Problem-1

a. Two vectors are represented by $\vec{A} = 2\hat{i} + 2\hat{j}, \vec{B} = 3\hat{i} + 4\hat{j} - 2\hat{k}$. Find the dot and cross products and the angle between the vectors. Show that $\vec{A} \times \vec{B}$ is at right angles to $\vec{A}$.

b. If $\phi(x, y, z) = 3x^2 y - y^3 z^2$, find $\nabla\phi$ at the point (1, -2, -1).

a. $\vec{A} = 2\hat{i} + 2\hat{j}, \quad \vec{B} = 3\hat{i} + 4\hat{j} - 2\hat{k}$

$$\therefore \; \vec{A}.\vec{B} = 2 \times 3 + 2 \times 4 + 0 \times (-2) = 14$$

$$\therefore \; \vec{A} \times \vec{B} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ 2 & 2 & 0 \\ 3 & 4 & -2 \end{vmatrix} = -4\hat{a}_x + 4\hat{a}_y + 2\hat{a}_z$$

Now, $A = \sqrt{2^2 + 2^2 + 0} = 2.83; \qquad B = \sqrt{3^2 + 4^2 + (-2)^2} = 5.39$

$$\because \; \vec{A}.\vec{B} = AB\cos\theta$$

$$\Rightarrow \quad \theta = \cos^{-1}\left(\frac{\vec{A}.\vec{B}}{AB}\right) = \cos^{-1}\left(\frac{14}{2.83 \times 5.39}\right) = 23.2°$$

For $\vec{A} \times \vec{B}$ to be at right angles to $\vec{A}$, $(\vec{A} \times \vec{B}).\vec{A}$ should be zero.

$(\vec{A} \times \vec{B}).\vec{A} = 2 \times (-4) + 2 \times 4 + 0 \times 2 = 0$ (Proved)

b. $\phi(x, y, z) = 3x^2 y - y^3 z^2$

$$\nabla\phi = \frac{\partial}{\partial x}(3x^2 y - y^3 z^2)\hat{a}_x + \frac{\partial}{\partial y}(3x^2 y - y^3 z^2)\hat{a}_y + \frac{\partial}{\partial z}(3x^2 y - y^3 z^2)\hat{a}_z$$

$$= 6xy\hat{a}_x + (3x^2 - 3y^2 z^2)\hat{a}_y - 2y^3 z\hat{a}_z$$

At the point (1, -2, -1), the gradient is given as,

$$\nabla\phi\big|_{(1,-2,-1)} = -12\hat{a}_x + -9\hat{a}_y - 16\hat{a}_z$$

Problems-2

a. For a vector field $\vec{A}$, show explicitly that $\nabla.\nabla\times\vec{A}=0$, i.e., the divergence of the curl of any vector field is zero.

b. For a scalar field V, show that $\nabla\times\nabla V=0$, i.e., the curl of the gradient of any scalar field is zero.

Solution

a. Let, $\vec{A}=A_x\hat{a}_x+A_y\hat{a}_y+A_z\hat{a}_z$

$$\therefore \quad \nabla\times\vec{A}=\begin{vmatrix}\hat{a}_x & \hat{a}_y & \hat{a}_z\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ A_x & A_y & A_z\end{vmatrix}=\left(\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}\right)\hat{a}_x+\left(\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x}\right)\hat{a}_y+\left(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right)\hat{a}_z$$

$$=G_x\hat{a}_x+G_y\hat{a}_y+G_z\hat{a}_z \quad \text{(Let)}$$

$$\therefore \quad \nabla.\nabla\times\vec{A}=\frac{\partial G_x}{\partial x}+\frac{\partial G_y}{\partial y}+\frac{\partial G_z}{\partial z}$$

$$=\frac{\partial}{\partial x}\left(\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}\right)+\frac{\partial}{\partial y}\left(\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x}\right)+\frac{\partial}{\partial z}\left(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right)$$

$$=\left(\frac{\partial^2 A_z}{\partial x\partial y}-\frac{\partial^2 A_y}{\partial x\partial z}\right)+\left(\frac{\partial^2 A_x}{\partial y\partial z}-\frac{\partial^2 A_z}{\partial y\partial x}\right)+\left(\frac{\partial^2 A_y}{\partial z\partial x}-\frac{\partial^2 A_x}{\partial z\partial y}\right)$$

$$=0 \qquad \left\{\because \frac{\partial^2 A_z}{\partial x\partial y}=\frac{\partial^2 A_z}{\partial y\partial x}, \text{ and so on}\right\}$$

$$\therefore \qquad \boxed{\nabla.\nabla\times\vec{A}=0}$$

b. $\nabla V=\frac{\partial V}{\partial x}+\frac{\partial V}{\partial y}+\frac{\partial V}{\partial z}$

$$\therefore \quad \nabla\times\nabla V=\begin{vmatrix}\hat{a}_x & \hat{a}_y & \hat{a}_z\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ \frac{\partial V}{\partial x} & \frac{\partial V}{\partial y} & \frac{\partial V}{\partial z}\end{vmatrix}=\left(\frac{\partial^2 V}{\partial y\partial z}-\frac{\partial^2 V}{\partial z\partial y}\right)\hat{a}_x+\left(\frac{\partial^2 V}{\partial z\partial x}-\frac{\partial^2 V}{\partial x\partial z}\right)\hat{a}_y+\left(\frac{\partial^2 V}{\partial x\partial y}-\frac{\partial^2 V}{\partial y\partial x}\right)\hat{a}_z=0$$

$$\therefore \qquad \boxed{\nabla\times\nabla V=0}$$

Note: These two vector identities are known as **null identities**.

Problem-3

If $\vec{v} = \vec{\omega} \times \vec{r}$, prove that $\vec{\omega} = \frac{1}{2}\text{curl}\,\vec{v}$, where ω is a constant vector.

