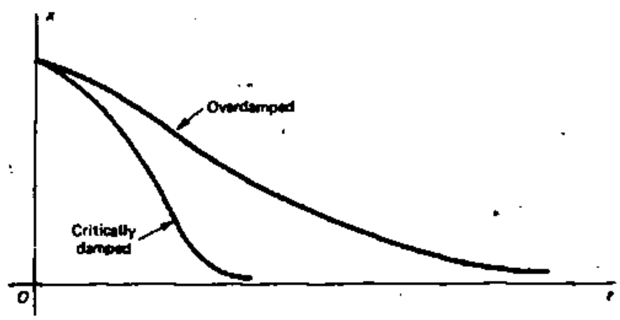
A second integration with respect to t then gives:

$$A_1 t = xe^{\gamma t} - A_2$$

Finally we can be rearranged to give:

$$x = (A_1 t + A_2)e^{-\gamma t} \qquad \text{------ 3}$$

For both over-damping and critical damping ,where the displacement x decaying to zero as a function of time, the motion are non-oscillatory as shown in figure bellow.



3- Under damping (q become imaginary): A mass initially displaced and then released from rest oscillates. The two roots in equation are conjugate complex number and the motion is given by the general solution:

$$x = A_+ e^{(-\gamma + i\omega_1)t} + A_- e^{(-\gamma - i\omega_1)t} \qquad \text{------- 4}$$

Where $\gamma = c/2m$ and

$$q = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m} = -\frac{c}{2m} \pm i\omega_1$$

that is

$$\omega_1 = \frac{\sqrt{4mk - c^2}}{2m} = \sqrt{\frac{k}{m} - \frac{c^2}{4m^2}} = \sqrt{\omega_0^2 - \gamma^2}$$

where w_o and w_1 are the angular frequencies of the undamped and underdamped harmonic oscillators respectively.

We now rewrite the general solution represented by equation 4 in terms of the factors described here,

$$x = e^{-\gamma t}(a \sin \omega_1 t + b \cos \omega_1 t)$$

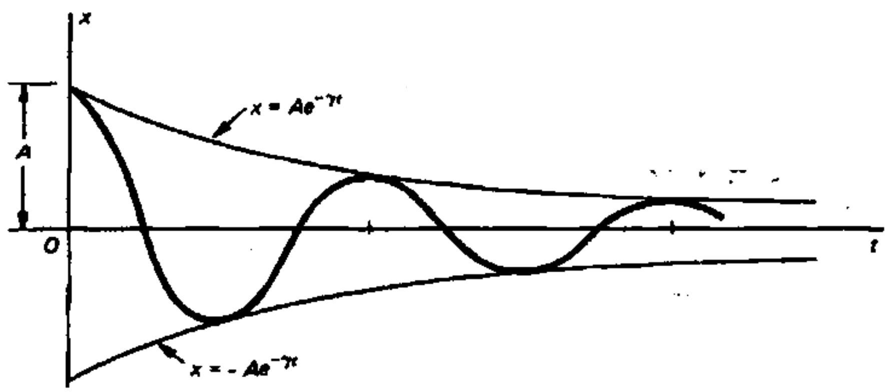
Where $a = i(A_+ - A_-)$ and $b = A_+ + A_-$.

We can also write the solution as:

$$x = Ae^{-\gamma t} \cos (\omega_1 t + \theta_0)$$

Where $A = (a^2+b^2)^{1/2}$ and $\theta_o = -\tan^{-1}(b/a)$.

The real from the solution shows that the motion is oscillatory, and the amplitude $Ae^{-\gamma t}$ decays exponentially with time and the angular frequancy of oscillation ω_1 is less than of the undamped oscillator ω_0 as shown in the figure. the frequancy ω_1 is called the natural frequancy.



2-7: Energy Considerations:

The total energy of the damped harmonic oscillator is given by the sum of the kinetic and potential energies:

$$E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$$

This is constant for the undamped oscillator. Let us differentiate the above expression with respect to t:

$$\frac{dE}{dt} = m\dot{x}\ddot{x} + kx\dot{x} = (m\ddot{x} + kx)\dot{x}$$

Now the differential equation of motion is:

$$m\ddot{x} + c\dot{x} + kx = 0, \text{ or } m\ddot{x} + kx = -c\dot{x}.$$

Thus we can write:

$$\frac{dE}{dt} = -c\dot{x}^2$$

For the time rate of change of total energy. We see that it is given by the product of the damping force and the velocity. Because this is always either zero or negative, the total energy continually decreases and, like the amplitude, eventually becomes negligibly small. The energy is dissipated[يتبدد] as frictional heat by virtue[استناد] of the viscous resistance to the motion.

Example: A particle of mass is attached to spring of stiffness k. the damping is such that $\gamma = \omega_0/4$. 1- Find the natural frequency. 2- Find the ratio of the amplitudes of two successive[متعاقبين] oscillations[اهتزازين].

Solution: 1-

$$\omega_1 = \sqrt{\frac{k}{m} - \frac{c^2}{4m^2}} = \sqrt{\omega_0^2 - \gamma^2}$$

$$\omega_0\sqrt{\frac{15}{16}} = \sqrt{\frac{k}{m}}\sqrt{\frac{15}{16}}$$

2- The ratio is given by

$$\frac{Ae^{-\gamma T_1}}{A} = e^{-\gamma T_1}$$

where

$$T_1 = \frac{1}{f_1} = \frac{2\pi}{\omega_1}$$

Hence, in our problem

$$T_1 = \frac{2\pi}{\omega_0}\sqrt{\frac{16}{15}} = \frac{2\pi}{4\gamma}\sqrt{\frac{16}{15}}$$

or

$$\gamma T_1 = \frac{\pi}{2}\sqrt{\frac{16}{15}} = 1.56$$

Hence the ratio of two successive swings is $e^{-1.56} = 0.21$.

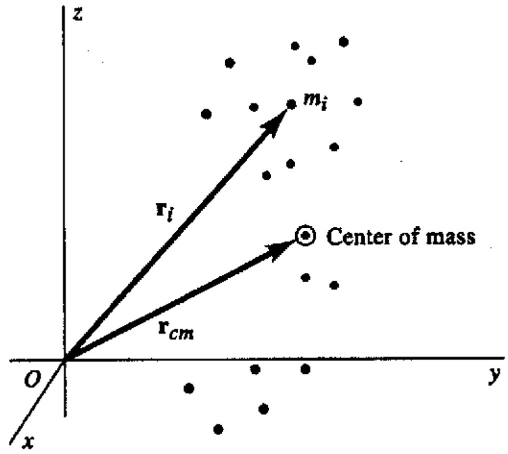
Chapter four

Dynamic of system of particles

4-1: Center of Mass and Linear Momentum of a System:

We now expand our study of mechanics of systems of many particles (two or more). These particles may or may not move independently of one another. Our general system consists of n particles of masses $m_1, m_2, \ldots, m_n$ whose position vectors are, respectively, $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n$. We define **the center of mass** of the system as the point whose position vector $\mathbf{r}_{cm}$(as shown in Figure bellow) is given by:

$$\mathbf{r}_{cm} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + \cdots + m_n\mathbf{r}_n}{m_1 + m_2 + \cdots + m_n} = \frac{\sum_i m_i\mathbf{r}_i}{m}$$



Where $m = \sum m_i$ the total mass of the system. The definition in Equation above is equivalent to the three equations

$$x_{cm} = \frac{\sum_i m_i x_i}{m} \qquad y_{cm} = \frac{\sum_i m_i y_i}{m} \qquad z_{cm} = \frac{\sum_i m_i z_i}{m}$$

We define the linear momentum **p** of the system as the vector sum of the linear momenta of the individual particles, namely,

$$\mathbf{p} = \sum_i \mathbf{p}_i = \sum_i m_i\mathbf{v}_i$$

On calculating $\dot{r}_{cm} = \mathbf{v}_{cm}$ from equations above we find

$$\mathbf{p} = m\mathbf{v}_{cm}$$

that is, the linear momentum of a system of particles is equal to the velocity of the center of mass multiplied by the total mass of the system.

Suppose now that there are external forces $\mathbf{F}_1, \mathbf{F}_2, \ldots, \mathbf{F}_n$ acting on the particles. In addition, there may be internal forces of interaction between any two particles of the system. We denote these internal forces by $\mathbf{F}_{ij}$ meaning the force exerted on particle i by particle j, with the understanding that $\mathbf{F}_{ii} = 0$.

The equation of motion of particle i is then:

$$\mathbf{F}_i + \sum_{j=1}^{n} \mathbf{F}_{ij} = m_i \ddot{\mathbf{r}}_i = \dot{\mathbf{p}}_i$$

Where F_i means the total external force acting on particle i. The second term in equation represents the vector sum of all the internal forces exerted on particle i by all other particles of the system. Adding equation above for the n particles, we have

$$\sum_{i=1}^{n} \mathbf{F}_i + \sum_{i=1}^{n}\sum_{j=1}^{n} \mathbf{F}_{ij} = \sum_{i=1}^{n} \dot{\mathbf{p}}_i$$

In the double summation in equation, for every force $\mathbf{F}_{ij}$ there is also a force F_{ji}, and these two forces are equal and opposite

$$\mathbf{F}_{ij} = -\mathbf{F}_{ji}$$

From the law of action and reaction, Newton's third law. The internal forces cancel in pairs, and the double sum vanishes. We can, therefore, write

$$\sum_i \mathbf{F}_i = \dot{\mathbf{p}} = m\mathbf{a}_{cm}$$

In words: The acceleration of the center of mass of a system of particles is the same as that of a single particle having a mass equal to the total mass of the system and acted on by the sum of the external forces.

