Calculus

Presented By

Dr. Amal Jasim Mohammed

College of Education For Pure Sciences

Mathematics Department

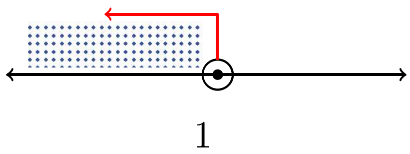
University of Mosul

Contains

College of Education For Pure Sciences, Mathematics Department,
University of Mosul

Inequalities المتباينات

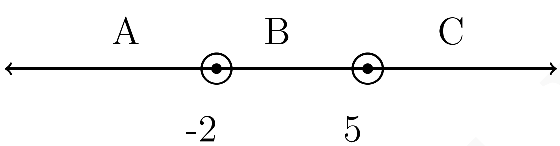
Definition 1. Inequalities: The real numbers can be ordered by size حجم: if $a - b$ is positive, then we write it either $a > b$ or $b < a$.



Example 1. Solve the inequality: $2x + 1 < 4 - x$

$$\begin{aligned} 2x + 1 &< 4 - x \\ 2x + x &< 4 - 1 \\ 3x &< 3 \end{aligned}$$

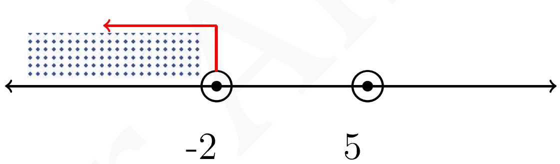
Then, the solution set $S = \{x : x < 1\}$ or $S = (-\infty, 1)$



Example 2. Solve the inequality: $x^2 - 3x > 10$

$$\begin{aligned} x^2 - 3x - 10 &> 0 \\ (x-5)(x+2) &> 0 \\ x - 5 = 0 &\text{ and } x + 2 = 0 \\ x = 5 &\text{ and } x = -2 \end{aligned}$$

We need to check the correct set of solution S

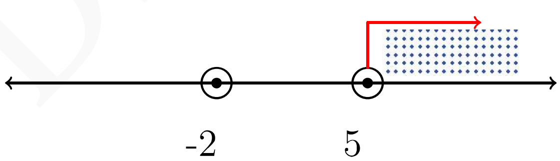


Region A:

take $x = -3$ substitut in $(x + 2)(x - 5) =$

$-1 \times -8 = 8 > 0$, which is satisfy our condition.

Then, $S_1 = \{x : x < -2\}$

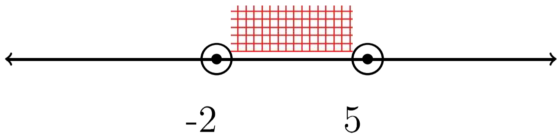


Region C:

take $x = 7$ substitut in $(x + 2)(x - 5) =$

$9 \times 2 = 18 > 0$, which is satisfy our condition.

Then, $S_2 = \{x : x > 5\}$



Region B:

take $x = 0$ substitut in $(x+2)(x-5) =$ $2 \times -5 = -10 < 0$, which is not satisfy our condition.

Then, the solution set $S = S_1 \cup S_2 = \{x : x < -2\} \cup \{x : x > 5\}$ or $S = (-\infty, -2) \cup (5, \infty)$.

Example 3. Find the set of solution of $\frac{2}{3x-2} \leq 1$

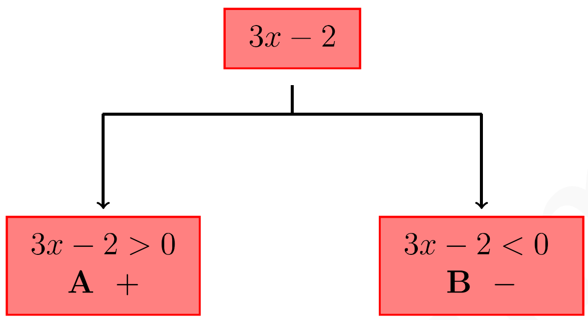
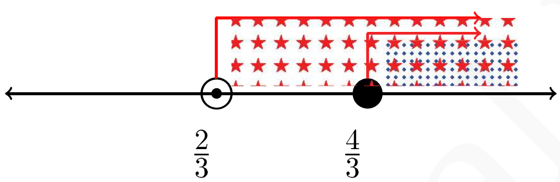
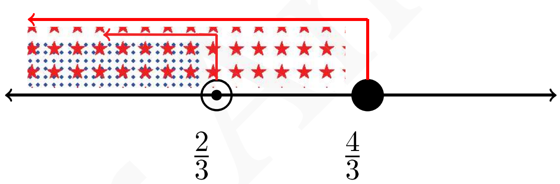


Figure 1: To avoid the denominator المقام not to be zero



A $+$**:** First, take $3x - 2 > 0$ simplify, $3x > 2$ then $x > \frac{2}{3}$. Now, from the question $\frac{2}{3x-2} \leq 1$, then $2 \leq 3x - 2$, which is $x \geq \frac{4}{3}$. Then, the intersection between two solutions gives $S_1 = \{x : x \geq \frac{4}{3}\}$



B $-$**:** First, take $3x - 2 < 0$ simplify, then $x < \frac{2}{3}$. Now, from the question $\frac{2}{3x-2} \leq 1$, then $2 \geq 3x - 2$, which is $x \leq \frac{4}{3}$. Then, the intersection between two solutions gives $S_2 = \{x : x < \frac{2}{3}\}$

Then, the solution set $S = S_1 \cup S_2 = \{x : x \geq \frac{4}{3}\} \cup \{x : x < \frac{2}{3}\}$

Homework

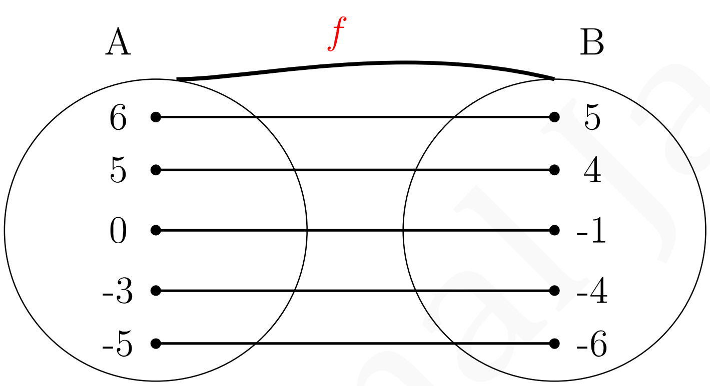
Question 1. *Find the set of solution*

$$1)\ -9 < 7 - 3x \leq 12$$

$$2)\qquad \frac{2x+5}{x+4} < 1$$

Functions الدوال

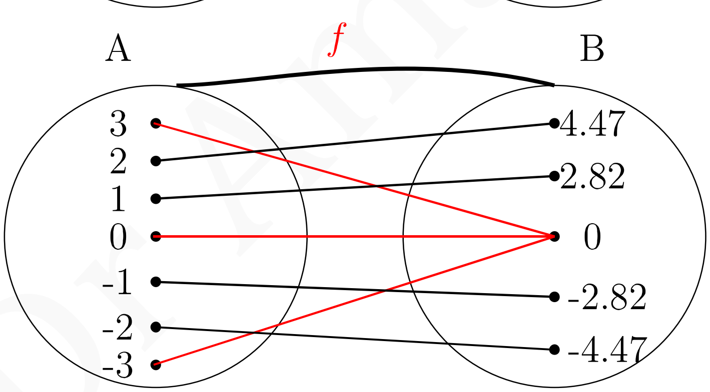
Definition 2. Function: If a variable y depends on a variable x in such a way that each value of x determines exactly one value of y is a function of x. So, If $f(x) = y$, then the set of all possible input (x- values) is called the domain of f (D_f) and the set of output (y- values) is called the range of f (R_f)



Example of a function:
$f : A \to B, \quad f(x) = y = x - 1, \quad x \in A, \ y \in B.$

f: Mapping, D_f: Domain of f,

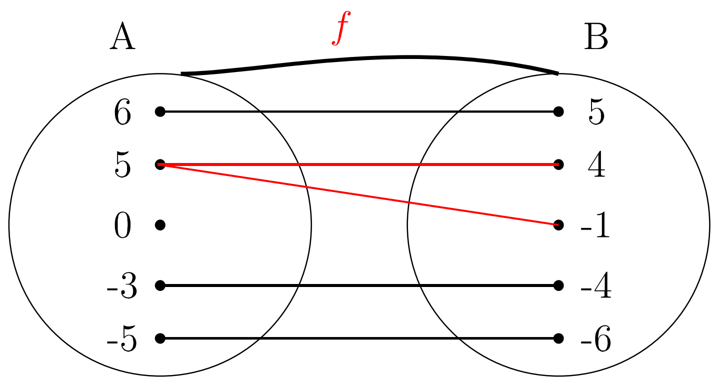
A: Domain and B: Co-domain



Example of a function:

$f : A \to B, \quad f(x) = y = x\sqrt{9 - x^2},$

$x \in -3 \leq x \leq 3 = A, \quad y \in B.$



Example of f which is not a function:

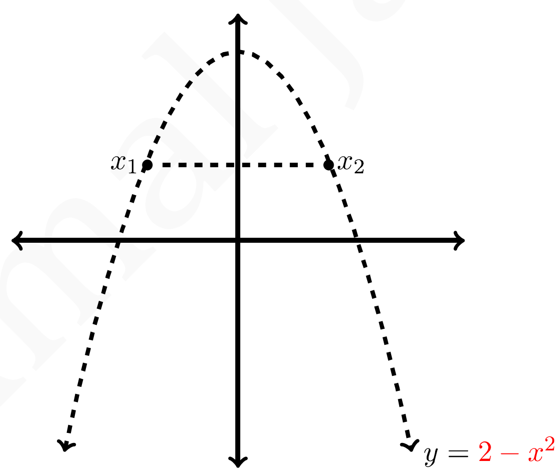
There are two points $y = 4, -1 \in B$ which are related to a one point $x = 5 \in A$.

Homework

Question 2. *Check the following mapping. Is it represent a function?* $y = 3x - 2$

Definition 3. Surjective mapping: if the range = the co-domain.

Definition 4. Injective mapping: If each element in B connected with only one element in A.



Not injective mapping

Definition 5. Bijective mapping: If the mapping is surjective and injective at the same time.

Homework

Question 3. *What is the type of the mapping, where* Z *is represent the set of integer numbers?* $f : Z \rightarrow Z$, *where* $f(x) = 2x^2 - 3$.

College of Education For Pure Sciences, Mathematics Department,
University of Mosul

Definition 1. The absolute value or magnitude of a real number $x \in \mathbb{R}$ is defined by:

$$|5| = 5, \quad \left|\frac{-4}{7}\right| = \frac{4}{7}.$$

Properties of Absolute value: if x, a and b are real numbers, then

1. $|-a| = |a|$,
2. $|ab| = |a||b|$,
3. $|a/b| = |a|/|b|, b \neq 0$,
4. $|a+b| \leq |a| + |b|$,
5. $|a| \geq 0$,
6. $|a-b| = |b-a|$,
7. $|a-b| \geq |a| - |b|$,
8. $|x| \leq a, \qquad -a \leq x \leq a$,
9. $|x| > a, \qquad x > a \quad \text{or} \quad x < -a$,
10. $-1 \times (x \geq a) \to -x \leq -a, \qquad -1(x < a) \to -x > -a.$

Note 1. If $f(x) = |x|$ and $a \in \mathbb{R}$, then we have two important cases that the question can be:

A) $|f(x)| \leq a$ or $|f(x)| < a$. The set of solution will be $S = \{x : -a \leq x \leq a\}$ or $S = \{x : -a < x < a\}$, respectively.

B) $|f(x)| \geq a$ or $|f(x)| > a$. The set of solution will be $S = \{x : x \geq a\} \cup \{x : x \leq -a\}$ or $S = \{x : x > a\} \cup \{x : x < -a\}$, respectively.

Example 1. Find the set of solution for this inequality: $|x + 6| < 3$.

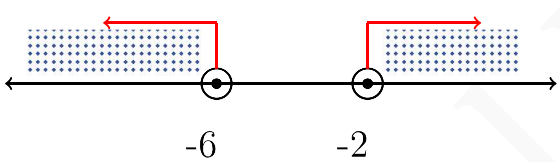
Solution: We will use number 8 in the above note, then

$$-3 < x + 6 < 3$$

$$-3 - 6 < x + 6 - 6 < 3 - 6$$

$$-9 < x < -3$$

then, the set of solution $S = \{x : -9 < x < -3\}$.



Example 2. Find the set of solution for this inequality: $|x + 4| > 2$.

We will use number 9 in the above note, then

$$x + 4 > 2 \quad \text{or} \quad x + 4 < -2$$

$$x > 2 - 4 \quad ** \quad x < -2 - 4$$

$$x > -2 \quad ** \quad x < -6.$$

Then, the set of solution $S = \{x : x > -2\} \cup \{x : x < -6\}$.

Example 3. Find the set of solution of $\left|\frac{2x+5}{x+4}\right| < 1$.

This example looks like number 8 in the above note, then

$$-1 < \frac{2x + 5}{x + 4} < 1$$

A +: First, take $x + 4 > 0 \Rightarrow$, $x > -4$. Now, from the question

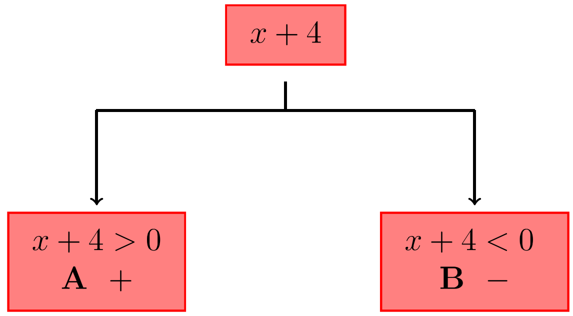


Figure 1: To avoid the denominator المقام not to be zero

$-1 < \frac{2x+5}{x+4} < 1$, multiply it by $(x+4)$ then,

$$-(x+4) < 2x+5 < x+4$$

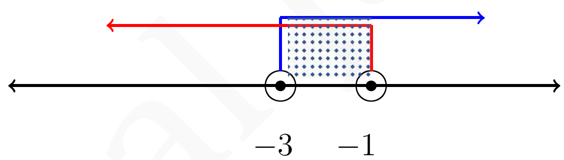
$$-x-4 < 2x+5 < x+4$$

Find the possible solution

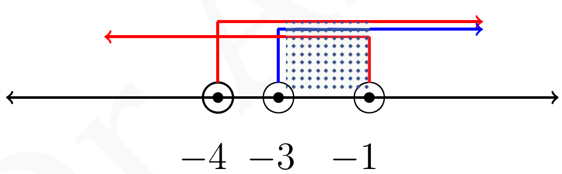
$$-x-4 < 2x+5 \quad ** \quad 2x+5 < x+4$$

$$2x+x > -4-5 \quad ** \quad 2x-x < 4-5$$

$$3x > -9 \Rightarrow x > -3 \quad ** \quad x < -1$$



Then, there is an intersection region $\{-3 < x < -1\}$.



A $+$**:** we had from the denomenator $\{x > -4\}$, from the question we had $\{-3 < x < -1\}$. Then, the intersection will be, $S_1 = \{x : -3 < x < -1\}$

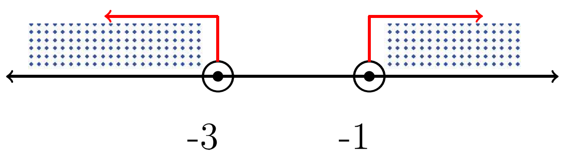
B $-$: First, take $x+4<0 \Rightarrow$, $x < -4$. Now, from the question $-1 < \frac{2x+5}{x+4} < 1$, multiply it by $(x+4)$ then,

$$-(x+4) > 2x+5 > x+4$$

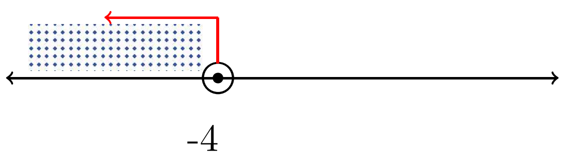
$$-x-4 > 2x+5 > x+4$$

Find the possible solution

$$-x-4>2x+5 \quad ** \quad 2x+5>x+4$$
$$2x+x<-4-5 \quad ** \quad 2x-x>4-5$$
$$3x>-9 \Rightarrow x<-3 \quad ** \quad x>-1$$



Then there is no intersection, Which means $\{x : x<-3\} \cap \{x : x>-1\} = \phi$



B –: we had from the denominator $\{x<-4\}$, from the question we had ϕ. Then the intersection between them $S_2 = \phi$

Finally, the set of solution $S = S_1 \cup S_2 = \{x : -3<x<-1\}$.

