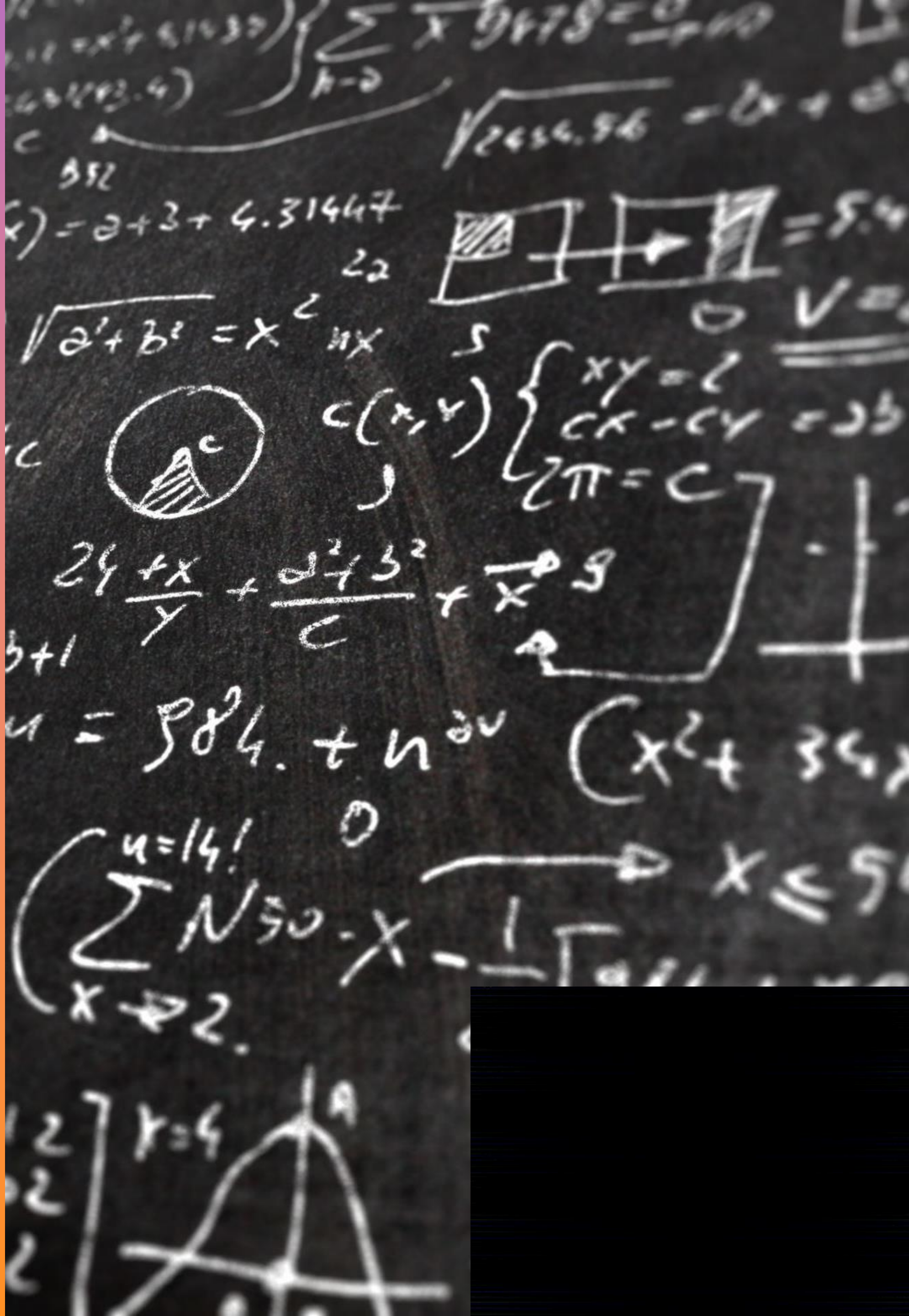
ADVANCED CALCULUS

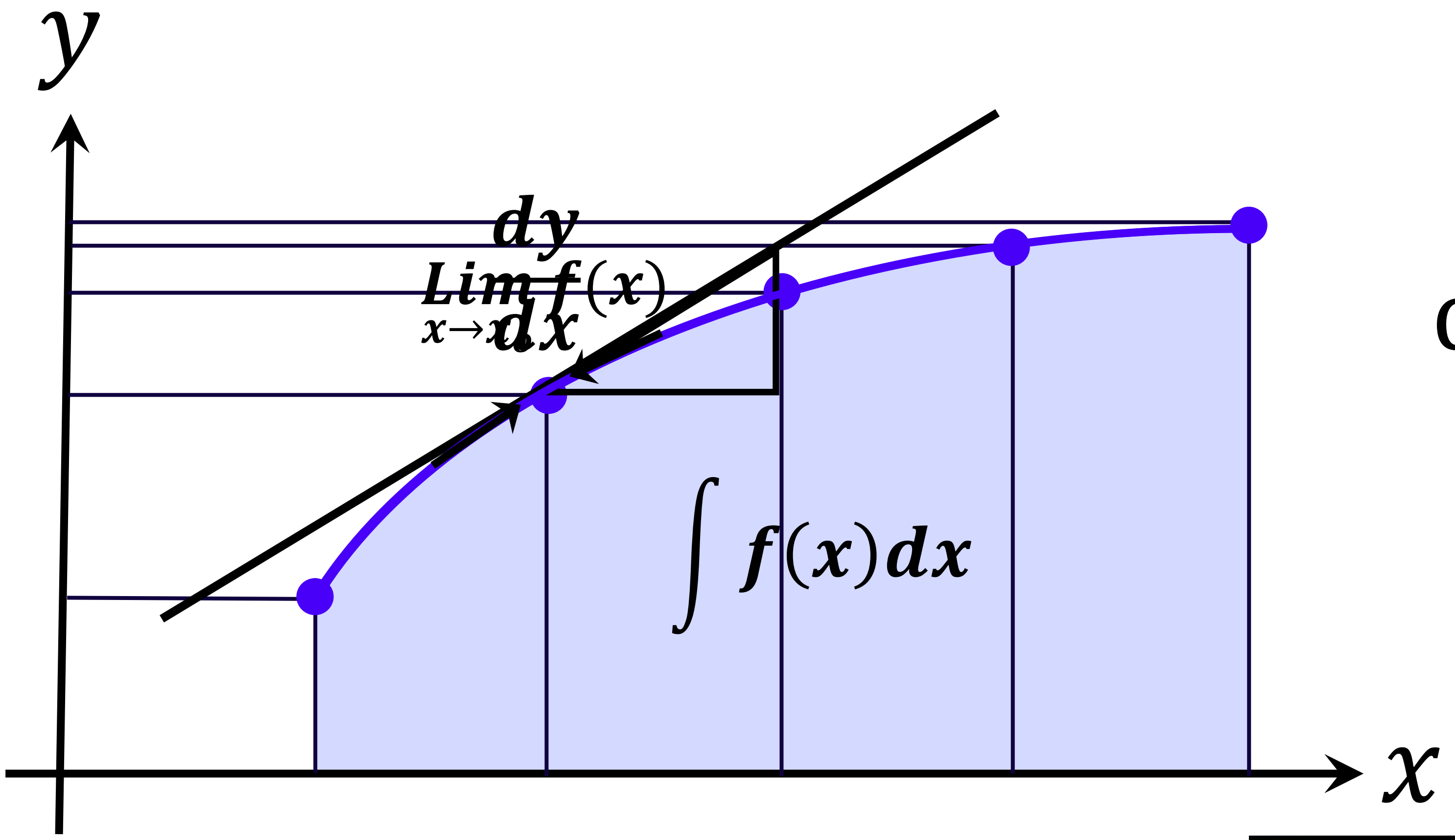
Introduction

Dr. Sohaib Al-Ramadhani



Calculus

$y = f(x)$



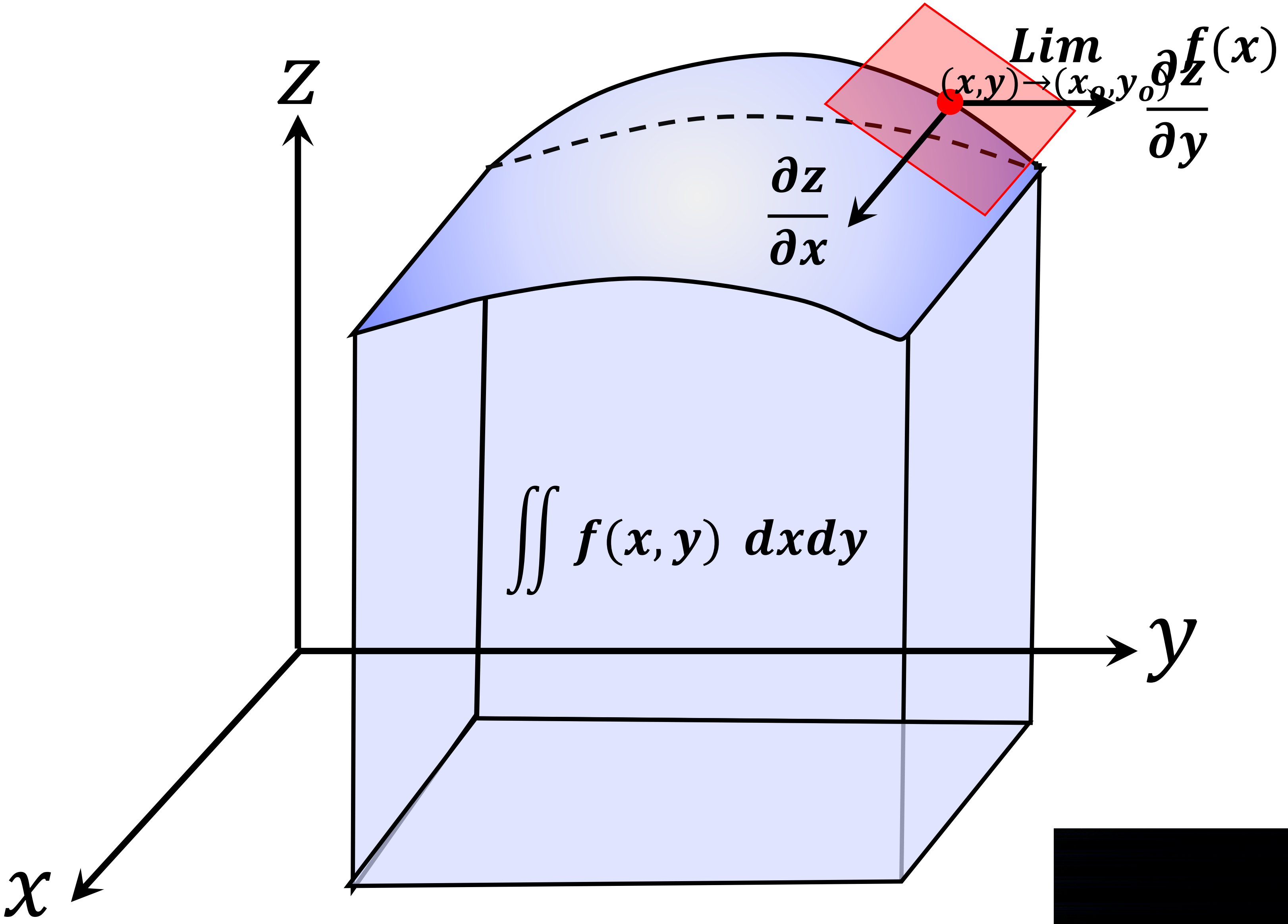
Cont.

Advanced Calculus

$$z = f(x, y)$$

$\Downarrow$

Cont.



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Other Topics

Sequences

Series

Vectors in space

Line and Plane

Polar Coordinates

Sequence

Part 1

$\{a_n\}_1^{\infty}$

$\langle a_1, a_2, a_3, \ldots, a_n, \ldots \rangle$

What is a sequence?

- Consider a set $A = \{a\ \ , a\ \ , a\ \ , a\ \ , \ldots , a\ \ , \ldots\}$
- Adding order $a\ \ ,\quad a\ \ ,\quad a\ \ ,\quad a\ \ , \ldots ,\quad a\quad \ldots ,$

$$\begin{array}{ccccc} a & a & a & a & a \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 1 & 2 & 3 & 4 & n \end{array}$$

- The order $\langle a_1, a_2, a_3, \ldots , a_n, \ldots \rangle$ is called a **sequence**

What is a sequence?