For example, a swarm of particles moving in a uniform gravitational field. Then, because $\mathbf{F} = m_i g$ for each particle,

$$\sum_i \mathbf{F}_i = \sum m_i \mathbf{g} = m\mathbf{g}$$

The last step follows from the fact that g is constant. Hence,

$$\mathbf{a}_{cm} = \mathbf{g}$$

This is the same as the equation for a single particle or projectile. In the special case in which no external forces are acting on a system (or if $\sum F_i = 0$), then $\mathbf{a}_{cm} = 0$ and $\mathbf{v}_{cm}$= constant; thus, the linear momentum of the system remains constant:

$$\sum_i \mathbf{p}_i = \mathbf{p} = m\mathbf{v}_{cm} = \text{constant}$$

This is the principle of conservation of linear momentum.

4-2: Angular Momentum and Kinetic Energy of a System:

The angular momentum of a single particle is defined as the cross product $\mathbf{r} \times m\mathbf{v}$. The angular momentum L of a system of particles is defined accordingly, as the vector sum of the individual angular momenta, namely,

$$\mathbf{L} = \sum_{i=1}^{n} (\mathbf{r}_i \times m_i \mathbf{v}_i)$$

Let us calculate the time derivative of the angular momentum, we find:

$$\frac{d\mathbf{L}}{dt} = \sum_{i=1}^{n} (\mathbf{v}_i \times m_i \mathbf{v}_i) + \sum_{i=1}^{n} (\mathbf{r}_i \times m_i \mathbf{a}_i)$$

Now the first term on the right vanishes, because, $\mathbf{v}_i \times \mathbf{v}_i = 0$ and, because $m_i\mathbf{a}_i$ is equal to the total force acting on particle i, we can write

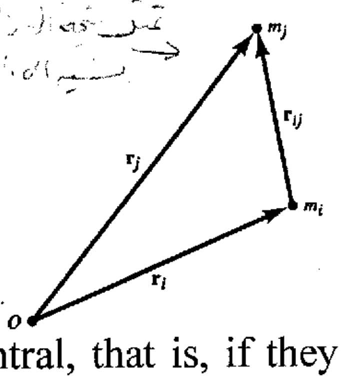
$$\frac{d\mathbf{L}}{dt} = \sum_{i=1}^{n} \left[\mathbf{r}_i \times \left(\mathbf{F}_i + \sum_{j=1}^{n} \mathbf{F}_{ij} \right) \right]$$

$$= \sum_{i=1}^{n} \mathbf{r}_i \times \mathbf{F}_i + \sum_{i=1}^{n} \sum_{j=1}^{n} \mathbf{r}_i \times \mathbf{F}_{ij} \quad \text{..........(1)}$$

where $\mathbf{F}_i$ denotes the total external force on particle i, and $\mathbf{F}_{ij}$ denotes the (internal) force exerted on particle i by any other particle j. Now the double summation on the right consists of pairs of terms of the form

$$(\mathbf{r}_i \times \mathbf{F}_{ij}) + (\mathbf{r}_j \times \mathbf{F}_{ji}) \quad \text{........ (2)}$$

Denoting the vector displacement of particle j relative to particle i by we see from the triangle shown in Figure



$$\mathbf{r}_{ij} = \mathbf{r}_j - \mathbf{r}_i$$

Therefore, because $\mathbf{F}_{ji} = -\mathbf{F}_{ij}$ expression (2) reduces to

$$-\mathbf{r}_{ij} \times \mathbf{F}_{ij}$$

Which clearly vanishes if the internal forces are central, that is, if they act along the lines connecting pairs of particles. Hence, the double sum in Equation 2 vanishes. Now the cross product $\mathbf{r}_i \times \mathbf{F}_i$ is the moment of the external force F_i. The $\sum \boldsymbol{r}_i \times \boldsymbol{F}_i$ is, therefore, the total moment of all the external forces acting on the system.

If we denote the total external torque, or moment of force, by **N**, equation (1) takes the form

$$\frac{d\mathbf{L}}{dt} = \mathbf{N}$$

That is, **the time rate of change of the angular momentum of a system is equal to the total moment of all the external forces acting on the system.** If a system is isolated, then **N** =0, and the angular momentum remains constant in both magnitude and direction:

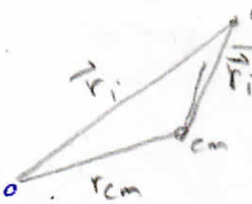
$$\mathbf{L} = \sum_i \mathbf{r}_i \times m_i \mathbf{v}_i = \text{constant vector}$$

This is a statement of the principle of conservation of angular momentum.

4-3-Kinetic Energy of a System of particles:

The total kinetic energy T of a system of particles is given by the sum of the individual energies, namely,

$$T = \sum_i \tfrac{1}{2} m_i v_i^2 = \sum_i \tfrac{1}{2} m_i (\mathbf{v}_i \cdot \mathbf{v}_i)$$



we can express the velocities relative to the mass center giving

$$T = \sum_i \tfrac{1}{2} m_i (\mathbf{v}_{cm} + \bar{\mathbf{v}}_i) \cdot (\mathbf{v}_{cm} + \bar{\mathbf{v}}_i)$$

$$= \sum_i \tfrac{1}{2} m_i v_{cm}^2 + \sum_i m_i (\mathbf{v}_{cm} \cdot \bar{\mathbf{v}}_i) + \sum_i \tfrac{1}{2} m_i \bar{v}_i^2$$

$$= \tfrac{1}{2} v_{cm}^2 \sum_i m_i + \mathbf{v}_{cm} \cdot \sum_i m_i \bar{\mathbf{v}}_i + \sum_i \tfrac{1}{2} m_i \bar{v}_i^2$$

Because the second summation $\sum_i m_i \bar{\mathbf{v}}_i$ vanishes, we can express the kinetic energy as follows:

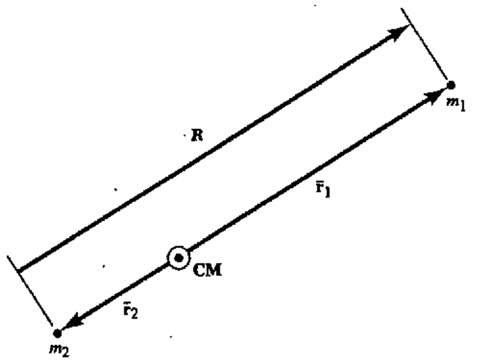
$$T = \tfrac{1}{2} m v_{cm}^2 + \sum_i \tfrac{1}{2} m_i \bar{v}_i^2$$

The first term is the kinetic energy of translation of the whole system, and the second is the kinetic energy of motion relative to the mass center.

4-4: Motion of Two Interacting Bodies: The Reduced Mass

Let us consider the motion of a system consisting of two bodies, treated here as particles, that interact with each other by a central force. We assume the system is isolated, and, hence, the center of mass moves with constant velocity. For simplicity, we take the center of mass as the origin. We have then

$$m_1\bar{\mathbf{r}}_1 + m_2\bar{\mathbf{r}}_2 = 0$$



where, as shown in Figure, the vectors $\vec{r}_1$ and $\vec{r}_2$ represent the positions of the particles m_1 and m_2, respectively, relative to the center of mass. Now, if R is the position vector of particle 1 relative to particle 2, then

$$\mathbf{R} = \bar{\mathbf{r}}_1 - \bar{\mathbf{r}}_2 = \bar{\mathbf{r}}_1\left(1 + \frac{m_1}{m_2}\right)$$

The differential equation of motion of particle 1 relative to the center of mass is

$$m_1\frac{d^2\bar{\mathbf{r}}_1}{dt^2} = \mathbf{F}_1 = f(R)\frac{\mathbf{R}}{R}$$

in which $|f_R|$ is the magnitude of the mutual force between the two particles. By using equation **R**, we can write

$$\mu\frac{d^2\mathbf{R}}{dt^2} = f(R)\frac{\mathbf{R}}{R}$$

The new equation of motion gives the motion of particle 1 relative to particle 2, and an exactly similar equation gives the motion of particle 2 relative to particle 1. This equation is precisely the same as the ordinary equation of motion of a single particle of mass μ moving in a central field of force given by f (R). Thus, the fact that both particles are moving relative to the center of mass is automatically accounted for by replacing m1 by the reduced mass μ where

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

The quantity μ is called the reduced mass. If the bodies are of equal mass m, then μ= m/2. On the other hand, if m_2 is very much greater than m_1, so that m_1/m_2 is very small, then μ is nearly equal to m_1.