Present Different Type of Functions Graphing and Analysing their Domain and Range

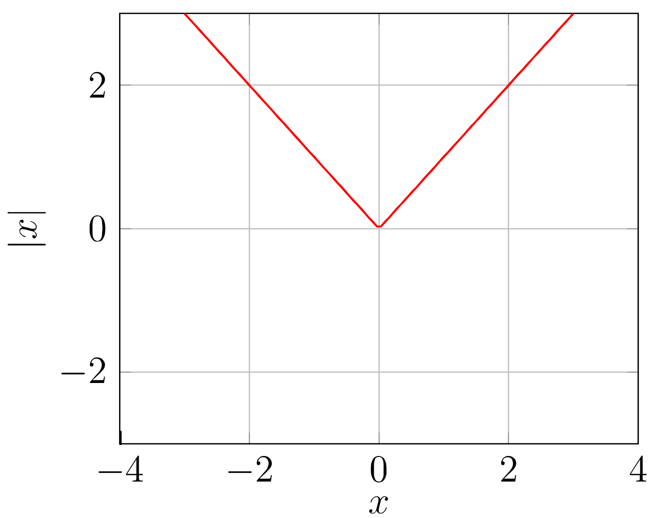
1) The absolute value function المجال والمدى لدالة القيمة المطلقة

The graph of the function $f(x)=|x|$ can be obtained by the two parts of equation:

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x<0. \end{cases}$$

x	$f(x) = \lvert x \rvert$
-2	$-(-2) = 2$
-1	$-(-1) = 1$
0	0
2	2
1	1

$D_f = \mathbb{R}, \ R_f = \{y : y \geq 0\}$

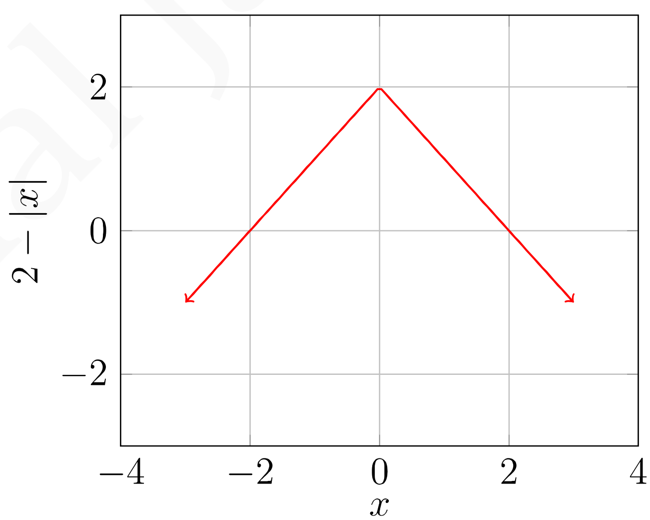


Example 4. *Graph the equation and find the* D_f *and* R_f $y = f(x) = 2 - |x|$

$$2 - |x| = \begin{cases} 2 - x & x \geq 0 \\ 2 + x & x < 0. \end{cases}$$

x	$f(x) = 2 - \lvert x \rvert$
-2	0
-1	1
0	2
2	0
1	1

$D_f = \mathbb{R}, \ R_f = \{y : y \leq 2\}$



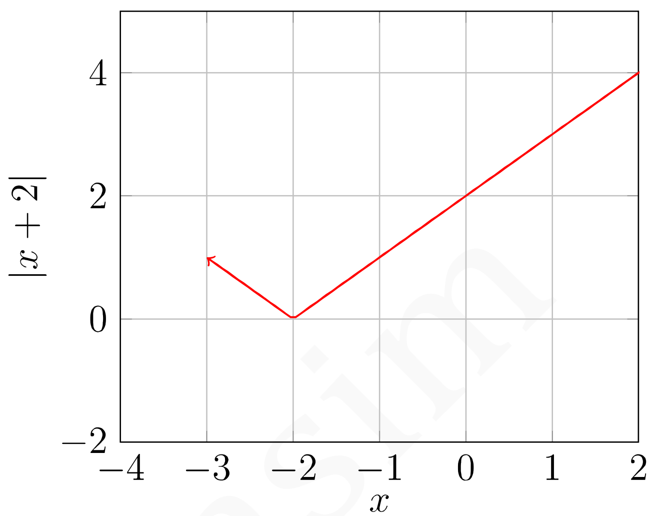
Example 5. *Graph the equation and find the* D_f *and* R_f $y = f(x) = |x+2|$

$$|x+2| = \begin{cases} x+2 & x+2 \geq 0 \\ -(x+2) & x+2 < 0. \end{cases}$$

$$|x+2| = \begin{cases} x+2 & x \geq -2 \\ -(x+2) & x < -2. \end{cases}$$

x	$f(x) = \|x+2\|$
-3	1
-2	0
-1	1
0	2
1	3

$D_f = \mathbb{R}, \ R_f = \{y : y \geq 0\}$

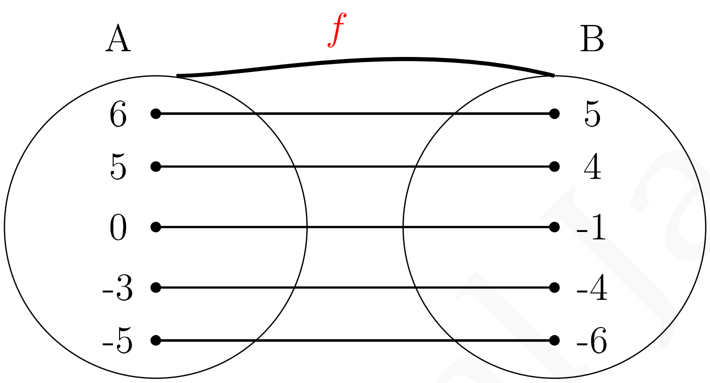


College of Education For Pure Sciences, Mathematics Department,
University of Mosul

Functions الدوال

Definition 1. Function: If a variable y depends on a variable x in such a way that each value of x determines exactly one value of y is a function of x. So, If $f(x) = y$, then the set of all possible input (x- values) is called the domain of f (D_f) and the set of output (y- values) is called the range of f (R_f). We can write the mapping as:

$$f : \text{domain} \longrightarrow \text{co-domain}.$$



Example 1. Example of a function:

$f : A \to B$, where $f(x) = y = x - 1$

$A = \{6, 5, 0, -3, -5\}$

$B = \{5, 4, -1, -4, -6\}$,

$D_f = A$, $R_f = B$.

f is surjective and injective. Then, f is bijective.

Definition 2. Surjective mapping: if the range = the co-domain.

Definition 3. Injective mapping: If each element in B (range) connected with only one element in A (domain).

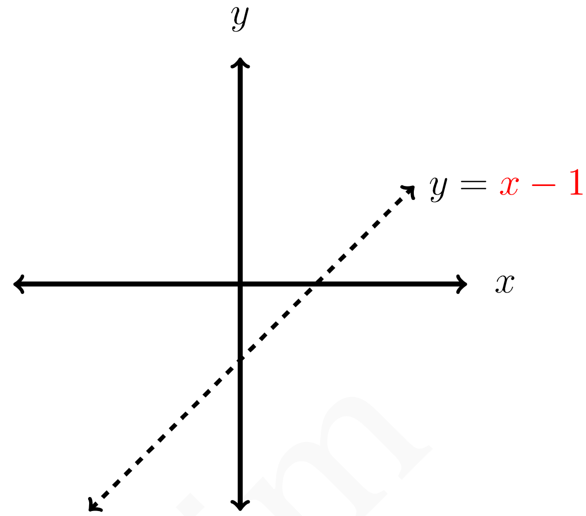
Definition 4. Bijective mapping: If the mapping is surjective and injective at the same time.

Example 2. Find the D_f and R_f to this function

$$y = f(x) = x - 1,$$

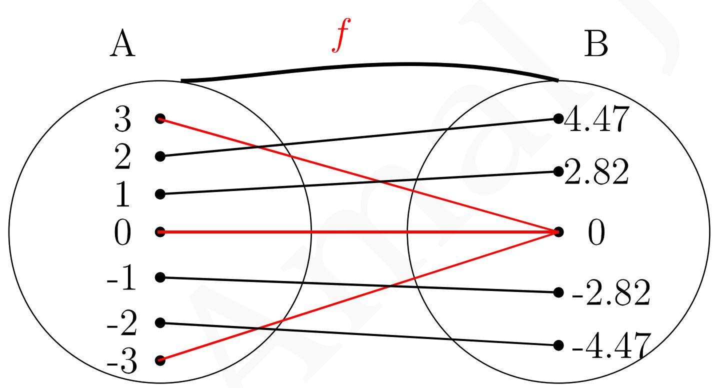
where $f : \mathbb{R} \to \mathbb{R}$. In this case $D_f = \mathbb{R}$ and the co-domain is $\mathbb{R}$.

x	$f(x) = x - 1$
$\vdots$	$\vdots$
2	1
1	0
-1	-2
$\vdots$	$\vdots$



f : surjective, injective, bijective.

$D_f = \mathbb{R}$, $R_f = \mathbb{R}$



Example 3. Example of a function:

$f : A \to B$, $f(x) = y = x\sqrt{9 - x^2}$,

$x \in -3 \le x \le 3 = A$, $y \in B$.

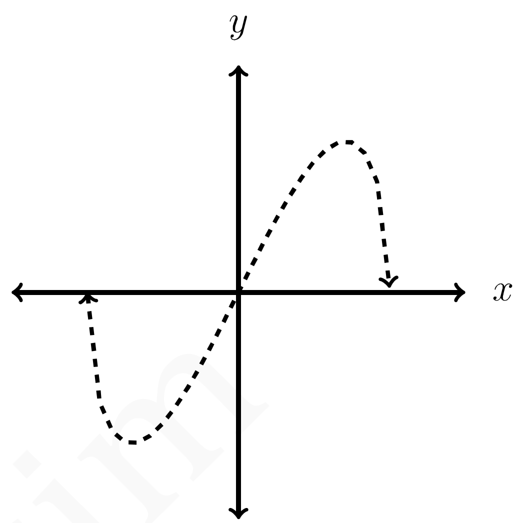
f is surjective, not injective. Then, f it is not bijective.

Example 4. Find the D_f and R_f to this function

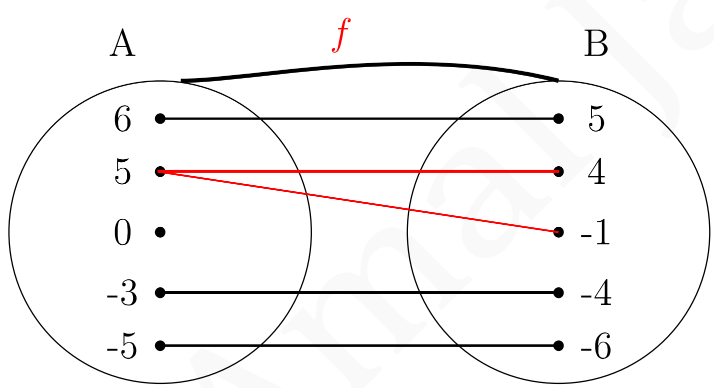
$$y = f(x) = x\sqrt{4 - x^2},$$

where $f : \{-2 \leq x \leq 2\} \to \mathbb{R}$. In this case $D_f = \{-2 \leq x \leq 2\}$ and the co-domain is $\mathbb{R}$.

x	$f(x) = x\sqrt{4 - x^2}$
2	0
1	1.73
0	0
-1	-1.73
-2	0
$\sqrt{2}$	2



not(surjective, injective, bijective). $D_f = \{-2 \leq x \leq 2\}$, $R_f = \{-2 \leq y \leq 2\}$



Example 5. Example of f which is not a function:

There are two points $y = 4, -1 \in B$ which are related to a one point $x = 5 \in A$.

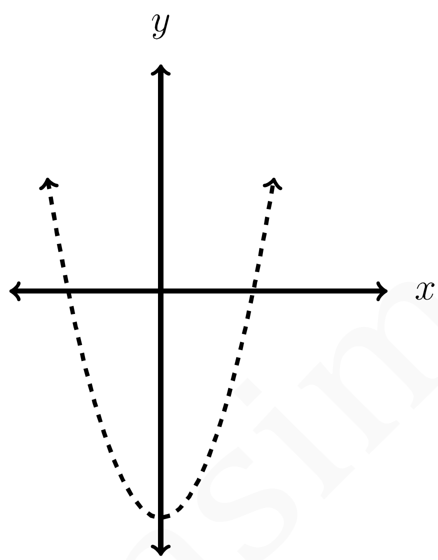
Homework

Question: What is the type of the mapping, where Z is represent the set of integer numbers? $f : Z \to Z$, where $f(x) = 2x^2 - 3$.

$D_f = Z$ and co-domain$= Z$. Then, f : is not surjective, not injective. Then it is not bijective.

x	$f(x) = 2x^2 - 3$
$\vdots$	$\vdots$
2	5
1	−1
0	−3
−1	−1
−2	5
$\vdots$	$\vdots$

$D_f = Z,\ R_f = \{y \geq -3\}$

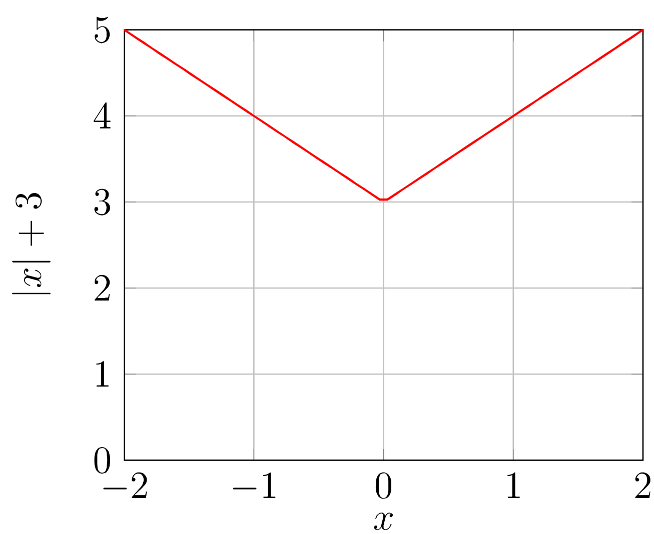


Example 6. Graph the equation and find the D_f and R_f of this function $f(x) = y = |x| + 3$.

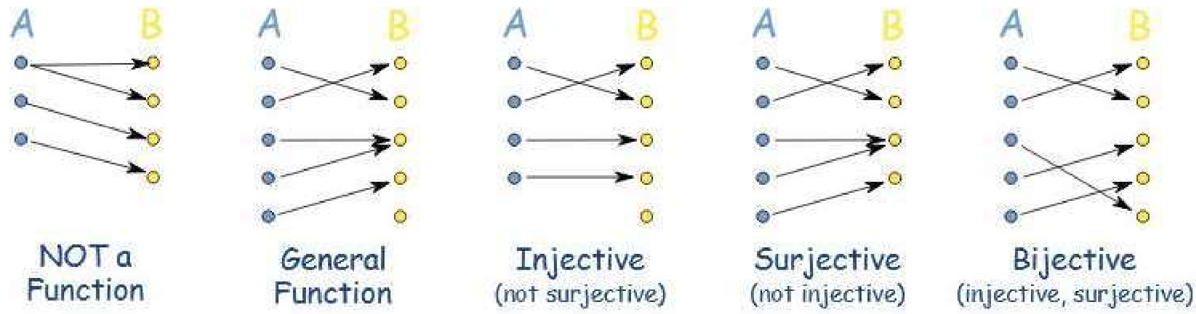
$$|x| + 3 = \begin{cases} x + 3 & x \geq 0 \\ -x + 3 & x < 0. \end{cases}$$

x	$f(x) = \lvert x\rvert + 3$
−2	5
−1	4
0	3
1	4
2	5

$D_f = \mathbb{R},\ R_f = \{y : y \geq 3\}$



Note: **A** is domain, **B** is co-domain in the following figures.



