2- Gamma and Beta functions:

2-1. Gamma function $\Gamma(n)$:

Gamma function $\Gamma(n)$ was introduced by the famous mathematican L. Euler (1729), as a natural extension of factorial operation $(n!)$, for positive integer n to real and even complex value of the argument n.

The Gamma function $\Gamma(n)$ is used in mathematical and applied science, such as in the solution of differential equ.s (such as Bessel's and Legendre's functions) the probability theory in statistical mechanics calculated what related to the factorial.

Gamma function can be introduced in the following definitions:

1. $\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx \equiv \int_0^\infty e^{-t} t^{n-1} dt$

2. $\Gamma(n) = 2\int_0^\infty e^{-x^2} x^{2n-1} dx$

3. $\Gamma(n) = \int_0^1 \left[\ln\left(\frac{1}{x}\right)\right]^{n-1} dx$

4. $\Gamma(n+1) = n\,\Gamma(n)$

5. $\Gamma(n+\frac{1}{2}) = \dfrac{1.3.5\ldots\ldots(2n-1)}{2^n}\sqrt{\pi}$

<u>Solutions:</u>

2. Prove that $\Gamma(n) = 2\int_0^\infty e^{-x^2} x^{2n-1}\,dx$

$\because \Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\,dx$

Let $x = t^2$

$\therefore dx = 2t\,dt$

limit: $x = t^2$

at $x = 0 \quad t = 0$

$x \to \infty \quad t \to \infty$

$$\Gamma(n) = \int_0^\infty e^{-t^2} (t^2)^{n-1} . (2t\,dt)$$

$$= 2\int_0^\infty e^{-t^2} t^{2(n-1)} . t\,dt$$

$$= 2\int_0^\infty e^{-t^2} t^{2n-1}\,dt \equiv 2\int_0^\infty e^{-x^2} x^{2n-1}\,dx$$

Note:

$$\boxed{\begin{array}{l} 1\times2\times3\times4\times5\times6 = 6! \\ 1\times3\times5 = 5!! \\ 2\times4\times6 = 6!! \end{array}}$$

4. Prove that $\Gamma(n+1) = n\Gamma(n)$

$$\because \Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx$$

$$\therefore \Gamma(n+1) = \int_0^\infty e^{-x} x^{(n+1)-1} dx$$

$$\Gamma(n+1) = \int_0^\infty e^{-x} x^{n} dx$$

$$\because \boxed{\begin{aligned} d(uv) &= u\,dv + v\,du \\ (uv)\Big|_0^\infty &= \int_0^\infty u\,dv + \int_0^\infty v\,du \\ I = \int_0^\infty u\,dv &= uv\Big|_0^\infty - \int_0^\infty v\,du \end{aligned}}$$

$$\therefore \text{Let } u = x^n \longrightarrow du = n x^{n-1} dx$$

$$dv = e^{-x} dx \longrightarrow v = -e^{-x}$$

$$\therefore \Gamma(n+1) \equiv I = -x^n e^{-x}\Big|_0^\infty - \int_0^\infty (-e^{-x})(n x^{n-1}) dx$$

$$= -x^n e^{-x}\Big|_0^\infty + n \underbrace{\int_0^\infty e^{-x} x^{n-1} dx}_{\Gamma(n)}$$

$$= -\left[(\infty)^n e^{-\infty} - x^n e^{-0}\right] + n\Gamma(n)$$

$$= -\left[0 - 0 \times e^{1}\right] + n\Gamma(n)$$

$$= 0 + n\Gamma(n) = n\Gamma(n)$$

3. Prove that

$$\Gamma(n) = \int_0^1 \left[\ln\left(\frac{1}{x}\right)\right]^{n-1} dx$$

$$\because \Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx$$

let $x = \ln\left(\frac{1}{u}\right) \longrightarrow dx = \frac{1}{\left(\frac{1}{u}\right)} \frac{d}{dx}\left(\frac{1}{u}\right)$

$$= u \frac{d}{dx}(u^{-1})$$

$$= \cancel{u}(-1\, u^{-2} du)$$

$$dx = -\frac{du}{u}$$

$$\because x = \ln\left(\frac{1}{u}\right)$$

$$e^x = \cancel{e}^{\ln\left(\frac{1}{u}\right)}$$

$$e^x = \frac{1}{u} \longrightarrow u = e^{\frac{1}{x}} = e^{-x}$$

$$\therefore \Gamma(n) = \int_1^0 \cancel{u}\left(\ln\frac{1}{u}\right)^{n-1}\left(-\frac{du}{\cancel{u}}\right)$$

$$= -\int_1^0 \left[\ln\left(\frac{1}{u}\right)\right]^{n-1} du$$

$$\Gamma(n) = \int_0^1 \left[\ln\left[\frac{1}{u}\right]\right]^{n-1} du \equiv \Gamma(n) = \int_0^1 \left[\ln\left(\frac{1}{x}\right)\right]^{n-1} dx$$