Solution

Here, $\vec{r} = x\hat{a}_x + y\hat{a}_y + z\hat{a}_z$ and let $\vec{\omega} = \omega_x\hat{a}_x + \omega_y\hat{a}_y + \omega_z\hat{a}_z$

$\therefore$ $\text{curl}\,\vec{v} = \nabla \times \vec{v} = \nabla \times (\vec{\omega} \times \vec{r})$

$$= \nabla \times \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \omega_x & \omega_y & \omega_z \\ x & y & z \end{vmatrix}$$

$$= \nabla \times \left[(\omega_y z - \omega_z y)\hat{a}_x + (\omega_z x - \omega_x z)\hat{a}_y + (\omega_x y - \omega_y x)\hat{a}_z\right]$$

$$= \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (\omega_y z - \omega_z y) & (\omega_z x - \omega_x z) & (\omega_x y - \omega_y x) \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(\omega_x y - \omega_y x) - \frac{\partial}{\partial z}(\omega_z x - \omega_x z)\right]\hat{a}_x + \left[\frac{\partial}{\partial z}(\omega_y z - \omega_z y) - \frac{\partial}{\partial x}(\omega_x y - \omega_y x)\right]\hat{a}_y$$

$$+ \left[\frac{\partial}{\partial x}(\omega_z x - \omega_x z) - \frac{\partial}{\partial y}(\omega_y z - \omega_z y)\right]\hat{a}_z$$

$$= 2\omega_x\hat{a}_x + 2\omega_y\hat{a}_y + 2\omega_z\hat{a}_z$$

$$= 2(\omega_x\hat{a}_x + \omega_y\hat{a}_y + \omega_z\hat{a}_z)$$

$$= 2\vec{\omega}$$

$$\therefore \boxed{\vec{\omega} = \frac{1}{2}\text{curl}\,\vec{v}}$$

Problem-4

Express the following vectors in Cartesian coordinates:

(a) $\vec{A} = rz\sin\phi\hat{a}_r + 3r\cos\phi\hat{a}_\phi + r\cos\phi\sin\phi\hat{a}_z$

(b) $\vec{B} = \rho^2\hat{a}_\rho + \sin\theta\hat{a}_\phi$

(a) $\vec{A} = rz\sin\phi\hat{a}_r + 3r\cos\phi\hat{a}_\phi + r\cos\phi\sin\phi\hat{a}_z$

Here, $A_r = rz\sin\phi, \quad A_\phi = 3r\cos\phi, \quad A_z = r\cos\phi\sin\phi$

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_r \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} rz\sin\phi \\ 3r\cos\phi \\ r\cos\phi\sin\phi \end{bmatrix}$$

$$A_x = rz\sin\phi\cos\phi - 3r\sin\phi\cos\phi$$

$$A_y = rz\sin^2\phi + 3r\cos^2\phi$$

$$A_z = r\cos\phi\sin\phi$$

But, $r = \sqrt{x^2+y^2}, \quad \cos\phi = \dfrac{x}{\sqrt{x^2+y^2}}, \quad \sin\phi = \dfrac{y}{\sqrt{x^2+y^2}}$

Substituting these values,

$$A_x = \sqrt{x^2+y^2}\,z\frac{xy}{x^2+y^2} - 3\sqrt{x^2+y^2}\,\frac{xy}{x^2+y^2} = \frac{xyz}{\sqrt{x^2+y^2}} - \frac{3xy}{\sqrt{x^2+y^2}}$$

$$A_y = \sqrt{x^2+y^2}\,z\frac{y^2}{x^2+y^2} + 3\sqrt{x^2+y^2}\,\frac{x^2}{x^2+y^2} = \frac{y^2z}{\sqrt{x^2+y^2}} + \frac{3x^2}{\sqrt{x^2+y^2}}$$

$$A_z = \sqrt{x^2+y^2}\,\frac{xy}{x^2+y^2} = \frac{xy}{\sqrt{x^2+y^2}}$$

Hence, the vector in Cartesian coordinates is written as,

$$\vec{A} = \frac{1}{\sqrt{x^2+y^2}}[(xyz - 3xy)\hat{a}_x + (y^2z + 3x^2)\hat{a}_y + xy\hat{a}_z]$$

(b) $\vec{B} = \rho^2 \hat{a}_\rho + \sin\theta \hat{a}_\phi$

Here, $B_\rho = \rho^2, \quad B_\theta = 0, \quad B_\phi = \sin\theta$

$$\begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} \rho^2 \\ 0 \\ \sin\theta \end{bmatrix}$$

$$B_x = \rho^2 \sin\theta\cos\phi - \sin\theta\sin\phi$$

$$B_y = \rho^2 \sin\theta\sin\phi + \sin\theta\cos\phi$$

$$B_z = \rho^2 \cos\theta$$

But,

$$\rho = \sqrt{x^2+y^2+z^2}, \quad \sin\theta = \frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}, \quad \cos\theta = \frac{z}{\sqrt{x^2+y^2+z^2}},$$

$$\sin\phi = \frac{y}{\sqrt{x^2+y^2}}, \quad \cos\phi = \frac{x}{\sqrt{x^2+y^2}}$$

Substituting these values,

$$B_x = (x^2+y^2+z^2)\frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}\frac{x}{\sqrt{x^2+y^2}} - \frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}\frac{y}{\sqrt{x^2+y^2}}$$

$$= x\sqrt{x^2+y^2+z^2} - \frac{y}{\sqrt{x^2+y^2+z^2}}$$

$$= \frac{1}{\sqrt{x^2+y^2+z^2}}[x(x^2+y^2+z^2) - y]$$

$$B_y = (x^2+y^2+z^2)\frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}\frac{y}{\sqrt{x^2+y^2}} + \frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}\frac{x}{\sqrt{x^2+y^2}}$$

$$= y\sqrt{x^2+y^2+z^2} + \frac{x}{\sqrt{x^2+y^2+z^2}}$$

$$= \frac{1}{\sqrt{x^2+y^2+z^2}}[y(x^2+y^2+z^2) + x]$$

$$B_z = (x^2+y^2+z^2)\frac{z}{\sqrt{x^2+y^2+z^2}} = z\sqrt{x^2+y^2+z^2}$$

Hence, the vector in Cartesian coordinates is written as,

$$\vec{B} = \frac{1}{\sqrt{x^2+y^2+z^2}}[\{x(x^2+y^2+z^2) - y\}\hat{a}_x + \{y(x^2+y^2+z^2) + x\}\hat{a}_y + z(x^2+y^2+z^2)\hat{a}_z]$$

Problem-5

Express the field $\vec{E} = 2xyz\hat{a}_x - 3(x+y+z)\hat{a}_z$, in cylindrical coordinates, and calculate $|\vec{E}|$ at the point P (r = 2, φ = 60°, z = 3).