- **Definition**: A ***Sequence*** is a function whose **domain** is the set of **positive integers** $\mathbb{Z}_+ = \{1,2,3,4,\cdots\}$.
- We denote to sequences as $\{\boldsymbol{a_n}\}_{\boldsymbol{1}}^{\infty}$ or $\langle \boldsymbol{a_n} \rangle$

$$\{a_n\}_1^{\infty} = \langle a_n \rangle = \langle a_1, a_2, a_3, \dots, a_n, \dots \rangle$$

- **Example**: $\{n^2\}_1^{\infty} = \langle 1, 4, 9, 16, \dots, n^2, \dots \rangle$

Monotonicity

$$a_1 < a_2 < \cdots < a_n < a_{n+1} < \cdots \quad \forall n \geq 1 \Rightarrow \text{ increasing seq.}$$

$$a_1 \leq a_2 \leq \cdots \leq a_n \leq a_{n+1} \leq \cdots \quad \forall n \geq 1 \Rightarrow \text{ nondecreasing seq.}$$

$$a_1 > a_2 > \cdots > a_n > a_{n+1} > \cdots \quad \forall n \geq 1 \Rightarrow \text{ decreasing seq.}$$

$$a_1 \geq a_2 \geq \cdots \geq a_n \geq a_{n+1} \geq \cdots \quad \forall n \geq 1 \Rightarrow \text{ nonincreasing seq.}$$

