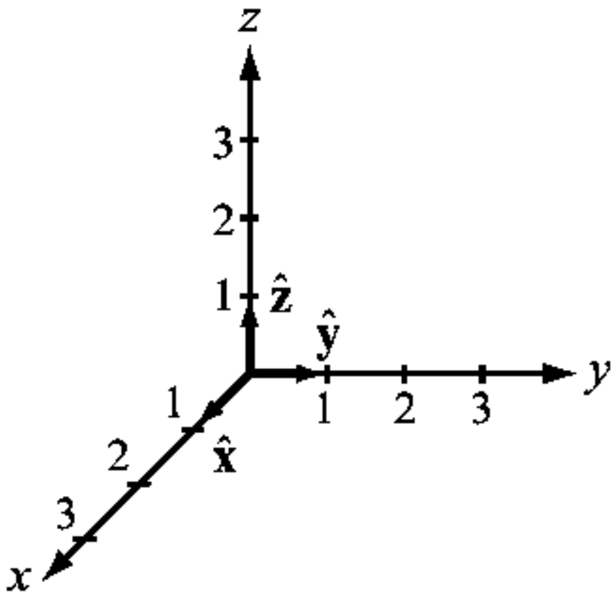
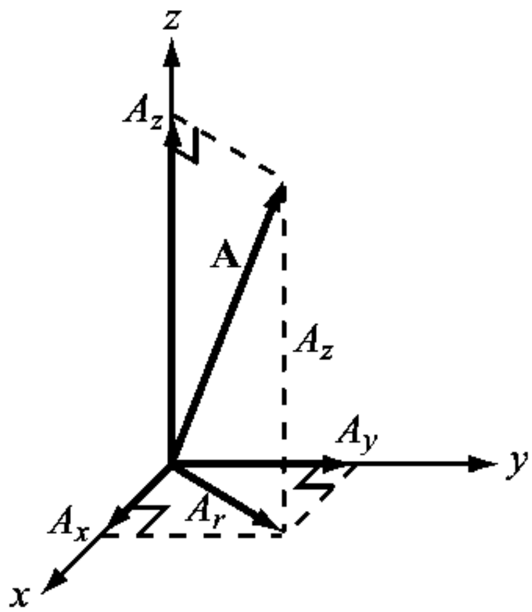
Vector Analysis

Basic Laws of Vector Algebra



(a) Base vectors



(b) Components of A

- The Cartesian coordinate system should be familiar to you from earlier math and physics courses
- The vector $\mathbf{A}$ is readily written in terms of the cartesian unit vectors $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$

$$\mathbf{A} = \hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z$$

- In linear algebra $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ are known as basis vectors, each having unit length, i.e., $|\hat{\mathbf{x}}|$ and mutually orthogonal
- Also, the length of $\mathbf{A}$ is

$$A = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

and the unit vector in the $\mathbf{A}$ direction is

$$\hat{\mathbf{a}} = \frac{\mathbf{A}}{A} = \frac{\hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z}{\sqrt{A_x^2 + A_y^2 + A_z^2}}$$

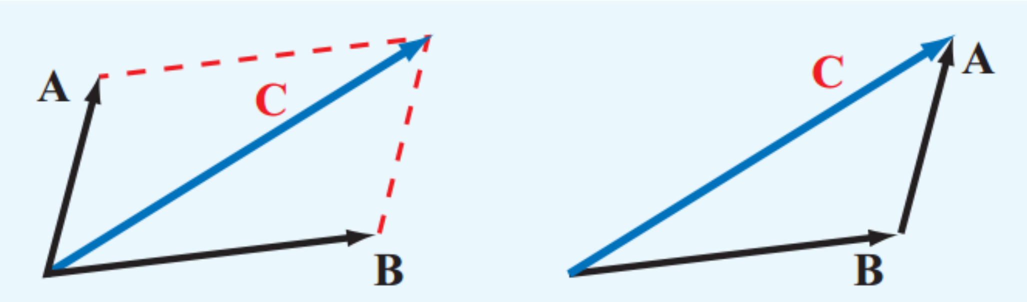
- Vectors $\mathbf{A}$ and $\mathbf{B}$ are equal if their components are equal, i.e., $A_x = B_x$, etc.

- Addition of vectors means that the individual components are added together, that is

$$\begin{aligned}\mathbf{C} &= \mathbf{A} + \mathbf{B} \\ &= \hat{\mathbf{x}}(A_x + B_x) + \hat{\mathbf{y}}(A_y + B_y) + \hat{\mathbf{z}}(A_z + B_z),\end{aligned}$$

thus $C_x = A_x + B_x$, etc.