Solution

In the cylindrical system,

$$\begin{bmatrix} E_r \\ E_\phi \\ E_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2xyz \\ 0 \\ -3(x+y+z) \end{bmatrix}$$

$$E_r = 2xyz\cos\phi$$

$$E_\phi = -2xyz\sin\phi$$

$$E_z = -3(x+y+z)$$

But, $x = r\cos\phi, \quad y = r\sin\phi, \quad z = z$

Substituting these values,

$$E_r = 2r\cos\phi r\sin\phi z\cos\phi = 2r^2 z\sin\phi\cos^2\phi$$

$$E_\phi = -2r\cos\phi r\sin\phi z\sin\phi = -2r^2 z\sin^2\phi\cos\phi$$

$$E_z = -3(r\cos\phi + r\sin\phi + z)$$

Hence, the vector in Cartesian coordinates is written as,

$$\vec{E} = r^2 z\sin 2\phi\cos\phi\hat{a}_r - r^2 z\sin 2\phi\sin\phi\hat{a}_\phi - 3(r\cos\phi + r\sin\phi + z)\hat{a}_z$$

At P (r = 2, φ = 60°, z = 3), the vector is given as,

$$E_r = 2\times 2^2 \times 3\sin 60°\cos^2 60° = 24\frac{\sqrt{3}}{2}\times\frac{1}{4} = 3\sqrt{3} = 5.196$$

$$E_\phi = -2\times 2^2\times 3\sin^2 60°\cos 60° = -24\times\frac{3}{4}\times\frac{1}{2} = -9$$

$$E_z = -3(2\cos 60° + 2\sin 60° + 3) = -3\left(2\times\frac{1}{2} + 2\times\frac{\sqrt{3}}{2} + 3\right) = -3(4+\sqrt{3}) = -17.196$$

Hence, at P, $\vec{E} = 5.196\hat{a}_r - 9\hat{a}_\phi - 17.196\hat{a}_z$

$$\therefore |\vec{E}| = \sqrt{(5.196)^2 - 9^2 - (17.196)^2} = 20.092$$

Problem-6

The electric field at a point *P*, expressed in the cylindrical coordinate system is given by,

$\vec{E} = 16r^2 \sin\phi \hat{a}_r + 3r^2 \cos\phi \hat{a}_\phi$

Find the value of divergence of the field if the location of the point *P* is given by (1, 2, 3) m in Cartesian coordinates.

Solution

Here, $x = 1, \quad y = 2, \quad z = 3$

$$\therefore r = \sqrt{x^2 + y^2} = \sqrt{5}, \quad \phi = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{2}{1}\right) = 63.43°, \quad z = 3$$

For the electric field, $\vec{E} = 16r^2 \sin\phi \hat{a}_r + 3r^2 \cos\phi \hat{a}_\phi$, the divergence in cylindrical coordinates is given as,

$$\nabla.\vec{E} = \frac{1}{r}\frac{\partial}{\partial r}(rE_r) + \frac{1}{r}\frac{\partial E_\phi}{\partial \phi} + \frac{\partial E_z}{\partial z} = \frac{1}{r}\frac{\partial}{\partial r}(16r^3 \sin\phi) + \frac{1}{r}\frac{\partial}{\partial \phi}(3r^2 \cos\phi) + \frac{\partial}{\partial z}(0)$$

$$= 48r\sin\phi - 3r\sin\phi$$

$$= 45r\sin\phi$$

Hence, the divergence of the field at the point *P* is given as,

$$\nabla.\vec{E} = 45r\sin\phi = 45 \times \sqrt{5}\sin(63.43°) = 90$$

Problem-8

If a scalar potential is given by the expression *φ* = *xyz*, determine the potential gradient and also prove that the vector $\vec{F} = \text{grad}\ \phi$ is irrotational.

Solution

Here, the potential is, $\phi = xyz$

The potential gradient is given as,

$$\nabla\phi = \frac{\partial}{\partial x}(xyz)\hat{a}_x + \frac{\partial}{\partial y}(xyz)\hat{a}_y + \frac{\partial}{\partial z}(xyz)\hat{a}_z = (yz\hat{a}_x + xz\hat{a}_y + xy\hat{a}_z)$$

Now, we let, $\vec{F} = \nabla\phi = (yz\hat{a}_x + xz\hat{a}_y + xy\hat{a}_z)$

$$\nabla \times \vec{F} = \nabla \times \nabla\phi = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz & xy \end{vmatrix}$$

$$= (1-1)\hat{a}_x + (1-1)\hat{a}_y + (1-1)\hat{a}_z = 0$$

Hence, the vector $\vec{F} = \nabla\phi$ is irrotational.

Problem-7

(a) Determine the constant *c* such that the vector
$\vec{F} = (x+ay)\hat{i} + (y+bz)\hat{j} + (x+cz)\hat{k}$ will be solenoidal.
(b) Find the value of the constant *Q* to make
$\vec{V} = (x+3y)\hat{i} + (y-2z)\hat{j} + (x+Qz)\hat{k}$, solenoidal.
(c) Find the constants *a, b, c* so that the vector,
$\vec{V} = (x+2y+az)\hat{i} + (bx-3y-z)\hat{j} + (4x+cy+2z)\hat{k}$ is irrotational.