Injective, Surjective and Bijective : tells us about how a function behaves.

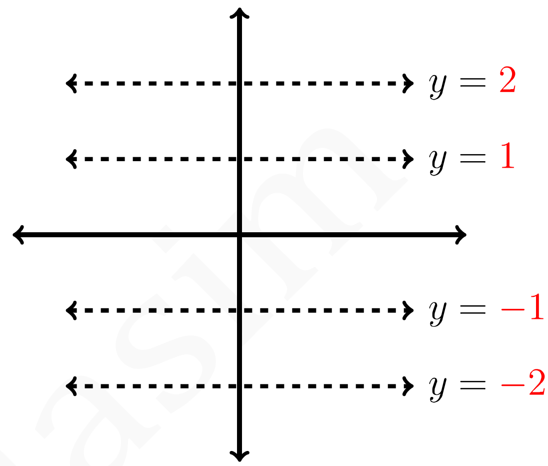
College of Education For Pure Sciences, Mathematics Department,
University of Mosul

2) Constant function المجال والمدى للدالة الثابته

Definition 1. Constant function: A function f whose values are all the same.

Example: Find the domain and the range for $f(x) = -2$, in this function the range R_f is always -2 for all $x \in D_f$.

x	$f(x) = -2$
-2	-2
-1	-2
0	-2
2	-2



$D_f = \mathbb{R}, \ R_f = \{y : y = -2\}$

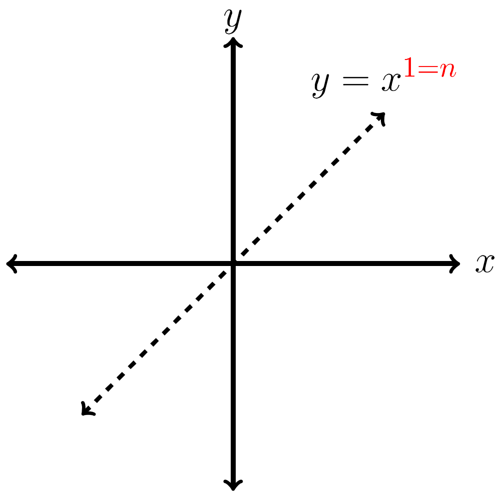
3) A power function دالة القوى

Definition 2. A power function : A function of the form $f(x) = x^p$, where p is constant.

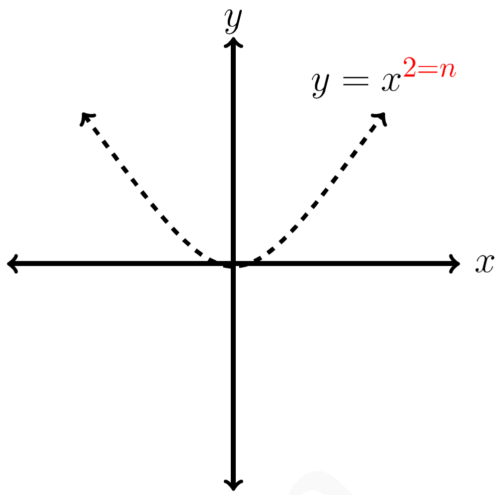
We can represent the power function as $f(x) = x^p$. In this case we have three kind of p: when $p > 0, p < 0$ and p is a fraction كسر.

3-A) دالة القوى عندما القوى موجبة

If we say $p = n$ and n is a positive integer.



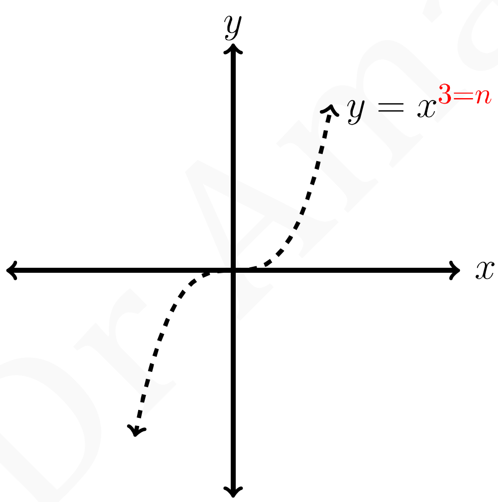
$D_f = \mathbb{R},\ R_f = \mathbb{R}$



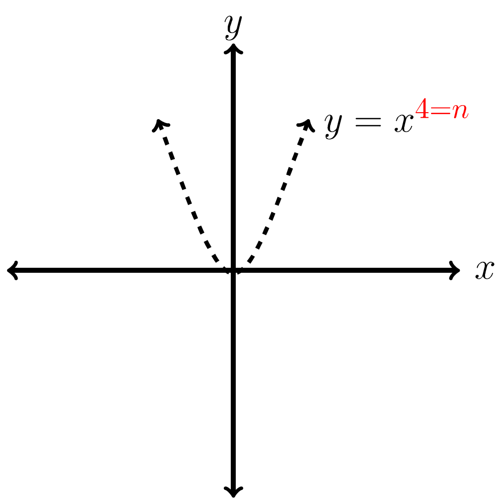
$D_f = \mathbb{R},\ R_f = \{y : y \geq 0\}$

x	$f(x) = x$
-2	-2
-1	-1
0	0
2	2

x	$f(x) = x^2$
-2	4
-1	1
0	0
2	4



$D_f = \mathbb{R},\ R_f = \mathbb{R}$



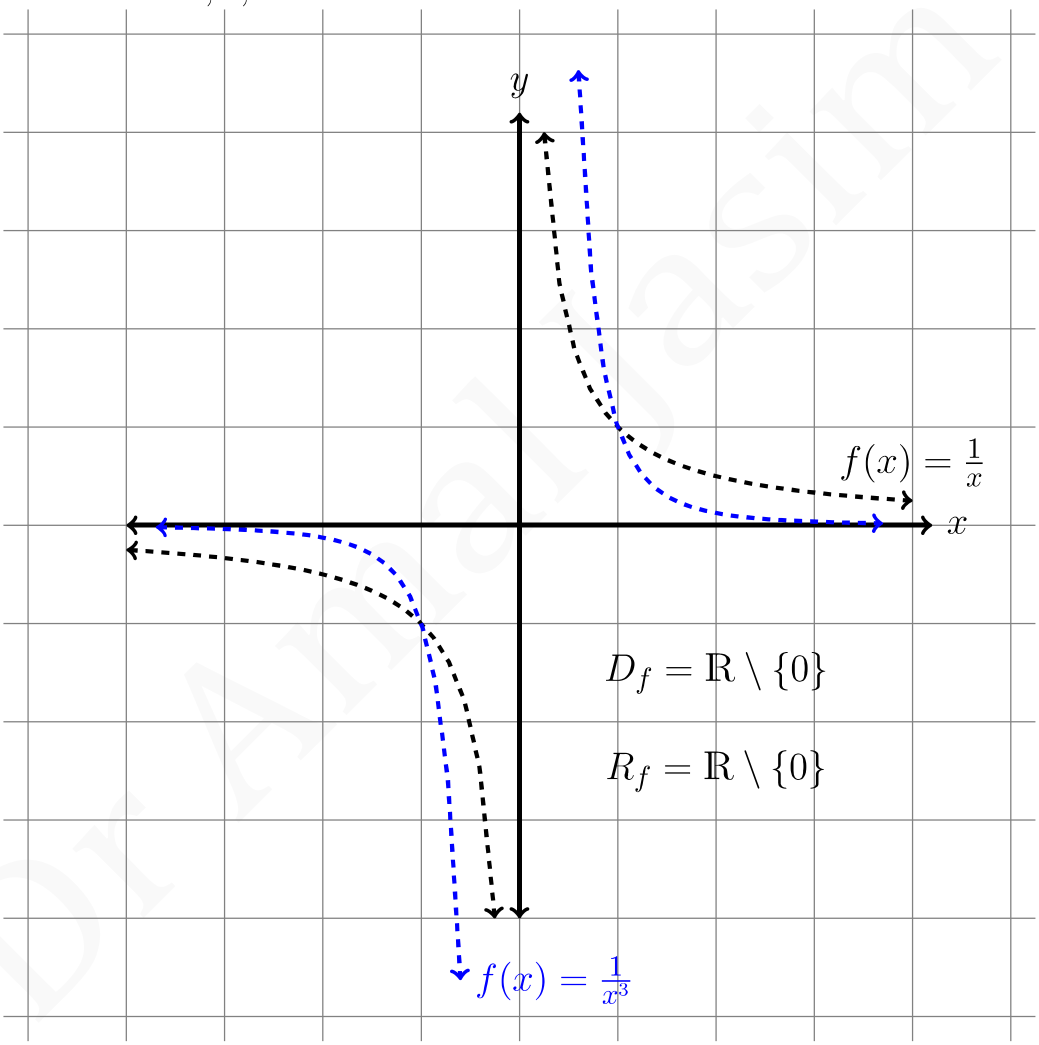
$D_f = \mathbb{R},\ R_f = \{y : y \geq 0\}$

Note 1. For even value of n, the function $f(x) = x^n$ are even, and their graphs are symmetric about the $y - axis$. For odd values of n the functions $f(x) = x^n$, are odd and their graphs are symmetric about the origin, that means about the point $(0, 0)$.

3-B) دالة القوى عندما القوى سالبة

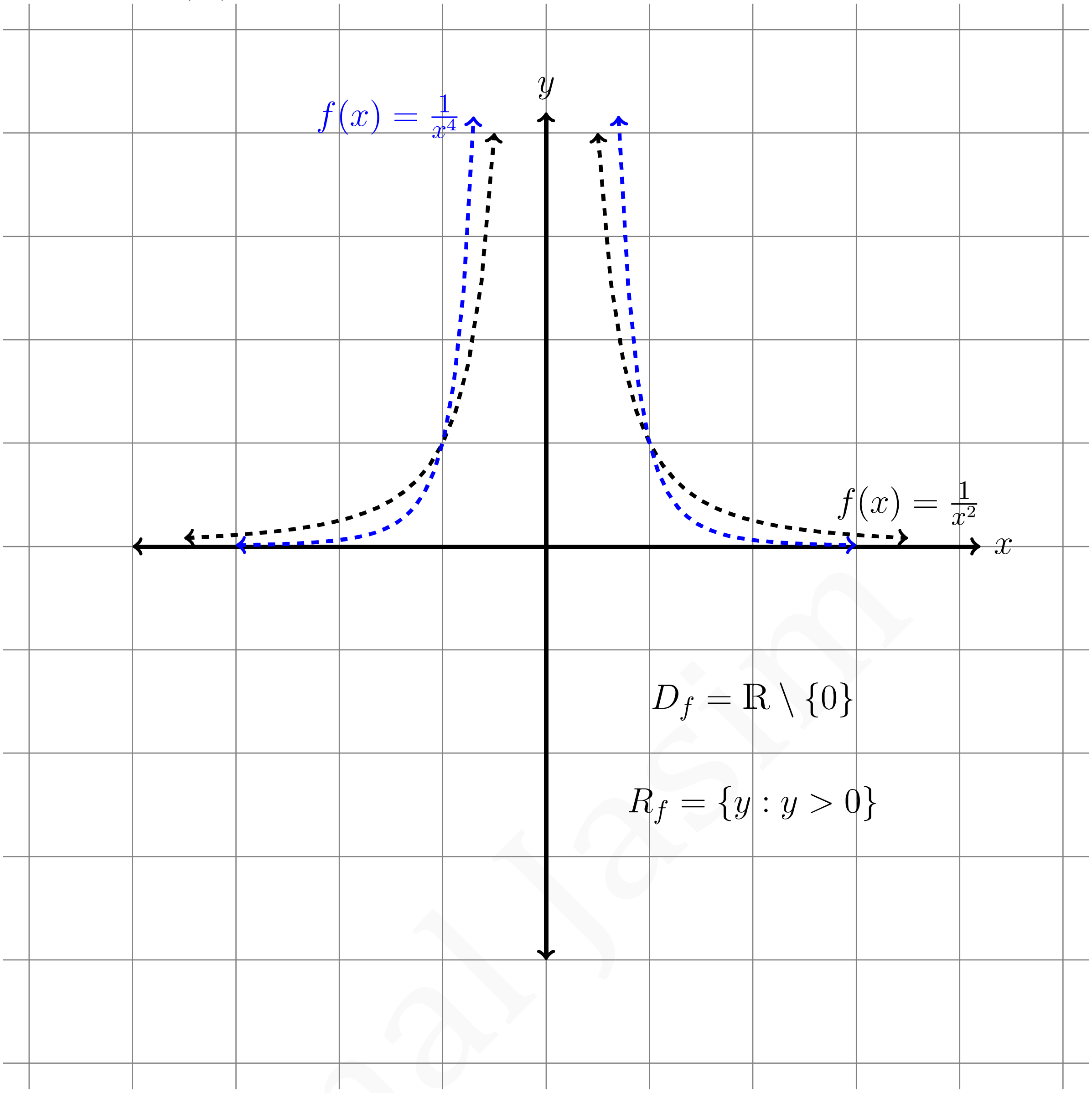
If we say $p = n$ and p is a negative integer. Say $p = -n$, n is integer, then $f(x) = x^{-n} = \frac{1}{x^n}$.

When $n = 1, 3, \ldots$ is odd:



Symmetric about the origin $(0, 0)$.

When $n = 2, 4, \ldots$ is even:

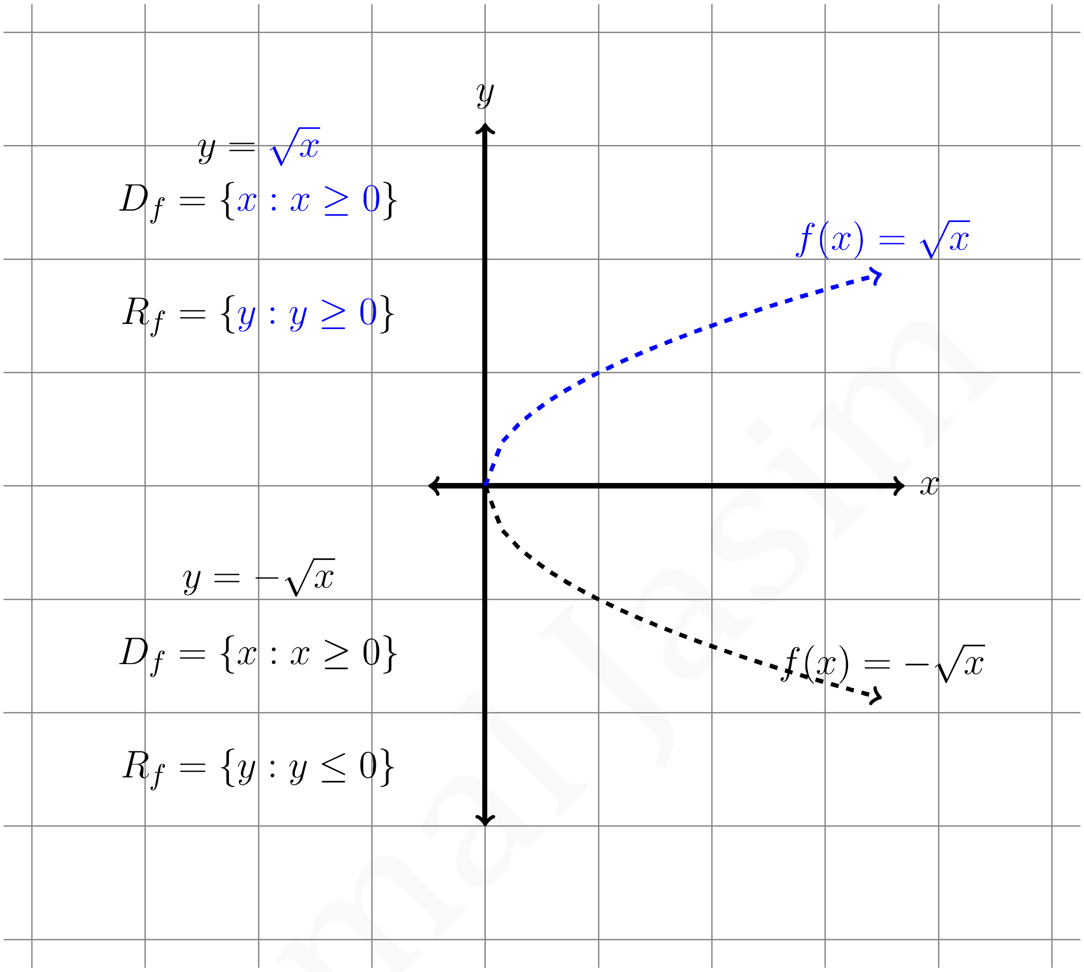


Symmetric about the $y - axis$.