- Visually you can utilize the *head-to-tail* or *parallelogram* rules

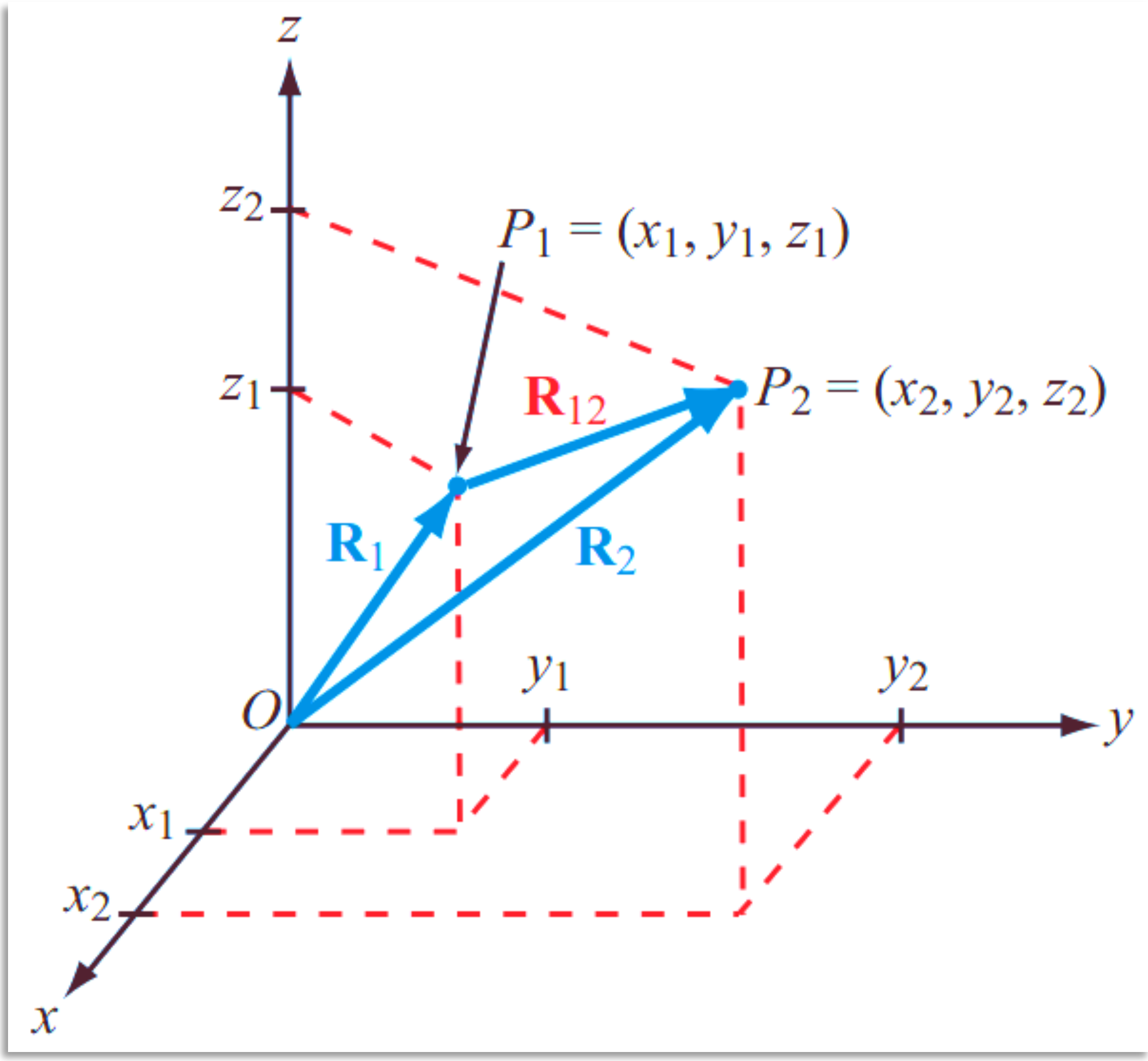


(a) Parallelogram rule (b) Head-to-tail rule

- Vector subtraction is similar

$$\begin{aligned}\mathbf{D} &= \mathbf{A} - \mathbf{B} \\ &= \hat{\mathbf{x}}(A_x - B_x) + \hat{\mathbf{y}}(A_y - B_y) + \hat{\mathbf{z}}(A_z - B_z),\end{aligned}$$

thus $D_x = A_x - B_x$, etc.



- Formally a position vector starts at the origin, so we use the notation

$$\mathbf{R}_i = \overrightarrow{OP_i} = \hat{\mathbf{x}}x_i + \hat{\mathbf{y}}y_i + \hat{\mathbf{z}}z_i$$

where x_i, y_i, and z_i correspond to the point $P_i = (x_i, y_i, z_i)$

- The scara distance between two points is just $d = |\mathbf{R}_{ij}|$

$$d = \sqrt{(x_j - x_i)^2 + (y_j - y_i)^2 + (z_j - z_i)^2}$$

- Vector multiplication takes the form
  - *scalar* × *vector*:

$$\mathbf{B} = k\mathbf{A} = \text{element-by-element multiply by } k$$

  - *scalar product* or *dot product*:

$$\mathbf{A} \cdot \mathbf{B} = AB\cos\theta_{AB}$$

  where θ_{AB} is the angle between the vectors (as in linear algebra)
  - **Note**: $A\cos\theta_{AB}$ is the component of **A** along **B** and $B\cos\theta_{AB}$ is the component of **B** along **A**
  - Also,

$$\mathbf{A} \cdot \mathbf{A} = |\mathbf{A}|^2 = A^2$$
$$A = |\mathbf{A}| = \sqrt{\mathbf{A} \cdot \mathbf{A}}$$

  - Using the inverse cosine

$$\theta_{AB} = \cos^{-1}\left[\frac{\mathbf{A} \cdot \mathbf{B}}{\sqrt{\mathbf{A} \cdot \mathbf{A}}\sqrt{\mathbf{B} \cdot \mathbf{B}}}\right]$$