4. $\Gamma(n+1) = n\,\Gamma(n)$

$= n!$

$$\Gamma(n+1) = n\,\Gamma(n)$$
$$= n\,\Gamma(n+1-1)$$
$$= n\,\Gamma((n-1)+1)$$
$$= n(n-1)\,\Gamma(n-1)$$
$$= n(n-1)\,\Gamma(n-2+1)$$
$$= n(n-1)(n-2)\,\Gamma(n-2)$$
$$= n(n-1)(n-2)\,\Gamma((n-3)+1)$$
$$\Gamma(n+1) = n(n-1)(n-2)(n-3)\,\Gamma(n-3)$$
$$= n(n-1)(n-2)(n-3)\cdots\Gamma(n-(n-1))$$
$$= n(n-1)(n-2)(n-3)\cdots\Gamma(1)$$
$$= n(n-1)(n-2)(n-3)\cdots\times 1$$
$$= n!$$

5 _ prove that

$$\Gamma(2) = 1$$

$$\Gamma(n) = \int_0^\infty e^{-x}\, x^{n-1}\, dx$$

$n=2$

$$\Gamma(2) = 1 \longrightarrow \Gamma(1+1) = 1\,\Gamma(1) = \Gamma(1)$$

6- Prove that

$$\Gamma(\tfrac{1}{2}) = \sqrt{\pi}$$

$$\because \Gamma(n) = 2\int_0^\infty e^{-x^2} x^{2n-1}\, dx$$

$$\Gamma(\tfrac{1}{2}) = 2\int_0^\infty e^{-x^2} x^{2\frac{1}{2}-1}\, dx$$

$$= 2\int_0^\infty e^{-x^2} dx = 2\int_0^\infty e^{-y^2} dy$$

$$\therefore [\Gamma(\tfrac{1}{2})]^2 = 2\int_0^\infty e^{-x^2} dx \cdot 2\int_0^\infty e^{-y^2} dy$$

$$= 4\int_0^\infty e^{-(x^2+y^2)}\, dx\, dy$$

now using the polar coordinates

$$x = r\cos\theta \qquad y = r\sin\theta \qquad r^2 = x^2 + y^2$$

$$dx\, dy = \underbrace{\text{Jacobian}}_{=r}\, dr\, d\theta = r\, dr\, d\theta$$

$$J\left(\frac{x,y}{r,\theta}\right) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$\frac{\partial x}{\partial r} = \frac{\partial r}{\partial r}\cos\theta = \cos\theta$$

$$\frac{\partial y}{\partial r} = \sin\theta$$

$$\frac{\partial x}{\partial \theta} = r\frac{\partial \cos\theta}{\partial \theta}$$

$$= -r\sin\theta$$

$$\frac{\partial y}{\partial \theta} = r\frac{\partial \sin\theta}{\partial \theta}$$

$$= r\cos\theta$$

$$J = \begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta$$

$$= r(\cos^2\theta + \sin^2\theta)$$

$$= r$$

$$\therefore \left[\Gamma\left(\tfrac{1}{2}\right)\right]^2 = 4\int_0^\infty e^{-(x^2+y^2)}\,dx\,dy$$

$$\left[\Gamma\left(\tfrac{1}{2}\right)\right]^2 = 4\int_0^\infty\int_0^{\pi/2} e^{-r^2}\,r\,dr\,d\theta$$

$$= 4\int_0^\infty re^{-r^2}\,dr\int_0^{\pi/2} d\theta$$

$$= 4\cdot\left|\theta\right|_0^{\pi/2}\int_0^\infty re^{-r^2}\,dr$$

$$= \cancel{4}\,\frac{\pi}{2}\cdot\left(-\frac{1}{2}\right)\left|e^{-r^2}\right|_0^\infty$$

$$= -\pi\left[e^{-\infty} - e^{-0}\right]$$

$$= -\pi\,[0-1]$$

$$= \pi$$

$$\left[\Gamma\left(\tfrac{1}{2}\right)\right]^2 = \pi$$

$$\therefore \Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}$$

- Find the integral

$$\int_0^\infty \sqrt{y}\; e^{-y^3}\, dy$$

Using $\Gamma(n)$ function

Basic properties of $\Gamma(n)$:

1) $\Gamma(1) = 1$; $\Gamma(2) = 1$

$\Gamma(3) = \Gamma(1+2) = 2\,\Gamma(2) = 2$

2) $\Gamma(n+1) = n\,\Gamma(n)$

3) If n is a positive integer, then

$\Gamma(n+1) = n!$

4) $\Gamma(\frac{1}{2}) = \int_0^\infty x^{-\frac{1}{2}}\, e^{-x}\, dx = \sqrt{\pi}$

5) If n a positive integer, then

$\Gamma(n+\frac{1}{2}) = ?$

If $n=0$

$\Gamma(0+\frac{1}{2}) = \Gamma(\frac{1}{2}) = \boxed{\sqrt{\pi}}$

$n=1$

$\Gamma(1+\frac{1}{2}) = \frac{1}{2}\,\Gamma(\frac{1}{2}) = \boxed{\frac{1}{2}\sqrt{\pi}}$

$n=2$

$$\Gamma\left(2+\frac{1}{2}\right)=\Gamma\left(1+\left(1+\frac{1}{2}\right)\right)$$

$$=\Gamma\left(1+\frac{3}{2}\right)=\frac{3}{2}\,\Gamma\left(\frac{3}{2}\right)$$

$$=\frac{3}{2}\,\Gamma\left(1+\frac{1}{2}\right)$$

$$=\frac{3}{2}\,\frac{1}{2}\,\Gamma\left(\frac{1}{2}\right)$$

$$=\frac{1}{2}\,\frac{3}{2}\,\Gamma\left(\frac{1}{2}\right)$$

$$=\boxed{\frac{1}{2}\,\frac{3}{2}\,\sqrt{\pi}}$$

$n=3$

$$\Gamma\left(3+\frac{1}{2}\right)=\Gamma\left(1+\frac{5}{2}\right)=\frac{5}{2}\,\Gamma\left(\frac{5}{2}\right)$$

$$=\frac{5}{2}\,\frac{3}{2}\,\Gamma\left(\frac{3}{2}\right)$$

$$\Gamma\left(3+\frac{1}{2}\right)=\frac{5}{2}\,\frac{3}{2}\,\frac{1}{2}\,\Gamma\left(\frac{1}{2}\right)$$

$$=\boxed{\frac{5.3.1}{2.2.2}\sqrt{\pi}}$$

or in general

$$\therefore\ \Gamma\left(n+\frac{1}{2}\right)=\frac{1.3.5.7\ldots.(2n-1)}{2^n}\sqrt{\pi}$$

$$\boxed{\Gamma\left(n+\frac{1}{2}\right)=\frac{(2n-1)!!}{2^n}\sqrt{\pi}}$$

6) For integer n

$$\boxed{\Gamma\left(\frac{1}{2}-n\right) = \frac{(-1)^n\, 2^n}{(2n-1)!!}\sqrt{\pi}}$$

for

$n=0$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

$n=1$

$$\Gamma\left(\frac{1}{2}-1\right) = \frac{(-1)^1\, 2^1}{(2\times 1-1)!!}\sqrt{\pi}$$

$$= -2\sqrt{\pi}$$

$n=2$

$$\Gamma\left(\frac{1}{2}-2\right) = \frac{(-1)^2\, 2^2}{(2\times 2-1)!!}\sqrt{\pi}$$

$$= \frac{4}{3!!}\sqrt{\pi} = \frac{4}{3}\sqrt{\pi}$$

$n=3$

$$\Gamma\left(\frac{1}{2}-3\right) = \frac{(-1)^3\, 2^3}{(2\times 3-1)!!}\sqrt{\pi}$$