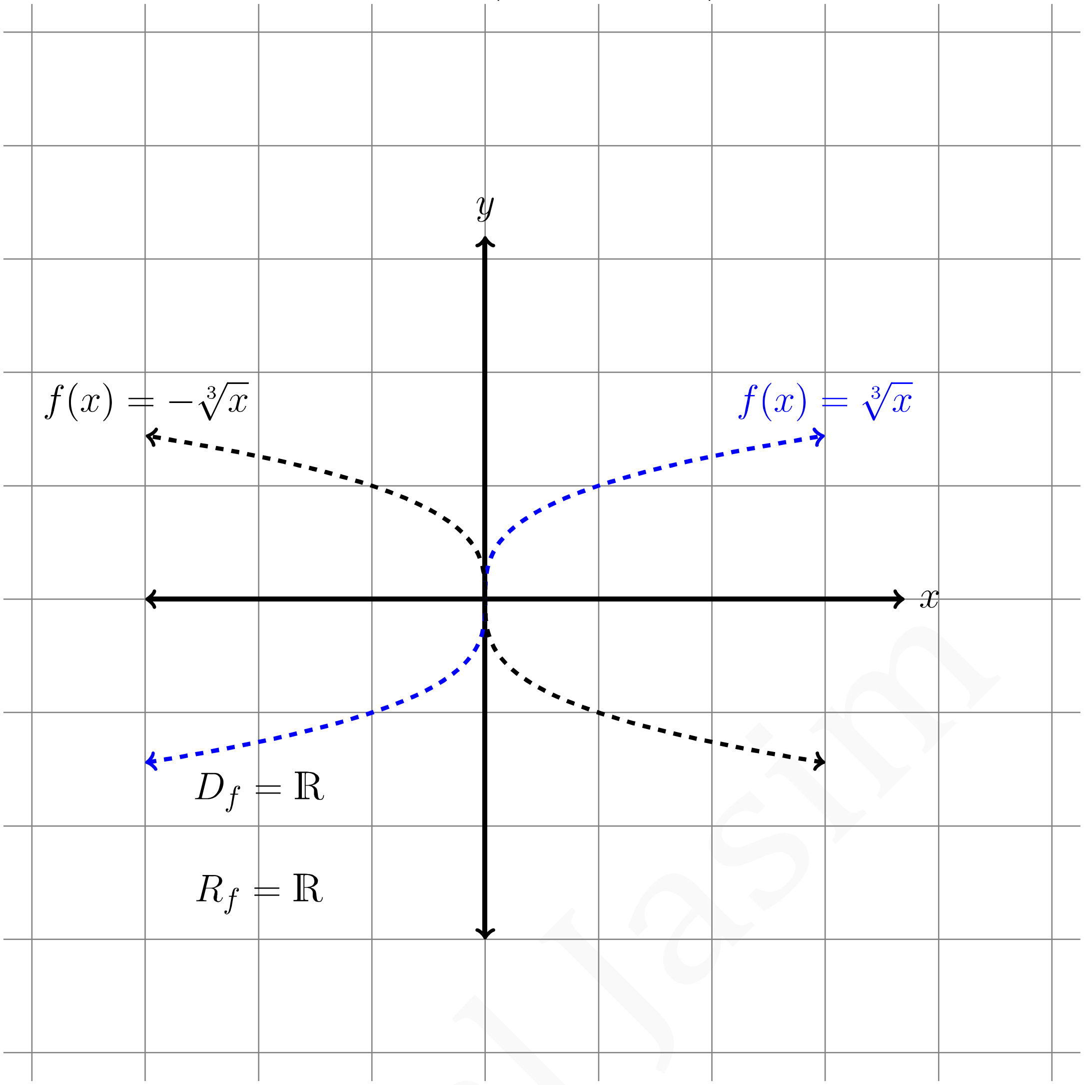
3-C) دالة القوى عندما القوى عبارة عن كسر

If we say $p = \frac{1}{n}$, n is positive integer. Then $f(x) = x^p = x^{\frac{1}{n}} = \sqrt[n]{x}$.

When $n = 2$, this means $y = \sqrt{x}$ or $y = -\sqrt{x}$:



When $n = 3$, this means $y = \sqrt[3]{x}$ or $y = -\sqrt[3]{x}$:



4) Polynomial function متعددة حدود

Definition 3. The polynomial function has this form: $y = f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n$. $a_0, a_1 \ldots a_n$ are constants and n is non-negative integer.

$$\begin{aligned} 3+5x &\quad \text{has degree1 (linear),} \\ x^2-3x+1 &\quad \text{has degree2 (quadratic),} \\ 2x^3-7 &\quad \text{has degree3 (cubic)} \\ 8x^4-9x^3+5x-3 &\quad \text{has degree4 (quartic)} \end{aligned}$$

Note 2. **مهمة**

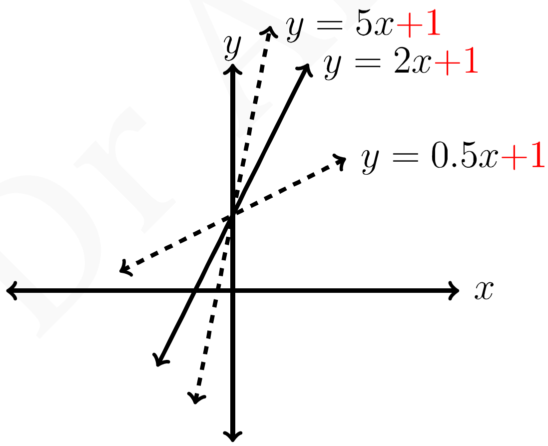
When the roots of a function in $\mathbb{R}$, then

- Linear function (degree 1) cross the x-axis on a point.
- Quadratic function (degree 2) cross the x-axis on two points.
- Cubic function (degree 3) cross the x-axis on three points.
- Quartic function (degree 4) cross the x-axis on four points.

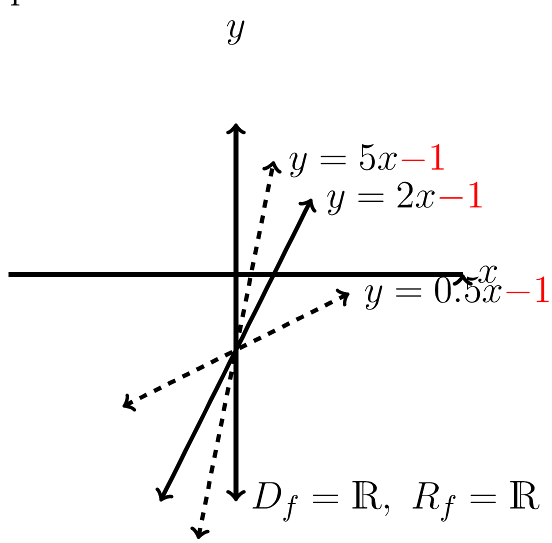
4-A) Linear function

The general form of a linear equation is $y = mx + b$, where $m, b \in \mathbb{R}, m \neq 0$.

If we keep b fixed and vary مختلفة the parameter m in the equation $y = mx + b$, then we obtain a family of lines. b is the point of intersection.

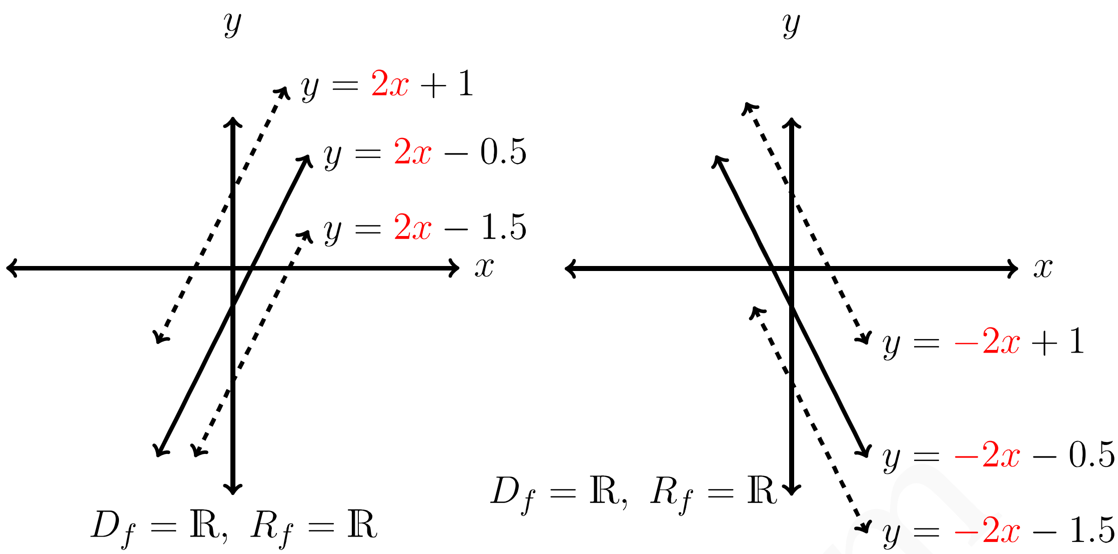


$D_f = \mathbb{R}, \; R_f = \mathbb{R}$



$D_f = \mathbb{R}, \; R_f = \mathbb{R}$

If we keep m fixed and vary مختلفة the parameter b, then we obtain a family of parallel line and they all have same slope m.



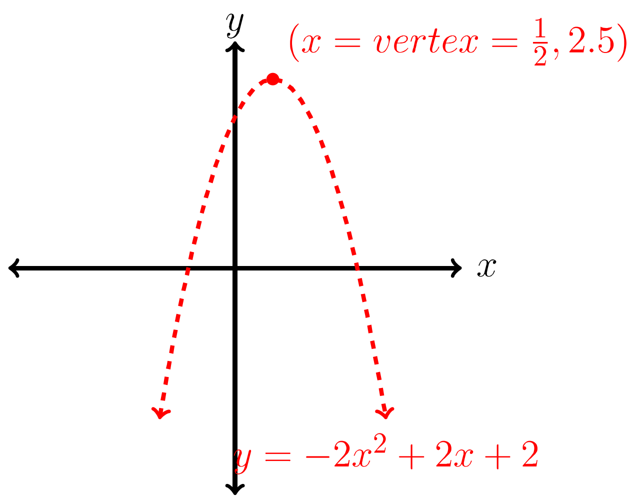
Homework

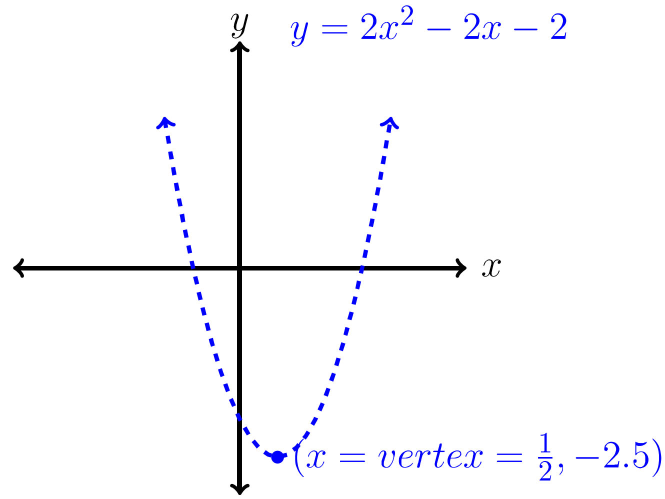
Question 1. We have $y(x) = \frac{1}{2}x - 1$.

- Find a parallel line for y and find the D_f, R_f.
- Find a line that cross يقطع y, and share يشارك it with a point.

4-B) Quadratic function

An equation of the form $y = ax^2 + bx + c, a \neq 0$, where $b, c \in \mathbb{R}$, is called a quadratic equation in x. Depending on a is positive or negative.





$D_f = \mathbb{R},\ R_f = \{y : y \leq 2.5\}$ $\qquad$ $D_f = \mathbb{R},\ R_f = \{y : y \geq -2.5\}$

To find the R_f substitute $x =$ vertex into the y function to find the $y(x = \text{vertex} = \frac{1}{2})$.

Note 3. **مهمة**

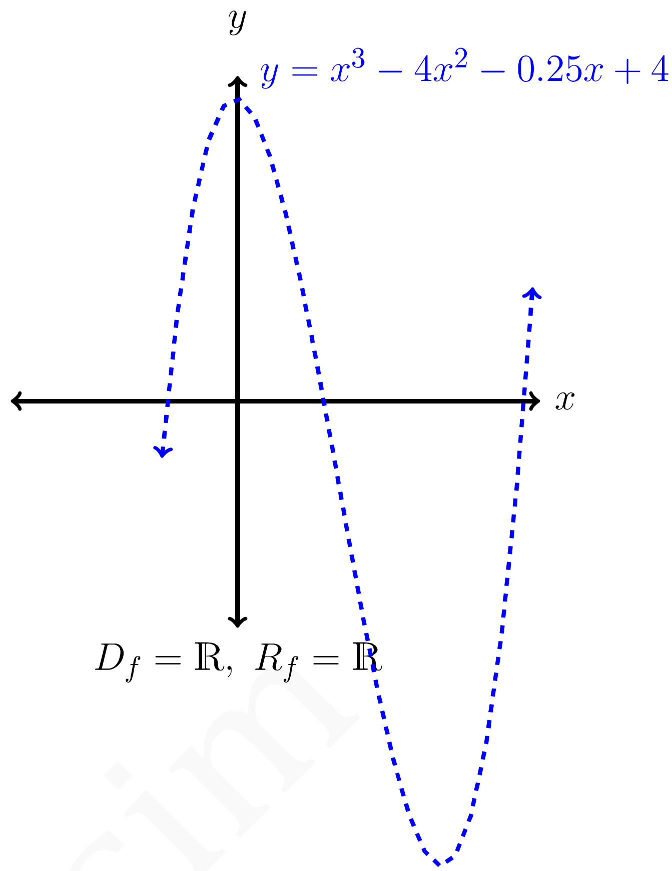
To find the vertex of $y = ax^2 + bx + c : x = \frac{-b}{2\times a}$.

Homework

Question 2. Sketch the graph and find the D_f and R_f and find the vertex of y.