  - Finally,

$$\mathbf{A} \cdot \mathbf{A} = A_x B_x + A_y B_y + A_z B_z$$

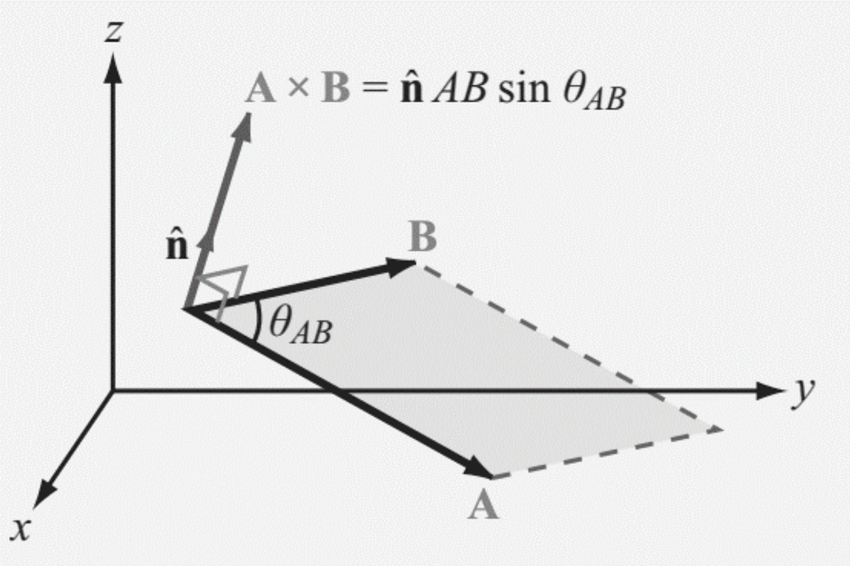
  - Commutative and Distributive

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}$$
$$\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C}$$

- *Vector product* or *cross product*:

$$\mathbf{A} \times \mathbf{B} = \hat{\mathbf{n}} AB \sin\theta_{AB}$$

where $\hat{\mathbf{n}}$ is a unit vector normal to the plane containing **A** and **B** (see picture below for details)



– The cross product is *anticommuntative*

$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$$

– The cross product is *distributive*

$$\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C}$$

– To calculate use the determinant formula

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \hat{\mathbf{x}}(A_y B_z - A_z B_y) + \hat{\mathbf{y}}(A_z B_x - A_x B_z) + \hat{\mathbf{z}}(A_x B_y - A_y B_x)$$

- Definition:

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

- Definition

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C})$$

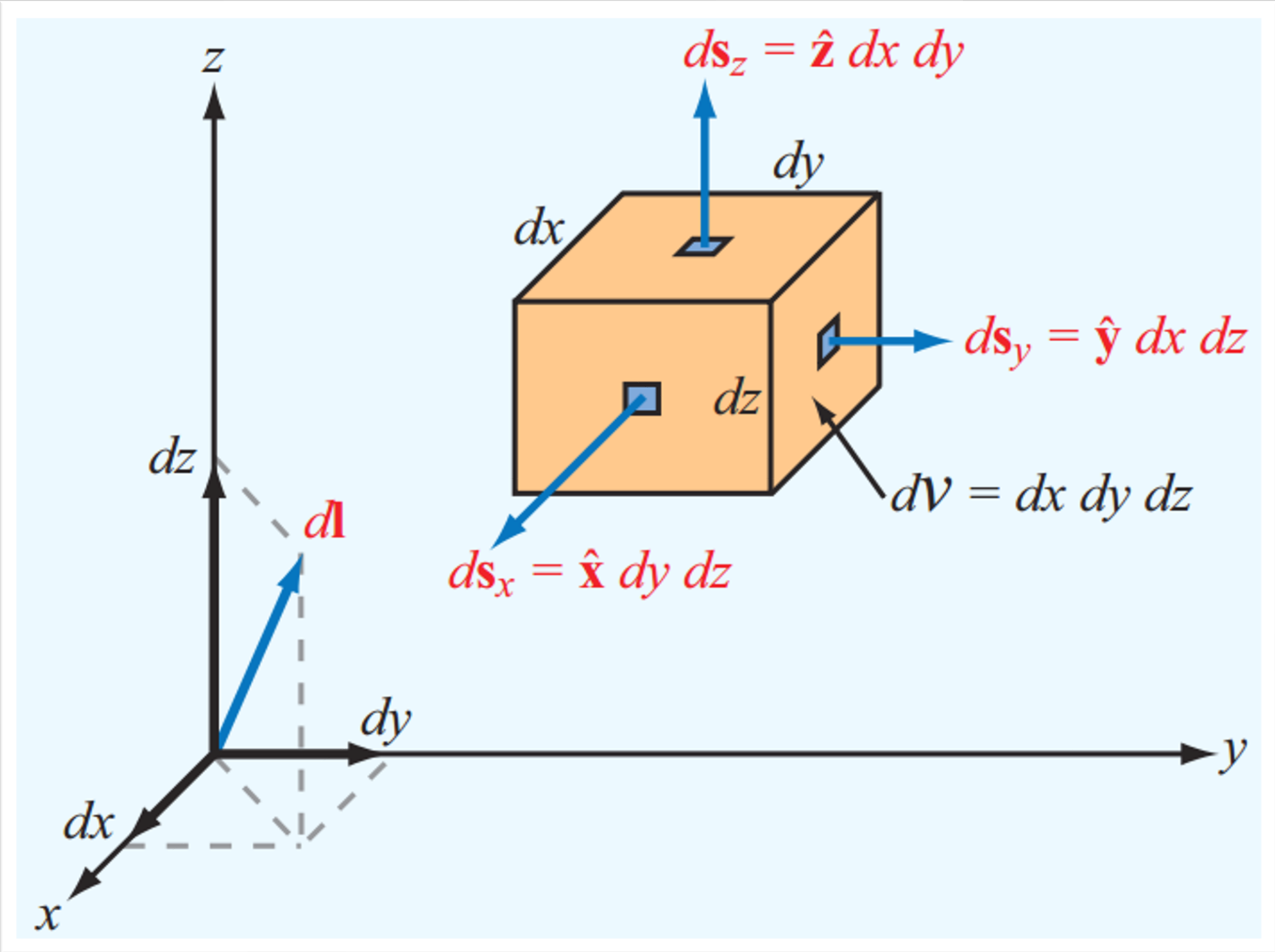
- **Note**:

