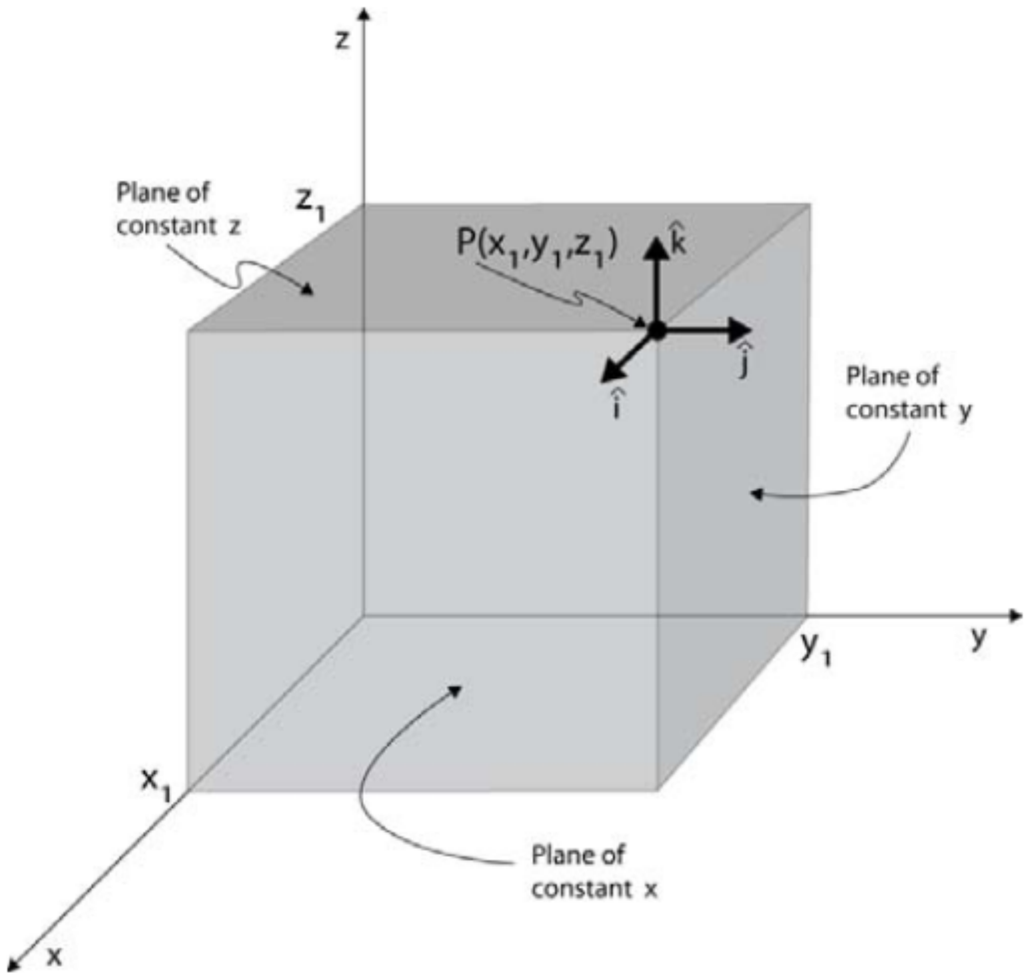
Introduction to Coordinate Systems

Rectangular, Cylindrical and Spherical Coordinate System.