increasing ⇒ nondecreasing ⇒ **monotonic** ⇐ nondecreasing ⇐ decreasing

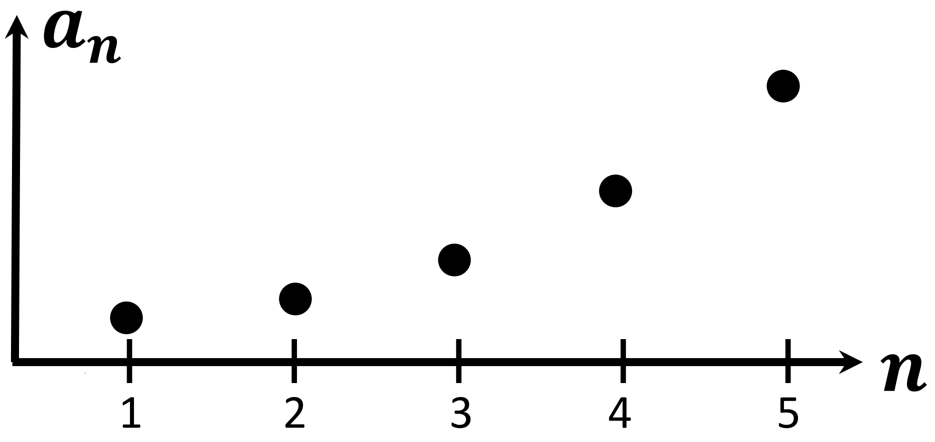
Not monotonic

Examples

- $\{n^2\}_1^\infty = \langle 1, 4, 9, \dots, n^2, \dots \rangle$

$$1 < 4 < 9 < 16 < 25 < \cdots$$

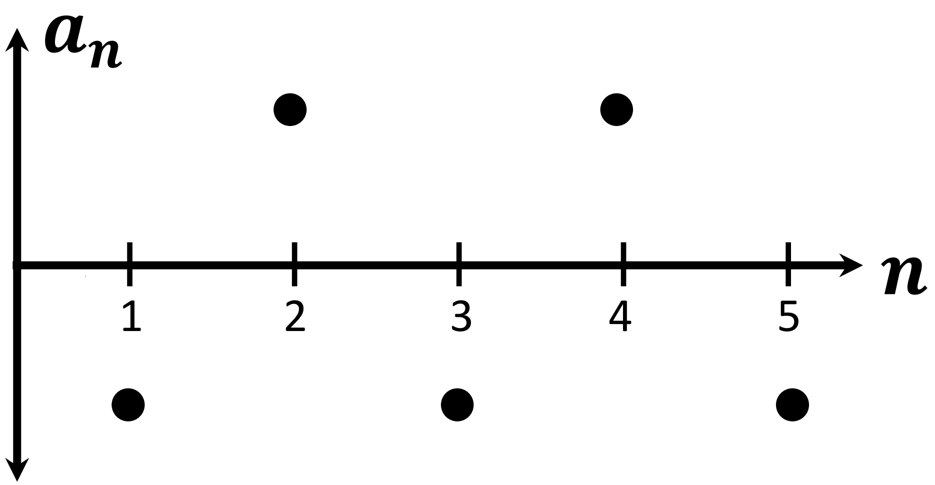
$$n^2 < (n+1)^2 \quad \forall n \geq 1$$



Increasing $\Rightarrow$ Monotonoic

- $\langle (-1)^n \rangle = \langle -1, +1, -1, \dots \rangle$

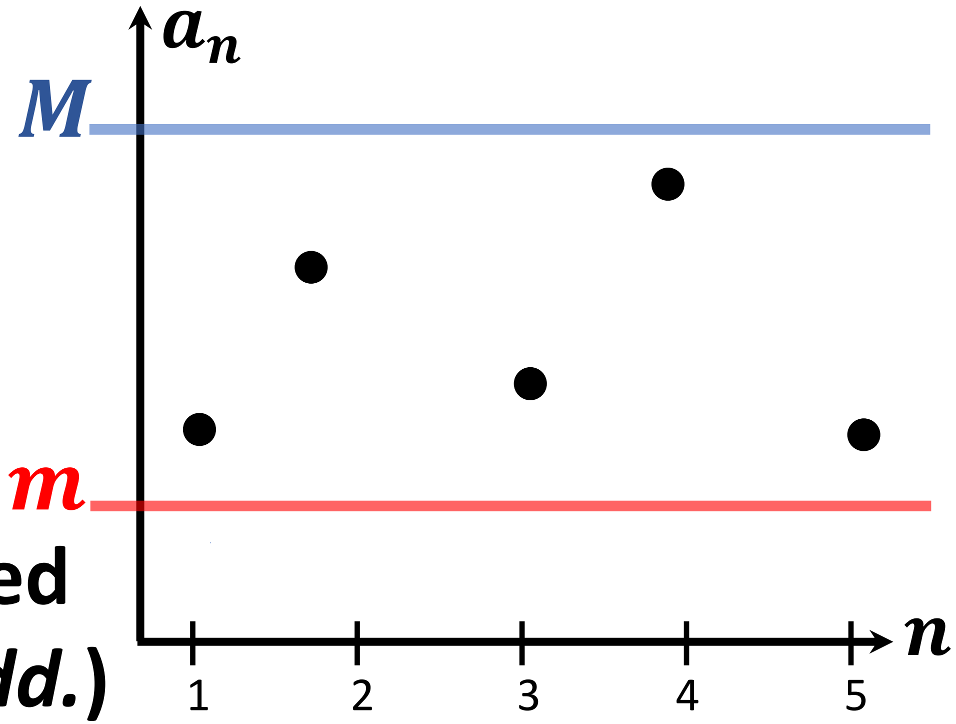
$$-1 < +1 > -1 < +1 > \cdots$$



not Monotonoic

Bounded sequence

- If $\exists \boldsymbol{M} \in \mathbb{R}$ such that $\boldsymbol{a_n} \leq \boldsymbol{M} \;\; \forall n \geq 1$ then $\langle a_n \rangle$ is called ***bounded above***
- If $\exists \boldsymbol{m} \in \mathbb{R}$ such that $\boldsymbol{a_n} \geq \boldsymbol{m} \;\; \forall n \geq 1$ then $\langle a_n \rangle$ is called ***bounded below***
- If $\langle a_n \rangle$ is **bounded below AND bounded above** then $\langle a_n \rangle$ is called ***bounded* (*bdd.*)**
- If $\langle a_n \rangle$ is **not bounded below OR not bounded above** then $\langle a_n \rangle$ is called ***not bounded*** **(not *bdd.*)**



Examples

- $\{n^2\}_1^\infty = \langle 1, 4, 9, \dots, n^2, \dots \rangle$

$\because\ n^2 \geq 1 \quad \forall n \geq 1 \implies$ **bdd. below** by $\boldsymbol{m} = \mathbf{1}$

$\because\ n^2 \to \infty$ as $n \to \infty \implies$ **not bdd. above**

$\implies \{n^2\}_1^\infty$ **not bdd.**

- $\{1/n\}_1^\infty = \langle 1, 1/2, 1/3, \dots, 1/n, \dots \rangle$

$\because\ 1/n \leq 1 \quad \forall n \geq 1 \implies$ **bdd. above** by $\boldsymbol{M} = \mathbf{1}$

$\because\ 1/n \geq 0 \quad \forall n \geq 1 \implies$ **bdd. below** by $\boldsymbol{m} = \mathbf{0}$

$\implies \{1/n\}_1^\infty$ **bdd.**

Convergence

- A sequence $\langle a_n \rangle$ is called ***convergent* (*conv.*)** if $\exists \boldsymbol{L} \in \mathbb{R}$ s.t.

$$\boldsymbol{\lim_{n\to\infty} a_n = L}$$

- ***Definition:*** $\lim_{n\to\infty} a_n = L$ if and only if $\forall \boldsymbol{\epsilon} > \boldsymbol{0}\ \exists \boldsymbol{N} \in \mathbb{Z}_+$ *s.t.*

$$\boldsymbol{|a_n - L| < \epsilon \quad \forall n > N}$$

- A sequence $\langle a_n \rangle$ is called ***divergent* (*div.*)** if it has **no limit**

$$\lim_{n\to\infty} a_n = \pm\infty \quad \textbf{OR} \quad \lim_{n\to\infty} a_n \text{ is } \textbf{not unique}$$

Examples

- $\{1/n\}_1^{\infty} = \langle 1, 1/2, 1/3, \ldots, 1/n, \ldots \rangle$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} 1/n = 1/\infty = 0 \quad \Longrightarrow \textbf{has a limit } L = 0$$

$\therefore\ \{1/n\}_1^{\infty}$ is **conv.**

- $\langle (-1)^n \rangle = \langle -1, +1, -1, \ldots \rangle$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (-1)^n = \begin{Bmatrix} +1 & n \text{ is even} \\ -1 & n \text{ is odd} \end{Bmatrix} \Longrightarrow \textbf{has no limit}$$

$\therefore\ \{(-1)^n\}_1^{\infty}$ is **div.**

Exercises

- Show whether the following sequences are monotonic, bounded, convergent or not:

$$\{-1/n\}_1^\infty, \qquad \{1+(-1)^n\}_1^\infty, \qquad \{(-1)^n/n\}_1^\infty$$

$$\left\langle\left[\frac{n+1}{2}\right]\right\rangle, \qquad \left\langle\frac{n}{n+1}\right\rangle, \qquad \left\langle\sin\left(\frac{1}{n}\right)\right\rangle$$

- Give examples of sequences satisfying the following:
- Bounded and not monotonic.
- Increasing and convergent.
- Monotonic and divergent.