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) \neq (\mathbf{A}\times\mathbf{B})\times\mathbf{C})$$

- It can however be shown that

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B}),$$

Differential length, area, and volume



1. Differential Length:

$$d\mathbf{l} = \hat{\mathbf{x}}\, dl_x + \hat{\mathbf{y}}\, dl_y + \hat{\mathbf{z}}\, dl_z = \hat{\mathbf{x}}\, dx + \hat{\mathbf{y}}\, dy + \hat{\mathbf{z}}\, dz$$

2. Differential Area: There are three different differential areas, d**s**, to consider:

$$\begin{aligned} d\mathbf{s}_x &= \hat{\mathbf{x}}\, dl_y\, dl_z = \hat{\mathbf{x}}\, dy\, dz \quad (y - z\text{-plane}) \\ d\mathbf{s}_y &= \hat{\mathbf{x}}\, dx\, dz \quad (x - z\text{-plane}) \\ d\mathbf{s}_z &= \hat{\mathbf{x}}\, dx\, dy \quad (x - y\text{-plane}) \end{aligned}$$

3.Differential Volume

$$d\mathcal{V} = dx\, dy\, dz$$

Gradient:

$$\vec{\nabla}\psi \equiv \hat{i}\frac{\partial\psi}{\partial x} + \hat{j}\frac{\partial\psi}{\partial y} + \hat{k}\frac{\partial\psi}{\partial z}$$

The gradient operates on a scalar field and produces a vector result that indicates the rate of spatial change of the field at a point and the direction of steepest increase from that point.

Del:

$$\vec{\nabla} \equiv \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

Del (nabla) represents a multipurpose differential operator that can operate on scalar or vector fields and produce scalar or vector results.

The gradient involves partial derivatives taken in three orthogonal direction. Gradient indicate how quickly the field is changing over space and the direction of the gradient indicates the direction in that the field is changing most quickly with distance

Therefore, although the gradient operates on a scalar field, the result of the gradient operation is a vector, with both magnitude and direction. Thus, if the scalar field represents terrain height, the magnitude of the gradient at any location tells you how steeply the ground is sloped at that location, and the direction of the gradient points *uphill* along the steepest slope.

The definition of the gradient of the scalar field ψ is

- From basic calculus it follows that

$$\nabla(U+V) = \nabla U + \nabla V$$
$$\nabla(UV) = U\,\nabla U + V\,\nabla U$$
$$\nabla V^n = nV^{n-1}\,\nabla V, \text{ for any } n$$

Example

- Consider the scalar function

$$V = x^2y + xy^2 + xz^2$$

- The gradient is simply

$$\nabla V = \hat{\mathbf{x}}(2xy + y^2 + z^2) + \hat{\mathbf{y}}(x^2 + 2xy) + \hat{\mathbf{z}}(2xz)$$

- At the point $P_1 = (1, -1, 2)$ the gradiant vector is

$$\nabla V(1, -1, 2) = \hat{\mathbf{x}}3 - \hat{\mathbf{y}} + \hat{\mathbf{z}}4$$

Divergence:

- The divergence of a vector field is in a sense complementary to the gradient:

Gradient of a scalar function $\Rightarrow$ Vector function

Divergence of a vector function $\Rightarrow$ Scalar function

$$\nabla \cdot \mathbf{E} = \text{div}\, \mathbf{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$$

- If $\nabla \cdot \mathbf{E} > 0$ a *source* if present, while $\nabla \cdot \mathbf{E} < 0$ means a *sink* is present, and $\nabla \cdot \mathbf{E} = 0$ means the field is *divergenceless*

Divergence Theorem

$$\int_{\mathcal{V}} \nabla \cdot \mathbf{E}\, d\mathcal{V} = \oint_S \mathbf{E} \cdot d\mathbf{s}$$

Example

- Consider $\mathbf{E} = \hat{\mathbf{x}}3x^2 + \hat{\mathbf{y}}2z + \hat{\mathbf{z}}x^2z$ at the point $P_1 = (2, -2, 0)$
- Using the definition in Cartesian coordinates

$$\nabla \cdot \mathbf{E} = \frac{\partial 3x^2}{\partial x} + \frac{\partial 2z}{\partial y} + \frac{\partial x^2 z}{\partial z}$$
$$= 6x + 0 + x^2 = x^2 + 6x$$