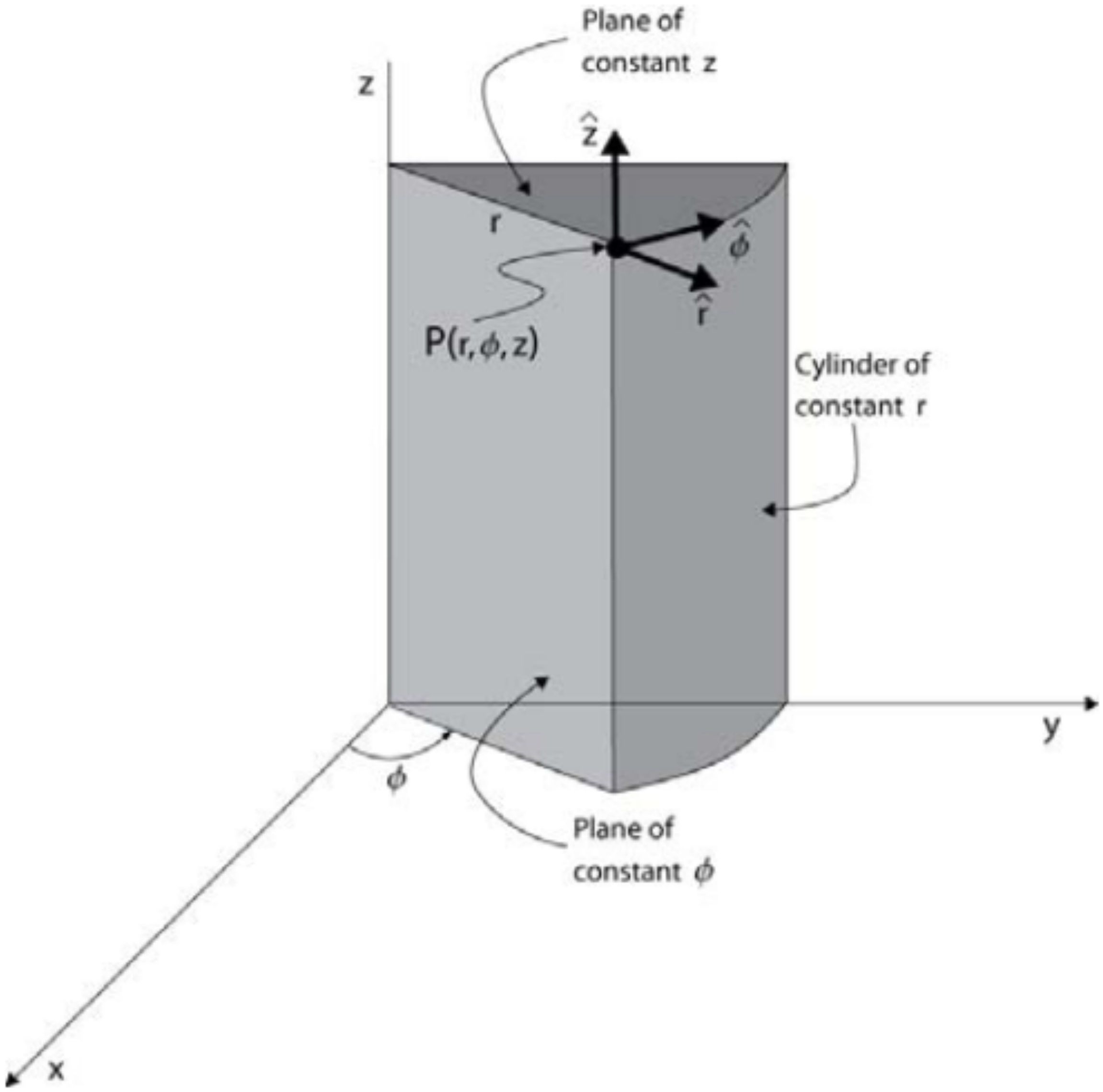
Rectangular Coordinate System

In *rectangular coordinates* a point P is specified by *x, y,* and *z*, where these values are all measured from the origin (see figure). A vector at the point P is specified in terms of three mutually perpendicular components with unit vectors $\hat{i}, \hat{j}$, and $\hat{k}$ (also called $\hat{x}, \hat{y}$, and $\hat{z}$). The unit vectors $\hat{i}, \hat{j}$, and $\hat{k}$ form a right-handed set; that is, if you push $\hat{i}$ into $\hat{j}$ with your right hand, your right thumb will point along $\hat{k}$ direction.