ترجمة مصطلحات

- Sequence متتابعة
- Function دالة
- Domain منطلق
- Positive integers الأعداد الصحيحة الموجبة
- Denote نرمز
- Monotonic رتيبة
- Increasing متزايدة
- Nonincreasing غير متزايدة
- Decreasing متناقصة
- Nondecreasing غير متناقصة
- Bounded مقيدة
- Bounded above مقيدة من الأعلى
- Bounded below نقيدة من الأدنى
- Convergent متقاربة
- Divergent متباعدة
- Even زوجي
- Odd فردي

Sequence

Part 2

$$\lim_{n\to\infty} a_n = L$$

Determinate forms

- Let $r > 0$, the following forms are determinate:
  $\frac{r}{0} = \infty, \infty \mp r = \infty, \infty \cdot r = \infty, \frac{\infty}{r} = \infty, \frac{r}{\infty} = 0, \infty^r = \infty$
- the following forms are also determinate:

$$\infty + \infty = \infty, \infty \cdot \infty = \infty, \infty^\infty = \infty, \frac{\infty}{0} = \infty, \frac{0}{\infty} = 0$$

- Let $r \geq 0$, then

$$\lim_{n \to \infty} r^n = r^\infty = \begin{Bmatrix} \infty & r > 1 \\ 1 & r = 1 \\ 0 & 0 \leq r < 1 \end{Bmatrix}$$

Indeterminate forms

Indeterminate forms: $\frac{0}{0}, \frac{\infty}{\infty}, 0 \cdot \infty, \infty - \infty, 0^0, \infty^0, 1^\infty$

$$\frac{0}{0} \stackrel{?}{=} 0,\ 1,\ \infty$$

$$\frac{\infty}{\infty} \stackrel{?}{=} \infty,\ 1,\ 0$$

$$0 \cdot \infty \stackrel{?}{=} 0,\ \infty$$

$$\infty^0 \stackrel{?}{=} \infty,\ 1$$

$$0^0 \stackrel{?}{=} 0,\ 1$$

$$\infty - \infty \stackrel{?}{=} \infty,\ 0,\ -\infty$$

(1) $\frac{0}{0}, \frac{\infty}{\infty}, 0 \cdot \infty$

Example: $\left\langle \frac{n^2+2n-1}{3n^3+n} \right\rangle \Rightarrow \lim_{n\to\infty} \frac{n^2+2n-1}{3n^3+n} = \frac{\infty^2+2\infty-1}{3\infty^3+\infty} = \frac{\infty}{\infty}$?

By cancelling the greatest power n^3

$\div n^3$

$$\lim_{n\to\infty} \frac{n^2+2n-1}{3n^3+n} = \lim_{n\to\infty} \frac{\frac{n^2}{n^3}+2\frac{n}{n^3}-\frac{1}{n^3}}{3\frac{n^3}{n^3}+\frac{n}{n^3}} = \lim_{n\to\infty} \frac{\frac{1}{n}+2\frac{1}{n^2}-\frac{1}{n^3}}{3+\frac{1}{n^2}}$$

$$= \frac{\frac{1}{\infty}+2\frac{1}{\infty^2}-\frac{1}{\infty^3}}{3+\frac{1}{\infty^2}} = \frac{0+0-0}{3+0} = 0 \quad \Rightarrow \text{Conv.}$$

(1) $\frac{0}{0}, \frac{\infty}{\infty}, 0 \cdot \infty$

Example: $\langle n \sin(\frac{1}{n}) \rangle \Rightarrow \lim_{n\to\infty} n \sin(\frac{1}{n}) = \infty \sin(\frac{1}{\infty}) = \infty \sin(0) = \infty \cdot 0$?

L'Hôpital's Rule: $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \frac{0}{0}$ or $\frac{\infty}{\infty} \Longrightarrow \lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{f'(n)}{g'(n)}$

$$\lim_{n\to\infty} n \sin\left(\frac{1}{n}\right) = \lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} = \frac{\sin\left(\frac{1}{\infty}\right)}{\frac{1}{\infty}} = \frac{\sin(0)}{0} = \frac{0}{0}$$?

L'Hôpital's Rule

d/dn

$$\lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} = \lim_{n\to\infty} \frac{\cos\left(\frac{1}{n}\right)\cdot\left(\frac{-1}{n^2}\right)}{\left(\frac{-1}{n^2}\right)} = \lim_{n\to\infty} \cos\left(\frac{1}{n}\right) = \cos\left(\frac{1}{\infty}\right) = \cos(0) = 1$$ ✓

$\Longrightarrow \langle n \sin(\frac{1}{n}) \rangle$ Conv.

(2) $\infty - \infty$

Example: $\langle n^3 - 2n \rangle \Rightarrow \lim_{n\to\infty} n^3 - 2n = \infty^3 - 2\infty = \infty - \infty$?

Factorizing: factoring out the greatest common

$$\lim_{n\to\infty} n^3 - 2n \overset{\text{Factorizing}}{=} \lim_{n\to\infty} n(n^2 - 2) = \infty\,(\infty^2 - 2)$$

$$= \infty \cdot \infty = \infty \checkmark$$

$$\Rightarrow \langle n^3 - 2n \rangle \text{ div.}$$

(2) $\infty - \infty$

Example: $\langle \sqrt{n+1} - \sqrt{n} \rangle \Rightarrow \lim_{n\to\infty} \sqrt{n+1} - \sqrt{n} = \sqrt{\infty+1} - \sqrt{\infty} = \infty - \infty$?

conjugate: $(a+b) \xleftrightarrow{\text{cojugate}} (a-b) \Longrightarrow (a+b)\cdot(a-b) = a^2 - b^2$

×÷ conjugate

$$\lim_{n\to\infty} \sqrt{n+1} - \sqrt{n} \times \frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} = \lim_{n\to\infty} \frac{\left(\sqrt{n+1}\right)^2 - \left(\sqrt{n}\right)^2}{\sqrt{n+1}+\sqrt{n}}$$

$$= \lim_{n\to\infty} \frac{(n+1)-n}{\sqrt{n+1}+\sqrt{n}} = \lim_{n\to\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}} = \frac{1}{\sqrt{\infty+1}+\sqrt{\infty}}$$

$$= \frac{1}{\infty+\infty} = \frac{1}{\infty} = 0$$ ✓ $\Longrightarrow \langle \sqrt{n+1} - \sqrt{n} \rangle$ Conv.

(3) $0^0, \infty^0, 1^\infty$

Example: $\left\langle \left(\frac{n+1}{n}\right)^n \right\rangle \Rightarrow \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = \left(1+\frac{1}{\infty}\right)^\infty = 1^\infty$?