- Evaluating at $(2, -2, 0)$ we have

$$\nabla \cdot \mathbf{E}\bigg|_{(2,-2,0)} = 16$$

Example

- A volume charge density

$$\rho_v = 4\cos^2\theta \quad (\text{C/m}^3)$$

is present in a sphere of radius 2 cm

- To find the total charge in the sphere we integrate the charge density over the volume

$$\begin{aligned} Q &= \int_{\mathcal{V}} \rho_v \, d\mathcal{V} \\ &= \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{R=0}^{0.02} (4\cos^2\theta) R^2 \sin\theta \, dR \, d\theta \, d\phi \\ &= 4 \int_0^{2\pi} \int_0^{\pi} \left(\frac{R^3}{3}\right)\bigg|_0^{0.02} \sin\theta\cos^2\theta \, d\theta \, d\phi \\ &= \frac{32}{3} \times 10^{-6} \int_0^{2\pi} \left(-\frac{\cos^3\theta}{3}\right)\bigg|_0^{\pi} d\phi \\ &= \frac{64}{9} \times 10^{-6} \int_0^{2\pi} d\phi = \frac{128\pi}{9} \times 10^{-6} \\ &= 44.68 \quad (\mu\text{C}) \end{aligned}$$

Just a little calculus review, especially the anti-derivative of $\sin\theta\cos^2\theta$

Properties of the Gradient Operator:

- From basic calculus it follows that

$$\nabla(U+V) = \nabla U + \nabla V$$
$$\nabla(U\,V) = U\,\nabla U + V\,\nabla U$$
$$\nabla V^n = nV^{n-1}\,\nabla V, \text{ for any } n$$

Example

- Consider the scalar function

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- The gradient is simply

$$\nabla V = \hat{\mathbf{x}}(2xy + y^2 + z^2) + \hat{\mathbf{y}}(x^2 + 2xy) + \hat{\mathbf{z}}(2xz)$$

- At the point $P_1 = (1, -1, 2)$ the gradiant vector is

$$\nabla V(1, -1, 2) = \hat{\mathbf{x}}3 - \hat{\mathbf{y}} + \hat{\mathbf{z}}4$$

Divergence of a Vector Field:

- The divergence of a vector field is in a sense complementary to the gradient:

$$\text{Gradient of a scalar function} \Rightarrow \text{Vector function}$$
$$\text{Divergence of a vector function} \Rightarrow \text{Scalar function}$$

$$\nabla \cdot \mathbf{E} = \text{div } \mathbf{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$$

- If $\nabla \cdot \mathbf{E} > 0$ a *source* if present, while $\nabla \cdot \mathbf{E} < 0$ means a *sink* is present, and $\nabla \cdot \mathbf{E} = 0$ means the field is *divergenceless*

Divergence Theorem:

- Moving forward into Chapter 4 we will quickly bump into the *divergence theorem*, which states that

$$\int_{\mathcal{V}} \nabla \cdot \mathbf{E}\, d\mathcal{V} = \oint_S \mathbf{E} \cdot d\mathbf{s}$$

Example

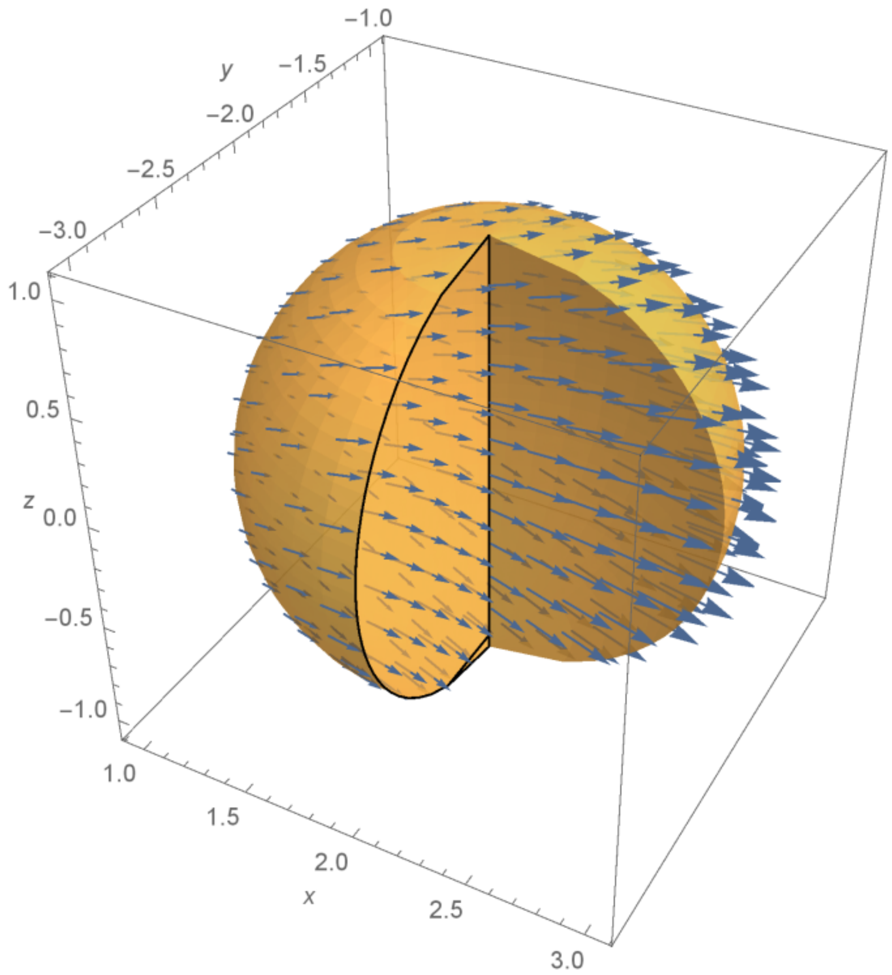
- Consider $\mathbf{E} = \hat{\mathbf{x}}3x^2 + \hat{\mathbf{y}}2z + \hat{\mathbf{z}}x^2z$ at the point $P_1 = (2, -2, 0)$
- Using the definition in Cartesian coordinates

$$\begin{aligned}\nabla \cdot \mathbf{E} &= \frac{\partial 3x^2}{\partial x} + \frac{\partial 2z}{\partial y} + \frac{\partial x^2 z}{\partial z} \\ &= 6x + 0 + x^2 = x^2 + 6x\end{aligned}$$