4-5: Motion of a Body with Variable Mass: Rocket Motion

Consider the general case of the motion of a body with changing mass. Let $\mathbf{F}_{ext}$ denote the external force acting on the body at a given time, and let Δm denote the increment of the mass of the body that occurs in a short time interval Δt. Then

$F_{ext}\Delta t$ is the impulse الدفع delivered by the external force, and we have for the total linear momentum of the system

$$\mathbf{F}_{ext}\Delta t = (\mathbf{p}_{total})_{t+\Delta t} - (\mathbf{p}_{total})_t$$

if **v** denotes the velocity of the body and V the velocity of the mass increment Δm relative to the body, then we can write

$$\mathbf{F}_{ext}\Delta t = (m + \Delta m)(\mathbf{v} + \Delta\mathbf{v}) - [m\mathbf{v} + \Delta m(\mathbf{v} + \mathbf{V})]$$

this reduce to

$$\mathbf{F}_{ext}\Delta t = m\Delta\mathbf{v} + \Delta m\Delta\mathbf{v} - \mathbf{V}\Delta m$$

Dividing by Δt, we can write

$$\mathbf{F}_{ext} = (m + \Delta m)\frac{\Delta\mathbf{v}}{\Delta t} - \mathbf{V}\frac{\Delta m}{\Delta t}$$

Thus, in the limit as Δt approach zero, we have the general equation

$$\mathbf{F}_{ext} = m\dot{\mathbf{v}} - \mathbf{V}\dot{m}$$

The force $\mathbf{F}_{ext}$ represents any external force, such as gravity, air resistance, and so forth. In the case of rockets the $V\dot{m}$ represents the thrust الدفع.

Let apply this equation of motion to two special cases in which mass is added to or lost from the moving body. First, suppose that the body is falling through a fog or mist so that it collects mass as it goes. In this case the initial velocity of the accumulated matter is zero. Hence, V = –v, and we obtain for the equation of motion.

$$\mathbf{F}_{ext} = m\dot{\mathbf{v}} + \mathbf{v}\dot{m} = \frac{d}{dt}(m\mathbf{v})$$

It applies only if the initial velocity of the matter that is being swept us is zero

For the second case, consider the motion of a rocket. The sign of $\dot{m}$ is negative because the rocket is losing mass in the form of ejected fuel. Here, we solve the equation of motion for the simplest case of rocket motion in which the external force on it is zero; that is, the rocket is not subject to any force of gravity, air resistance, and so on. Thus, F_{ext}= 0, and we have

$$m\dot{\mathbf{v}} = \mathbf{V}\dot{m}$$

We can now separate the variables and integrate to find v as follows:

$$\int d\mathbf{v} = \int \frac{\mathbf{V}\,dm}{m}$$

If we assume that **V** is constant, then we can integrate between limits to find the speed as function of m:

$$\int_{v_0}^{v} dv = -V\int_{m_0}^{m} \frac{dm}{m}$$

$$v = v_0 + V\ln\frac{m_0}{m}$$

Here m_o is the initial mass of the rocket plus unburned[غير محترق] fuel[وقود], m is the mass at any time, and V is the speed of the ejected fuel relative to the rocket. Owing[نظرا] to the nature of the logarithmic function[الدالة], the rocket must have a large fuel-to-payload ratio to attain[تحقيق] the large speeds needed for launching[لإطلاق] satellites into space.

Problem:

Find the equation of motion for a rocket fired vertically upward, assuming g is constant. **Find** the ratio of fuel to payload to achieve a final speed equal to the escape speed; from the Earth if the speed of the exhaust gas is kv_e, where k is a given constant, and the fuel burning rate is $|\dot{m}|$.

Solution:

$$-m\bar{g} = m\dot{\bar{v}} - \bar{V}\dot{m}$$

since $\bar{V}$ is opposite in direction from $\vec{v}$.

$$-mg = m\dot{v} + V\dot{m}$$

$$-mg\,dt = m\,dv + V\,dm$$

$$\dot{m} = \frac{dm}{dt} \qquad \text{so } dt = \frac{dm}{\dot{m}}$$

$$-mg\frac{dm}{\dot{m}} = m\,dv + V\,dm$$

$$dv = -dm\left(\frac{g}{\dot{m}} + \frac{V}{m}\right)$$

$$\int_0^{v_e} dv = -\int_{m_0}^{m_p} dm\left(\frac{g}{\dot{m}} + \frac{V}{m}\right)$$

$$v_e = \frac{g}{\dot{m}}(m_o - m_p) + V\ln\frac{m_o}{m_p}$$

m_p = payload mass

$$m_f = m_o - m_p \approx m_o \qquad m_f = \text{fuel mass}$$

$$v_e = \frac{g}{\dot{m}} m_f + V \ln\left(1 + \frac{m_f}{m_p}\right)$$

$$\ln\left(1 + \frac{m_f}{m_p}\right) = \frac{v_e}{V} - \frac{g}{V}\frac{m_o}{\dot{m}}$$

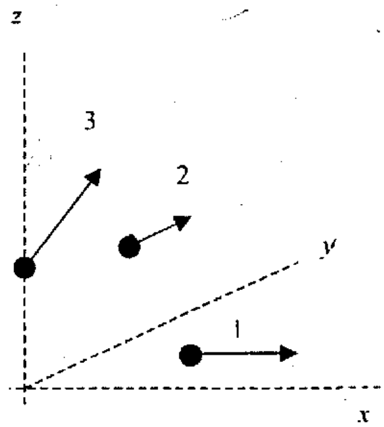
$$\frac{m_f}{m_p} = \exp\left(\frac{v_e}{V} - \frac{g}{V}\frac{m_o}{\dot{m}}\right) - 1$$

For $V = kv_e$, $\quad \dfrac{m_f}{m_p} = \exp\left(\dfrac{1}{k} - \dfrac{g}{kv_e}\dfrac{m_o}{\dot{m}}\right) - 1$

Problems:

4-1: A system consists of three particles, each of unit mass, with positions and velocities as follows:

$$\mathbf{r}_1 = \mathbf{i} + \mathbf{j} \qquad \mathbf{v}_1 = 2\mathbf{i}$$
$$\mathbf{r}_2 = \mathbf{j} + \mathbf{k} \qquad \mathbf{v}_2 = \mathbf{j}$$
$$\mathbf{r}_3 = \mathbf{k} \qquad \mathbf{v}_3 = \mathbf{i} + \mathbf{j} + \mathbf{k}$$



(a) Find the position and velocity of the center of mass.

(b) Find also the linear momentum of the system.

(c) Find the kinetic energy of the system.

(d) Find the value of $mv_{cm}^2/2$.

(e) Find the angular momentum about the origin.

Solution: from equation $\vec{r}_{cm} = \frac{1}{m}\sum_i m_i \vec{r}_i$

a-

$$\vec{r}_{cm} = \frac{1}{3}(\vec{r}_1 + \vec{r}_2 + \vec{r}_3) = \frac{1}{3}\left(\hat{i} + \hat{j} + \hat{j} + \hat{k} + \hat{k}\right)$$

$$\vec{r}_{cm} = \frac{1}{3}\left(\hat{i} + 2\hat{j} + 2\hat{k}\right)$$

$$\vec{v}_{cm} = \frac{d}{dt}\vec{r}_{cm} = \frac{1}{3}(\vec{v}_1 + \vec{v}_2 + \vec{v}_3) = \frac{1}{3}\left(2\hat{i} + \hat{j} + \hat{i} + \hat{j} + \hat{k}\right)$$

$$\vec{v}_{cm} = \frac{1}{3}\left(3\hat{i} + 2\hat{j} + \hat{k}\right)$$

b: From equation $\vec{p} = \sum_i m_i \vec{v}_i = \vec{v}_1 + \vec{v}_2 + \vec{v}_3$

$$\vec{p} = 3\hat{i} + 2\hat{j} + \hat{k}$$

c: $$T = \sum_i \frac{1}{2} m_i v_i^2$$

$$T = \frac{1}{2}\left[2^2 + 1^2 + \left(1^2 + 1^2 + 1^2\right)\right] = 4$$

d: $$\vec{v}_{cm} = \frac{1}{3}\left(3\hat{i} + 2\hat{j} + \hat{k}\right)$$

$$\frac{1}{2} m v_{cm}^2 = \frac{1}{2} \times 3 \times \frac{1}{9}\left(3^2 + 2^2 + 1^2\right) = 2\frac{1}{3}$$

e: $$\vec{L} = \sum_i \vec{r}_i \times m\vec{v}_i$$

$$\vec{L} = \left[\left(\hat{i} + \hat{j}\right) \times 2\hat{i}\right] + \left[\left(\hat{j} + \hat{k}\right) \times \hat{j}\right] + \left[\hat{k} \times \left(\hat{i} + \hat{j} + \hat{k}\right)\right]$$

$$\vec{L} = \left(-2\hat{k}\right) + \left(-\hat{i}\right) + \left(\hat{j} - \hat{i}\right) = -2\hat{i} + \hat{j} - 2\hat{k}$$

4-2: Show that the kinetic energy of a two-particle system $\frac{1}{2} m v_{cm}^2 + \frac{1}{2}\mu v^2$, where $m = m_1 + m_2$, v is the relative speed, and μ is the reduced mass.