Logarithm: $\ln(a^b) = b\ln(a)$, $e^{\ln(a)} = a$

Let $a_n = \left(1+\frac{1}{n}\right) \overset{ln}{\Longrightarrow} \ln(a_n) = \ln\left(1+\frac{1}{n}\right)^n = n\ln\left(1+\frac{1}{n}\right)$

$\lim_{n\to\infty} \ln(a_n) = \lim_{n\to\infty} n\ln\left(1+\frac{1}{n}\right) = \infty \ln\left(1+\frac{1}{\infty}\right) = \infty \overset{?}{\cdot} 0$ L'Hôpital's Rule

$\lim_{n\to\infty} \ln(a_n) = \lim_{n\to\infty} \frac{\ln\left(1+\frac{1}{n}\right)}{\frac{1}{n}} = \lim_{n\to\infty} \frac{1/\left(1+\frac{1}{n}\right)\cdot\frac{-1}{n^2}}{\frac{-1}{n^2}} = 1/\left(1+\frac{1}{\infty}\right) = 1$

$\Longrightarrow \lim_{n\to\infty}(a_n) = e^{\lim_{n\to\infty}\ln(a_n)} = e^1 = e$ ✓ $\Longrightarrow \left\langle \left(\frac{n+1}{n}\right)^n \right\rangle \to e$ Conv.

Exercises

- Show whether the following sequences are convergent or divergent:

$$\left\{\left(-\frac{1}{2}\right)^n\right\}_{n=1}^{\infty} \qquad \left\{\frac{2n^4 - n^3 + n}{n^4 + n - 1}\right\}_{n=1}^{\infty} \qquad \left\langle\frac{\sin(n)}{n}\right\rangle \qquad \left\langle\frac{\ln(n)}{n^2}\right\rangle$$

$$\langle e^n - 2^n\rangle \qquad \left\{\left(\frac{n}{n+1}\right)^n\right\}_{n=1}^{\infty} \qquad \left\langle\frac{n^2}{e^n}\right\rangle \qquad \langle\sqrt[n]{n}\,\rangle$$

ترجمة مصطلحات

Determinate forms	صيغ معيّنة (محددة الاجابة)
Indeterminate forms	صيغ غير معيّنة (غير محددة الاجابة)
Cancelling	حذف، اختزال
Greatest power	أعلى درجة، أعظم أس
L'Hôpital's Rule	قاعدة لوبتال
Factorizing	تحليل الى عوامل
greatest common factor	العامل المشترك الأكبر
Conjugate	مُرافق الحدانية
Logarithm	لوغاريتم

Example 1. Show that whether the following sequences (seqs.) are bounded, monotonic, convergent or not?

A. $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$

Solution: We have $\langle\frac{1}{n}\rangle = \langle 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots\rangle$,

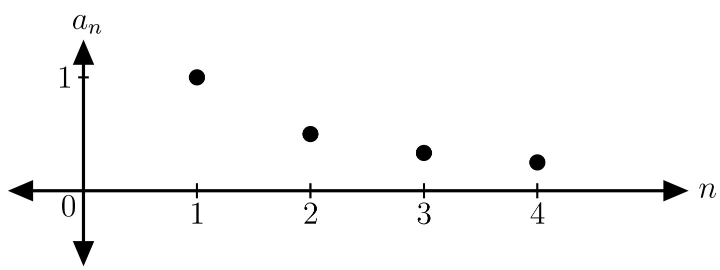


Figure 1: The points represent the terms of the sequence $\langle\frac{1}{n}\rangle$

then,

1. $\because 1 > \frac{1}{2} > \frac{1}{3} > \frac{1}{4} > \cdots$ i.e. $\frac{1}{n} > \frac{1}{n+1}$, $\forall n \geq 1$.
   So, $\langle\frac{1}{n}\rangle$ is decreasing $\Rightarrow$ **monotonic** (See figure (1)).

2. $\left.\begin{cases} \because \frac{1}{n} \leq 1, \ \forall n \geq 1 \Rightarrow \text{bdd. above by } M = 1, \\ \because \frac{1}{n} \geq 0, \ \forall n \geq 1 \Rightarrow \text{bdd. below by } m = 0. \end{cases}\right\} \Rightarrow \langle\frac{1}{n}\rangle$ is **bounded**.

3. $\langle\frac{1}{n}\rangle$ is **convergent** because

$$\lim_{n\to\infty} \frac{1}{n} = \frac{1}{\infty} = 0.$$

B. $\left\{\frac{-1}{n}\right\}_{n=1}^{\infty}$

Solution: We have $\langle\frac{-1}{n}\rangle = \langle -1, \frac{-1}{2}, \frac{-1}{3}, \frac{-1}{4}, \cdots\rangle$,

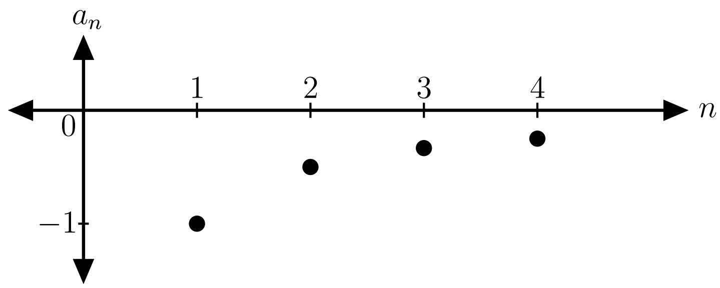


Figure 2: The points represent the terms of the sequence $\langle\frac{-1}{n}\rangle$

then,

1. $\because -1 < \frac{-1}{2} < \frac{-1}{3} < \frac{-1}{4} < \cdots$ i.e. $\frac{-1}{n} < \frac{-1}{n+1}, \quad \forall n \geq 1.$

   So, $\langle\frac{-1}{n}\rangle$ is increasing $\Rightarrow$ **monotonic** (See figure (2)).

2. $\left\{\begin{array}{r} \because \frac{-1}{n} \leq 0, \forall n \geq 1 \Rightarrow \text{bdd. above by } M = 0, \\ \because \frac{-1}{n} \geq -1, \forall n \geq 1 \Rightarrow \text{bdd. below by } m = -1. \end{array}\right\} \Rightarrow \langle\frac{-1}{n}\rangle$ is **bounded**.

3. $\langle\frac{-1}{n}\rangle$ is **convergent** because

$$\lim_{n\to\infty} \frac{-1}{n} = \frac{-1}{\infty} = 0.$$

C. $\{1 + (-1)^n\}_{n=1}^{\infty}$

Solution: We have $\langle 1 + (-1)^n\rangle = \langle 0, 2, 0, 2, \cdots\rangle$,

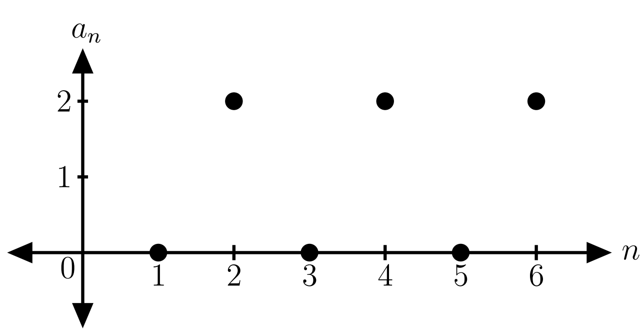


Figure 3: The points represent the terms of the sequence $\langle 1 + (-1)^n\rangle$

then,

1. $\because 0 < 2 > 0 < 2 > \cdots$

   $\Rightarrow \langle 1 + (-1)^n \rangle$ is **not monotonic** (See figure (3)).

2. $\left\{ \begin{array}{l} \because 1 + (-1)^n \leq 2, \forall n \geq 1 \Rightarrow \text{bdd. above by } M = 2, \\ \because 1 + (-1)^n \geq 0, \forall n \geq 1 \Rightarrow \text{bdd. below by } m = 0. \end{array} \right\} \Rightarrow \langle 1+(-1)^n \rangle$ is **bounded**.