- Evaluating at $(2, -2, 0)$ we have

$$\nabla \cdot \mathbf{E}\Big|_{(2,-2,0)} = 16$$

- The positive diverge at $(2, -2, 0)$ can be seen in a 3D vector slice plot from Mathematica



3D vector field plot from Mathematica with a cut-sphere centered at (2, -2,0/; the positive divergence is clear.

Example

- Working a diverge calculation in cylindical or spherical requires the formulas inside the back cover of the text
- For the problem at hand we have

$$\mathbf{E} = \hat{\mathbf{R}}(a^3 \cos\theta/R^2) - \hat{\boldsymbol{\theta}}(a^3 \sin\theta/R^2),$$

  which is in spherical coordinates

- Find the divergence at $P_2 = (a/2, 0, \pi)$

$$\nabla \cdot \mathbf{E} = \frac{1}{R^2}\frac{\partial}{\partial R}(R^2 E_R) + \frac{1}{R\sin\theta}\frac{\partial}{\partial\theta}(E_\theta \sin\theta) + \frac{1}{R\sin\theta}\frac{\partial E_\phi}{\partial\phi}$$

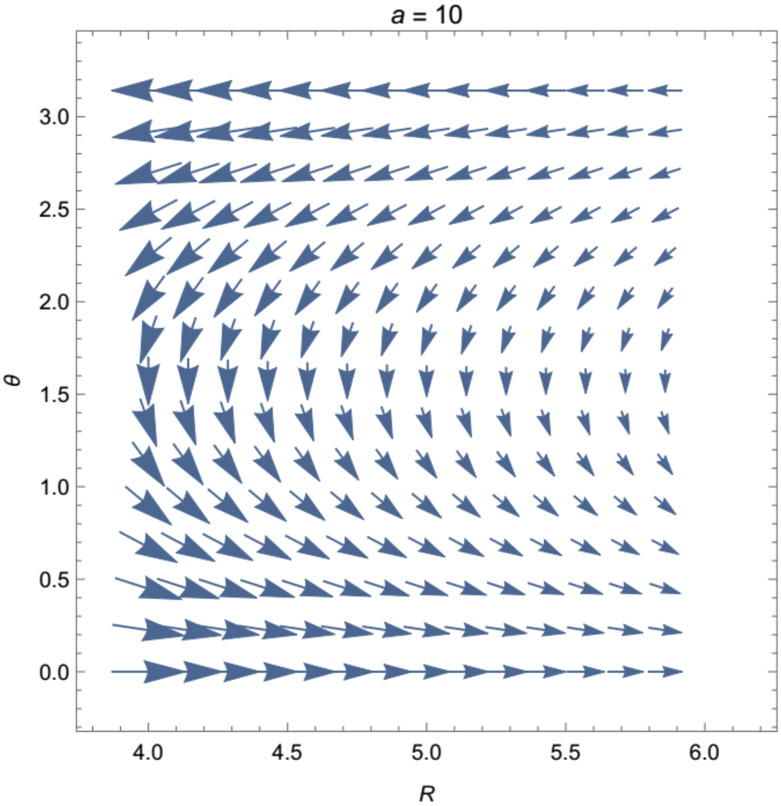
$$= \frac{1}{R^2}\frac{\partial}{\partial R}(a^3\cos\theta) + \frac{1}{R\sin\theta}\frac{\partial}{\partial\theta}\left(-\frac{a^3\sin^2\theta}{R^2}\right)$$

$$= -\frac{2a^3\cos\theta}{R^3}$$

- At the point $(a/2, 0, \pi)$ we have

$$\nabla \cdot \mathbf{E}\Big|_{(a/2,0,\pi)} = -16$$

- Since the divergence is negative at this point, we conclude that a field sink is present



2D vector field plot for $a = 10$ in just the R and θ axes making the negative divergence at $(5, 0, \pi)$ clear.