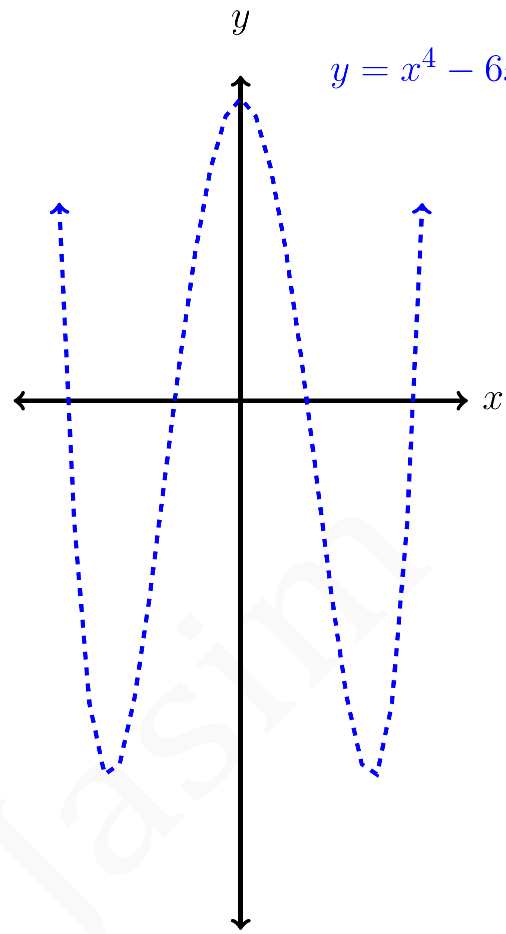
$$y = x^2 - 2x - 2$$

4-C) Cubic function للاطلاع



x	$y = x^3 - 4x^2 - 0.25x + 4$
-2	-19.50
-1	-0.75
0	4
1	0.75
2	-4.50

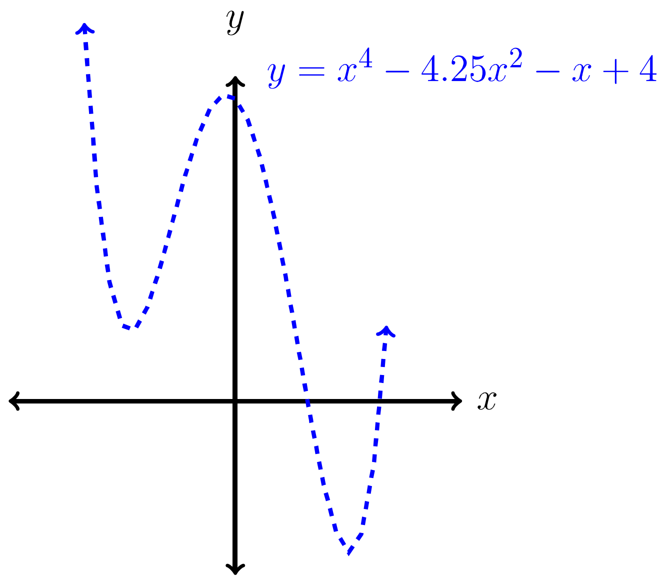
4-D) Quartic function للاطلاع



x	$y = x^4 - 6x^2 + 4$
-2	-4
-1	-1
0	4
1	-1
2	-4

$D_f = \mathbb{R},\ R_f = \{y : y \geq -5\}$

The roots are $-2.288245611, -.8740320489, .8740320489, 2.288245611$.



$D_f = \mathbb{R},\ R_f = \{y : y \geq -2\}$

Example of a quartic function:

has two real roots and two imagenary roots

College of Education For Pure Sciences, Mathematics Department,
University of Mosul

Homework 1

Express the number 0.351351351 as the rational of two integers p/q.

Let

$$n = 0.351351351 \qquad \ldots \tag{1}$$

$$[n = 0.351351351] \times 1000$$

$$1000 \times n = 351.351351 \qquad \ldots \tag{2}$$

subtract (1) from (2)

$$999 \times n = 351.000000$$
$$n = \frac{351}{999}$$

Homework 2

Solve the compound inequality a:

$$\begin{aligned} -3y - 5 &> 4 && \text{or} && 4 - y \leq 6 \\ -3y &> 4 + 5 && \text{or} && -y \leq 6 - 4 \\ [-3y > 9] &\div (-3) && \text{or} && [-y \leq 2] \times (-1) \\ y &< -3 && \text{or} && y \geq -2 \\ S_1 &= \{y : y < -3\} && \text{or} && S_2 = \{y : y \geq -2\} \end{aligned}$$

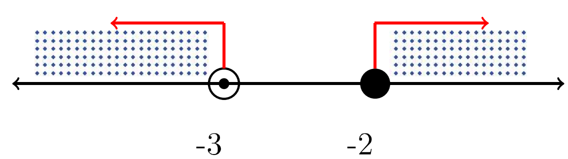


Figure 1: The set of solution $S = S_1 \cup S_2 = (-\infty, -3) \cup [-2, \infty)$

Homework 3

Solve the compound inequality b:

$$4x + 3 < 16 \qquad \text{or} \qquad [-2x < 3] \div (-2)$$

$$4x < 16 - 3 \qquad \text{or} \qquad x > -\frac{3}{2}$$

$$[4x < 13] \div (4)$$

$$x < \frac{13}{4}$$

$$S_1 = \{x : x < \frac{13}{4}\} \qquad \text{or} \qquad S_2 = \{x : x > -\frac{3}{2}\}$$

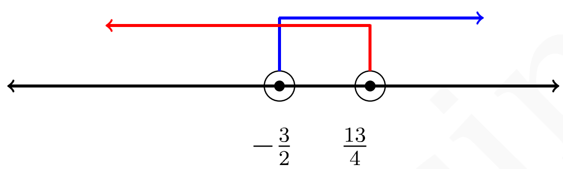


Figure 2: The set of solution $S = S_1 \cup S_2 = (-\infty, \infty)$

Homework 4

Solving absolute value inequalities b:

$$3 \leq 1 + \left|\frac{1}{2}t - 5\right|$$

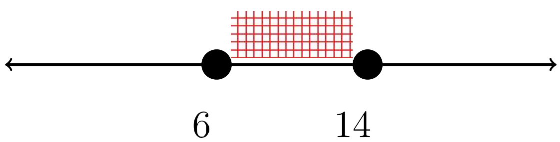
$$-1 + 3 \leq -1 + 1 + \left|\frac{1}{2}t - 5\right|$$

$$2 \leq \left|\frac{1}{2}t - 5\right|$$

$$\left|\frac{1}{2}t - 5\right| \geq 2$$

This is as note (9) in Lecture (3).

$$
\begin{aligned}
\frac{1}{2}t - 5 &\geq 2 & \text{or} & \quad \frac{1}{2}t - 5 \leq -2 \\
\frac{1}{2}t &\geq 2 + 5 & \text{or} & \quad \frac{1}{2}t \leq -2 + 5 \\
[\frac{1}{2}t \geq 7] &\times (2) & \text{or} & \quad [\frac{1}{2}t \leq 3] \times 2 \\
t &\geq 14 & \text{or} & \quad t \leq 6 \\
S_1 = \{t : t &\geq 14\} & \text{or} & \quad S_2 = \{t : t \leq 6\}
\end{aligned}
$$



take $t = 8$ substitut in $\left|\frac{8}{2} - 5\right| \geq 2$, which is $1 < 2$, not satisfy our condition.

Then the set of solution $S = S_1 \cup S_2 = \{t : t \geq 14\} \cup \{t : t \leq 6\}$

Homework 5

Find the set of solution 1)

$$-9 < 7 - 3x \leq 12$$

$$-9 - 7 < -7 + 7 - 3x \leq -7 + 12$$

$$[-16 < -3x \leq 5] \div (-3)$$

$$\frac{16}{3} > x \geq -\frac{5}{3}$$

Then the set of solution $S = \{x : \frac{16}{3} > x \geq -\frac{5}{3}\}$

Homework 6

Find the set of solution of 2)

$$\frac{2x+5}{x+4} < 1$$

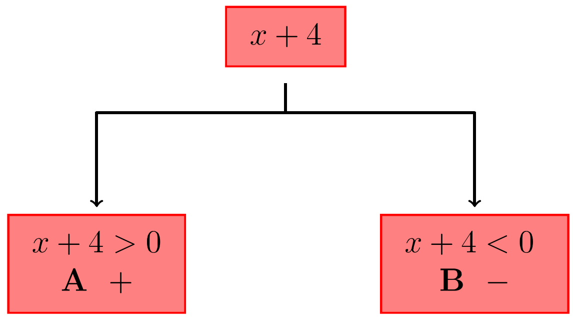
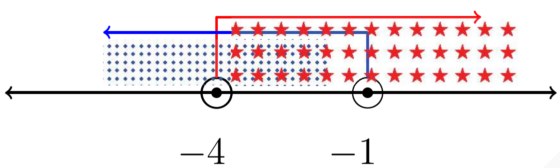


Figure 3: To avoid the denominatorالمقام not to be zero



A $+$**:** First, take $x+4>0$

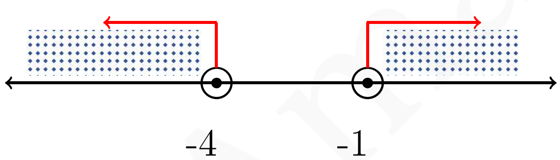
then $x>-4$. Now, from the question

$\frac{2x+5}{x+4}<1$, then $2x+5<x+4$,

simplify $2x-x<4-5$ which is $x<-1$.

Then, the intersection between

two solutions gives $S_1=\{x:-4<x<-1\}$



B $-$**:** First, take $x+4<0$ then $x<-4$.

Now, from the question $\frac{2x+5}{x+4}<1$,

then $2x+5>x+4$, simplify $2x-x>4-5$

which is $x>-1$. Then,

the intersection between two solutions gives

$S_2=\phi$.

Then, the solution set $S=S_1\cup S_2=\{x:-4<x<-1\}\cup\phi$.

Then, $S=\{x:-4<x<-1\}$.

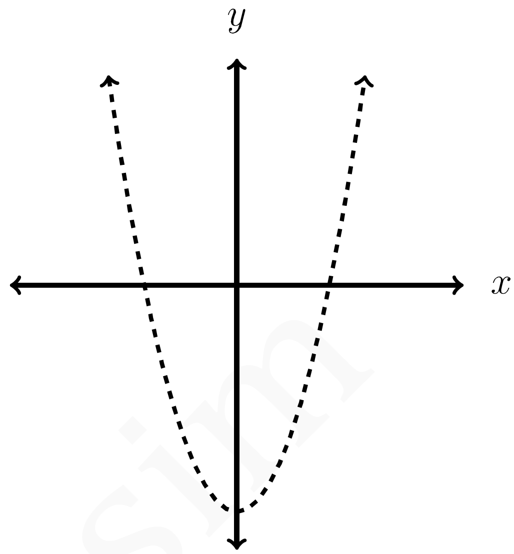
Homework 7

Find the D_f and R_f to this function and the type of the mapping

$$y = f(x) = 2x^2 - 3,$$

where $f : Z \longrightarrow Z$. In this case $D_f = Z$ and the co-domain is Z.

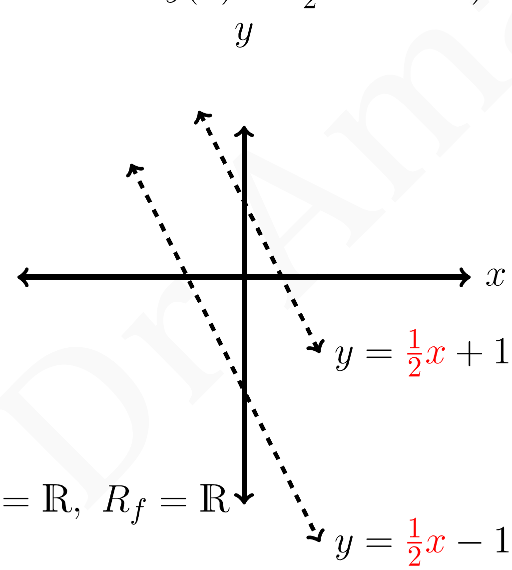
x	$f(x) = 2x^2 - 3$
2	5
1	-1
0	-3
-1	-1
-2	5



not(surjective, injective, bijective). $D_f = Z,\ R_f = \{y : y \geq -3\}$

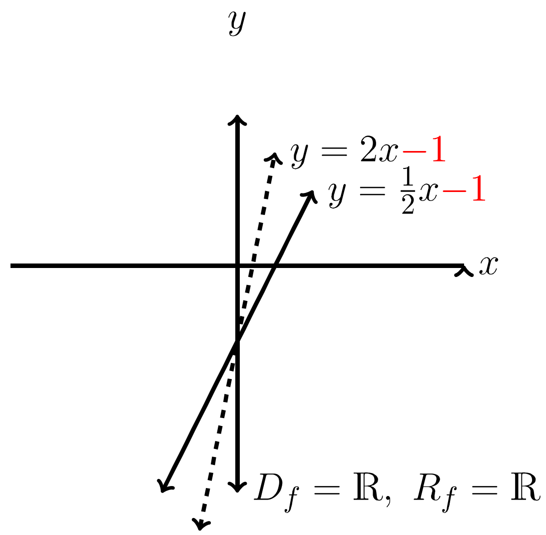
Homework 8

We have $y(x) = \frac{1}{2}x - 1$. 1) Find a parallel line for y and find the D_f, R_f.



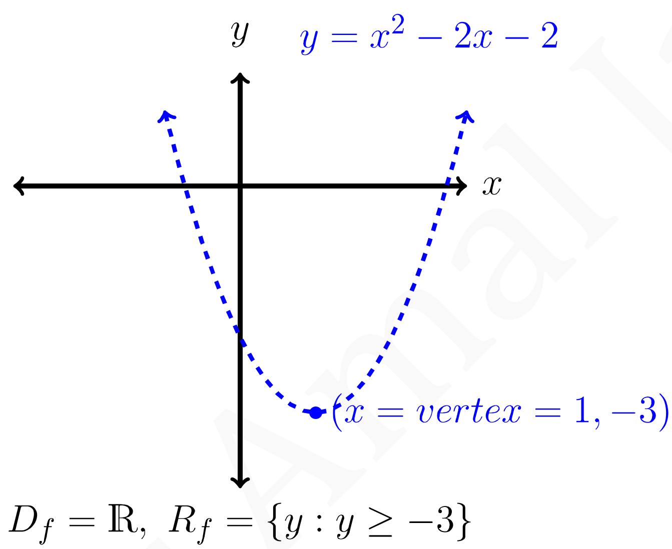
$D_f = \mathbb{R},\ R_f = \mathbb{R}$

2) Find a line that cross يقطع y, and share يشارك it with a point.



Homework 9

Sketch the graph of $y = x^2 - 2x - 2$ and find the D_f and R_f and find the vertex of y



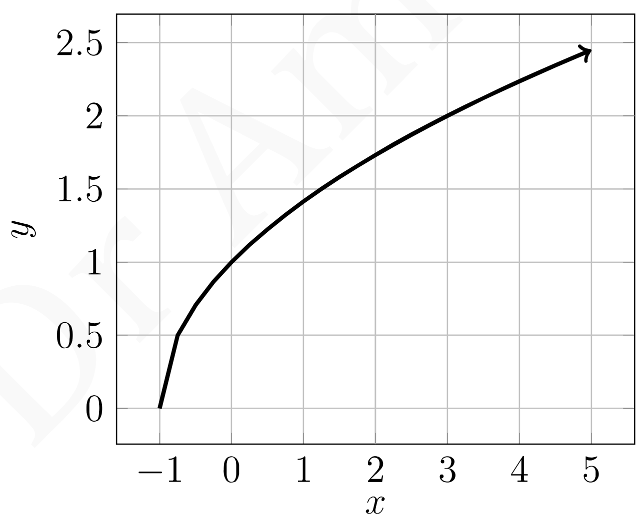
we have $a = 1, b = -2$, then the vertex of y, $x = \frac{-b}{2a} = \frac{-(-2)}{2\times 1} = 1$. To find the R_f, we need to subtitute $x = 1$ in $y = x^2 - 2x - 2$, we get $y = -3$.