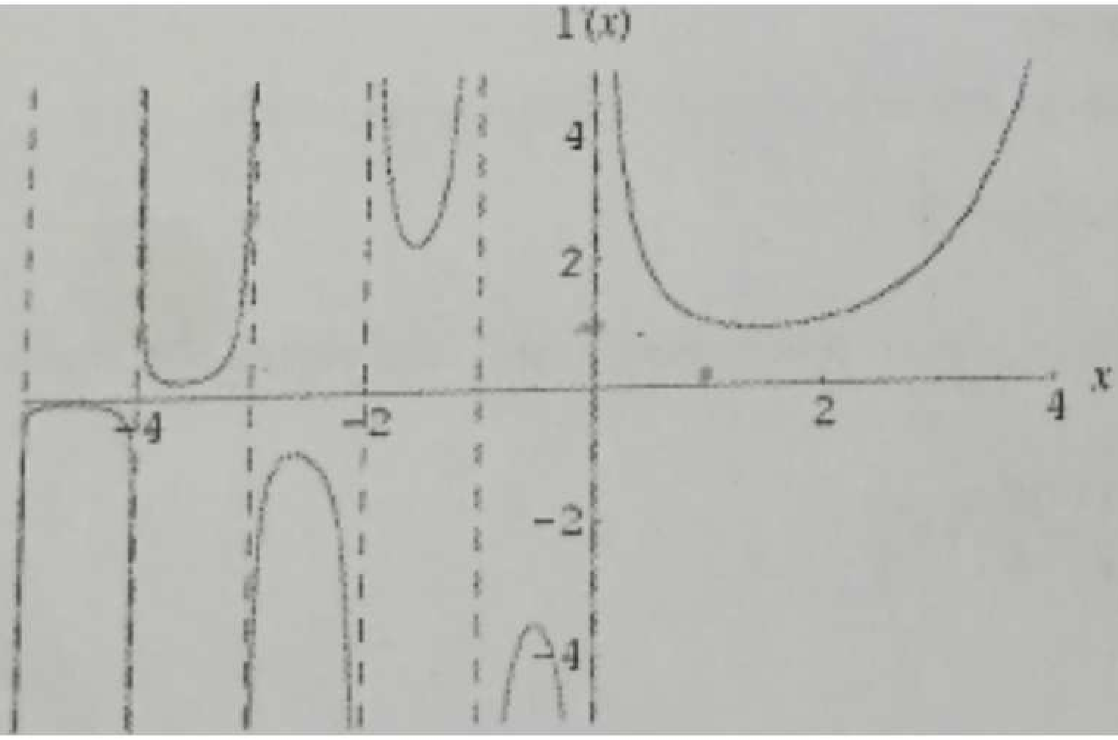
$$= \frac{-8}{5!!}\sqrt{\pi}$$

$$= \frac{-8}{5\times 3\times 1}\sqrt{\pi}$$

$$= \frac{-8}{15}\sqrt{\pi}$$

A table of the values of $\Gamma(n)$ $n = 1 \longrightarrow 2$

n	$\Gamma(n)$
1	1
1.1	0.9514
1.2	0.9182
1.3	0.8975
1.4	0.8873
1.5	0.8835
1.6	0.8935
1.7	0.9286
1.8	0.9314
1.9	0.9618
2.0	1



- Home Work. Evaluate the following.

1. $\frac{\Gamma(6)}{2\Gamma(3)}$

2. $\frac{\Gamma(\frac{5}{2})}{\Gamma(\frac{1}{2})}$

3. $\frac{\Gamma(3)\,\Gamma(2.5)}{\Gamma(5.5)}$

4. $\frac{\Gamma(2)\,\Gamma(-\frac{1}{2})}{\Gamma(3)}$

5. Evaluate the following integral (using the definitions of $\Gamma(n)$)

$$I = \int_0^\infty \sqrt{y}\; e^{-y^3}\, dy$$

Let $x = y^3$ in order to reform the shape of the function which will be integrated simillar to Gamma function shape

$$\left(\Gamma(n) = \int_0^\infty e^{-x}\, x^{n-1}\, dx \right)$$

i.e $x = y^3 \longrightarrow y^{3\cdot\frac{1}{3}} = x^{1/3}$

$$\boxed{y = x^{1/3}}$$

$$dy = \frac{1}{3} x^{\frac{1}{3}-1}\, dx$$

$$\boxed{dy = \frac{1}{3} x^{-\frac{2}{3}}\, dx}$$

$$\because I = \int_0^\infty y^{\frac{1}{2}} e^{-y^3} dy$$

Limits: $y = x^{1/3}$

at $y=0 \quad x=0$

$y=\infty \quad x=\infty$

$$= \int_0^\infty (x^{\frac{1}{3}})^{1/2} \cdot e^{-x} \left(\frac{1}{3} x^{-2/3} dx\right)$$