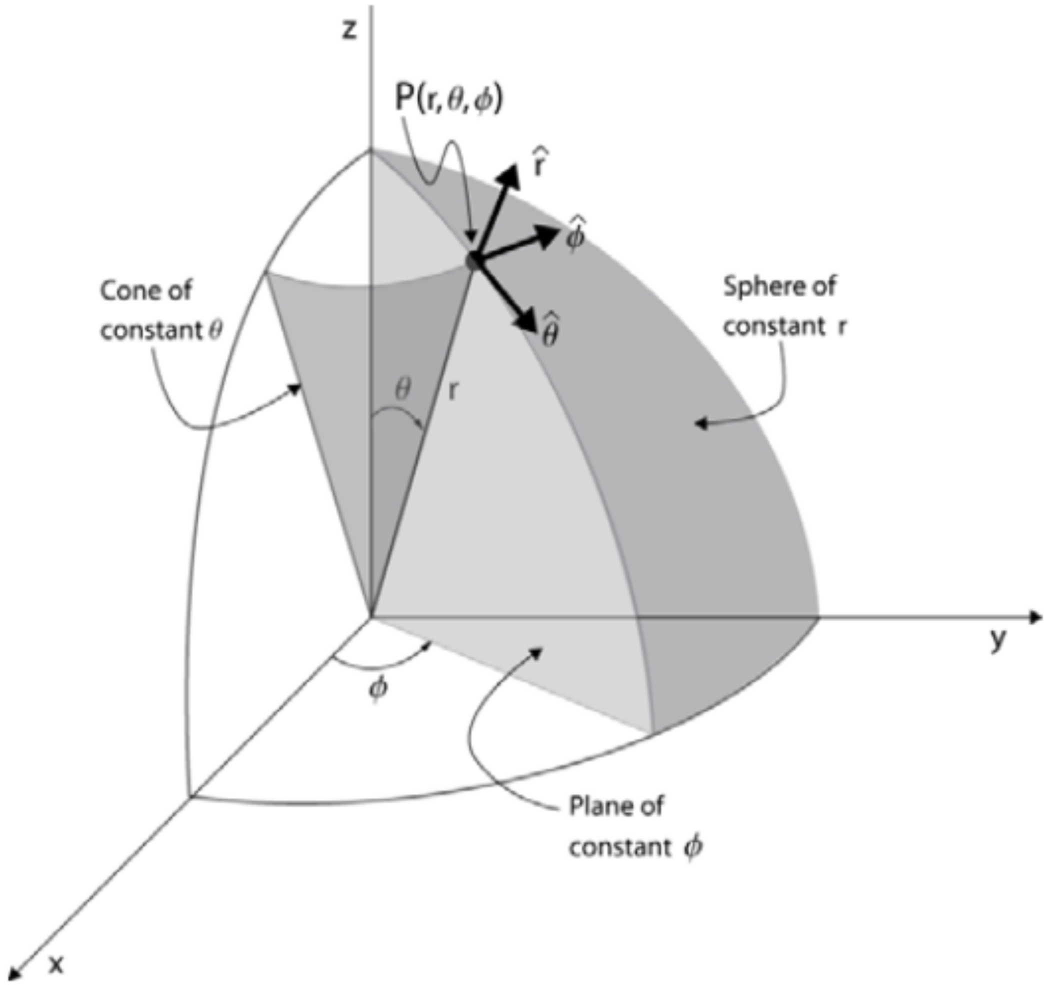
Cylindrical Coordinate System

In *cylindrical coordinates* a point P is specified by r, ϕ, z, where ϕ is measured from the x axis (or x-z plane) (see figure). A vector at the point P is specified in terms of three mutually perpendicular components with unit vectors $\hat{\mathbf{r}}$ perpendicular to the cylinder of radius r, $\hat{\boldsymbol{\varphi}}$ perpendicular to the plane through the z axis at angle ϕ, and $\hat{\mathbf{z}}$ perpendicular to the x-y plane at distance z. The unit vectors $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\varphi}}$, $\hat{\mathbf{z}}$ form a right-handed set.