Solution: from equation

$$T = \sum_i \tfrac{1}{2} m_i v_i^2 = \sum_i \tfrac{1}{2} m_i (\mathbf{v}_i \cdot \mathbf{v}_i)$$

$$= \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2$$

Meanwhile:

$$\frac{1}{2} m v_{cm}^2 + \frac{1}{2}\mu v^2 = \frac{1}{2} m \left(\frac{m_1\vec{v}_1 + m_2\vec{v}_2}{m}\right)^2 + \frac{1}{2}\frac{m_1 m_2}{m_1 + m_2}\left(\vec{v}_1 - \vec{v}_2\right)^2$$

$$= \frac{1}{2m}\left[m_1^2 v_1^2 + m_2^2 v_2^2 + 2m_1 m_2 \vec{v}_1 \cdot \vec{v}_2 + m_1 m_2\left(v_1^2 + v_2^2 - 2\vec{v}_1 \cdot \vec{v}_2\right)\right]$$

$$= \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2$$

Therefore, $T = \frac{1}{2} m v_{cm}^2 + \frac{1}{2}\mu v^2$

4-3: Show that the angular momentum of a two-particle system is:

$$\mathbf{r}_{cm} \times m\mathbf{v}_{cm} + \mathbf{R} \times \mu\mathbf{v}$$

where $m = m_1 + m_2$, μ is the reduced mass, R is the relative position vector, and **v** is the relative velocity of the two particles.

Solution: from equation

$$\vec{L} = \vec{r}_{cm} \times m\vec{v}_{cm} + \sum_i \bar{\vec{r}}_i \times m_i \bar{\vec{v}}_i$$

$$\sum_i \bar{\vec{r}}_i \times m_i \bar{\vec{v}}_i = \bar{\vec{r}}_1 \times m_1 \bar{\vec{v}}_1 + \bar{\vec{r}}_2 \times m_2 \bar{\vec{v}}_2$$

From equation:

$$\mathbf{R} = \bar{\mathbf{r}}_1 - \bar{\mathbf{r}}_2 = \bar{\mathbf{r}}_1\left(1 + \frac{m_1}{m_2}\right) = \bar{\vec{r}}_1\left(\frac{m_1 + m_2}{m_2}\right) = \frac{m_1}{\mu}\bar{\vec{r}}_1$$

Since from equation

$$\bar{\vec{r}}_1 = -\frac{m_2}{m_1}\bar{\vec{r}}_2$$

$$\vec{R} = -\frac{m_2}{\mu}\bar{\vec{r}}_2$$

$$\sum_i \bar{\vec{r}}_i \times m_i \bar{\vec{v}}_i = \frac{\mu}{m_1}\vec{R} \times m_1 \bar{\vec{v}}_1 + \left(-\frac{\mu}{m_2}\right)\vec{R} \times m_2 \bar{\vec{v}}_2$$

$$= \mu\vec{R} \times \left(\bar{\vec{v}}_1 - \bar{\vec{v}}_2\right) = \vec{R} \times \mu\vec{v}$$

$$\vec{L} = \vec{r}_{cm} \times m\vec{v}_{cm} + \vec{R} \times \mu\vec{v}$$

4-4: A rocket traveling through the atmosphere experiences a linear air resistance $-kv$. Find the differential equation of motion when all other external forces are negligible. Integrate the equation and show that if the rocket starts from rest, the final speed is given by

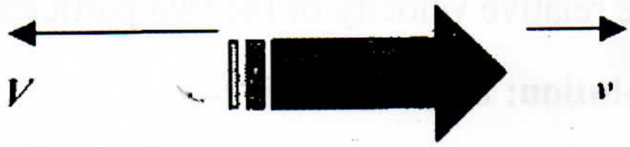
$$v = V\alpha[1 - (m/m_0)^{1/\alpha}]$$

Where V is the relative speed of the exhaust fuel, $\alpha = |\dot{m}/k| =$ constant, m_o is the initial mass of the rocket plus fuel, and m is the final mass of the rocket.

Solution: from equation $\mathbf{F}_{ext} = \dot{\mathbf{P}} = m\dot{\mathbf{v}} - \mathbf{V}\dot{m}$

$$-k\vec{v} = m\dot{\vec{v}} - \vec{V}\dot{m}$$

Since $\vec{V}$ is opposite in direction from $\vec{v}$



$$-kv = m\dot{v} + V\dot{m}$$

$$v = -\frac{m}{k}\dot{v} - \frac{\dot{m}}{k}V$$

$$\alpha = \left|\frac{\dot{m}}{k}\right| \text{ and since } \dot{m} < 0,\ \alpha = -\frac{\dot{m}}{k}$$

$$v - \alpha V = \alpha\frac{m}{\dot{m}}\dot{v} = \alpha\frac{m}{\dot{m}}\frac{dv}{dt} = \alpha\frac{m}{\dot{m}}\frac{dv}{dm}\frac{dm}{dt} = \alpha m\frac{dv}{dm}$$

$$\frac{dm}{\alpha m} = \frac{dv}{v - V\alpha}$$

$$\frac{1}{\alpha}\int_{m_o}^{m}\frac{dm}{m} = \int_0^{v}\frac{dv}{v - V\alpha}$$

$$\frac{1}{\alpha}\ln\left(\frac{m}{m_o}\right) = \ln\left(\frac{v - V\alpha}{-V\alpha}\right)$$

$$\left(\frac{m}{m_o}\right)^{\frac{1}{\alpha}} = -\frac{v}{V\alpha} + 1$$

$$v = V\alpha\left[1 - \left(\frac{m}{m_o}\right)^{\frac{1}{\alpha}}\right]$$

Homework:

1- locate the center of mass of three particles of mass m_1=1 kg, m_2= 2kg, and m_3=3 kg at the center of triangle 1 m on side

2- find the center of mass, the velocity of the center of mass, the momentum, and the kinetic energy of the following system:

m_1= 1 kg $\quad \vec{r_1} = \hat{\imath} + 2\hat{\jmath} + 3\hat{k} \quad \vec{v}_1 = 2\hat{\imath} + 3\hat{\jmath}$

m_2= 1 kg $\quad \vec{r_2} = \hat{\imath} - \hat{\jmath} + \hat{k} \quad \vec{v}_2 = 2\hat{\jmath} + 3\hat{k}$

4-6: Mechanics of Rigid Bodies: Planer Motion

For idealized a rigid body may be regarded as a system of particles whose relative positions are fixed, in other word, the distance between any two particles is constant.

4-6-1: Center of Mass of a Rigid Body:

We have already defined the center of mass of a system of particles as :

$$x_{cm} = \frac{\sum_i x_i m_i}{\sum_i m_i} \qquad y_{cm} = \frac{\sum_i y_i m_i}{\sum_i m_i} \qquad z_{cm} = \frac{\sum_i z_i m_i}{\sum_i m_i}$$

For a rigid extended body, we can replace the summation by, an integration over the volume of the body, namely,

$$x_{cm} = \frac{\int_v \rho x \, dv}{\int_v \rho \, dv} \qquad y_{cm} = \frac{\int_v \rho y \, dv}{\int_v \rho \, dv} \qquad z_{cm} = \frac{\int_v \rho z \, dv}{\int_v \rho \, dv}$$

Where ρ is the density and dv is the element of volume. If a rigid body is in the form of a thin shell, the equations for the center of mass become:

$$x_{cm} = \frac{\int_s \rho x \, ds}{\int_s \rho \, ds} \qquad y_{cm} = \frac{\int_s \rho y \, ds}{\int_s \rho \, ds} \qquad z_{cm} = \frac{\int_s \rho z \, ds}{\int_s \rho \, ds}$$

Where *ds* is the element of area and ρ is the mass per unit area, the integration extending over the area of the body. Similarly, if the body is in the form of a thin wire, we have

$$x_{cm} = \frac{\int_l \rho x \, dl}{\int_l \rho \, dl} \qquad y_{cm} = \frac{\int_l \rho y \, dl}{\int_l \rho \, dl} \qquad z_{cm} = \frac{\int_l \rho z \, dl}{\int_l \rho \, dl}$$

In this case, ρ is the mass per unit length and *dl* is the element of length. For uniform homogeneous bodies, the density factors **ρ** are constant in each case and, therefore, may be canceled out in each of the preceding equations.

If a body is composite, that is, if it consists of two or more parts whose centers of mass are known, then it is clear, from the definition of the center of mass, that we can write

$$x_{cm} = \frac{x_1 m_1 + x_2 m_2 + \cdots}{m_1 + m_2 + \cdots}$$

With similar equations for y_{cm} and z_{cm}. Here (x_1, y_1, z_1) is the center of mass of the part m_1, and so on.

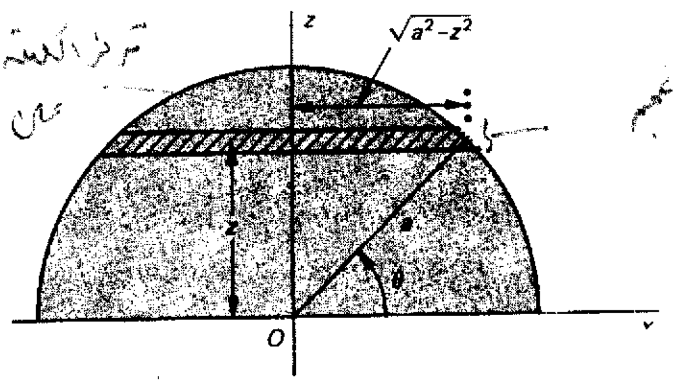
If a body possesses symmetry, for example, the body has a plane of symmetry, that is, if each particle m_i has a mirror image of itself m'_i relative to some plane, then the center of mass lies in that plane. To prove this, let us suppose that the xy plane is a plane of symmetry. We have then

$$z_{cm} = \frac{\sum_i (z_i m_i + z'_i m'_i)}{\sum_i (m_i + m'_i)}$$

But $m_i = m'_i$ and $z_i = -z'_i$ Hence, the terms in the numerator cancel in pairs, and so $z_{cm} = 0$; that is, the center of mass lies in the xy plane. Similarly, if the body has a line of symmetry, it is easy to show that the center of mass lies on that line.