$$= \frac{1}{3} \int_0^\infty x^{1/6} \cdot x^{-2/3} e^{-x} dx$$

$$= \frac{1}{3} \underbrace{\int_0^\infty x^{-1/2} e^{-x} dx}_{\Gamma(\frac{1}{2})} = \frac{1}{3} \Gamma(\tfrac{1}{2}) = \frac{\sqrt{\pi}}{3}$$

6. Find the integral

$$I = \int_0^\infty e^{-\sqrt{x}} dx \text{ ------ } (1)$$

$$\because \Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx \text{ ------ } (2)$$

Let $y = \sqrt{x}$

i.e. $y = x^{1/2}$

$dy = \frac{1}{2} x^{-1/2} dx$

$dx = 2y\, dy$

substitute $y = \sqrt{x}$ & dx in equ. (1)

$$I = \int_0^\infty e^{-y} (2y\,dy)$$

$$= 2 \underbrace{\int_0^\infty e^{-y} y^1 dy}_{\Gamma(2)} \text{ ---- } (3)$$

Compare equ. ③ with equ. ②

$\therefore n-1 = 1 \qquad n = 2$

$\therefore I = 2\Gamma(2) = 2$

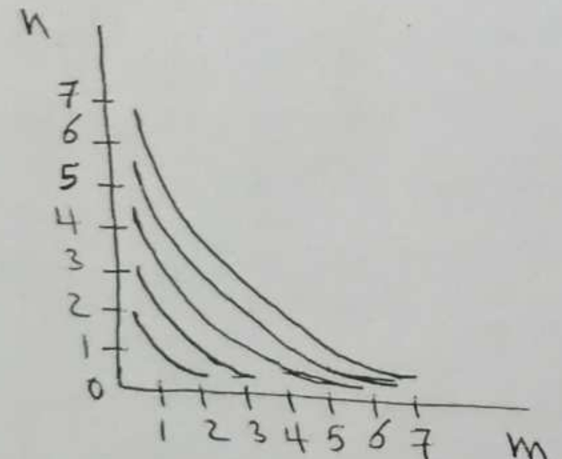
2. Beta function :-

Definition of Beta function

a function in the following form is called Beta-function

1) $\beta(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx \qquad m \,\&\, n > 0$

the following diagram represent β-function



note that β-function is symmetric

i.e. $\beta(m,n) = \beta(n,m)$

Proof: setting

$y = 1 - x \longrightarrow dy = -dx$

$\therefore \beta(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1}\, dx$

2) $\beta(m,n) = -\int_1^0 (1-y)^{m-1}\, y^{n-1}\, dy$

$$= \int_0^1 y^{n-1} (1-y)^{m-1}\, dy = \beta(n,m)$$

3) Trigonometric form of Beta function

$$\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \, \cos^{2n-1}\theta \, d\theta$$

The proof:

$\because \beta(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1}\, dx$ ------- (1)

setting $x = \sin^2\theta$

$dx = 2\sin\theta \cos\theta \, d\theta$

Limits: $\because x = \sin^2\theta$

at $x = 0 \longrightarrow \theta = 0$

$x = 1 \longrightarrow \theta = \pi/2$

substitute in equ. (1)

$$\beta(m,n) = \int_0^{\pi/2} (\sin^2\theta)^{m-1} \underbrace{(1-\sin^2\theta)}_{\cos^2\theta}{}^{n-1} (2\sin\theta\cos\theta\, d\theta)$$

$$= 2\int_0^{\pi/2} \sin^{2m-2}\theta\,(\cos^2\theta)^{n-1}\sin\theta\cos\theta\,d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2m-1}\theta\,\cos^{2n-1}\theta\,d\theta$$

4) $$\beta(m,n) = \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}}\,dy = \int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}}\,dy$$