Spherical Coordinate System

In *spherical coordinates* a point P is specified by r, θ, ϕ, where r is measured from the origin, θ is measured from the z axis, and ϕ is measured from the x axis (or x-z plane) (see figure). With z axis up, θ is sometimes called the *zenith* angle and ϕ the *azimuth* angle. A vector at the point P is specified in terms of three mutually perpendicular components with unit vectors $\hat{\mathbf{r}}$ perpendicular to the sphere of radius r, $\hat{\boldsymbol{\theta}}$ perpendicular to the cone of angle θ, and $\hat{\boldsymbol{\varphi}}$ perpendicular to the plane through the z axis at angle ϕ. The unit vectors $\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\varphi}}$ form a right-handed set.



Infinitesimal lengths and volumes

An infinitesimal length in the *rectangular* system is given by

$$d\mathbf{L} = \sqrt{dx^2 + dy^2 + dz^2} \tag{1}$$

and an infinitesimal volume by

$$dv = dx\,dy\,dz \tag{2}$$

In the *cylindrical* system the corresponding quantities are

$$d\mathbf{L} = \sqrt{dr^2 + r^2 d\phi^2 + dz^2} \tag{3}$$

and

$$dv = dr\; r\,d\phi\; dz \tag{4}$$

In the *spherical* system we have

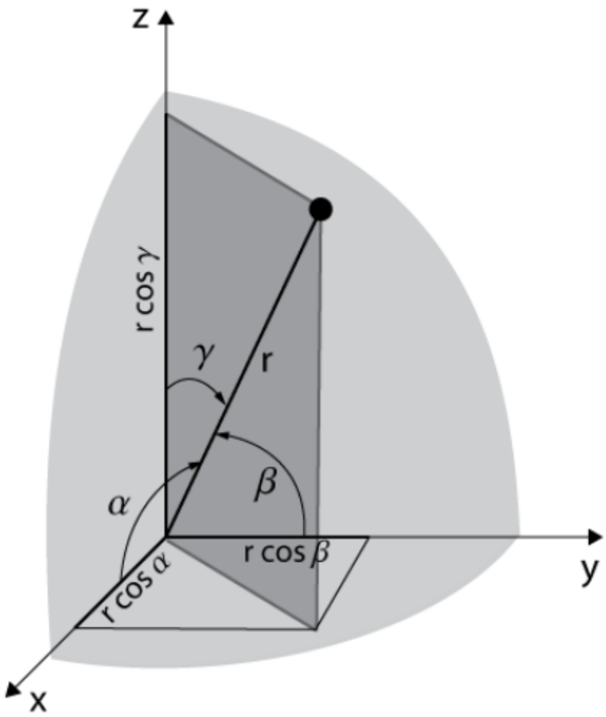
$$d\mathbf{L} = \sqrt{dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \; d\phi^2} \tag{5}$$

and

$$dv = dr\, r\, d\theta\;\; r\, \sin\theta\, d\phi \tag{6}$$

Direction cosines and coordinate-system transformation

As shown in the figure on the right, the projection x of the scalar distance r on the x axis is given by $r\cos\alpha$ where α is the angle between r and the x axis. The projection of r on the y axis is given by $r\cos\beta$, and the projection on the z axis by $r\cos\gamma$. Note that $\gamma = \theta$ so $\cos\gamma = \cos\theta$.



The quantities $\cos\alpha$, $\cos\beta$, and $\cos\gamma$ are called the *direction cosines*. From the theorem of Pythagoras,

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1 \qquad (7)$$

The scalar distance r of a spherical coordinate system transforms into rectangular coordinate distance

$$x = r\cos\alpha = r\sin\theta\cos\phi \qquad (8)$$
$$y = r\cos\beta = r\sin\theta\sin\phi \qquad (9)$$
$$z = r\cos\gamma = r\cos\theta \qquad (10)$$

from which

$$\left.\begin{aligned} \cos\alpha &= \sin\theta\cos\phi \\ \cos\beta &= \sin\theta\sin\phi \\ \cos\gamma &= \cos\theta \end{aligned}\right\}\text{direction cosines} \qquad \begin{aligned}(11)\\(12)\\(13)\end{aligned}$$