Solution

(a) $\vec{F} = (x+ay)\hat{i} + (y+bz)\hat{j} + (x+cz)\hat{k}$
The vector will be solenoidal if its divergence is zero.

$$\nabla.\vec{F} = \frac{\partial}{\partial x}(x+ay) + \frac{\partial}{\partial y}(y+bz) + \frac{\partial}{\partial z}(x+cz) = 0$$

$$\Rightarrow \quad 1+1+c=0$$

$$c=-2$$

(b) $\vec{V} = (x+3y)\hat{i} + (y-2z)\hat{j} + (x+Qz)\hat{k}$
The vector will be solenoidal if its divergence is zero.

$$\nabla.\vec{V} = \frac{\partial}{\partial x}(x+3y) + \frac{\partial}{\partial y}(y-2z) + \frac{\partial}{\partial z}(x+Qz) = 0$$

$$\Rightarrow \quad 1+1+Q=0$$

$$Q=-2$$

(c) $\vec{V} = (x+2y+az)\hat{i} + (bx-3y-z)\hat{j} + (4x+cy+2z)\hat{k}$
The vector will be irrotational if its curl is zero.

$$\nabla \times \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x+2y+az) & (bx-3y-z) & (4x+cy+2z) \end{vmatrix} = 0$$

$$\Rightarrow \quad (c+1)\hat{i} + (a-4)\hat{j} + (b-2)\hat{k} = 0$$

This vector will be zero if and only if its components are individually zero.

$(c+1)=0 \quad \Rightarrow \quad c=-1$

$(a-4)=0 \quad \Rightarrow \quad a=4$

$(b-2)=0 \quad \Rightarrow \quad b=2$

and
Hence, *a* = 4, *b* = 2, *c* = -1.

Problem-9

Using the concept of volume integral, find the volume of a sphere of radius *a*.

Solution

In spherical coordinates, the differential volume is given as,

$$dv = \rho^2 \sin\theta \, d\rho \, d\theta \, d\phi$$

where,

$$0 \le \rho \le a$$

$$0 \le \theta \le \pi$$

$$0 \le \phi \le 2\pi$$

By volume integral, the volume of the sphere is obtained as,

$$v = \int_v dv = \int_{\rho=0}^{a} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \rho^2 \sin\theta \, d\rho \, d\theta \, d\phi = 2\pi \times (-\cos\theta)_0^{\pi} \times \int_{\rho=0}^{a} \rho^2 d\rho = 4\pi \times \frac{a^3}{3} = \frac{4}{3}\pi a^3$$

Problem-10

Use the spherical coordinate system to find the area of the strip $\alpha \le \theta \le \beta$ on the spherical shell of radius '*a*'. What results when $\alpha = 0$ and $\beta = \pi$?

Solution

For a fixed radius of a, the elemental surface is,

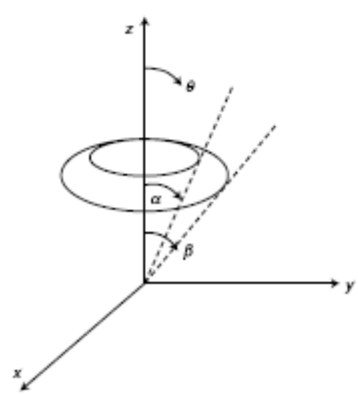
$$d\vec{S} = \rho^2 \sin\theta \, d\theta \, d\phi \, \hat{a}_\rho \Big|_{\rho=a} = a^2 \sin\theta \, d\theta \, d\phi \, \hat{a}_\rho$$

Hence, the area of the strip is given as,

$$S = \int_S d\vec{S}.\hat{a}_\rho = \int_{\theta=\alpha}^{\beta} \int_{\phi=0}^{2\pi} a^2 \sin\theta \, d\theta \, d\phi$$

$$= 2\pi \times a^2 \times (-\cos\theta)_\alpha^\beta$$

$$= 2\pi a^2 (\cos\alpha - \cos\beta)$$



For $\alpha = 0$ and $\beta = \pi$, the area is,

$$S = 2\pi a^2 (\cos 0 - \cos\pi) = 4\pi a^2$$

This is the area of a sphere of radius *a*.

Problem-11

Given that $\vec{F}=10e^{-r}\hat{a}_r-2z\hat{a}_z$, evaluate both sides of the divergence theorem for the volume enclosed by r = 2, z = 0 and z = 5.

Solution

Here, $\vec{F}=10e^{-r}\hat{a}_r-2z\hat{a}_z$

$d\vec{S}=rd\phi dz\hat{a}_r+drdz\hat{a}_\phi+rdrd\phi\hat{a}_z$

$\vec{F}.d\vec{S}=(10e^{-r}\hat{a}_r-2z\hat{a}_z).(rd\phi dz\hat{a}_r+drdz\hat{a}_\phi+rdrd\phi\hat{a}_z)=10re^{-r}d\phi dz-2rzdrd\phi$

The cylinder has three surfaces as follows.

$$\oint_S \vec{F}.d\vec{S}=\int_{top}\vec{F}.d\vec{S}+\int_{bottom}\vec{F}.d\vec{S}+\int_{curved}\vec{F}.d\vec{S}$$

For the top surface, $z=5$

$$\int_{top}\vec{F}.d\vec{S}=\int_{r=0}^{2}\int_{\phi=0}^{2\pi}-2rzdrd\phi\Bigg|_{z=5}=-10\int_{r=0}^{2}\int_{\phi=0}^{2\pi}rdrd\phi=-10\times2\pi\times\frac{2^2}{2}=-40\pi$$

For the bottom surface, $z=0$

$$\int_{bottom}\vec{F}.d\vec{S}=\int_{r=0}^{2}\int_{\phi=0}^{2\pi}-2rzdrd\phi\Bigg|_{z=0}=0$$

For the curved surface, $r=2$

$$\int_{curved}\vec{F}.d\vec{S}=\int_{\phi=0}^{2\pi}\int_{z=0}^{5}10re^{-r}d\phi dz\Bigg|_{r=2}=20e^{-2}\times2\pi\times5=200\pi e^{-2}$$