College of Education For Pure Sciences, Mathematics Department,
University of Mosul

New functions from old: Arithmetic operation on functions. If f and g are functions and $x \in \mathbb{R}$, then

1. $(f \pm g)(x) = f(x) \pm g(x)$.
$$D_{f+g} = D_{f-g} = D_f \cap D_g.$$
2. (f g)(x)=f(x) g(x).
$$D_{fg} = D_f \cap D_g.$$
3. $(f/g)(x)$ and $(g/f)(x)$.
$$\begin{aligned}(f/g)(x) =& f(x)/g(x), \quad g(x) \neq 0,\\ D_{f/g} =& D_f \cap D_g - \{g(x) = 0\},\end{aligned}$$
$$\begin{aligned}(g/f)(x) =& g(x)/f(x), \quad f(x) \neq 0,\\ D_{g/f} =& D_g \cap D_f - \{f(x) = 0\},\end{aligned}$$

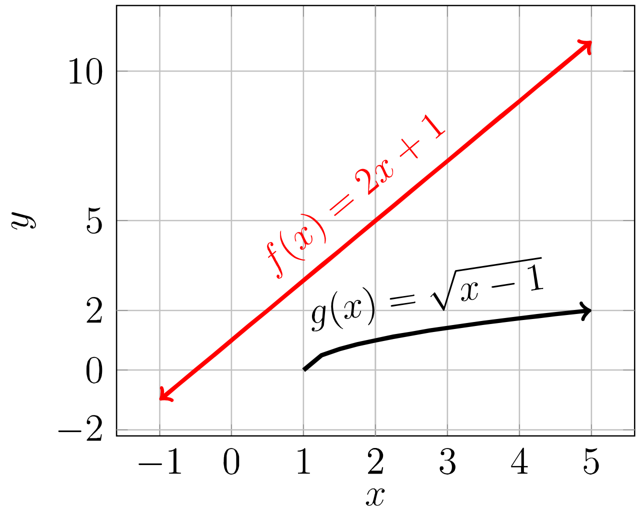
Example 1. Find the D_f and R_f of $f(x) = \sqrt{x+1}$



$D_f = \{x : x \geq -1\}$

and $R_f = \{y : y \geq 0\}$

Example 2. Let $f(x) = 2x+1$ and $g(x) = \sqrt{x-1}$. Find $D_{f\pm g}$, $D_{f/g}$ and $D_{g/f}$



Since $D_f = \mathbb{R}$ and $D_g = \{x : x \geq 1\}$, then

$$\begin{aligned} D_{f\pm g} &= D_f \pm D_g \\ &= D_f \cap D_g \\ &= \mathbb{R} \cap \{x : x \geq 1\} \\ &= \{x : x \geq 1\} \end{aligned}$$

$\frac{f(x)}{g(x)} = \frac{2x+1}{\sqrt{x-1}}$. To find $D_{f/g}$,

$$\begin{aligned} D_{f/g} &= D_f \cap D_g, \quad g(x) \neq 0, \\ &= \mathbb{R} \cap \{x : x \geq 1\} - \{g(x) = 0\}, \\ &= \{x : x \geq 1\} - \{x = 1\}, \\ &= \{x : x > 1\}. \end{aligned}$$

$\frac{g(x)}{f(x)} = \frac{\sqrt{x-1}}{2x+1}$. To find $D_{g/f}$

$$\begin{aligned} D_{g/f} &= D_g \cap D_f, \quad f(x) \neq 0, \\ &= \{x : x \geq 1\} \cap \mathbb{R} - \{f(x) = 0\}, \\ &= \{x : x \geq 1\} - \{x = \frac{-1}{2}\}, \\ &= \{x : x \geq 1\}. \end{aligned}$$

Composition of functions: Given functions f and g, the composition of f and g, denoted by $f \circ g$ or $g \circ f$, is the function defined by :

$$\begin{aligned}(f \circ g)(x) =& f(g(x))\\ D_{f\circ g} =& \{x : x \in D_g, \quad g(x) \in D_f\}\end{aligned}$$

$$\begin{aligned}(g \circ f)(x) =& g(f(x))\\ D_{g\circ f} =& \{x : x \in D_f, \quad f(x) \in D_g\}\end{aligned}$$

Example 3. We have $f(x) = x^2$ and $g(x) = 2x + 1$. Find $D_{f\circ g}$ and $D_{g\circ f}$. Since $D_f = \mathbb{R}$ and $D_g = \mathbb{R}$.

$$\begin{aligned}(f \circ g)(x) =& f(g(x)),\\ =& f(2x + 1),\\ =& (2x + 1)^2.\end{aligned}$$

Then, $D_{f\circ g} = \mathbb{R}$.

$$\begin{aligned}(g \circ f)(x) =& g(f(x)),\\ =& g(x^2),\\ =& 2x^2 + 1.\end{aligned}$$

Then, $D_{g\circ f} = \mathbb{R}$.

Example 4. We have $f(x) = x + 1$ and $g(x) = \sqrt{x - 1}$. Find $D_{f\circ g}$ and $D_{g\circ f}$. Since $D_f = \mathbb{R}, R_f = \mathbb{R}$ and $D_g = \{x : x \geq 1\}, R_g = \{y : y \geq 0\}$.

$$\begin{aligned}(f \circ g)(x) =& f(g(x)),\\ =& f(\sqrt{x - 1}),\\ =& \sqrt{x - 1} + 1. \qquad \text{Then,} \quad D_{f\circ g} = \{x : x \geq 1\}.\end{aligned}$$

$$\begin{aligned}(g\circ f)(x) =& g(f(x)),\\ =& g(x+1),\\ =& \sqrt{x+1-1},\\ =& \sqrt{x}. \qquad \text{Then,} \quad D_{g\circ f} = \{x : x \geq 0\}.\end{aligned}$$

Example 5. We have $f(x) = x^2 + 1$ and $g(x) = \sqrt{x}$. Find $D_{f\circ g}$ and $D_{g\circ f}$. Since $D_f = \mathbb{R}$ and $D_g = \{x : x \geq 0\}$.

$$\begin{aligned}(f\circ g)(x) =& f(g(x)),\\ =& f(\sqrt{x}),\\ =& x+1. \qquad \text{Then,} \quad D_{f\circ g} = \{x : x \geq 0\}.\end{aligned}$$

$$\begin{aligned}(g\circ f)(x) =& g(f(x)),\\ =& g(x^2+1),\\ =& \sqrt{x^2+1}, \qquad \text{Then,} \quad D_{g\circ f} = \mathbb{R}.\end{aligned}$$

Homework

Question 1. *Find the domain and range of $f\circ g$ and $g\circ f$ where*

$$f(x) = x^2 - 3, \qquad g(x) = \sqrt{x}.$$

9) Rational function المجال والمدى للدالة الكسرية

Definition 1. A function that expressed as a ratio of two polynomials called a rational function.

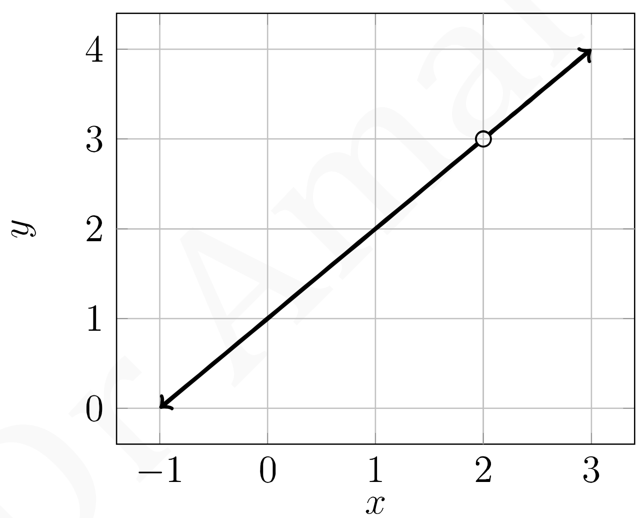
If $P(x)$ and $Q(x)$ are polynomials, then the domain of $\frac{P(x)}{Q(x)}$ is defined as in the arithmetic operation on functions (3).

Example 6. Find the domain and range for this function: $f(x) = \frac{x^2-x-2}{x-2}$.

Since $f(x)$ is rational then, we need to avoid the zero of the denominator : $x = 2$.

$$f(x) = \frac{(x-2)(x+1)}{x-2}$$
$$f(x) = y = x + 1$$

The graph of $y = x + 1$ is a linear function but, the graph will have a hole فجوة:شكل دائرة فارغة at $x = 2, y = 3$. At the same time, if we change the function $x \to y$, this will be $x = y - 1$ has no problem.



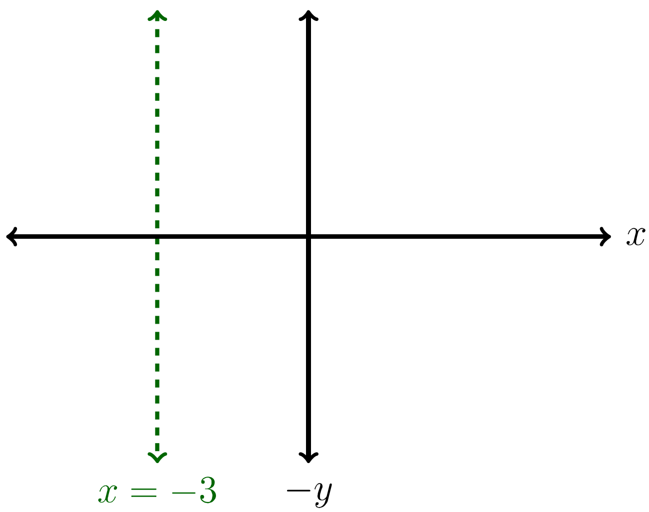
$D_f = \mathbb{R} \setminus \{2\}$

and $R_f = \mathbb{R} \setminus \{3\}$

Example 7. Find the domain and range for this function: $f(x) = \frac{1}{x+3} - 5$.

First, simplify:

$$f(x) = \frac{1 - 5(x+3)}{x+3},$$
$$y = \frac{-14 - 5x}{x+3}.$$

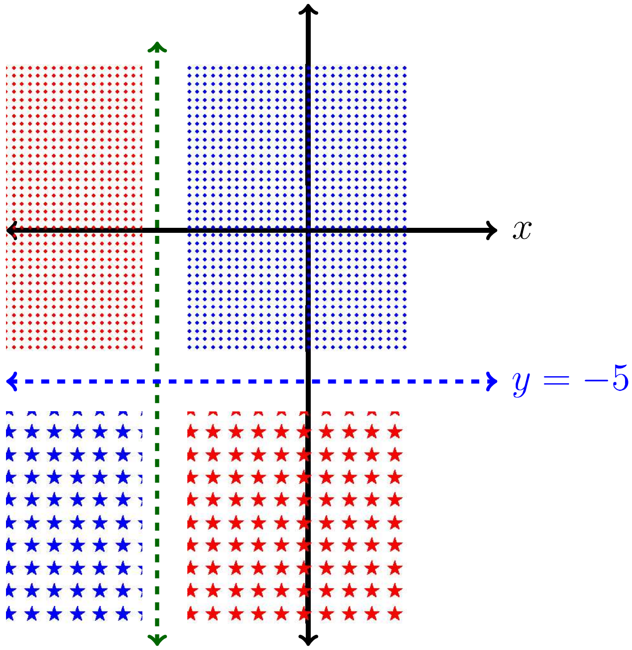


In this case, $D_f = \mathbb{R} \setminus \{-3\}$.

So we have a vertical line $x = -3$.

If we change the function $x \to y$, this will be

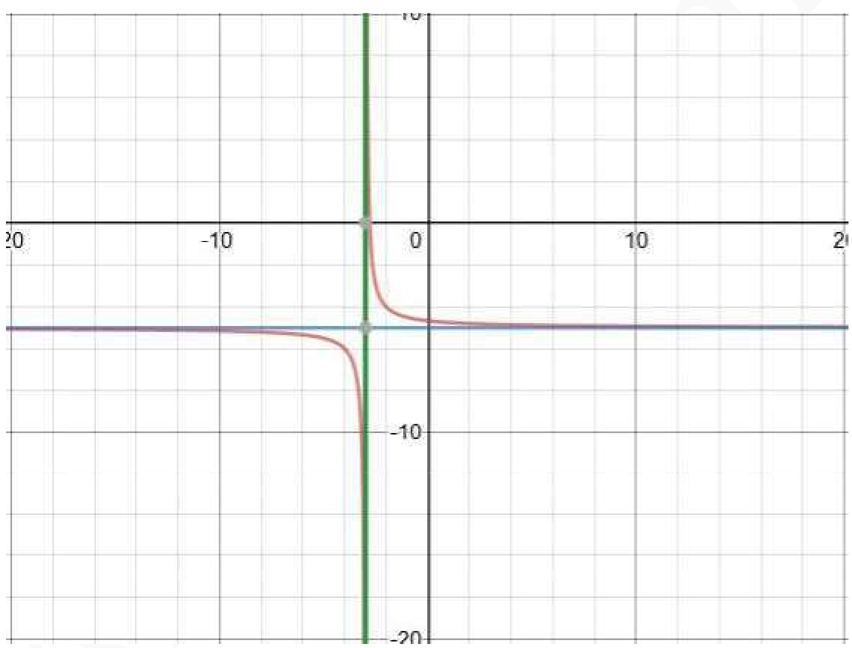
$$y(x+3) = -14 - 5x,$$
$$yx + 3y = -14 - 5x,$$
$$yx + 5x = -14 - 3y,$$
$$x(y+5) = -14 - 3y,$$
$$x = \frac{-14 - 3y}{y+5}.$$



In this case, $y=-5$

is a problem.

نتوقع بان يكون الرسم في
المناطق الاربعة (ليس شرط في جميعها)
حسب قيم x



$D_f = \mathbb{R} \setminus \{x=-3\}$ and $R_f = \mathbb{R} \setminus \{y=-5\}$

Homework

Question 2. *Find the domain and range for this function:*

$$f(x) = \frac{x^2-3x-4}{x+1}$$

College of Education For Pure Sciences, Mathematics Department,
University of Mosul

Limit and Continuity الغاية والاستمرارية

Definition 1. Limits: we write it as

$$\lim_{x\to a} f(x) = L$$

which is read تقرأ: the limit of $f(x)$ as x approaches تقترب to a is L

Theorem: Let a and k be numbers,

$$\lim_{x\to a} f(x) = \lim_{x\to a} k = k$$

$$\lim_{x\to a} f(x) = \lim_{x\to a} x = a$$

Theorem: Let a and k be numbers and suppose that

$$\lim_{x\to a} f(x) = L_1, \qquad \lim_{x\to a} g(x) = L_2$$

$$1) \lim_{x\to a}[f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$$
$$= L_1 \pm L_2$$

$$\lim_{x\to 2}[x \pm 3] = \lim_{x\to 2} x \pm \lim_{x\to 2} 3 = 2 \pm 3 = 5 \quad \text{or} \quad -1$$

$$2) \lim_{x\to a}[f(x) \times g(x)] = \lim_{x\to a} f(x) \times \lim_{x\to a} g(x)$$
$$= L_1 \times L_2$$

$$\lim_{x\to 0}[x \times e^x] = \lim_{x\to 0} x \times \lim_{x\to 0} e^x = 0 \times 1 = 0$$

$$3)\ \lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}=\frac{L_1}{L_2},\quad L_2\neq 0$$

$$4)\ \lim_{x\to a}\sqrt[n]{f(x)}=\sqrt[n]{\lim_{x\to a} f(x)}$$

$$=\sqrt[n]{L_1},\qquad L_1>0,\quad \text{when } n \text{ is even}$$

$$\lim_{x\to 1}\sqrt[4]{\frac{1}{x}}=\sqrt[4]{\lim_{x\to 1}\frac{1}{x}}=\sqrt[4]{1}=1$$

$$5)\ \lim_{x\to a}\ k\times f(x)=k\times\lim_{x\to a} f(x)=k\ L_1$$

$$\lim_{x\to 1} 3\times x=3\times\lim_{x\to 1} x=3\times 1=3$$

The above statements are true for one-side limit as $x\to a^+$ or $x\to a^-$.

Remark: If n is a positive integer, then

$$\lim_{x\to a} x^n=\left(\lim_{x\to a} x\right)^n=a^n$$

$$\lim_{x\to 3} x^4=\left(\lim_{x\to 3} x\right)^4=3^4=81$$

Theorem: For any polynomial $P(x)=c_0+c_1x+\cdots+c_nx^n$ for any real number a

$$\lim_{x\to a} P(x)=c_0+c_1a+\cdots+c_na^n=P(a).$$

Example 1. Find

$$\lim_{x\to 5} P(x), \qquad \text{where} \qquad P(x) = x^2 - 4x + 3$$

$$\begin{aligned}\lim_{x\to 5} P(x) &= \lim_{x\to 5}(x^2 - 4x + 3)\\ &= \lim_{x\to 5} x^2 - 4\lim_{x\to 5} x + \lim_{x\to 5} 3\\ &=5^2 - 4\times 5 + 3 = 8\end{aligned}$$

Theorem: Consider the rational function $f(x) = \frac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomials.