4-6-1-1: Solid Hemisphere

To find the center of mass of a solid homogeneous hemisphere (نصف كرة) of radius a, we know from that the center of mass lies on the radius that is normal to the plane face. Choosing coordinate axes as shown in figure we see that the center of mass lies on the z-axis. To calculate z_{cm} we use a circular element of volume of thickness dz and radius $= (a^2 - z^2)^{1/2}$, as shown. Thus,

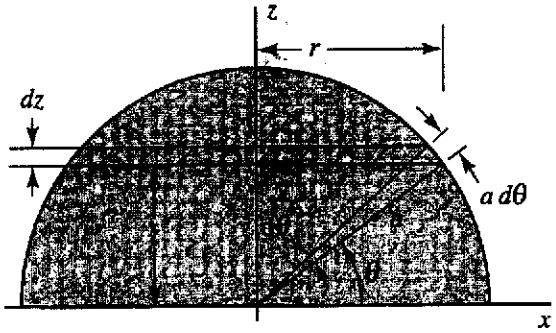


$$dv = \pi(a^2 - z^2)dz$$

$$z_{cm} = \frac{\int_0^a \rho\pi z(a^2 - z^2)dz}{\int_0^a \rho\pi(a^2 - z^2)dz} = \frac{3}{8}a$$

4-6-1-2: Hemispherical Shell قشرة:

For a hemispherical shell of radius a, we use the same axes as in Figure. Again, from symmetry, the center of mass is located on the z-axis. For our element of surface ds, we choose a circular strip of width $dl = a d\theta$. Hence,



$$ds = 2\pi r\,dl = 2\pi(a^2 - z^2)^{1/2} a\,d\theta$$

$$\theta = \sin^{-1}\left(\frac{z}{a}\right) \qquad d\theta = (a^2 - z^2)^{-1/2} dz$$

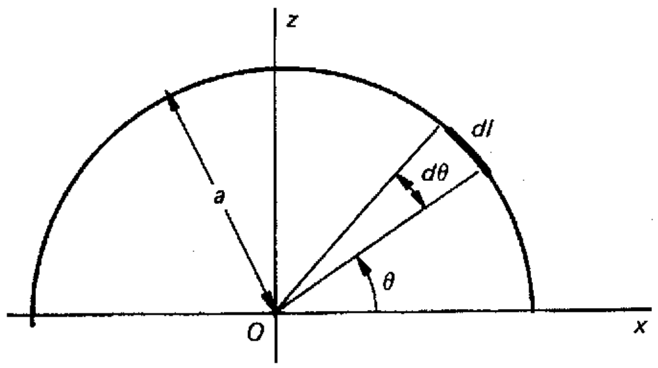
$$\therefore ds = 2\pi a\,dz$$

The location of the center of mass is accordingly given by:

$$z_{cm} = \frac{\int_0^a \rho 2\pi a z\,dz}{\int_0^a \rho 2\pi a\,dz} = \frac{1}{2}a$$

4-6-1-3: Semicircle

To find the center of mass of a thin wire bent into the form of a semicircle of radius a, we use axes as shown in figure. We have



$$dl = a\,d\theta$$

and

$$z = a \sin\theta$$

Hence,

$$z_{cm} = \frac{\int_0^{\pi} \rho(a\sin\theta)a\,d\theta}{\int_0^{\pi} \rho a\,d\theta} = \frac{2a}{\pi}$$

4-6-1-4: Semicircular Lamina

In the case of a uniform semicircular lamina, the center of mass is on the z-axis (Figure above).

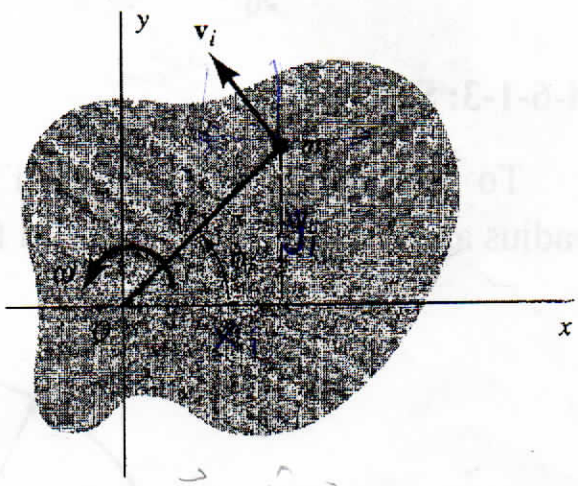
$$z_{cm} = \frac{4a}{3\pi}$$

4-6-2: Rotation of a Rigid Body about a Fixed Axis: Moment of Inertia

The simplest type of rigid-body motion is that in which the body is constrained to rotate about a fixed axis. Let us choose the z- axis of an appropriate coordinate system as the axis of rotation. The path of a representative particle m_i located at the point (x_i, y_i, z_i) is then a circle of radius $\left(x_i^2 + y_i^2\right)^{1/2} = r_i =$ centered on the z-axis. A representative cross section parallel to the xy plane is shown in figure.

The speed v_i of particle i is given by:

$$v_i = r_i\omega = \left(x_i^2 + y_i^2\right)^{1/2}\omega$$



where ω is the angular speed of rotation. From a study of figure, we see that the velocity has components as follows:

$$\dot{x}_i = -v_i \sin\phi_i = -\omega y_i$$
$$\dot{y}_i = v_i \cos\phi_i = \omega x_i$$
$$\dot{z}_i = 0$$

The three equations above can also be obtained by extracting the components of the vector equation:

$$\mathbf{v}_i = \boldsymbol{\omega} \times \mathbf{r}_i$$

where $\boldsymbol{\omega} = \mathbf{k}\omega$.

Let us calculate the kinetic energy of rotation of the body. We have

$$T_{rot} = \sum_i \tfrac{1}{2} m_i v_i^2 = \tfrac{1}{2}\left(\sum_i m_i r_i^2\right)\omega^2 = \tfrac{1}{2} I_z \omega^2$$

$$I_z = \sum_i m_i r_i^2 = \sum_i m_i\left(x_i^2 + y_i^2\right)$$

The quantity I_z is called the moment of inertia about the z-axis.

Let us next calculate the angular momentum about the axis of rotation. Because the angular momentum of a single particle is $\mathbf{r}_i \times m_i \mathbf{v}_i$ the z-component is:

$$m_i(x_i \dot{y}_i - y_i \dot{x}_i) = m_i\left(x_i^2 + y_i^2\right)\omega = m_i r_i^2 \omega$$

The total z-component of the angular momentum, which we call L_z is then given by summing over all the particles, namely,

$$L_z = \sum_i m_i r_i^2 \omega = I_z \omega$$

the rate of change of angular momentum for any system is equal to the total moment of the external forces. For a body constrained to rotate about a fixed axis, taken here as the z-axis, then

$$N_z = \frac{dL_z}{dt} = \frac{d(I_z\omega)}{dt}$$

Where N_z is the total moment of all the applied forces about the axis of rotation (the component of N along the z-axis). If the body is rigid, then I_z is constant, and we can write

$$N_z = I_z \frac{d\omega}{dt}$$

The analogy between the equations for translation and for rotation about a fixed axis is shown in the following table:

Translation along x-axis		***Rotation about z-axis***	
Linear momentum	$p_x = mv_x$	Angular momentum	$L_z = I_z\omega$
Force	$F_x = m\dot{v}_x$	Torque	$N_z = I_z\dot{\omega}$
Kinetic energy	$T = \frac{1}{2}mv^2$	Kinetic energy	$T_{rot} = \frac{1}{2}I_z\omega^2$

Thus, the moment of inertia is analogous to mass; it is a measure of the rotational inertia of a body relative to some fixed axis of rotation, just as mass is a measure of translational inertia of a body.

4-7: Calculation of the Moment of Inertia:

In calculations of the moment of inertia $\sum_i m_i r_i^2$ for extended bodies, we can replace the summation by an integration over the body, just as we did in calculation of the center of mass. Thus, we may write for any axis

$$I = \int r^2 dm$$

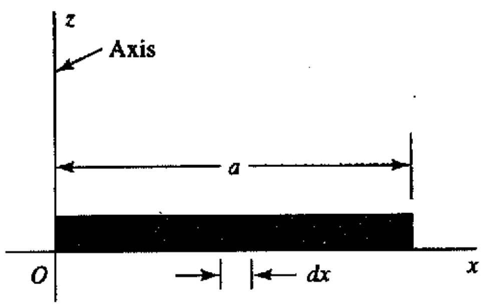
Where the element of mass dm is given by a density factor multiplied by an appropriate differential (volume, area, or length), and r is the perpendicular distance from the element of mass to the axis of rotation. In the case of a composite body, from the definition of the moment of inertia, we may write

$$I = I_1 + I_2 + \cdots$$

where I_1, I_2 and soon, are the moments of inertia of the various parts about the particular axis chosen. Let us calculate the moments of inertia for some important special cases.