By addition, total surface area of the closed cylinder is given as,

$$\oint_S \vec{F}.d\vec{S}=-40\pi+0+200\pi e^{-2}=-40.63$$

Also,

$$\nabla.\vec{F}=\frac{1}{r}\frac{\partial}{\partial r}(rF_r)+\frac{1}{r}\frac{\partial F_\phi}{\partial\phi}+\frac{\partial F_z}{\partial z}=\frac{1}{r}\frac{\partial}{\partial r}(10re^{-r})+\frac{\partial}{\partial z}(-2z)=\frac{10e^{-r}}{r}-10e^{-r}-2$$

$$\int_v(\nabla.\vec{F})dv=\int_{r=0}^{2}\int_{\phi=0}^{2\pi}\int_{z=0}^{5}\left(\frac{10e^{-r}}{r}-10e^{-r}-2\right)rdrd\phi dz$$

$$=2\pi\times5\times\int_{r=0}^{2}(10e^{-r}-10re^{-r}-2r)dr=-40.63$$

$\oint_s \vec{F}.d\vec{S}=\int_v(\nabla.\vec{F})dv$ and hence, divergence theorem is verified.

Problem-12

Evaluate both sides of the divergence theorem

$$\oint_S \vec{A}.d\vec{S} = \int_v \nabla.\vec{A}dv$$

For each of the following cases:

(a) $\vec{A} = xy^2\hat{a}_x + y^3\hat{a}_y + y^2z\hat{a}_z$ and S is the surface of the cuboid defined by $0 < x < 1, 0 < y < 1,$
$0 < z < 1$

(b) $\vec{A} = 2rz\hat{a}_r + 3z\sin\phi\hat{a}_\phi - 4r\cos\phi\hat{a}_z$ and S is the surface of the wedge $0 < r < 2, 0 < \varphi < 45°, 0 < z < 5$

(a) Here, $\vec{A} = xy^2\hat{a}_x + y^3\hat{a}_y + y^2z\hat{a}_z$

We evaluate the surface integrals for the six surfaces as follows.

$$\oint_S \vec{A}.d\vec{S} = \int_{x=0}^{1}\int_{y=0}^{1} A_z\big|_{z=1} dxdy + \int_{x=0}^{1}\int_{y=0}^{1} (-A_z)\big|_{z=0} dxdy + \int_{z=0}^{1}\int_{x=0}^{1} A_y\big|_{y=1} dxdz$$

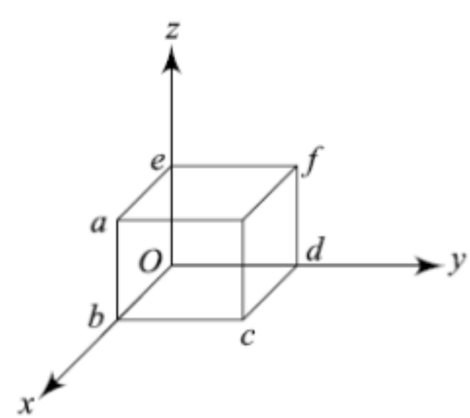
$$+ \int_{z=0}^{1}\int_{x=0}^{1} (-A_y)\big|_{y=0} dxdz + \int_{y=0}^{1}\int_{z=0}^{1} A_x\big|_{x=1} dydz + \int_{y=0}^{1}\int_{z=0}^{1} (-A_x)\big|_{x=0} dydz$$

where $A_X = xy^2$; $A_Y = y^3$; $A_Z = y^2z$

$$\oint_S \vec{A}.d\vec{S} = \int_{x=0}^{1}\int_{y=0}^{1} (y^2 - 0)\,dxdy + \int_{z=0}^{1}\int_{x=0}^{1} (1-0)\,dxdz + \int_{y=0}^{1}\int_{z=0}^{1} (y^2-0)\,dydz$$

$$= \frac{1}{3} + 1 + \frac{1}{3}$$

$$= \frac{5}{3}$$



Also, $\nabla.\vec{A} = \frac{\partial}{\partial x}(xy^2) + \frac{\partial}{\partial y}(y^3) + \frac{\partial}{\partial z}(y^2z) = y^2 + 3y^2 + y^2 = 5y^2$

$$\int_v \nabla.\vec{A}dv = \int_{z=0}^{1}\int_{y=0}^{1}\int_{x=0}^{1} 5y^2\,dxdydz = \int_{y=0}^{1} 5y^2dy = \frac{5}{3}$$

Since, $\int_v \nabla.\vec{A}dv = \oint_S \vec{A}.d\vec{S}$, divergence theorem is verified.

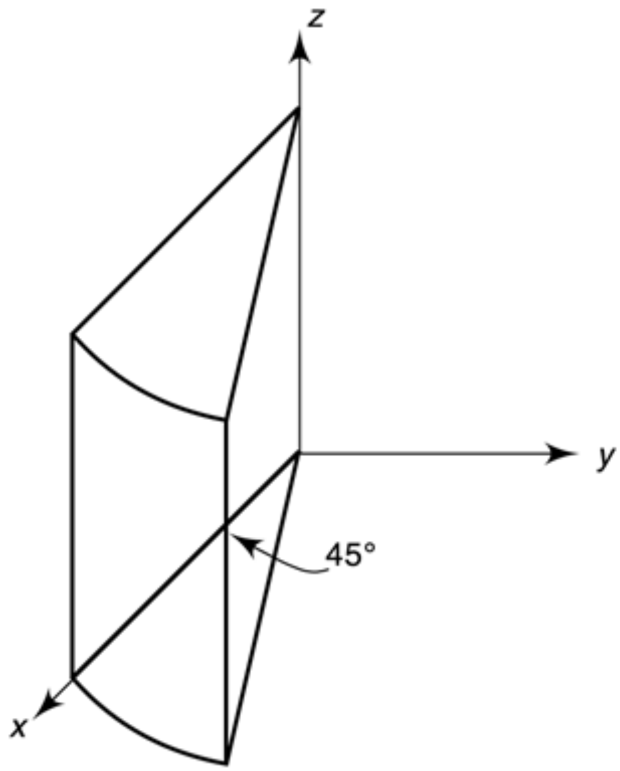
(b) Here, $\vec{A} = 2rz\hat{a}_r + 3z\sin\phi\,\hat{a}_\phi - 4r\cos\phi\,\hat{a}_z$