As the converse of (8), (9), and (10), the spherical coordinate values (r, θ, ϕ) may be expressed in terms of rectangular coordinate distances as follows:

$$r = \sqrt{x^2 + y^2 + z^2} \qquad r \geq 0 \qquad (14)$$

$$\theta = \cos^{-1}\frac{z}{\sqrt{x^2 + y^2 + z^2}} \qquad (0 \leq \theta \leq \pi) \qquad (15)$$

$$\phi = \tan^{-1}\frac{y}{x} \qquad (16)$$

From these and similar coordinate transformations of spherical to rectangular and rectangular to spherical coordinates, we may express a vector **A** at some point P with spherical components A_r, A_θ, A_ϕ as the rectangular components A_x, A_y, and A_z, where

$$A_x = A_r \sin\theta \cos\phi + A_\theta \cos\theta \cos\phi - A_\phi \sin\phi \qquad (17)$$
$$A_y = A_r \sin\theta \sin\phi + A_\theta \cos\theta \sin\phi + A_\phi \cos\phi \qquad (18)$$
$$A_z = A_r \cos\theta - A_\theta \sin\theta \qquad (19)$$

Note that the direction cosines are simply the dot products of the spherical unit vector $\hat{\mathbf{r}}$ with the rectangular unit vectors $\hat{\mathrm{x}}$, $\hat{\mathrm{y}}$, and $\hat{\mathrm{z}}$:

$$\hat{\mathbf{r}} \cdot \hat{\mathbf{x}} = \sin\theta \cos\phi = \cos\alpha \qquad (20)$$
$$\hat{\mathbf{r}} \cdot \hat{\mathbf{y}} = \sin\theta \sin\phi = \cos\beta \qquad (21)$$
$$\hat{\mathbf{r}} \cdot \hat{\mathbf{z}} = \cos\theta = \cos\gamma \qquad (22)$$

These and other dot product combinations are listed in the following table:

		Rectangular			*Cylindrical*			*Spherical*		
	Σ	$\hat{\mathbf{x}}$	$\hat{\mathbf{y}}$	$\hat{\mathbf{z}}$	$\hat{\mathbf{r}}$	$\hat{\boldsymbol{\varphi}}$	$\hat{\mathbf{z}}$	$\hat{\mathbf{r}}$	$\hat{\boldsymbol{\theta}}$	$\hat{\boldsymbol{\varphi}}$
Rectangular	$\hat{\mathbf{x}}$	1	0	0	$\cos\phi$	$-\sin\phi$	0	$\sin\theta\cos\phi$	$\cos\theta\cos\phi$	$-\sin\phi$
	$\hat{\mathbf{y}}$	0	1	0	$\sin\phi$	$\cos\phi$	0	$\sin\theta\sin\phi$	$\cos\theta\sin\phi$	$\cos\phi$
	$\hat{\mathbf{z}}$	0	0	1	0	0	1	$\cos\theta$	$-\sin\theta$	0
Cylindrical	$\hat{\mathbf{r}}$	$\cos\phi$	$\sin\phi$	0	1	0	0	$\sin\theta$	$\cos\theta$	0
	$\hat{\boldsymbol{\varphi}}$	$-\sin\phi$	$\cos\phi$	0	0	1	0	0	0	1
	$\hat{\mathbf{z}}$	0	0	1	0	0	1	$\cos\theta$	$-\sin\theta$	0
Spherical	$\hat{\mathbf{r}}$	$\sin\theta\cos\phi$	$\sin\theta\sin\phi$	$\cos\theta$	$\sin\theta$	0	$\cos\theta$	1	0	0
	$\hat{\boldsymbol{\theta}}$	$\cos\theta\cos\phi$	$\cos\theta\sin\phi$	$-\sin\theta$	$\cos\theta$	0	$-\sin\theta$	0	1	0
	$\hat{\boldsymbol{\varphi}}$	$-\sin\phi$	$\cos\phi$	0	0	1	0	0	0	1

Note that the unit vectors $\hat{\mathbf{r}}$ in the cylindrical and spherical systems are *not* the same. For example,