4-7-1: Thin Rod

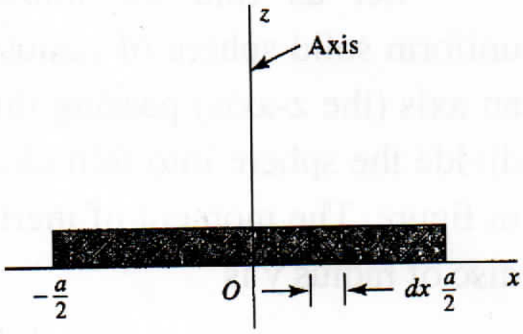
For a thin, uniform rod of length a and mass m, we have, for an axis perpendicular to the rod at one end



$$I_z = \int_0^a x^2 \rho dx = \tfrac{1}{3}\rho a^3 = \tfrac{1}{3} m a^2$$

The last step follows from the fact that m = ρa.

If the axis is taken at the center of the rod, we have



$$I_z = \int_{-a/2}^{a/2} x^2 \rho\, dx = \tfrac{1}{12}\rho a^3 = \tfrac{1}{12} m a^2$$

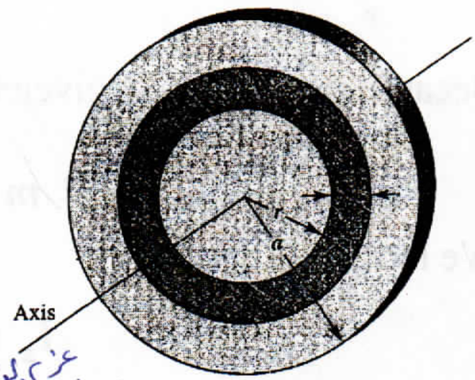
4-7-2: Circular Disc or Cylindrical Shell

In the case of a thin circular disc or cylindrical shell, for the central, or symmetry, axis, all particles lie at the same distance from the axis. Thus,

$$I_{axis} = m a^2$$

Where a is the radius and m is the mass.

To calculate the moment of inertia of a uniform circular disc of radius a and mass m, we use polar coordinates. The element of mass, a thin ring of radius r and thickness dr, is given by



$$dm = \rho 2\pi r\, dr$$

Where ρ is the mass per unit area. The moment of inertia about an axis through the center of the disc normal to the plane faces (as shown in figure) is obtained as follows:

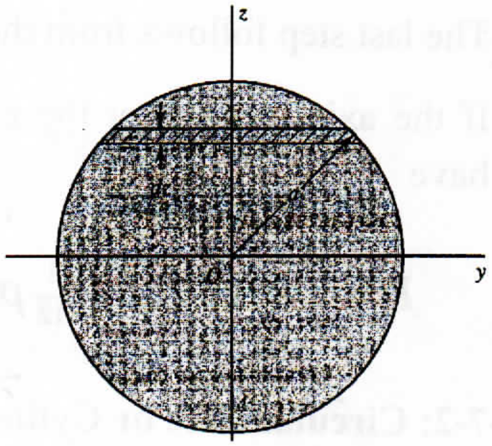
$$I_{axis} = \int_0^a r^2 \rho\, 2\pi r\, dr = 2\pi\rho \frac{a^4}{4} = \tfrac{1}{2} m a^2$$

The last step results from the relation $m = \rho\pi a^2$.

Equation above also applies to a uniform right-circular cylinder of radius a and mass m, the axis being the central axis of the cylinder.

4-7-3: Sphere:

Let us find the moment of inertia of a uniform solid sphere of radius a and mass m about an axis (the z-axis) passing through the center. We divide the sphere into thin circular discs, as shown in figure. The moment of inertia of a representative disc of radius y is



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$$\frac{1}{2}y^2 dm$$

$$dm = \rho\pi y^2 dz;$$

But

$$I_z = \int_{-a}^{a} \frac{1}{2}\pi\rho y^4 \, dz = \int_{-a}^{a} \frac{1}{2}\pi\rho(a^2 - z^2)^2 \, dz = \frac{8}{15}\pi\rho a^5$$

Because the mass m is given by

$$m = \frac{4}{3}\pi a^3 \rho$$

We have

$$I_z = \frac{2}{5} m a^2$$

For a solid uniform sphere, Clearly also, $I_x = I_y = I_z$.

4-7-4: Spherical Shell

The moment of inertia of a thin, uniform, spherical shell can be found very simply by differentiate equation ($I_z = \frac{8}{15}\pi\rho a^5$) with respect to a, namely,

$$dI_z = \frac{8}{3}\pi\rho a^4 \, da$$

the result is the moment of inertia of a shell of thickness *da* and radius a. The mass of the shell is $4\pi a^2 \rho \, da$. Hence, we can write

$$I_z = \frac{2}{3} m a^2$$

Problems

1- Find the center of mass of each of the following:

(a) A thin wire bent into the form of a three-sided, block-shaped "u" with each segment of equal length b.

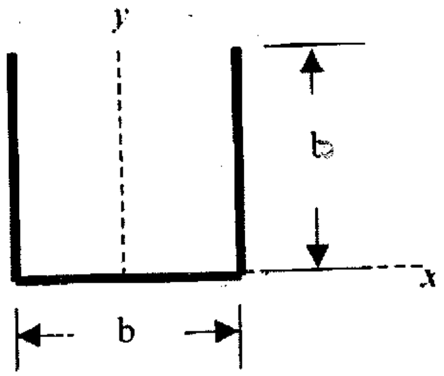
(b) A quadrant of a uniform circular lamina of radius b

(c) The area bounded by parabola $y = x^2/b$ and the line $y = b$

(d) The volume bounded by paraboloid of revolution $z = (x^2 + y^2)/b$ and the plane $z = b$.

Solution:

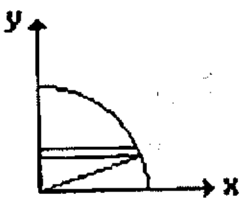
(a) For each portion of the wire having a mass $\frac{m}{3}$ and centered at $\left(-\frac{b}{2},\frac{b}{2}\right)$, $(0,0)$. and $\left(\frac{b}{2},\frac{b}{2}\right)$ ...



$$x_{cm} = \frac{1}{m}\left[-\left(\frac{b}{2}\right)\left(\frac{m}{3}\right)+0+\left(\frac{b}{2}\right)\left(\frac{m}{3}\right)\right]=0$$

$$y_{cm} = \frac{1}{m}\left[\left(\frac{b}{2}\right)\left(\frac{m}{3}\right)+0+\left(\frac{b}{2}\right)\left(\frac{m}{3}\right)\right]=\frac{b}{3}$$

(b)



$$ds = xdy = \left(b^2 - y^2\right)^{\frac{1}{2}} dy$$

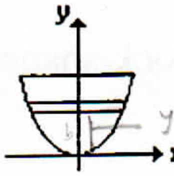
$$y_{cm} = \frac{1}{m}\int_0^b \rho y\left(b^2 - y^2\right)^{\frac{1}{2}} dy$$

$$y_{cm} = \frac{-\frac{\rho}{2}\int_{y=0}^{y=b}\left(b^2 - y^2\right)^{\frac{1}{2}} d\left(b^2 - y^2\right)}{\frac{1}{4}\pi b^2 \rho}$$

$$y_{cm} = \frac{4b}{3\pi}$$

From symmetry, $x_{cm} = \frac{4b}{3\pi}$

(c)



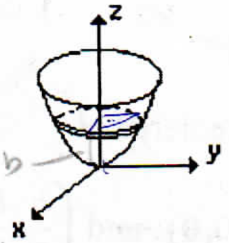
The center of mass is on the y-axis.

$$ds = 2x\,dy = 2(by)^{\frac{1}{2}}dy$$

$$y_{cm} = \frac{\int_0^b 2\rho y (by)^{\frac{1}{2}}dy}{\int_0^b 2\rho (by)^{\frac{1}{2}}dy} = \frac{\int_0^b y^{\frac{3}{2}}dy}{\int_0^b y^{\frac{1}{2}}dy}$$

$$y_{cm} = \frac{3b}{5}$$

(d)



The center of mass is on the z-axis.

$$dv = \pi r^2 dz = \pi\left(x^2 + y^2\right)dz = \pi bz\,dz$$

$$z_{cm} = \frac{\int_0^b \rho z \pi bz\,dz}{\int_0^b \rho \pi bz\,dz} = \frac{\int_0^b z^2 dz}{\int_0^b z\,dz}$$

$$z_{cm} = \frac{2}{3}b$$

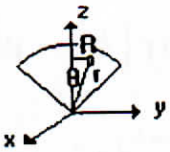
2- Find the moments of inertia of each of the objects in **Problem 1** about their symmetry axes.