For any real number a:

- if $q(a) \neq 0$,

$$\lim_{x\to a} f(x) = f(a)$$

- if $q(a) = 0$, but $p(a) \neq 0$, then

$$\lim_{x\to a} f(x), \qquad \text{DNE: does not exist.}$$

Example 2. Show that

$$\lim_{x\to 3} \frac{5x^3 + 4}{x - 3}$$

is not exist.

Solution:

$$\lim_{x\to 3} 5x^3 + 4 = 5\times 3^3 + 4 = 139$$

$$\lim_{x\to 3} x - 3 = 3 - 3 = 0!!$$

Then,

$$\lim_{x\to 3} \frac{5x^3 + 4}{x - 3}, \qquad \text{DNE}$$

Limit at infinity $\pm\infty$

Remark

$$\lim_{x\to+\infty} x^n = +\infty, \qquad n = 1, 2, 3, 4, \ldots$$

$$\lim_{x\to-\infty} x^n = \begin{cases} -\infty, & n = 1, 3, 5, \ldots \\ +\infty, & n = 2, 4, 6, \ldots \end{cases}$$

Example 3. Find the limit for the following functions

$$1)\ \lim_{x\to+\infty} 2\,x^2 = +\,\infty$$

$$2)\ \lim_{x\to+\infty} -7\,x^6 = -\,(+\infty) = -\infty$$

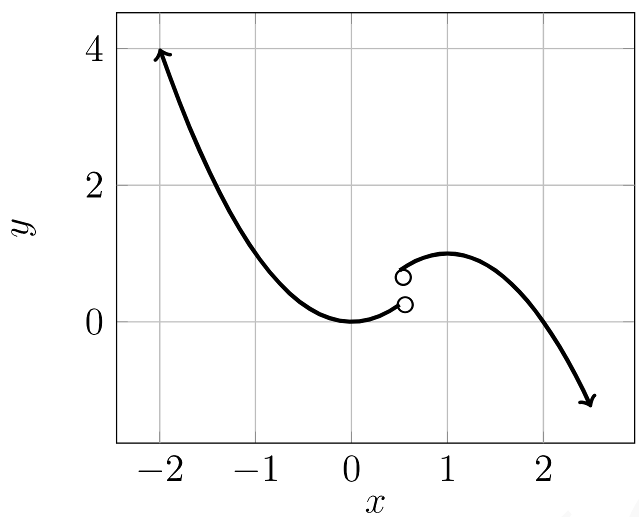
$$3)\ \lim_{x\to-\infty} 2\,x^5 = -\,\infty$$

$$4)\ \lim_{x\to-\infty} -3\,x^6 = -\,(+\infty) = -\infty$$

College of Education For Pure Sciences, Mathematics Department,
University of Mosul

Note: A function $f(x)$ has a limit at $x \to a$ if and only if

$$\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = L, \qquad L \quad \text{is constant.}$$



Example 1. For the function f in the picture, the one-side limits

$$\lim_{x\to x_0^+} f(x) \quad \text{and} \quad \lim_{x\to x_0^-} f(x)$$

both are exist but they are not the same. Then

$$\lim_{x\to x_0} f(x)$$

dose not exist (DNE)

Example 2. Show that $f(x)$ has limit at $x \to 2$

$$\lim_{x\to 2} f(x)? \quad \text{where} \quad f(x) = \begin{cases} \frac{x^2-4}{x-2} & x \geq 2 \\ 4 & x < 2 \end{cases}$$

First,

$$\begin{aligned} \lim_{x\to 2^+} f(x) &= \lim_{x\to 2^+} \frac{x^2-4}{x-2} \\ &= \lim_{x\to 2^+} \frac{(x-2)(x+2)}{x-2} \\ &= \lim_{x\to 2^+} (x+2) = \underline{\underline{4}} \end{aligned}$$

Second,

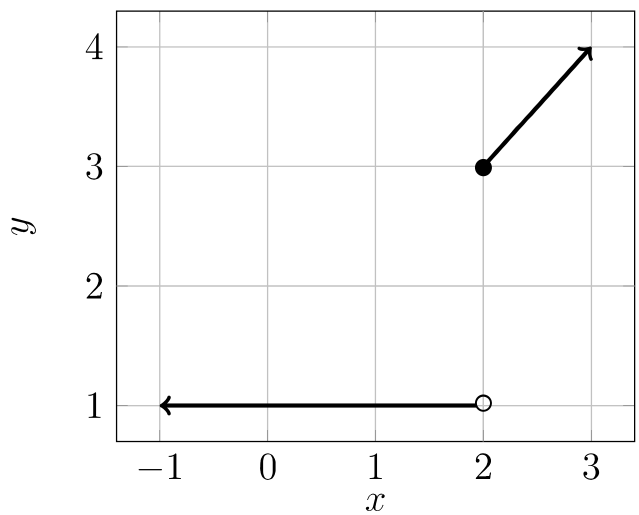
$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} 4$$
$$= \underline{\underline{4}}.$$

In this case,

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^-} f(x) = 4$$

Then, $f(x)$ has limit at $x \to 2$

$$\lim_{x\to 2} f(x) = 4$$



Example 3. Check whether $f(x)$ has limit at $x \to 2$

$$\text{where} \quad f(x) = \begin{cases} x+1 & x \geq 2 \\ 1 & x < 2 \end{cases}$$

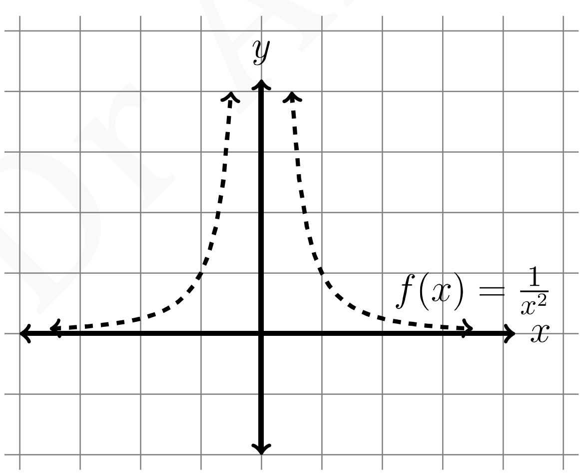
$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (x+1) = 3$$
$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} 1 = 1$$

In this case,

$$\lim_{x\to 2^+} f(x) \neq \lim_{x\to 2^-} f(x)$$

Then,

$$\lim_{x\to 2} f(x) \quad \text{DNE.}$$

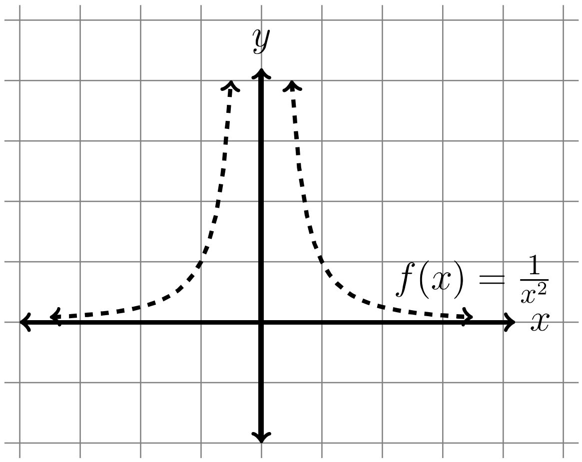


Example 4. Find

$$\lim_{x\to \infty^{\pm}} f(x), \quad \text{where} \quad f(x) = \frac{1}{x^2}$$

$D_f = \mathbb{R}\backslash\{0\}$ and $R_f = \{y : y > 0\}$

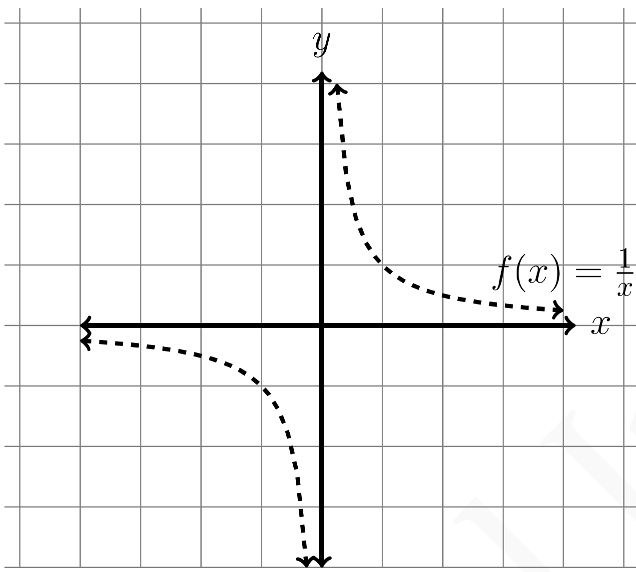
$$\lim_{x\to \infty^+} \frac{1}{x^2} = 0 = \lim_{x\to \infty^-} \frac{1}{x^2}$$



Example 5. Find

$$\lim_{x \to 0^{\pm}} f(x), \quad \text{where} \quad f(x) = \frac{1}{x^2}$$

$$\lim_{x \to 0^{+}} \frac{1}{x^2} = \infty^{+} = \lim_{x \to 0^{-}} \frac{1}{x^2}$$

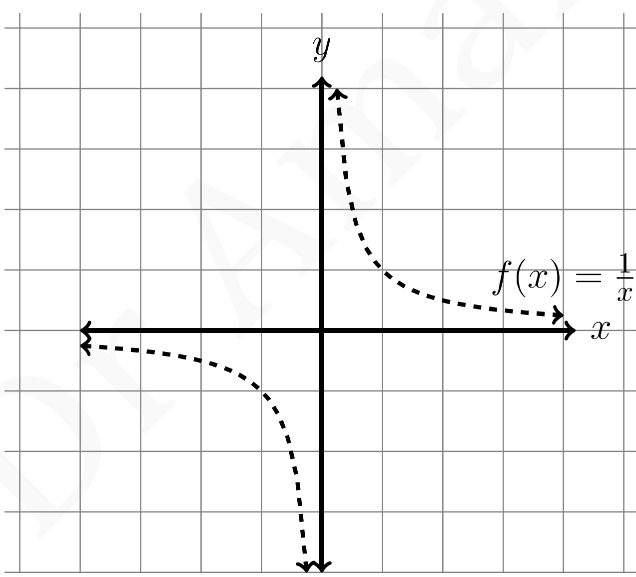


Example 6. Find

$$\lim_{x \to 0^{\pm}} f(x), \quad \text{where} \quad f(x) = \frac{1}{x}$$

$D_f = \mathbb{R} \backslash \{0\}$ and $R_f = \mathbb{R} \backslash \{0\}$

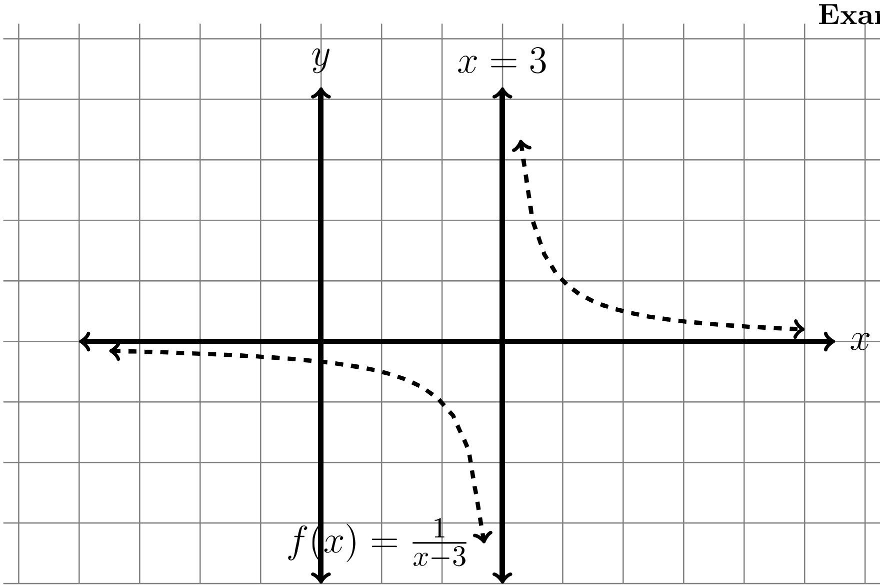
$$\lim_{x \to 0^{+}} \frac{1}{x} = \infty^{+} \qquad \lim_{x \to 0^{-}} \frac{1}{x} = \infty^{-}$$



Example 7. Find

$$\lim_{x \to \infty^{\pm}} f(x), \quad \text{where} \quad f(x) = \frac{1}{x}$$

$$\lim_{x \to \infty^{+}} \frac{1}{x} = 0 = \lim_{x \to \infty^{-}} \frac{1}{x}$$

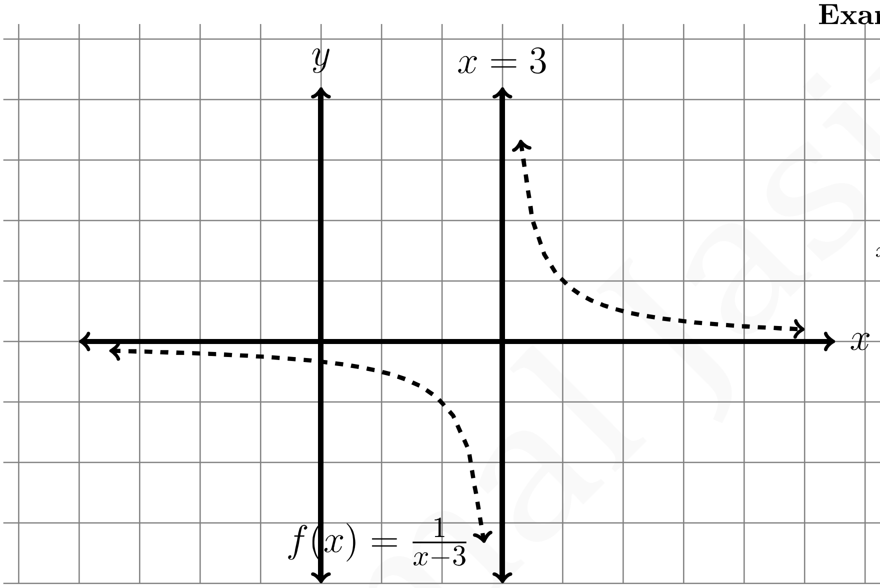


Example 8. Find

$$\lim_{x\to\infty^{\pm}} f(x), \quad \text{where} \quad f(x) = \frac{1}{x-3}$$

$$D_f = \mathbb{R}\backslash\{3\} \quad \text{and} \quad R_f = \mathbb{R}\backslash\{0\}$$

$$\lim_{x\to\infty^{+}} \frac{1}{x-3} = 0 = \lim_{x\to\infty^{-}} \frac{1}{x-3}$$



Example 9. Find

$$\lim_{x\to 3^{\pm}} f(x), \quad \text{where} \quad f(x) = \frac{1}{x-3}$$

$$\lim_{x\to 3^{+}} \frac{1}{x-3} = \infty^{+} \qquad \lim_{x\to 3^{-}} \frac{1}{x-3} = \infty^{-}$$