3. $\because$ the seq. oscillates between two values (0 and 2) because

$$\lim_{n\to\infty} 1 + (-1)^n = \begin{cases} 2, & \text{if n is even} \\ 0, & \text{if n is odd} \end{cases},$$

   $\Rightarrow$ the seq. $\langle 1 + (-1)^n \rangle$ is **divergent**.

D. $\left\{ \frac{(-1)^n}{n} \right\}_{n=1}^{\infty}$

Solution: We have $\langle \frac{(-1)^n}{n} \rangle = \langle -1, \frac{1}{2}, \frac{-1}{3}, \frac{1}{4}, \cdots \rangle$,

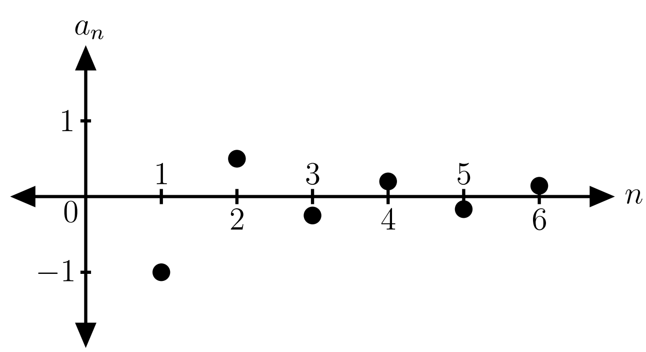


Figure 4: The points represent the terms of the sequence $\langle \frac{(-1)^n}{n} \rangle$

then,

1. $\because -1 < \frac{1}{2} > \frac{-1}{3} < \frac{1}{4} > \cdots$

   $\Rightarrow \langle \frac{(-1)^n}{n} \rangle$ is **not monotonic** (See figure (4)).

2. $\left\{ \begin{array}{l} \because \frac{(-1)^n}{n} \leq +1, \ \forall n \geq 1 \Rightarrow \text{bdd. above by } M = +1, \\ \because \frac{(-1)^n}{n} \geq -1, \ \forall n \geq 1 \Rightarrow \text{bdd. below by } m = -1. \end{array} \right\} \Rightarrow \langle \frac{(-1)^n}{n} \rangle$ is **bounded**.

3. $\langle \frac{(-1)^n}{n} \rangle$ is **convergent** because

$$\lim_{n\to\infty} \frac{(-1)^n}{n} = \left\{ \begin{array}{ll} \frac{1}{\infty} = +0, & \text{if n is even} \\ \frac{-1}{\infty} = -0, & \text{if n is odd} \end{array} \right\} = 0.$$

E. $\left\{ [\frac{n+1}{2}] \right\}_{n=1}^{\infty}$

Solution: We have $\langle [\frac{n+1}{2}] \rangle = \langle 1, 1, 2, 2, 3, 3, \cdots \rangle$,

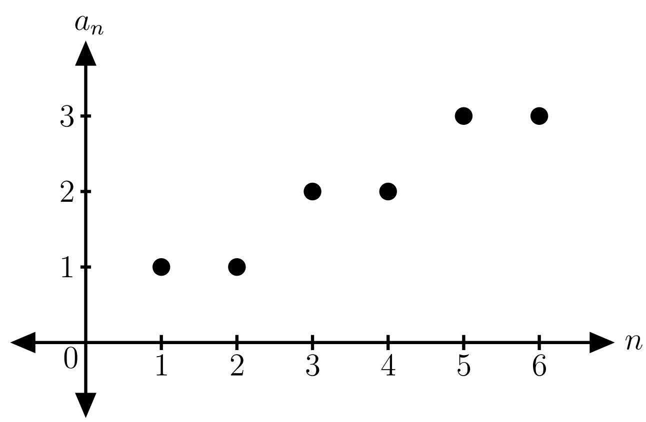


Figure 5: The points represent the terms of the sequence $\langle [\frac{n+1}{2}] \rangle$

then,

1. $\because 1 \leq 1 \leq 2 \leq 2 \leq 3 \leq 3 \cdots$ i.e. $[\frac{n+1}{2}] \leq [\frac{n+2}{2}], \quad \forall n \geq 1.$

   So, $\langle [\frac{n+1}{2}] \rangle$ is non decreasing $\Rightarrow$ **monotonic** (See figure (5)).

2. $\left\{ \begin{array}{l} \because [\frac{n+1}{2}] \to \infty \text{ as } n \to \infty, \Rightarrow \text{not bdd. above }, \\ \because [\frac{n+1}{2}] \geq 1, \ \forall n \geq 1 \Rightarrow \text{bdd. below by } m = 1. \end{array} \right\} \Rightarrow \langle [\frac{n+1}{2}] \rangle$ is **not bounded**.

3. $\langle [\frac{n+1}{2}] \rangle$ is **divergent** because

$$\lim_{n\to\infty} \left[\frac{n+1}{2} \right] = \left[\frac{\infty + 1}{2} \right] = [\infty] = \infty.$$

F. $\left\{\frac{n}{n+1}\right\}_{n=1}^{\infty}$

Solution: We have $\langle\frac{n}{n+1}\rangle = \langle\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \cdots\rangle$,

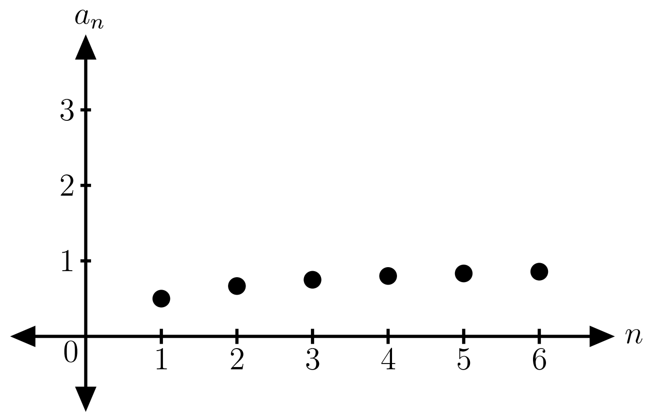


Figure 6: The points represent the terms of the sequence $\langle\frac{n}{n+1}\rangle$

then,

1. $\because \frac{1}{2} < \frac{2}{3} < \frac{3}{4} < \frac{4}{5} < \frac{5}{6} < \frac{6}{7} \cdots$ i.e. $\frac{n}{n+1} < \frac{n}{n+2}, \quad \forall n \geq 1$.
   So, $\langle\frac{n}{n+1}\rangle$ is increasing $\Rightarrow$ **monotonic** (See figure (6)).

2. $\left\{\begin{array}{l} \because \frac{n}{n+1} \leq 1, \ \forall n \geq 1 \ \Rightarrow \text{ bdd. above by } M = 1, \\ \because \frac{n}{n+1} \geq 0, \ \forall n \geq 1 \Rightarrow \text{bdd. below by } m = 0. \end{array}\right\} \Rightarrow \langle\frac{n}{n+1}\rangle$ is **bounded**.