We evaluate the surface integrals for the six surfaces as follows.

$$\oint_S \vec{A}.d\vec{S} = \int_{z=0}^{5}\int_{\phi=0}^{45^\circ} A_r\Big|_{r=2} r\,d\phi\,dz + \int_{z=0}^{5}\int_{r=0}^{2} A_\phi\Big|_{\phi=45^\circ} dr\,dz + \int_{z=0}^{5}\int_{r=0}^{2} (-A_\phi)\Big|_{\phi=0} dr\,dz$$

$$+ \int_{\phi=0}^{45^\circ}\int_{r=0}^{2} A_z\Big|_{z=5} r\,dr\,d\phi + \int_{\phi=0}^{45^\circ}\int_{r=0}^{2} (-A_z)\Big|_{z=0} r\,dr\,d\phi$$

where $A_r = 2rz$; $A_f = 3z \sin f$; $A_z = -4r\cos\varphi$



$$\oint_S \vec{A}.d\vec{S} = \int_{z=0}^{5}\int_{\phi=0}^{45^\circ} 8z\,d\phi\,dz + \int_{z=0}^{5}\int_{r=0}^{2} \frac{3}{\sqrt{2}} z\,dr\,dz + 0 + \int_{\phi=0}^{45^\circ}\int_{r=0}^{2} -4r\,r\,dr\,d\phi + \int_{\phi=0}^{45^\circ}\int_{r=0}^{2} 4r\,r\,dr\,d\phi$$

$$= 25\pi + \frac{75}{\sqrt{2}}$$

Also,

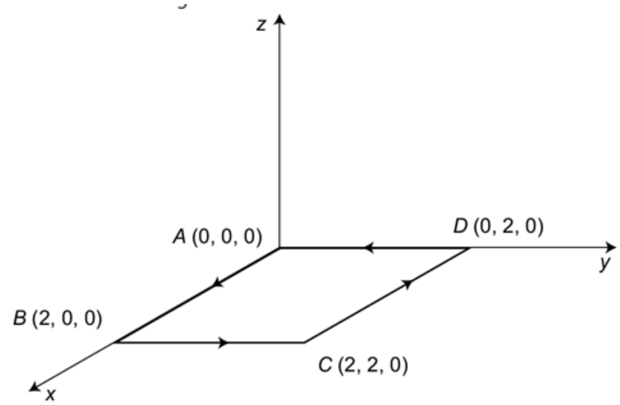
$$\nabla.\vec{A} = \frac{1}{r}\frac{\partial}{\partial r}(2r^2 z) + \frac{1}{r}\frac{\partial}{\partial \varphi}(3z\sin\phi) + \frac{\partial}{\partial z}(-4r\cos\phi) = 4z + \frac{3z}{r}\cos\phi$$

$$\therefore \int_v \nabla.\vec{A}\,dv = \int_{z=0}^{5}\int_{\phi=0}^{45^\circ}\int_{r=0}^{2}\left(4z + \frac{3z}{r}\cos\phi\right) r\,dr\,d\phi\,dz = 25\pi + \frac{75}{\sqrt{2}}$$

Since $\int_v \nabla.\vec{A}\,dv = \oint_S \vec{A}.d\vec{S}$, the divergence theorem is verified.

Problem-13

Evaluate the line integral of the vector function $\vec{F} = x\hat{i} + x^2y\hat{j} + y^2x\hat{k}$ around the square contour *ABCD* in the *xy*-plane as shown in the figure. Also integrate $\nabla \times \vec{F}$ over the surface bounded by *ABCD* and verify that Stokes' theorem holds good.



Line integral

Solution

Here, $\vec{F} = x\hat{i} + x^2y\hat{j} + y^2x\hat{k}$

$$d\vec{l} = dx\hat{i} + dy\hat{j} + dz\hat{k}$$

$$\vec{F}.d\vec{l} = (x\hat{i} + x^2y\hat{j} + y^2x\hat{k}).(dx\hat{i} + dy\hat{j} + dz\hat{k})$$

$$= xdx + x^2ydy + y^2xdz$$

The closed line integral is given as,

$$\oint_L \vec{F}.d\vec{l} = \int_A^B \vec{F}.d\vec{l} + \int_B^C \vec{F}.d\vec{l} + \int_C^D \vec{F}.d\vec{l} + \int_D^A \vec{F}.d\vec{l}$$

Along the path *AB, dy = dz = 0, y = 0, z = 0; x* varies from 0 to 2

$$\int_A^B \vec{F}.d\vec{l} = \int_0^2 xdx = \left[\frac{x^2}{2}\right]_0^2 = 2$$

Along the path *BC, dx = dz* = 0, *x* = 2, *z* = 0; y varies from 0 to 2

$$\int_B^C \vec{F}.d\vec{l} = \int_0^2 x^2ydy\Big|_{x=2} = 4\left[\frac{y^2}{2}\right]_0^2 = 8$$

Along the path *CD, dy = dz* = 0, *y* = 2, *z* = 0; x varies from 2 to 0

$$\int_B^C \vec{F}.d\vec{l} = \int_2^0 xdx = -2$$

Along the path *DA, dx = dz* = 0, *x* = 0, *z* = 0; y varies from 2 to 0

$$\int_B^C \vec{F}.d\vec{l} = \int_0^2 x^2ydy\Big|_{x=0} = 0$$

By addition, we get the closed line integral as,

$$\oint_L \vec{F}.d\vec{l} = 2 + 8 - 2 + 0 = 8$$

Now, the curl of the vector is,

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & x^2y & y^2x \end{vmatrix} = 2xy\hat{i} - y^2\hat{j} + 2xy\hat{k}$$

$$\int_S (\nabla \times \vec{F}).d\vec{S} = \int_{x=0}^2 \int_{y=0}^2 (2xy\hat{i} - y^2\hat{j} + 2xy\hat{k}).(dxdy\hat{k}) = \int_{x=0}^2 \int_{y=0}^2 2xydxdy = 2\left[\frac{x^2}{2}\right]_0^2 \times \left[\frac{y^2}{2}\right]_0^2 = 8$$

Thus, $\oint_L \vec{F}.d\vec{l} = \int_S (\nabla \times \vec{F}).d\vec{S}$; Stokes' theorem is verified.