Example

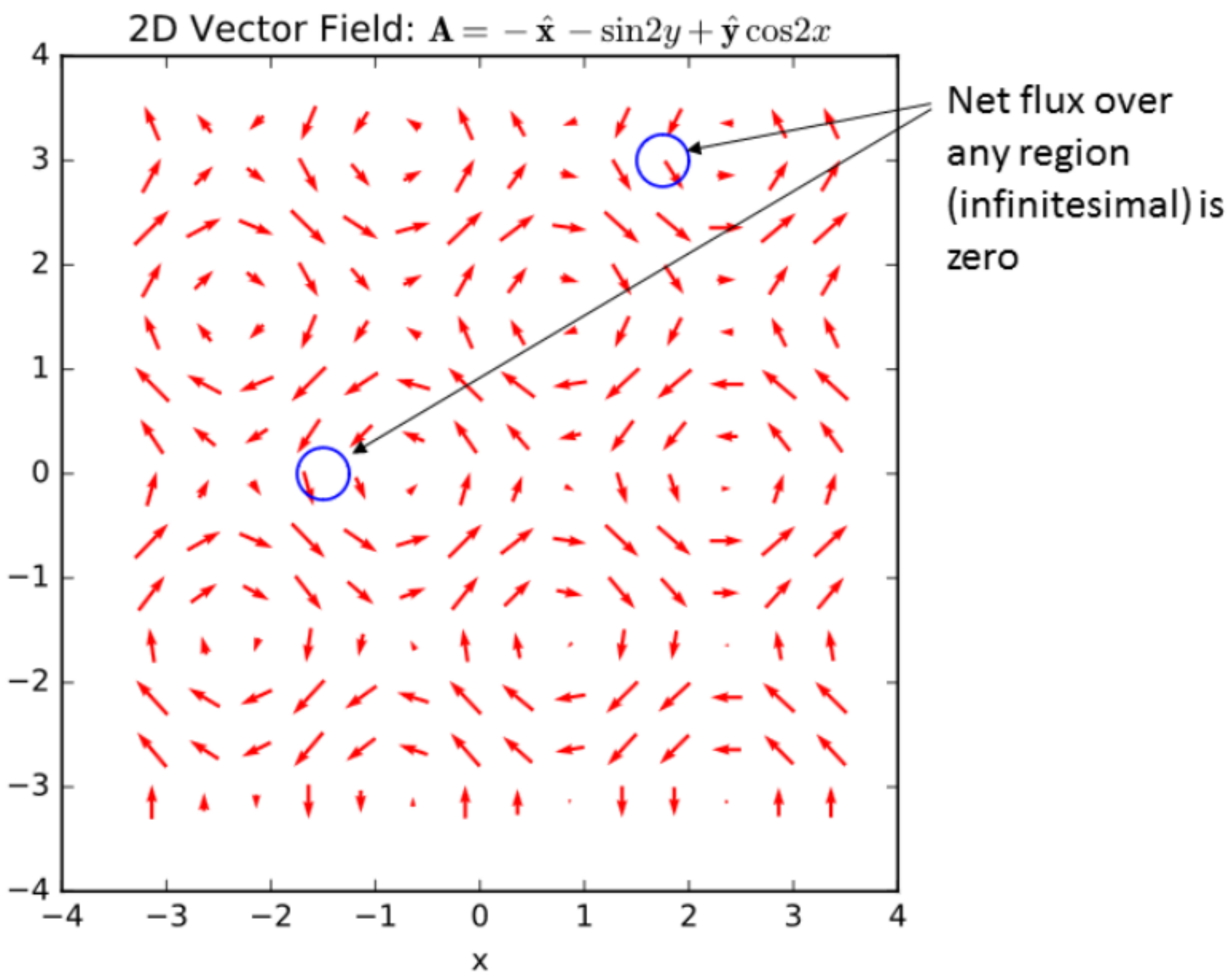
- Each of the following vector fields is displayed below in the form of a vector representation. Determine $\nabla \cdot \mathbf{A}$ analytically and then compare the results with your expectations on the basis of the displayed pattern.

$\mathbf{A} = -\hat{\mathbf{x}} \sin 2y + \hat{\mathbf{y}} \cos 2x$, for $-\pi \le x, y \le \pi$.

- The divergence calculation in Cartesian coordinates is just

$$\nabla \cdot \mathbf{E} = \frac{\partial}{\partial x}\big(- \sin 2y\big) + \frac{\partial}{\partial y}\big(\cos 2x\big) = 0$$

- The fact that $\nabla \cdot \mathbf{E} = 0$ means that the arrow/quiver plot should show everywhere that the net input and output of flux over \textit{any} 2D region is zero
- The quiver plot is reproduced below using Python (Mathematica or MATLAB also work):



Curl of a Vector Field:

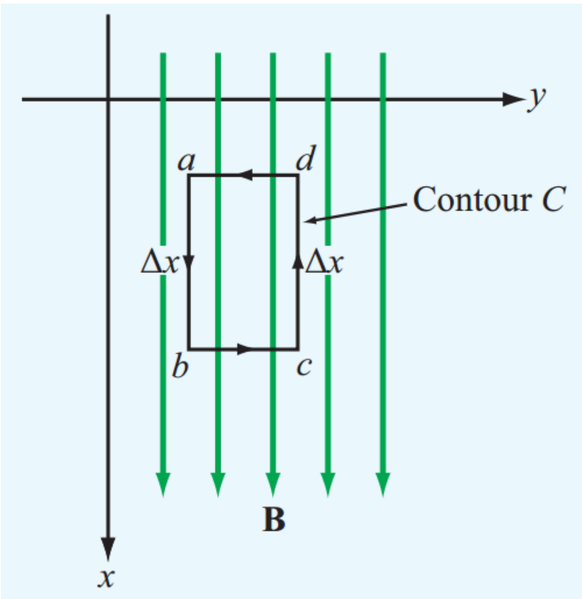
- The Curl describes the rotation of a 3D field, in an infinitesimal sense
- A field **B** has *circulation* if the line integral

$$\text{Circulation} = \oint_C \mathbf{B} \cdot d\mathbf{l} \neq 0$$

- For the case of a uniform field, e.g., $\mathbf{B} = \hat{\mathbf{x}}B_0$, forming a line integral around a closed rectangular contour in the $x - y$ plane yields zero, i.e.,

$$\begin{aligned}\text{Circulation} &= \int_a^b \hat{\mathbf{x}}B_0 \cdot \hat{\mathbf{x}}\,dx + \int_b^c \hat{\mathbf{x}}B_0 \cdot \hat{\mathbf{y}}\,dy \\ &\quad + \int_c^d \hat{\mathbf{x}}B_0 \cdot \hat{\mathbf{x}}\,dx + \int_d^a \hat{\mathbf{x}}B_0 \cdot \hat{\mathbf{y}}\,dy \\ &= B_0\,\Delta x - B_0\,\Delta x = 0\end{aligned}$$

- Futhermore, a small fictitious *paddle wheel* placed in the uniform field will not rotate, no matter the orientation of the wheel rotation axis



A uniform field, $\mathbf{B} = \hat{\mathbf{x}}B_0$ with circulation over C zero.

