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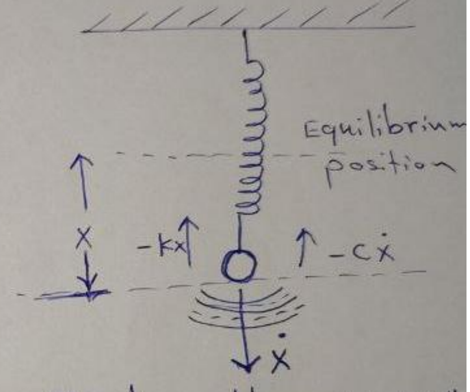
Analytical Mechanics

2023-2024

Lec.3: Damped harmonic motion

Damped Harmonic Motion :-

Let us consider, for example, the motion of an object that is supported by a spring of stiffness k.
We shall assume that there is a viscous retarding force varying linearly with speed



The damped harmonic oscilla

$$m\ddot{x} + c\dot{x} + kx = 0$$

$$m\frac{d^2}{dt^2}(Ae^{qt}) + c\frac{d}{dt}(Ae^{qt}) + k(Ae^{qt}) = 0$$

$$mq^2 + cq + k = 0$$

$$q = \frac{-c \pm (c^2 - 4mk)^{1/2}}{2m}$$

There are three physically distinct cases

I. $c^2 > 4mk$ over damping
II. $c^2 = 4mk$ critical damping
III. $c^2 < 4mk$ under damping

I.

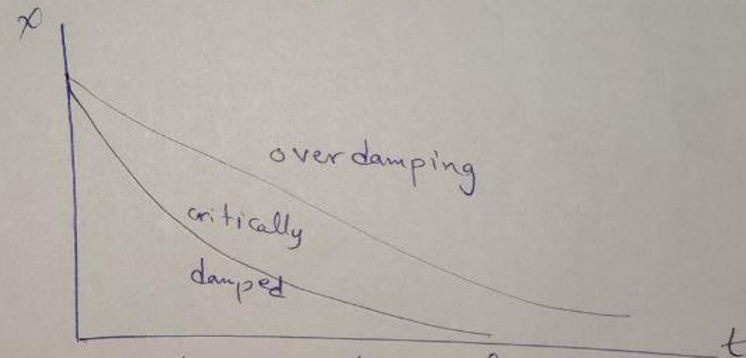
$$q = -\gamma_1 = \frac{-c + \sqrt{c^2 - 4mk}}{2m}$$

$$q = -\gamma_2 = \frac{-c - \sqrt{c^2 - 4mk}}{2m}$$

General solution is

$$x = A e^{-\gamma_1 t} + B e^{-\gamma_2 t} \quad \text{.........} \quad (1)$$

The motion is nonoscillatory, the displacement x decaying to zero in an exponential manner as time goes on as shown in Fig



Graphs of displacement versus time for the overdamped and the critically damped cases of the H.O.

Case II Critical damping

In this case there is only one root

$$\gamma = \frac{c}{2m}$$

To find the general solution for

$$m\ddot{x} + c\dot{x} + kx = 0$$

$$\frac{d^2x}{dt^2} + \frac{c}{m}\frac{dx}{dt} + \frac{k}{m}x = 0$$

$$\frac{d^2x}{dt^2} + 2\gamma\frac{dx}{dt} + \gamma^2 x = 0$$

$$\boxed{\begin{array}{l} \gamma = \dfrac{c}{2m} \longrightarrow 2\gamma = \dfrac{c}{m} \\ c^2 = 4mk \Longrightarrow \dfrac{c^2}{4m} = k \Longrightarrow \dfrac{c^2}{4m^2} = \dfrac{k}{m} \end{array}}$$

$$\left(\frac{d}{dt} + \gamma\right)\left(\frac{d}{dt} + \gamma\right)x = 0$$

Let $u = \gamma x + \frac{dx}{dt}$ which gives

$$\left(\frac{d}{dt} + \gamma\right)u = 0$$

$$\frac{du}{dt} + u\gamma = 0 \longrightarrow \frac{du}{u} = -\gamma\,dt$$

$$\ln\frac{u}{u_0} = -\gamma t \longrightarrow u = u_0 e^{-\gamma t}$$

$$u_0 = A \qquad u = Ae^{-\gamma t}$$

~~III~~ Case III under damping

If resistance c is small so $c^2 < 4mk$

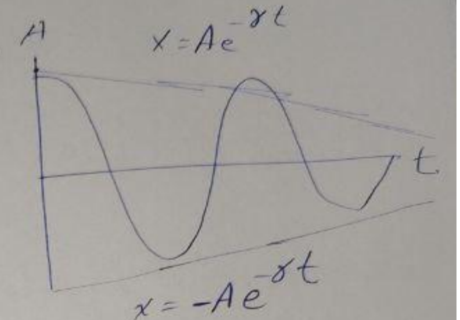
$q \Rightarrow$ Complex

The two roots ~~of~~ are

$$q_1 = -\frac{c}{2m} - i\,\frac{\sqrt{4km - c^2}}{2m}$$

$$q_2 = -\frac{c}{2m} + i\,\frac{\sqrt{4mk - c^2}}{2m}$$

$$x = A e^{-\gamma t} \cos(\omega_1 t + \theta_0)$$



$$W_1 = \sqrt{W_0^2 - \gamma^2}$$

~~Energy so~~