Problem-14

Compute the line integral of $\vec{F} = 6\hat{i} + yz^2\hat{j} + (3y+z)\hat{k}$ along the triangular path shown in the figure.

Solution

Here, $\vec{F} = 6\hat{i} + yz^2\hat{j} + (3y+z)\hat{k}$

$$d\vec{l} = dx\hat{i} + dy\hat{j} + dz\hat{k}$$

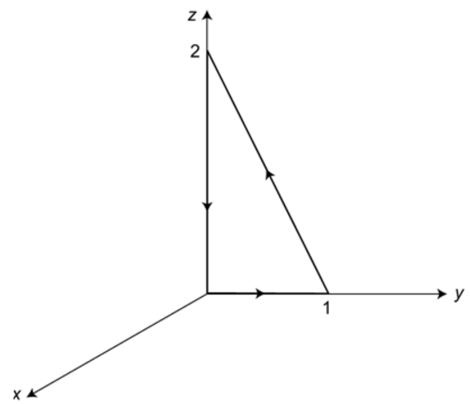
$$\vec{F}.d\vec{l} = \{6\hat{i} + yz^2\hat{j} + (3y+z)\hat{k}\}.(dx\hat{i} + dy\hat{j} + dz\hat{k}) = 6dx + yz^2dy + (3y+z)dz$$

The closed line integral is given as,

$$\oint_L \vec{F}.d\vec{l} = \int_A^B \vec{F}.d\vec{l} + \int_B^C \vec{F}.d\vec{l} + \int_C^A \vec{F}.d\vec{l}$$

Along the path *AB*, *dx* = *dz* = 0, *x* = *z* = 0; *y* varies from 0 to 1.

$$\int_A^B \vec{F}.d\vec{l} = \left[\int_0^1 yz^2dy\right]_{z=0} = 0$$



Line integral

Along the path *BC*, *dx* = 0, *x* = 0; *y* varies from 1 to 0 and *z* varies from 0 to 2.

Also, for this path, the equation relating y and *z* is obtained as,

$$\frac{y}{1} + \frac{z}{2} = 1 \quad \Rightarrow \quad z = 2(1-y)$$

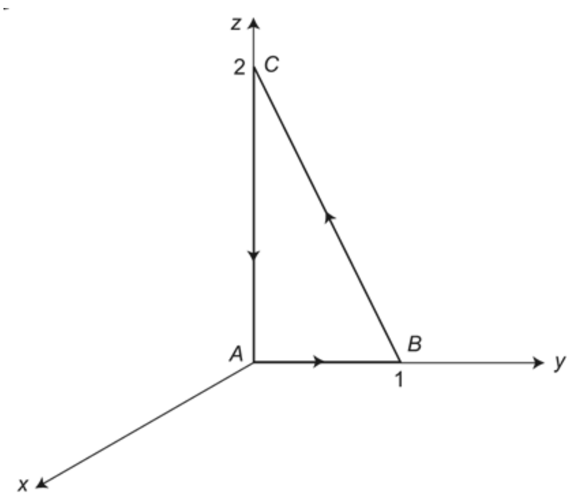
$$\int_B^C \vec{F}.d\vec{l} = \int_{y=1}^{0}\int_{z=0}^{2} yz^2dy + (3y+z)dz = \int_{y=1}^{0} y[2(1-y)]^2dy + \int_{z=0}^{2}\left[3\left(\frac{2-z}{2}\right) + z\right]dz$$

$$= \int_{y=1}^{0} 4(y^3 - 2y^2 + y)dy + \int_{z=0}^{2}\left(3 - \frac{z}{2}\right)dz$$

$$= 4\left[\frac{y^4}{4} - \frac{2y^3}{3} + \frac{y^2}{2}\right]_1^0 + \left[3z - \frac{z^2}{4}\right]_0^2$$

$$= 4\left[-\frac{1}{4} + \frac{2}{3} - \frac{1}{2}\right] + (6-1)$$

$$= \frac{14}{3}$$



Line integral along ABC

Along the path *CA*, *dx* = *dy* = 0, *x* = *y* = 0; z varies from 2 to 0.

$$\int_C^A \vec{F}.d\vec{l} = \left[\int_2^0 (3y+z)dz\right]_{y=0} = -2$$

By addition, we get the closed line integral as,

$$\oint_L \vec{F}.d\vec{l} = 0 + \frac{14}{3} - 2 = \frac{8}{3}$$

Summary

- A quantity that has only magnitude is said to be a scalar quantity, such as time, mass, distance, temperature, work, electric potential, etc. A quantity that has both magnitude and direction is called a *vector* quantity, such as force, velocity, displacement, electric field intensity, etc.
- If the value of the physical function at each point is a scalar quantity then the field is known as a *scalar field*, such as temperature distribution in a building. If the value of the physical function at each point is a vector quantity then the field is known as a *vector field*, such as the gravitational force on a body in space.
- A unit vector $\hat{a}_A$ along $\vec{A}$ is defined as a vector whose magnitude is unity and its direction is along $\vec{A}$. In general, any vector can be represented as

$$\vec{A} = A\hat{a}_A = |\vec{A}|\hat{a}_A$$

where A or $|\vec{A}|$ represents the magnitude of the vector and $\hat{a}_A$, direction of the vector $\vec{A}$.