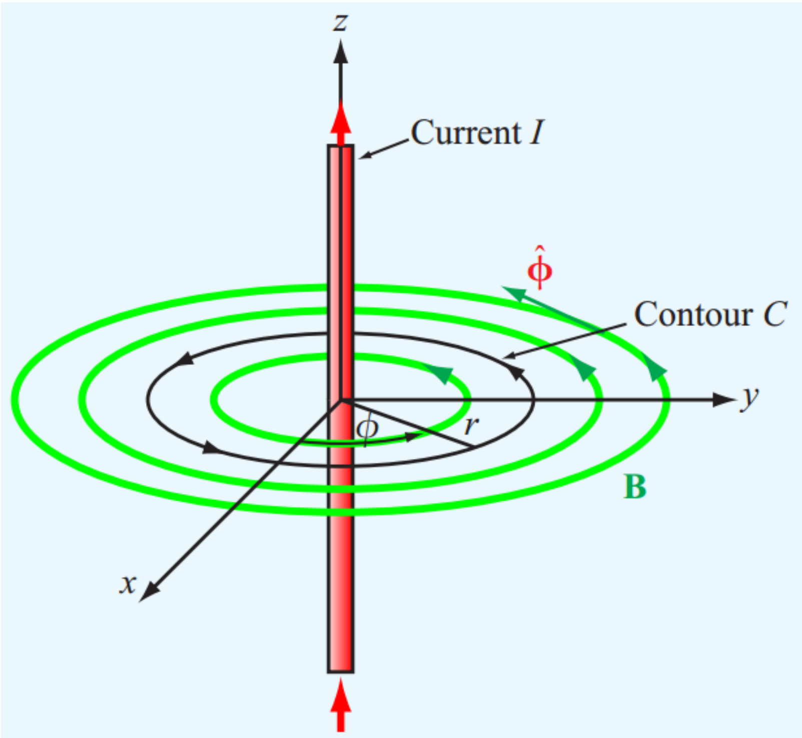
- Consider the azimuthal field of a wire carrying current I along the z-axis

- The magnetic flux in the$x - y$ plane follows $\hat{\boldsymbol{\phi}}$ with strength $\mu_0 I/(2\pi r)$

- To compute the circulation we consider differential length $d\mathbf{l} = \hat{\boldsymbol{\phi}} r\, d\phi$ and determine the circulation to be

$$\text{Circulation} = \int_0^{2\pi} \hat{\boldsymbol{\phi}} \frac{\mu_0 I}{2\pi r} \cdot \hat{\boldsymbol{\phi}} r\, d\phi = \mu_0 I$$

- Clearly a paddle wheel placed in this field will rotate!



An azimuthal field, $\mathbf{B} = \hat{\boldsymbol{\phi}} \mu_0 I/(2\pi r)$ with circulation around the z-axis.

we cab defined curl as:

$$\nabla \times \mathbf{B} = \text{curl}\,\mathbf{B} = \lim_{\Delta s \to 0} \frac{1}{\Delta s} \left[\hat{\mathbf{n}} \oint_C \mathbf{B} \cdot d\mathbf{l} \right]_{\max}$$

- **Note**: The contour C is oriented to given the maximum circulation; position the paddle wheel so it spins the fastest

- Since $\nabla \times \mathbf{B}$ is a vector, its direction is $\hat{\mathbf{n}}$, the unit normal of surface Δs (use the right-hand rule with the fingers curling in the direction of C and the thumb pointing along $\hat{\mathbf{n}}$)

- In rectangular coordinates we compute the curl via

$$\nabla \times \mathbf{B} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ B_x & B_y & B_z \end{vmatrix}$$

Stoke's Theorem

- *Stoke's theorem* converts a surface integral of the curl to a line integral of a vector along a contour C bounding surface S

$$\int_S (\nabla \times \mathbf{B}) \cdot d\mathbf{s} = \oint_C \mathbf{B} \cdot d\mathbf{l}$$

Laplacian Operator:

- The *Laplacian operator* shows up in a number of contexts
- The text mentions the divergence of the gradiant, $(\nabla \cdot (\nabla V))$ as one possibility
- The result is known as *del square*

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

Laplace's equation, $\nabla^2 V = 0$, arises when determining the electrostatic potential in 1D, 2D, and 3D problems

- **Vector identities involving curl**

$$(1) \quad \nabla \times (\mathbf{A} + \mathbf{B}) = \nabla \times \mathbf{A} + \nabla \times \mathbf{B}$$

$$(2) \quad \nabla \cdot (\nabla \times \mathbf{A}) = 0 \quad \text{for any vector } \mathbf{A}$$

$$(3) \quad \nabla \times (\nabla V) = 0 \quad \text{for any scalar function V}$$