3. $\langle\frac{n}{n+1}\rangle$ is **convergent** because

$$\lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1} = 1, \quad \text{(L'Hôpital's rule)}.$$

G. $\{\sin(\frac{1}{n})\}_{n=1}^{\infty}$

Solution: We have $\langle\sin(\frac{1}{n})\rangle = \langle 0.01745, 0.00872, 0.00581, 0.00436, 0.00349, \cdots\rangle$,

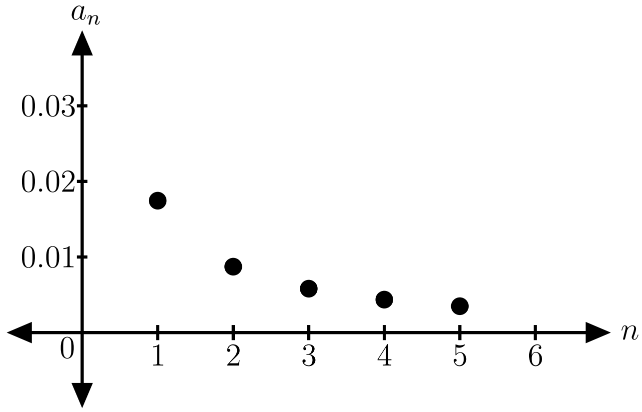


Figure 7: The points represent the terms of the sequence $\langle\sin(\frac{1}{n})\rangle$

then,

1. $\because 0.01745 \geq 0.00872 \geq 0.00581 \geq 0.00436 \geq 0.00349 \cdots$

   i.e. $\sin(\frac{1}{n}) \geq \sin(\frac{1}{n+1}), \quad \forall n \geq 1.$

   So, $\langle\sin(\frac{1}{n})\rangle$ is decreasing $\Rightarrow$ **monotonic** (See figure (7)).

2. $\left\{\begin{array}{l} \because \sin(\frac{1}{n}) \leq 1, \forall n \geq 1 \ \Rightarrow \text{ bdd. above by } \ M = 1, \\ \because \sin(\frac{1}{n}) \geq 0, \forall n \geq 1 \Rightarrow \text{bdd. below by } \ m = 0. \end{array}\right\} \Rightarrow \langle\sin(\frac{1}{n})\rangle$ is **bounded**.

3. $\langle\sin(\frac{1}{n})\rangle$ is **convergent** because

$$\lim_{n\to\infty} \sin(\frac{1}{n}) = \sin(\frac{1}{\infty}) = 0.$$

Example 2. Give examples of sequences satisfying the following:

A. Bounded and not monotonic.

Solution: The seq. is $\{(-1)^n\}_{n=1}^{\infty} = \langle -1, 1, -1, 1, \cdots \rangle$,

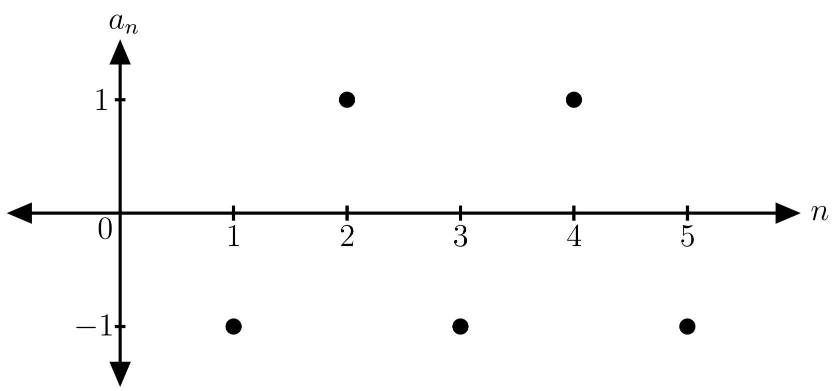


Figure 8: The points represent the terms of the sequence $\langle (-1)^n \rangle$

then,

1. **Bounded:**

$$\left\{ \begin{array}{c} \because (-1)^n \leq 1, \ \ \forall n \geq 1 \Rightarrow \text{bdd. above by } \ M = 1, \\ \because (-1)^n \geq -1, \ \ \forall n \geq 1 \Rightarrow \text{bdd. below by } \ m = -1. \end{array} \right\}$$

$\Rightarrow \ \langle (-1)^n \rangle$ is **bounded**.

2. **Not monotonic:**
$\because -1 < 1 > -1 < 1 > \cdots$
$\Rightarrow \langle (-1)^n \rangle$ is **not monotonic** (See figure (8)).

B. Increasing and convergent.

Solution: The seq. is $\left\{1-\frac{1}{n}\right\}_{n=1}^{\infty}=\langle 0,\frac{1}{2},\frac{2}{3},\frac{3}{4},\cdots\rangle$,

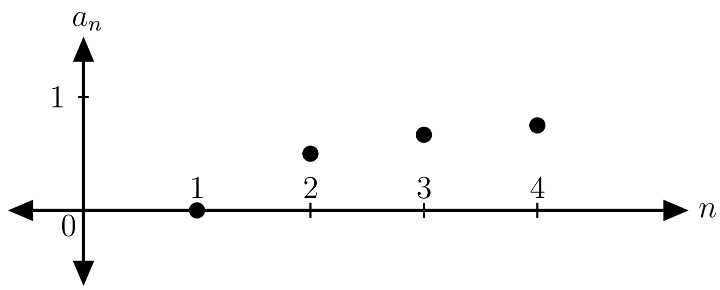


Figure 9: The points represent the terms of the sequence $\left\{1-\frac{1}{n}\right\}_{n=1}^{\infty}$

then,

1. **Increasing:**
   $\because 0<\frac{1}{2}<\frac{2}{3}<\frac{3}{4}<\cdots$ i.e. $1-\frac{1}{n}<1-\frac{1}{n+1}, \quad \forall n\geq 1.$
   $\Rightarrow \langle 1-\frac{1}{n}\rangle$ is **increasing** (See figure (9)).

2. **Convergent:**
   $\langle 1-\frac{1}{n}\rangle$ is **convergent** because

$$\lim_{n\to\infty} 1-\frac{1}{n}=1-\frac{1}{\infty}=1.$$

C. Monotonic and divergent.

Solution: The seq. is $\{-\sqrt{n}\}_{n=1}^{\infty}=\langle -1,-\sqrt{2},-\sqrt{3},-2,-\sqrt{5},\cdots\rangle$,

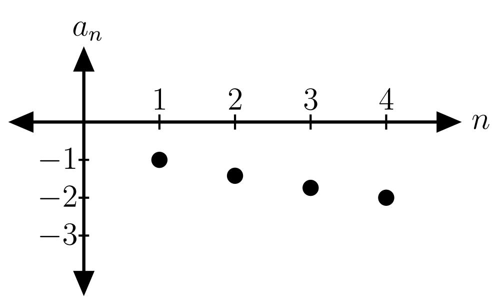


Figure 10: The points represent the terms of the sequence $\langle -\sqrt{n}\rangle$

then

1. **Monotonic:**
   $\because -1 > -\sqrt{2} > -\sqrt{3} > -2 > -\sqrt{5} > \cdots$
   i.e. $-\sqrt{n} > -\sqrt{n+1}, \quad \forall n \geq 1.$
   So, $\langle -\sqrt{n} \rangle$ is decreasing $\Rightarrow$ **monotonic** (See figure (10)).

2. **Divergent:**
   $\langle -\sqrt{n} \rangle$ is **divergent** becuase

$$\lim_{n\to\infty} -\sqrt{n} = -\infty.$$

Advanced calculus / Sequences / Lecture 4 / 24/12/2020

Dr. Junaid I. Mustafa.

Exercises: Show whether the following seqs. are convergent or divergent:

① $\left\{\left(-\frac{1}{2}\right)^n\right\}_{n=1}^{\infty}$

Solution:

$$\lim_{n\to\infty}\left(-\frac{1}{2}\right)^n = \lim_{n\to\infty}\left(\frac{(-1)^n}{(2)^n}\right) = \begin{cases} \frac{-1}{(2)^\infty}, & \text{if } n \text{ odd,} \\ \frac{+1}{(2)^\infty}, & \text{if } n \text{ even.} \end{cases}$$

$$= \begin{cases} \frac{-1}{\infty}, & \text{if } n \text{ odd} \\ \frac{+1}{\infty}, & \text{if } n \text{ even} \end{cases} = \mp 0 = \boxed{0}$$

$\Rightarrow \left\langle\left(-\frac{1}{2}\right)^n\right\rangle$ is convergent.