- Two vectors can be added together by the *triangle rule or parallelogram rule of vector addition*.
- The *dot product* of two vectors $\vec{A}$ and $\vec{B}$, written as $\vec{A}.\vec{B}$, is defined as,

$$\vec{A}.\vec{B} = AB\cos\theta_{AB}$$

where θ_{AB} is the smaller angle between $\vec{A}$ and $\vec{B}$, and $A = |\vec{A}|$ and $B = |\vec{B}|$ represent the magnitude of $\vec{A}$ and $\vec{B}$, respectively.

- The cross product of two vectors $\vec{A}$ and $\vec{B}$, written as $(\vec{A} \times \vec{B})$, is defined as,

$$\vec{A} \times \vec{B} = AB\sin\theta_{AB}\hat{a}_n$$

where $\hat{a}_n$ is the unit vector normal to the plane containing $\vec{A}$ and $\vec{B}$. The direction of the cross product is obtained from a common rule, called *right-hand rule*.

- Three orthogonal coordinate systems commonly used are Cartesian coordinates (*x, y, z*), cylindrical coordinates (*r, φ, z*) and spherical coordinates (*ρ, θ, φ*).
- The differential lengths in three coordinate systems are given respectively as,

$$d\vec{l} = dx\hat{a}_x + dy\hat{a}_y + dz\hat{a}_z$$
$$d\vec{l} = dr\hat{a}_r + rd\phi\hat{a}_\phi + dz\hat{a}_z$$
$$d\vec{l} = d\rho\hat{a}_\rho + \rho d\theta\hat{a}_\theta + \rho\sin\theta d\phi\hat{a}_\phi$$

- The differential areas in three coordinate systems are given respectively as,

$$d\vec{S} = dydz\hat{a}_x + dxdz\hat{a}_y + dxdy\hat{a}_z$$
$$d\vec{S} = rd\phi dz\hat{a}_r + drdz\hat{a}_\phi + rdrd\phi\hat{a}_z$$
$$d\vec{S} = \rho^2\sin\theta d\theta d\phi\hat{a}_\rho + \rho\sin\theta d\rho d\phi\hat{a}_\theta + \rho d\rho d\theta\hat{a}_\phi$$

- The differential volumes in three coordinate systems are given respectively as,

$$dV = dxdydz$$
$$dV = rdrd\phi dz$$
$$dV = \rho^2 \sin\theta \, d\rho d\theta d\phi$$

- For the vector $\vec{F}$ and a path *l*, the line integral is given by,

$$\int_l \vec{F}.d\vec{l} = \int_a^b \left|\vec{F}\right| \cos\theta dl$$

- If the path of integration is a closed curve, the line integral is the circulation of the vector around the path.
- If the line integration of a vector along a closed path is zero, i.e., $\oint_l \vec{F}.d\vec{l} = 0$ then the vector is known as *conservative* or *lamellar*.
- For a vector $\vec{F}$, continuous in a region containing a smooth surface *S*, the surface integral or the flux of $\vec{F}$ through *S* is defined as,

$$\psi = \int_S \vec{F}.d\vec{S} = \int_S \vec{F}.\hat{a}_n dS = \int_S \left|\vec{F}\right| \cos\theta dS$$

where, $\hat{a}_n$ is the unit normal vector to the surface *S*.

- If the surface is a closed surface, the surface integral is the net outward flux of the vector.
- If the surface integral of a vector over a closed surface is zero, i.e., $\oint_S \vec{F}.d\vec{S} = 0$ then the vector is known as a *solenoidal vector*.

- The volume integral of a scalar quantity *F* over a volume *V* is written as,

$$U = \int_V F dv$$

- The differential vector operator () or *del* or *nabla*, defined in Cartesian coordinates as,

$$\nabla = \frac{\partial}{\partial x}\hat{a}_x + \frac{\partial}{\partial y}\hat{a}_y + \frac{\partial}{\partial z}\hat{a}_z$$

is merely a *vector operator, but not a vector quantity*. It is a *vector space function operator*. It is used for performing vector differentiations.

- The *gradient* of a scalar function is both the magnitude and the direction of the maximum space rate of change of that function.
- The gradient of a scalar quantity in three different coordinate systems is expressed respectively as,

$$\nabla F = \frac{\partial F}{\partial x}\hat{a}_x + \frac{\partial F}{\partial y}\hat{a}_y + \frac{\partial F}{\partial z}\hat{a}_z \qquad \text{(Cartesian coordinates)}$$

$$= \frac{\partial F}{\partial r}\hat{a}_r + \frac{1}{r}\frac{\partial F}{\partial \phi}\hat{a}_\phi + \frac{\partial F}{\partial z}\hat{a}_z \qquad \text{(Cylindrical coordinates)}$$

$$= \frac{\partial F}{\partial \rho}\hat{a}_\rho + \frac{1}{\rho}\frac{\partial F}{\partial \theta}\hat{a}_\theta + \frac{1}{\rho \sin\theta}\frac{\partial F}{\partial \phi}\hat{a}_\phi \qquad \text{(Spherical coordinates)}$$

- The *divergence* of a three-dimensional vector field at a point is a measure of how much the vector diverges or converges from that point.
- If the divergence of a vector is zero then the vector is known as a solenoidal vector.
- The divergence of a vector quantity in three different coordinate systems is expressed respectively as,

$$\nabla.\vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \qquad \text{(Cartesian coordinates)}$$

$$= \frac{1}{r}\frac{\partial}{\partial r}(rF_r) + \frac{1}{r}\frac{\partial F_\phi}{\partial \phi} + \frac{\partial F_z}{\partial z} \qquad \text{(Cylindrical coordinates)}$$

$$= \frac{1}{\rho^2}\frac{\partial}{\partial \rho}(\rho^2 F_\rho) + \frac{1}{\rho \sin\theta}\frac{\partial}{\partial \theta}\left(F_\theta \sin\theta\right) + \frac{1}{\rho \sin\theta}\frac{\partial F_\phi}{\partial \phi} \qquad \text{(Spherical coordinates)}$$