Solution:

(a) $$I_z = \sum_i m_i R_i^2 = \frac{m}{3}\left[\left(\frac{b}{2}\right)^2 + 0 + \left(\frac{b}{2}\right)^2\right]$$

$$I_z = \frac{mb^2}{6}$$

(b)

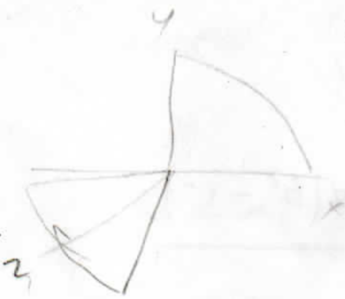


$$ds = r\,d\theta\,dr\,,\; R = r\sin\theta$$

$$I_z = \int R^2 \rho\,ds$$

$$I_z = \rho\int_{r=0}^{r=b} r^2\,r\,dr \int_{\theta=-\frac{\pi}{4}}^{\theta=\frac{\pi}{4}} \sin^2\theta\,d\theta$$

$$I_z = \frac{\rho b^4}{4}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sin^2\theta\,d\theta$$

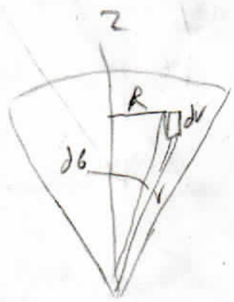


$$\left[\int \sin^2\theta\,d\theta = \frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]$$

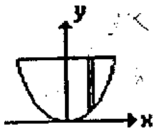
$$I_z = \frac{\rho b^4}{4}\left(\frac{\pi}{4} - \frac{1}{2}\right)$$

$$m = \frac{1}{4}\rho\pi b^2$$

$$I_z = \frac{mb^2}{4\pi}(\pi - 2)$$



(c)



$$ds = h\,dx = \left(b - \frac{x^2}{b}\right)dx$$

Where the parabola intersects the line $y = b$.

$$x = (by)^{\frac{1}{2}} = \pm b$$

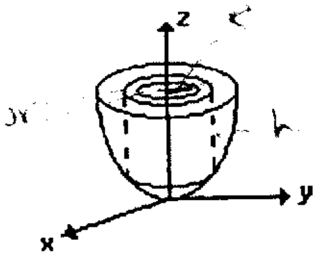
$$I_y = \int_{-b}^{b} x^2 \rho \left(b - \frac{x^2}{b}\right)dx = \rho \int_{-b}^{b} \left(bx^2 - \frac{x^4}{b}\right)dx$$

$$I_y = \frac{4}{15}\rho b^4$$

$$m = \int_{-b}^{b} \rho \left(b - \frac{x^2}{b}\right)dx = \frac{4}{3}\rho b^2$$

$$I_y = \frac{1}{5} m b^2$$

(d)



$$dv = 2\pi R h\,dR$$

$$h = b - z$$

$$R = \left(x^2 + y^2\right)^{\frac{1}{2}} = (bz)^{\frac{1}{2}}$$

$$dR = \frac{1}{2}\left(\frac{b}{z}\right)^{\frac{1}{2}} dz$$

$$I_z = \int R^2 \rho\, dv = \int_0^b bz\rho 2\pi (bz)^{\frac{1}{2}} (b - z) \frac{1}{2}\left(\frac{b}{z}\right)^{\frac{1}{2}} dz$$

$$I_z = \pi \rho b^2 \int_0^b \left(bz - z^2\right) dz = \frac{1}{6}\pi \rho b^5$$

$$m = \int \rho\, dv = \int_0^b \rho 2\pi (bz)^{\frac{1}{2}} (b - z) \frac{1}{2}\left(\frac{b}{z}\right)^{\frac{1}{2}} dz$$

$$m = \pi \rho b \int_0^b (b - z)\, dz = \frac{1}{2}\pi \rho b^3$$

$$I_z = \frac{1}{3} m b^2$$

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Chapter One
The Vectors

1-1: Scalar and vectors

1-2:Geometric Properties

1-3: Algebraic Properties of Vectors

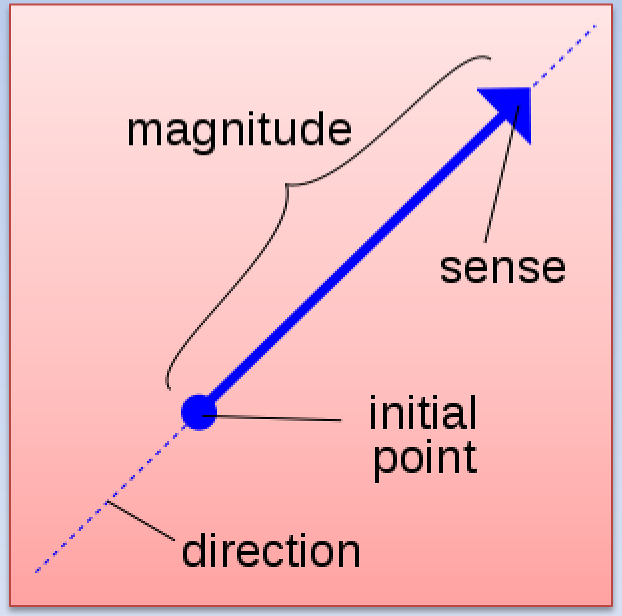
1-4: Unit Vectors

1-5: Components of A Vectors:

1-1: Scalar and vectors

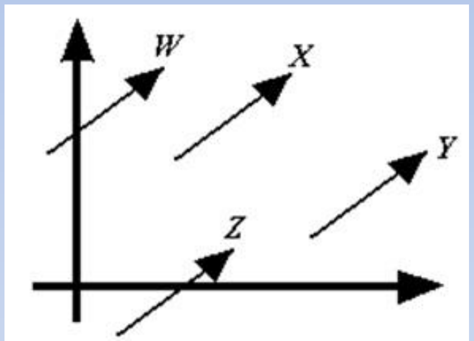
A **scalar** is any **positive** or **negative physical quantity** that can be completely specified by its **magnitude**. such as mass, length, density, volume and temperature.

A **vector** is any physical quantity that requires both a **magnitude** and **direction** for its complete description. A vector is shown **graphically** by an **arrow**. The **length** of the arrow represents the **magnitude** of the vector, and a fixed axis defines the **direction** of its line of action .The **head** of the arrow indicates the **sense** of **direction of the vector** (as shown in figure).



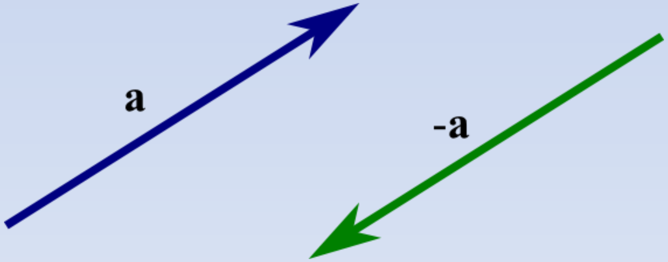
For handwritten work, it is often convenient to denote a vector quantity by simply drawing an arrow on top it A, ($\vec{A}$). In print, vector quantities are represented by bold face letters such as **A**, and its magnitude of the vector is italicized, *A*.

1-2:Geometric Properties :

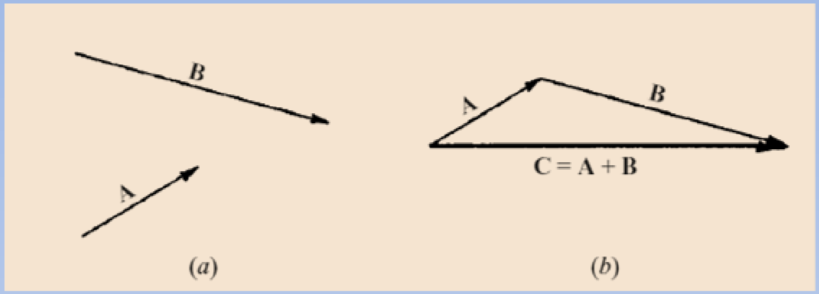


1. Two vectors **A** and **B** are equal if they have the same magnitude and direction regardless of their initial points. Thus **A** = **B**.

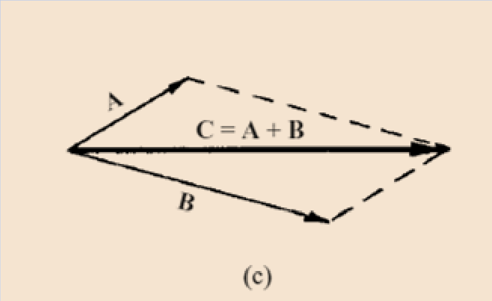
2. A vector having direction opposite to that of vector **a** but with the same magnitude is denoted by **-a** .