Spherical	*Cylindrical*
$\hat{\mathbf{r}} \cdot \hat{\mathbf{x}} = \sin\theta\cos\phi$	$\hat{\mathbf{r}} \cdot \hat{\mathbf{x}} = \cos\phi$
$\hat{\mathbf{r}} \cdot \hat{\mathbf{y}} = \sin\theta\sin\phi$	$\hat{\mathbf{r}} \cdot \hat{\mathbf{y}} = \sin\phi$
$\hat{\mathbf{r}} \cdot \hat{\mathbf{z}} = \cos\theta$	$\hat{\mathbf{r}} \cdot \hat{\mathbf{z}} = 0$

The fundamental parameters of the coordinate systems

Coordinate system	Coordinates	Range	Unit vectors	Length elements	Coordinate surfaces	
Rectangular	x	$-\infty$ to $+\infty$	$\hat{\mathbf{x}}$ or $\hat{\mathbf{i}}$	dx	Plane	x=constant
	y	$-\infty$ to $+\infty$	$\hat{\mathbf{y}}$ or $\hat{\mathbf{j}}$	dy	Plane	y=constant
	z	$-\infty$ to $+\infty$	$\hat{\mathbf{z}}$ or $\hat{\mathbf{k}}$	dz	Plane	z=constant
Cylindrical	r	0 to ∞	$\hat{\mathbf{r}}$	dr	Cylinder	r=constant
	ϕ	0 to 2π	$\hat{\boldsymbol{\varphi}}$	$r\,d\phi$	Plane	ϕ=constant
	z	$-\infty$ to $+\infty$	$\hat{\mathbf{z}}$	dz	Plane	z=constant
Spherical	r	0 to ∞	$\hat{\mathbf{r}}$	dr	Sphere	r=constant
	θ	0 to π	$\hat{\boldsymbol{\theta}}$	$r\,d\theta$	Cone	θ=constant
	ϕ	0 to 2π	$\hat{\boldsymbol{\varphi}}$	$r\sin\theta\,d\phi$	Plane	ϕ=constant

The following two tables give the unit vector dot products in rectangular coordinates for both rectangular-cylindrical and rectangular-spherical coordinates.

Σ	$\hat{\mathbf{x}}$	$\hat{\mathbf{y}}$	$\hat{\mathbf{z}}$
$\hat{\mathbf{r}}$	$\dfrac{x}{\sqrt{x^2+y^2}}$	$\dfrac{y}{\sqrt{x^2+y^2}}$	0
$\hat{\boldsymbol{\varphi}}$	$\dfrac{-y}{\sqrt{x^2+y^2}}$	$\dfrac{x}{\sqrt{x^2+y^2}}$	0
$\hat{\mathbf{z}}$	0	0	1

·	$\hat{\mathbf{x}}$	$\hat{\mathbf{y}}$	$\hat{\mathbf{z}}$
$\hat{\mathbf{r}}$	$\dfrac{x}{\sqrt{x^2+y^2+z^2}}$	$\dfrac{y}{\sqrt{x^2+y^2+z^2}}$	$\dfrac{z}{\sqrt{x^2+y^2+z^2}}$
$\hat{\boldsymbol{\theta}}$	$\dfrac{xz}{\sqrt{x^2+y^2}\sqrt{x^2+y^2+z^2}}$	$\dfrac{yz}{\sqrt{x^2+y^2}\sqrt{x^2+y^2+z^2}}$	$-\dfrac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}$
$\hat{\boldsymbol{\varphi}}$	$-\dfrac{y}{\sqrt{x^2+y^2}}$	$\dfrac{x}{\sqrt{x^2+y^2}}$	0

Example: $\hat{\boldsymbol{\varphi}}\cdot\hat{\mathbf{y}} = \cos\phi = \dfrac{x}{\sqrt{x^2+y^2}}$

Example: $\hat{\mathbf{x}}\cdot\hat{\mathbf{r}} = \sin\theta\,\cos\phi = \dfrac{x}{\sqrt{x^2+y^2+z^2}}$

Transformations of vector components between coordinate systems:

Rectangular to cylindrical

$$A_r = A_x \frac{x}{\sqrt{x^2+y^2}} + A_y \frac{x}{\sqrt{x^2+y^2}}$$

$$A_\phi = -A_x \frac{y}{\sqrt{x^2+y^2}} + A_y \frac{x}{\sqrt{x^2+y^2}}$$

$$A_z = A_z$$

Cylindrical to rectangular

$$A_x = A_r \cos\phi - A_\phi \sin\phi$$

$$A_y = A_r \sin\phi - A_\phi \cos\phi$$

$$A_z = A_z$$

Rectangular to spherical

$$A_r = A_x \frac{x}{\sqrt{x^2+y^2+z^2}} + A_y \frac{y}{\sqrt{x^2+y^2+z^2}} + A_z \frac{z}{\sqrt{x^2+y^2+z^2}}$$

$$A_\theta = A_x \frac{xz}{\sqrt{x^2+y^2}\sqrt{x^2+y^2+z^2}} + A_y \frac{yz}{\sqrt{x^2+y^2}\sqrt{x^2+y^2+z^2}} - A_z \frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}$$

$$A_\phi = -A_x \frac{y}{\sqrt{x^2+y^2}} + A_y \frac{x}{\sqrt{x^2+y^2}}$$

Spherical to rectangular

$$A_x = A_r \sin\theta\cos\phi + A_\theta \cos\theta\cos\phi - A_\phi \sin\phi$$

$$A_y = A_r \sin\theta\sin\phi + A_\theta \cos\theta\sin\phi + A_\phi \cos\phi$$

$$A_z = A_r \cos\theta - A_\theta \sin\theta$$

Electromagnetic Theory

Expressions for the gradient, divergence, and curl in all three coordinate systems:

Rectangular coordinates

$$\nabla f = \hat{\mathbf{x}}\frac{\partial f}{\partial x} + \hat{\mathbf{y}}\frac{\partial f}{\partial y} + \hat{\mathbf{z}}\frac{\partial f}{\partial z}$$

$$\nabla \bullet \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \mathbf{A} = \hat{\mathbf{x}}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{\mathbf{y}}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{\mathbf{z}}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

Cylindrical coordinates

$$\nabla f = \hat{\mathbf{r}}\frac{\partial f}{\partial r} + \hat{\boldsymbol{\varphi}}\frac{1}{r}\frac{\partial f}{\partial \phi} + \hat{\mathbf{z}}\frac{\partial f}{\partial z}$$

$$\nabla \bullet \mathbf{A} = \frac{1}{r}\frac{\partial}{\partial r} r A_r + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \mathbf{A} = \hat{\mathbf{r}}\left(\frac{1}{r}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right) + \hat{\boldsymbol{\varphi}}\left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right) + \hat{\mathbf{z}}\frac{1}{r}\left(\frac{\partial}{\partial r} r A_\phi - \frac{\partial A_r}{\partial \phi}\right) = \begin{vmatrix} \hat{\mathbf{r}}\frac{1}{r} & \hat{\boldsymbol{\varphi}} & \hat{\mathbf{z}}\frac{1}{r} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_r & rA_\phi & A_z \end{vmatrix}$$

Spherical coordinates

$$\nabla f = \hat{\mathbf{r}}\frac{\partial f}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial f}{\partial \theta} + \hat{\boldsymbol{\varphi}}\frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}$$

$$\nabla \bullet \mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r} r^2 A_r + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(A_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi}$$

$$\nabla \times \mathbf{A} = \hat{\mathbf{r}}\frac{1}{r\sin\theta}\left(\frac{\partial}{\partial \theta}(A_\phi \sin\theta) - \frac{\partial A_\theta}{\partial \phi}\right) + \hat{\boldsymbol{\theta}}\frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial A_r}{\partial \phi} - \frac{\partial}{\partial r} r A_\phi\right) + \hat{\boldsymbol{\varphi}}\frac{1}{r}\left(\frac{\partial}{\partial r} r A_\theta - \frac{\partial A_r}{\partial \theta}\right)$$