Homework

Question 1. *Find D_f, R_f, and check whether $f(x)$ has limit for the following functions:*

$$\lim_{x\to 3} f(x), \quad \textit{where} \quad f(x) = \frac{1}{(x-3)^2}$$

$$\lim_{x\to 0} f(x), \quad \textit{where} \quad f(x) = \frac{1}{|x|}$$

Trigonometric functions and their limits

Properties of the trigonometric functions :

$$\sin(A \pm B) = \sin(A)\cos(B) \pm \sin(B)\cos(A)$$
$$\cos(A \pm B) = \cos(A)\cos(B) \mp \sin(B)\sin(A)$$

$$\sin^2(x) + \cos^2(x) = 1$$
$$1 + \cot^2(x) = \csc^2(x)$$
$$\tan^2(x) + 1 = \sec^2(x)$$

$$\cos(2x) = 1 - 2\sin^2(x)$$
$$\cos(2x) = 2\cos^2(x) - 1$$

Theorem: If c is any number in the N domain of the stated trigonometric function, then

$$\lim_{x\to c} \sin(x) = \sin(c) \quad , \quad \lim_{x\to c} \cos(x) = \cos(c), \quad \lim_{x\to c} \tan(x) = \tan(c)$$
$$\lim_{x\to c} \csc(x) = \csc(c) \quad , \quad \lim_{x\to c} \sec(x) = \sec(c), \quad \lim_{x\to c} \cot(x) = \cot(c).$$

Theorem:

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1, \qquad \lim_{x\to 0} \frac{1-\cos(x)}{x} = 0$$

Example 10. Find :

$$\lim_{x\to 0} \frac{\tan(x)}{x}.$$

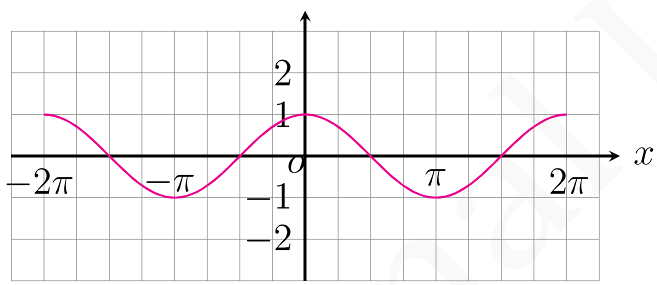
$$\lim_{x\to 0} \left(\tan(x) \times \frac{1}{x}\right) = \lim_{x\to 0} \left(\frac{\sin(x)}{\cos(x)} \times \frac{1}{x}\right)$$

$$= \lim_{x\to 0} \left(\frac{\sin(x)}{x} \times \frac{1}{\cos(x)}\right)$$

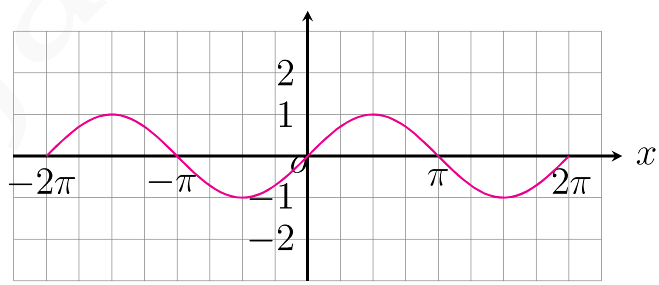
$$= \lim_{x\to 0} \frac{\sin(x)}{x} \times \lim_{x\to 0} \frac{1}{\cos(x)}$$

$$=1 \times 1 = 1$$

$y = f(x) = \cos(x)$



$y = f(x) = \sin(x)$



Example 11. Find :

$$\lim_{\theta\to 0} \frac{\sin(2\theta)}{\theta}.$$

$$\lim_{\theta\to 0} \frac{\sin(2\theta)}{\theta} = \frac{2}{2} \lim_{\theta\to 0} \frac{\sin(2\theta)}{\theta}$$

$$=2 \lim_{\theta\to 0} \frac{\sin(2\theta)}{2\theta}$$

$$=2 \times 1 = 2$$

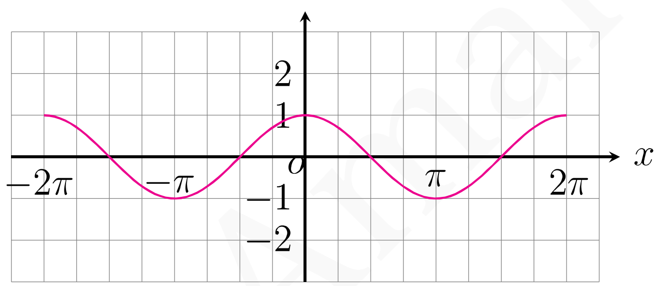
Example 12. Find

$$\lim_{x\to 0}\frac{\tan(x)-\sin(x)}{\sin^3(x)}$$

$$\begin{aligned}
\lim_{x\to 0}\frac{\tan(x)-\sin(x)}{\sin^3(x)} &= \lim_{x\to 0}\frac{\frac{\sin(x)}{\cos(x)}-\sin(x)}{\sin^3(x)}\\
&= \lim_{x\to 0}\left(\frac{\sin(x)-\sin(x)\cos(x)}{\cos(x)}\times\frac{1}{\sin^3(x)}\right)\\
&= \lim_{x\to 0}\left(\frac{\sin(x)(1-\cos(x))}{\cos(x)}\times\frac{1}{\sin^3(x)}\right)
\end{aligned}$$

$$\begin{aligned}
\lim_{x\to 0}\frac{\tan(x)-\sin(x)}{\sin^3(x)} &= \lim_{x\to 0}\left(\frac{1-\cos(x)}{\cos(x)}\times\frac{1}{\sin^2(x)}\right), \quad \sin^2(x)+\cos^2(x)=1\\
&= \lim_{x\to 0}\left(\frac{1-\cos(x)}{\cos(x)}\times\frac{1}{1-\cos^2(x)}\right)\\
&= \lim_{x\to 0}\left(\frac{1-\cos(x)}{\cos(x)}\times\frac{1}{(1-\cos(x))(1+\cos(x))}\right)\\
&= \lim_{x\to 0}\left(\frac{1}{\cos(x)}\times\frac{1}{(1+\cos(x))}\right) = 1\times\frac{1}{2}=\frac{1}{2}.
\end{aligned}$$

$y = f(x) = \cos(x)$



Then,

$$\lim_{x\to 0}\frac{\tan(x)-\sin(x)}{\sin^3(x)}=\frac{1}{2}.$$

College of Education For Pure Sciences, Mathematics Department, University of Mosul

Continuity

Definition 1. *A function $f(x)$ is said to be continuous at $x = c$, if the following conditions are hold* يجب ان تتحقق الشروط الثلاث ادناه:

1. *$f(x)$ is defined at $x = c$.*
2. *The limit of $f(x)$ at $x \to c$ must be exist, which means*

$$\lim_{x\to c^+} f(x) = \lim_{x\to c^-} f(x).$$

3. *The limit of $f(x)$ at $x \to c$ must be equal to $f(c)$, which means*

$$\lim_{x\to c} f(x) = f(c).$$

If one or more condition of the above definition fails يفشل to hold يتحقق then, we will say that $f(x)$ has a discontinuity غير مستمرة at $x = c$.

Example 1. Prove (show) that $f(x) = 3x^2 + 2x + 20$ is continuous at $x = 2$.
نحتاج ان نطبق الشروط الثلاث اعلاه

1. $f(x)$ at $x = 2$, $f(2) = 3 \times (2)^2 + 2 \times 2 + 20 = 36.$
2. لكون الدالة متعددة حدود: الغاية من اليمين والغاية من اليسار معرفة

$$\lim_{x\to 2^\pm} f(x) = 3 \times (2)^2 + 2 \times 2 + 20 = 36.$$

3.

$$\lim_{x\to 2} f(x) = f(2) = 36.$$

Then, $f(x)$ is continuous at $x = 2$.

Example 2. Show that

$$f(x) = \begin{cases} \frac{x^2-16}{x-4}, & x \neq 4 \to x \gtrless 4 \\ 10, & x = 4 \end{cases}$$

has a discontinuity at $x = 4$.

1. $f(4) = 10$.

2. We need to find the limit of $f(x)$ at $x \to 4$:

$$\begin{aligned} \lim_{x\to 4^{\pm}} f(x) &= \lim_{x\to 4^{\pm}} \frac{x^2-16}{x-4} \\ &= \lim_{x\to 4^{\pm}} \frac{(x-4)(x+4)}{x-4} \\ &= \lim_{x\to 4^{\pm}} (x+4) = 8 \end{aligned}$$

3.

$$\lim_{x\to 4} f(x) \neq f(4).$$

Then, $f(x)$ has a discontinuity at $x = 4$.

Example 3. Show that

$$f(x) = \begin{cases} \frac{x^3-1}{x-1}, & x \neq 1 \to x \gtrless 1 \\ 3, & x = 1 \end{cases}$$

is continuous at $x = 1$.

1. $f(1) = 3$

2. We need to find the limit of $f(x)$ at $x \to 1$:

$$\begin{aligned} \lim_{x\to 1^{\pm}} f(x) &= \lim_{x\to 1^{\pm}} \frac{x^3-1}{x-1} \\ &= \lim_{x\to 1^{\pm}} \frac{(x-1)(x^2+x+1)}{x-1} \\ &= \lim_{x\to 1^{\pm}} (x^2+x+1) = 3 \end{aligned}$$

3.

$$\lim_{x\to 1} f(x) = f(1) = 3.$$

Then, $f(x)$ is continuous at $x = 1$.

Example 4. Check whether this function is continuous or has a discontinuity at $x = 0$, where

$$f(x) = \begin{cases} \frac{x}{\sqrt{x+1}\ -1}, & x > 0 \\ x + 2, & x \le 0 \end{cases}$$

1. $f(0) = 0 + 2 = 2$
2. We need to find the limit of $f(x)$ at $x \to 0$:
   - the limit of $f(x)$ at $x \to 0^+$

$$\begin{aligned}\lim_{x\to 0^+} f(x) &= \lim_{x\to 0^+} \frac{x}{\sqrt{x+1}\ -1} \\ &= \lim_{x\to 0^+} \frac{x}{\sqrt{x+1}\ -1} \times \frac{\sqrt{x+1}\ +1}{\sqrt{x+1}\ +1} \\ &= \lim_{x\to 0^+} \frac{x(\sqrt{x+1}\ +1)}{x+1-1} \\ &= \lim_{x\to 0^+} \frac{x(\sqrt{x+1}\ +1)}{x} \\ &= \lim_{x\to 0^+} \sqrt{x+1}\ +1 = 2\end{aligned}$$

   - the limit of $f(x)$ at $x \to 0^-$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (x+2) = 2$$

3.

$$\lim_{x\to 0} f(x) = f(0) = 2.$$

Then, $f(x)$ is continuous at $x = 0$.

Example 5. Is $f(x) = [x + 0.5]$ continuous at $x = 0$?

1. $f(0) = [0 + 0.5] = [0.5] = 0.$

2. We need to find the limit of $f(x)$ at $x \to 0$:

   - the limit of $f(x)$ at $x \to 0^+$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} [x + 0.5]$$
$$= \lim_{x\to 0^+} [0 + 0.5] = [0.51] = 0$$

   - the limit of $f(x)$ at $x \to 0^-$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} [x + 0.5]$$
$$= \lim_{x\to 0^-} [0 + 0.5] = [0.49] = 0$$

3.
$$\lim_{x\to 0} f(x) = f(0) = 0.$$

Then, $f(x)$ is continuous at $x = 0$.

Homework

Question 1. *Is $f(x)$ continuous at $x = 9$?*

$$f(x) = \begin{cases} \frac{x-9}{\sqrt{x}\ -3}, & x > 9 \\ 6, & x \le 9 \end{cases}$$

Example 6. Find the value of k which makes the function $f(x)$ (if it is possible) continuous at $x = 1$, where k is a constant and

$$f(x) = \begin{cases} 7x - 2, & x \le 1 \\ k\ x^2, & x > 1 \end{cases}$$

Since $f(x)$ is continuous at $x = 1$, then the conditions must be hold which are

1. $f(1) = (7 \times 1 - 2) = 5.$

2. The limit of $f(x)$ is exist at $x = 1$ which means

$$\begin{aligned}\lim_{x\to 1^+} f(x) &= \lim_{x\to 1^-} f(x)\\ \lim_{x\to 1^+} k\ x^2 &= \lim_{x\to 1^-}(7x-2)\\ k &= 5.\end{aligned}$$

3. the limit of $f(x)$ when $x \to 1$ equal to $f(1)$ when $k = 5$.

Then, the function $f(x)$ is continuous at $x = 1$ when $k = 5$.

Example 7. Find the value of k which makes the function $f(x)$ (if it is possible) continuous at $x = 2$, where k is a constant and

$$f(x) = \begin{cases} k\ x^2, & x \le 2 \\ 2x + k, & x > 2 \end{cases}$$

Since $f(x)$ is continuous at $x = 2$, then the conditions must be hold which are

1. The limit of $f(x)$ is exist at $x \to 2$ which means

$$\begin{aligned}\lim_{x\to 2^+} f(x) &= \lim_{x\to 2^-} f(x)\\ \lim_{x\to 2^+}(2x+k) &= \lim_{x\to 2^-}(k\ x^2)\\ 4+k &= 4k\\ 4 &= 4k-k\\ k &= \frac{4}{3}.\end{aligned}$$

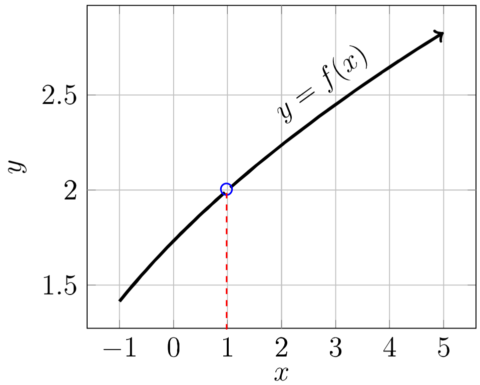
2. $f(2) = k\ x^2 = \frac{4}{3} \times 4 = \frac{16}{3}$.

3.

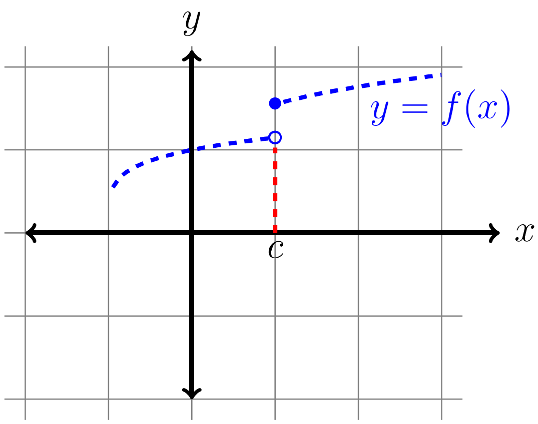
$$\begin{aligned}\lim_{x\to 2^+}(2x+k) &= \lim_{x\to 2^+} 2x + \frac{4}{3} = \frac{16}{3}.\\ \lim_{x\to 2^-}(k\ x^2) &= \lim_{x\to 2^-} \frac{4}{3} \times x^2 = \frac{16}{3}.\end{aligned}$$

Then, the function $f(x)$ is continuous at $x = 2$ when $k = \frac{4}{3}$.

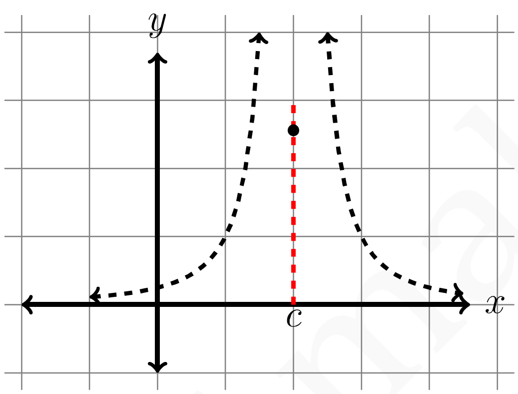
The following Figures illustrate a discontinuity at $x = c$:



The function is not defined at $x = 1$, then the first condition of the defenition does not satisfy. Then, the function has a discontinuity at $x = 1$.



The function is defined at $x = c$, but the limit of $f(x)$ when $x \to c$ DNE. Then the function has a discontinuity at $x = c$.



The function is defined at $x = c$, but the limit of $f(x)$ when $x \to c$ DNE. Then the function has a discontinuity at $x = c$.