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Analytical Mechanics

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Lec.10: Motion of a projectile in a uniform gravitational field

- Motion of a projectile in a Uniform Gravitational Field

b) Linear Air resistance :-

In this case motion is not conservative. The Total energy continually ~~diminishes~~ diminishes as a result of frictional loss

$$\text{Air resistance} \propto \vec{v}$$
$$= -\gamma m \vec{v}$$

$m\gamma \rightarrow$ constant of propotionality
$m \rightarrow$ mass of projectile

Thus there are two forces acting on the projectile
* air resistance $-\gamma m \vec{v}$
* the force of gravity $-mg\hat{k}$

The differential eq. of motion is

$$m \frac{d^2\vec{r}}{dt^2} = -m\gamma\vec{v} - mg\hat{k}$$

or

$$\frac{d^2\vec{r}}{dt^2} = -\gamma\vec{v} - g\hat{k}$$

by using principle forces of the separable type

$\ddot{x} = -\gamma \dot{x}$ ——— (a)

$\ddot{y} = -\gamma \dot{y}$ ——— (b)

$\ddot{z} = -\gamma \dot{z} - g$ ——— (c)

From equation (a

$$\frac{d\dot{x}}{dt} = -\gamma \dot{x} \Longrightarrow \int_{\dot{x}_0}^{\dot{x}} \frac{d\dot{x}}{\dot{x}} = -\int_0^t \gamma \, dt$$

$$\ln \frac{\dot{x}}{\dot{x}_0} = -\gamma t \longrightarrow \dot{x} = \dot{x}_0 e^{-\gamma t} \text{ ------ (1)}$$

and position canbe found from equ. (1)

$$\frac{dx}{dt} = \dot{x}_0 e^{-\gamma t} \longrightarrow x = \int_0^t \dot{x}_0 e^{-\gamma t} dt$$

$$x = -\frac{\dot{x}_0}{\gamma}(e^{-\gamma t} - 1)$$

$$x = \frac{\dot{x}_0}{\gamma}(1 - e^{-\gamma t}) \text{ ------ (2)}$$

from equ. (b)

$$y = \frac{\dot{y}_0}{\gamma}(1 - e^{-\gamma t}) \text{ ——— (3)}$$

From equ - C

$$\ddot{z} = -g - \gamma \dot{z}$$

$$\frac{d\dot{z}}{dt} = -g - \gamma \dot{z}$$

$$\frac{d\dot{z}}{g+\gamma\dot{z}} = -dt$$

~~multiply both sides~~

Multiply by γ both sides and integrals

$$\int_{\dot{z}_0}^{\dot{z}} \frac{\gamma\, d\dot{z}}{g+\gamma\dot{z}} = -\int_0^t \gamma\, dt$$

$$\ln \frac{g+\gamma\dot{z}}{g+\gamma\dot{z}_0} = -\gamma t \longrightarrow g+\gamma\dot{z} = (g+\gamma\dot{z}_0)e^{-\gamma t}$$

$$\dot{z} = \left(\frac{g}{\gamma} + \dot{z}_0\right)e^{-\gamma t} - \frac{g}{\gamma}$$

$$\dot{z} = \dot{z}_0 e^{-\gamma t} - \frac{g}{\gamma}(1-e^{-\gamma t}) \quad \text{-----} \quad (4)$$

and position canbe found

$$\frac{dz}{dt} = \dot{z}_0 e^{-\gamma t} - \frac{g}{\gamma} + \frac{g}{\gamma}e^{-\gamma t}$$

$$z = -\frac{\dot{z}_0}{\gamma}(e^{-\gamma t}-1) - \frac{g}{\gamma}t + \left(\frac{-g}{\gamma^2}\right)\left[e^{-\gamma t}\right]_0^t$$

$$z = \frac{\dot{z}_0}{\gamma}(1-e^{-\gamma t}) - \frac{g}{\gamma}t + \frac{g}{\gamma^2}(1-e^{-\gamma t})$$

$$z = \left(\frac{\dot{z}_0}{\gamma} + \frac{g}{\gamma^2}\right)(1 - e^{-\gamma t}) - \frac{g}{\gamma} t \quad \text{——} \quad (5)$$

from eq. (2), (3) and (5).

when

Zero air resistance, the motion remains entirely in plane $y = bx$ with $b = \frac{\dot{y}_0}{\dot{x}_0}$.

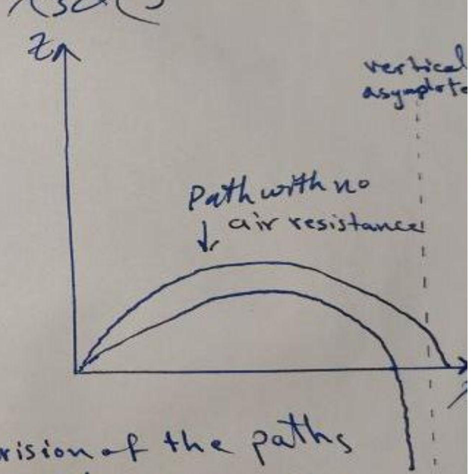
The path in this plane is not parabola, but a curve that lies below the corresponding parabolic trajectory as Fig.

as. $t \longrightarrow \infty$ from equ (2), (3) & (5)

$$x = \frac{\dot{x}_0}{\gamma}$$

$$y = \frac{\dot{y}_0}{\gamma}$$

$z = -\infty$ asymplot



Comparision of the paths of aprojectile moving with and without resistance

This means that the complete trajectory has a vertical asymplote as shown in figure.

The final solution with Linear resistance can be written vectorially in following way:—

$$\vec{r} = \left(\frac{v_0}{\gamma} + \frac{\hat{k} g}{\gamma^2}\right)(1 - e^{-\gamma t}) - \hat{k} \frac{g t}{\gamma}$$