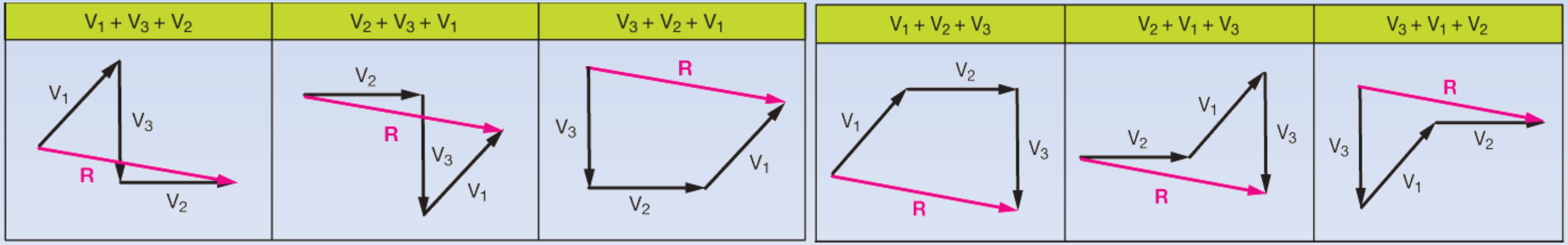


3. The sum or resultant of vectors **A** and **B** of Fig. 1-2(a) below is a vector **C** formed by placing the initial point of **B** on the terminal point of **A** and joining the initial point of **A** to the terminal point of **B** [see Fig. 1-2(b) below].

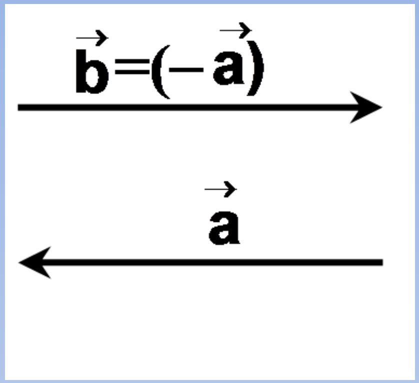


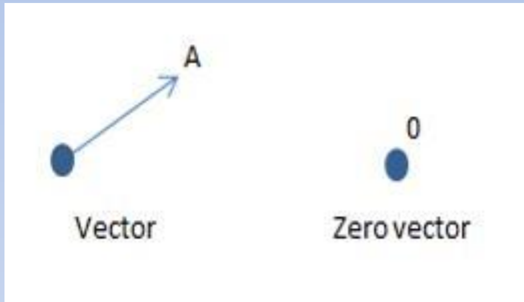
The sum C is written C = A + B. The definition here is equivalent to the parallelogram law for vector addition as indicated in Fig.1-2(c) below.



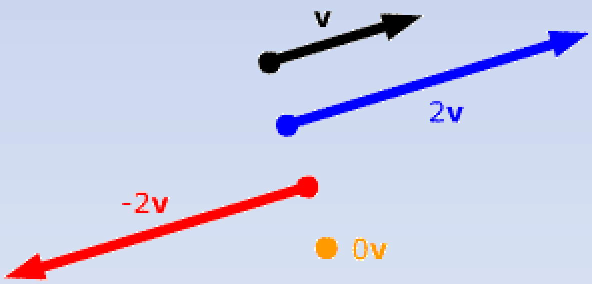


4. The difference of vectors **a** and **b**, represented by **A** - **B**, is that vector **C** which added to **B** gives **A**. Equivalently, **A** - **B** may be defined as **A** + (**-B**). If **A** = **B**, then **A−B** is defined as the null or zero vector and is represented by the symbol **0**. This has a magnitude of zero but its direction is not defined.





5. Multiplication of a vector **v** by a scalar m produces a vector m**v** with magnitude times the magnitude of **v** and direction the same as or opposite to that of **v** according as m is positive or negative. If m = 0, mv = 0, the null vector.



1-3: Algebraic Properties of Vectors:

The following algebraic properties are consequences of the geometric definition of a vector. If **A**, **B** and **C** are vectors, and m and n are scalars, then:

1. **A + B = B + A** — Commutative Law for Addition
2. **A +(B + C) = (A + B) + C** — Associative Law for Addition
3. m(n**A**)= **(**mn**)A =** n(m**A**) — Associative Law for Multiplication
4. (m + n)**A** = m**A** + n**A** — Distributive Law
5. m**(A + B) =** m**A** + m**B** — Distributive Law
6. **A+0=A**
7. **A+(-A)= 0**

Note that in these laws only multiplication of a vector by one or more scalars is defined.

1-4: Unit Vectors:

Unit vectors are vectors having unit length. If **A** is any vector with length $A > 0$, then A/A is a unit vector, denoted by **a**, having the same direction as A. Then **A** = A**a**. The rectangular unit vectors **i**, **j**, and **k** are unit vectors having the direction of the positive x, y, and z axes of a rectangular coordinate system [see Fig. 1-3].

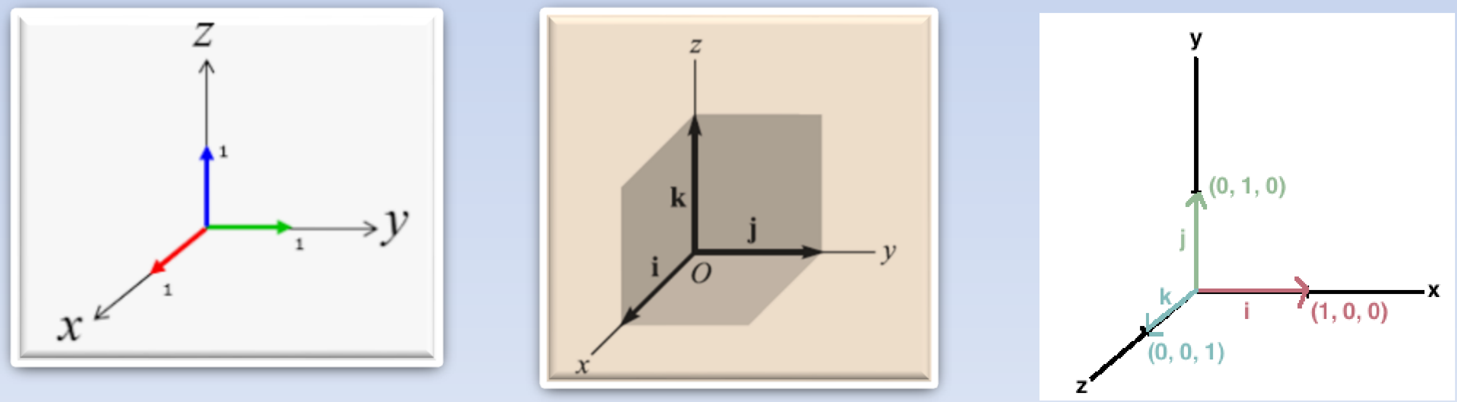
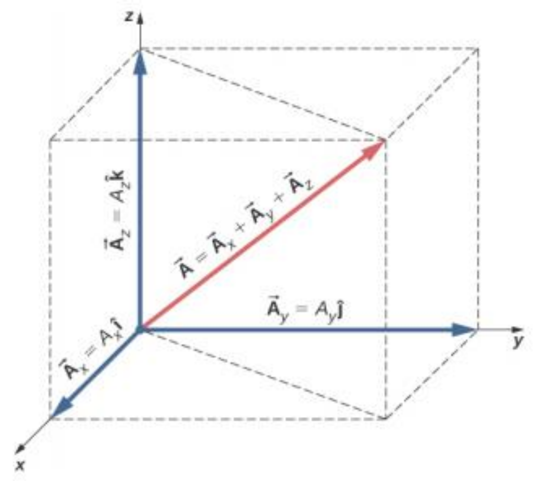


Fig. 1-3

1-5: Components of A Vectors:

Any vector **A** in 3 dimensions can be represented with initial point at the origin O of a rectangular coordinate system [see Fig. 1-3]. Let (A_x;A_y;A_z) be the rectangular coordinates of the terminal point of vector A with initial point at O. The vectors $\mathbf{A}_x$ **i**; $\mathbf{A}_y$ **j**; and $\mathbf{A}_z$ **k** are called the rectangular component vectors, or simply **component vectors**, of A in the x, y; and z directions respectively. The sum or resultant of $\mathbf{A}_x$ **i**; $\mathbf{A}_y$ **j**; and $\mathbf{A}_z$**k** is the vector **A,** so that we can write:

$$\mathbf{A} = \mathbf{A}_x\,\mathbf{i} + \mathbf{A}_y\,\mathbf{j} + \mathbf{A}_z\,\mathbf{k}$$

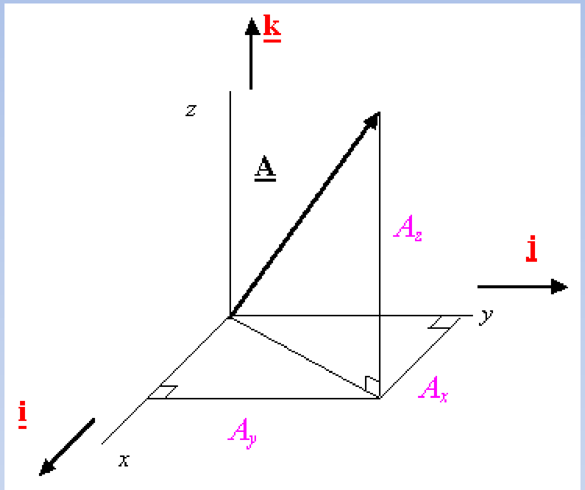


The magnitude of **A** is:

$$A = |\mathbf{A}| = \sqrt{A_1^2 + A_2^2 + A_3^2}$$

In particular, the position vector or radius vector r from O to the point (x; y; z) is written:

$$\mathbf{A} = \mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$



and has magnitude:

$$r = |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}.$$