---

② $\left\{\frac{2n^4-n^3+n}{n^4+n-1}\right\}_{n=1}^{\infty}$

$$\lim_{n\to\infty}\frac{2n^4-n^3+n}{n^4+n-1} = \frac{\infty}{\infty}\ ?$$

Solution: By cancelling the greatest power (n^4).

$$\lim_{n\to\infty}\left(\frac{2n^4-n^3+n}{n^4+n-1}\right)^{\div n^4} = \lim_{n\to\infty}\frac{2\frac{n^4}{n^4}-\frac{n^3}{n^4}+\frac{n}{n^4}}{\frac{n^4}{n^4}+\frac{n}{n^4}-\frac{1}{n^4}}$$

$$\therefore \lim_{n\to\infty} \frac{2n^4-n^3+n}{n^4+n-1} = \lim_{n\to\infty} \frac{2-\frac{1}{n}+\frac{1}{n^3}}{1+\frac{1}{n^3}-\frac{1}{n^4}}$$

$$= \frac{2-\frac{1}{\infty}+\frac{1}{\infty}}{1+\frac{1}{\infty}-\frac{1}{\infty}} = \frac{2-0+0}{1+0-0} = \boxed{2}$$

$$\Rightarrow \left\langle \frac{2n^4-n^3+n}{n^4+n-1} \right\rangle \text{ is } \boxed{\text{convergent}}$$

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(3) $\left\{ \frac{\sin(n)}{n} \right\}_{n=1}^{\infty}$

Solution: $\lim_{n\to\infty} \frac{\sin(n)}{n} = \frac{\sin(\infty)}{\infty} = \frac{K}{\infty}, \quad -1 \leq K \leq 1$

$$\Rightarrow \left\langle \frac{\sin(n)}{n} \right\rangle \text{ is } \boxed{\text{convergent.}}$$

---

(4) $\left\{ \frac{Ln(n)}{n^2} \right\}_{n=1}^{\infty}$

$\lim_{n\to\infty} \frac{Ln(n)}{n^2} = \frac{Ln(\infty)}{\infty^2} = \frac{\infty}{\infty}$?

Solution: By using L'Hôpital's Rule.

$$\lim_{n\to\infty} \frac{Ln(n)}{n^2} \overset{d/dn}{=} \lim_{n\to\infty} \frac{\frac{1}{n}}{2n} = \lim_{n\to\infty} \frac{1}{2n^2} = \frac{1}{2\cdot\infty} = \boxed{0}$$

$$\therefore \left\langle \frac{Ln(n)}{n^2} \right\rangle \text{ is convergent.}$$

⑤ $\langle e^n - 2^n \rangle$

$\lim_{n\to\infty} (e^n - 2^n) = e^\infty - 2^\infty = \boxed{\infty - \infty}$?

Solution: By using factoring ..

$$\lim_{n\to\infty} (e^n - 2^n) = \lim_{n\to\infty} e^n\left(1 - \frac{2^n}{e^n}\right) = \lim_{n\to\infty} e^n\left(1 - \left(\frac{2}{e}\right)^n\right)$$

$$\because 0 < \frac{2}{e} < 1 \Rightarrow \lim_{n\to\infty} \left(\frac{2}{e}\right)^n = 0$$

then,

$$\lim_{n\to\infty} (e^n - 2^n) = e^\infty (1 - 0) = e^\infty = \boxed{\infty}$$

$\Rightarrow \langle e^n - 2^n \rangle$ is **divergent.**

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⑥ $\left\{\left(\frac{n}{n+1}\right)^n\right\}_{n=1}^{\infty}$

$$\lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n = \lim_{n\to\infty} \left(\frac{n+1-1}{n+1}\right)^n = \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right)^n = \left(1 - \frac{1}{\infty}\right)^\infty = \boxed{1^\infty} \ ?$$

Solution: By using (Logarithm): $\ln(a^b) = b\ln(a)$, $e^{\ln a} = a$

Let $a_n = \left(1 - \frac{1}{n+1}\right) \overset{\ln}{\Rightarrow} \ln(a_n) = \ln\left(1 - \frac{1}{n+1}\right)^n = n\ln\left(1 - \frac{1}{n+1}\right)$

$$\lim_{n\to\infty} \ln(a_n) = \lim_{n\to\infty} n\ln\left(1 - \frac{1}{n+1}\right) = \infty \cdot \ln\left(1 - \frac{1}{\infty}\right) = \boxed{\infty \cdot 0} \ ?$$

Now, By using L'Hôpital Rule.

$$\lim_{n\to\infty} Ln(a_n) = \lim_{n\to\infty} \frac{Ln\left(1-\frac{1}{n+1}\right)}{\frac{1}{n}} \overset{d/dn}{=} \lim_{n\to\infty} \frac{\frac{1}{1-1/n+1} \times \frac{+1}{(n+1)^2}}{-\frac{1}{n^2}}$$

$$\Rightarrow \lim_{n\to\infty} Ln(a_n) = \lim_{n\to\infty} \frac{\frac{1}{\frac{n}{n+1}} \times \frac{1}{(n+1)^2}}{-\frac{1}{n^2}} = \lim_{n\to\infty} \frac{-n^2(n+1)}{n(n+1)}$$

$$= -\lim_{n\to\infty} \left(\frac{n}{n+1}\right) \overset{\boxed{d/dn}}{=} -\lim_{n\to\infty} 1 = \boxed{-1}$$

$$\Rightarrow \lim_{n\to\infty}(a_n) = e^{\lim_{n\to\infty} Ln(a_n)} = \boxed{e^{-1}} = \boxed{\frac{1}{e}}$$

$\therefore \left\langle \left(\frac{n}{n+1}\right)^n \right\rangle$ is **convergent**.

---

⑦ $\left\langle \frac{n^2}{e^n} \right\rangle$ $\lim_{n\to\infty}\left(\frac{n^2}{e^n}\right) = \frac{\infty^2}{e^\infty} = \boxed{\frac{\infty}{\infty}}$?

Solution By using L'Hôpital Rule

$$\lim_{n\to\infty} \frac{2^n}{e^n} \overset{\boxed{d/dn}}{=} \lim_{n\to\infty} \frac{2n}{e^n} \overset{\boxed{d/dn}}{=} \lim_{n\to\infty} \frac{2}{e^n} = \frac{2}{e^\infty} = \frac{2}{\infty} = 0.$$

$\therefore \left\langle \frac{n^2}{e^n} \right\rangle$ is **convergent**.

---

⑧ $\langle \sqrt[n]{n} \rangle$ $\lim_{n\to\infty} \sqrt[n]{n} = \lim_{n\to\infty} n^{\frac{1}{n}} = \boxed{\infty^0}$?

Solution: By using Logarithem

Let $a_n = n^{1/n} \Rightarrow Ln(a_n) = Ln(n)^{1/n} = \frac{1}{n} Ln(n) = \frac{Ln(n)}{n}$

$$\lim_{n\to\infty} Ln(a_n) = \lim_{n\to\infty} \frac{Ln(n)}{n} \overset{\boxed{d/dn}}{=} \lim_{n\to\infty} \frac{1/n}{1} = \frac{1/\infty}{1} = \frac{1}{\infty} = [$$

$$\Rightarrow \lim_{n\to\infty}(a_n) = e^{\lim_{n\to\infty} Ln(a_n)} = e^0 = \boxed{1} \Rightarrow \langle \sqrt[n]{n} \rangle \text{ is } \boxed{\text{Convergent}}$$