<u>The proof:</u>

$$\because\ \beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

setting $x = \dfrac{y}{1+y}$

$$\therefore\ y = \frac{x}{1-x}$$

$$dx = \frac{(1+y)-y}{(1+y)^2}\,dy$$

$$dx = \frac{dy}{(1+y)^2}$$

$$\therefore\ dy = (1+y)^2\,dx$$

Limits: $\because\ x = \dfrac{y}{1+y}$

at $x = 0 \longrightarrow y = 0$

$x = 1 \longrightarrow y = \infty$

Then

$$\beta(m,n) = \int_0^\infty \left(\frac{y}{1+y}\right)^{m-1} \left(1-\frac{y}{1+y}\right)^{n-1} \frac{dy}{(1+y)^2}$$

$$= \int_0^\infty \left(\frac{y}{1+y}\right)^{m-1} \left(\frac{1}{1+y}\right)^{n-1} \frac{dy}{(1+y)^2}$$

$$= \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}}\, dy = \int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}}\, dy$$

$(\beta(m,n) = \beta(n,m))$

5) $$\beta(m,n) = \int_0^1 \frac{y^{m-1} + y^{n-1}}{(1+y)^{m+n}}\, dy$$

6) $$\beta(m,n) = a^m b^n \int_0^\infty \frac{y^{m-1}}{(ay+b)^{m+n}}\, dy$$

<u>The proof:</u>

Let $X = \frac{ay}{ay+b}$

<u>or</u>

$$\beta(m,n) = \frac{1}{a^{m-1}\, a^{n-1}} \int_0^a y^{m-1} (a-y)^{n-1}\, dy$$

Let $X = \frac{y}{a}$

2.2. Relation between Beta function and Gamma function.

$$\beta(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)} \quad \text{-------} (7)$$

$$= \beta(n,m)$$

<u>proof:</u>

From the definition of $\Gamma(n)$ function

$$\Gamma(n) = 2\int_0^\infty e^{-t^2}\, t^{2n-1}\, dt.$$

<u>or</u> $$\Gamma(m) = 2\int_0^\infty e^{-x^2}\, x^{2m-1}\, dx$$

$$\Gamma(n) = 2\int_0^\infty e^{-y^2}\, y^{2n-1}\, dy$$

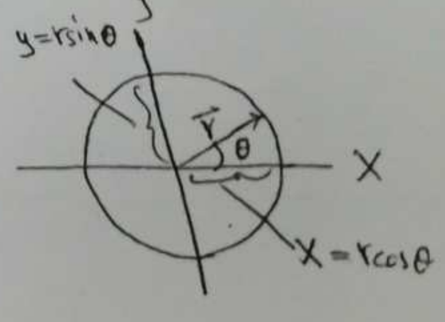
$$\therefore\ \Gamma(m)\,\Gamma(n) = \left(2\int_0^\infty e^{-x^2}\, x^{2m-1}\, dx\right)\left(2\int_0^\infty e^{-y^2}\, y^{2n-1}\, dy\right) \text{-----} (1)$$

$$= 4\int_0^\infty\int_0^\infty x^{2m-1}\, y^{2n-1}\, e^{-(x^2+y^2)}\, dx\, dy$$

$\because\ x = r\cos\theta\ ;\ y = r\sin\theta$

$\therefore\ r = \sqrt{x^2+y^2}$

$dx\,dy = r\,dr\,d\theta$



Limit: $r = 0 \longrightarrow \infty$

$\theta = 0 \longrightarrow \pi/2$

substitute in equ. (1)

$$\Gamma(m)\Gamma(n) = 4\int_0^{\pi/2}\int_0^{\infty} \underbrace{r^{2m-1}\cos^{2m-1}\theta\, r^{2n-1}\sin^{2n-1}\theta\, e^{-r^2}}_{\underline{\underline{H.W.}}}\, r\, d\theta\, dr$$

$$= 4\int_0^{\pi/2}\int_0^{\infty} r^{2m-\cancel{1}+2n-1+\cancel{1}}\cos^{2m-1}\theta\,\sin^{2n-1}\theta\, e^{-r^2}\, d\theta\, dr$$

$$\Gamma(m)\Gamma(n) = \Big[\underbrace{2\int_0^{\pi/2}\cos^{2m-1}\theta\,\sin^{2n-1}\theta\, d\theta}_{\beta(m,n)}\Big]\Big[\underbrace{2\int_0^{\infty} r^{2(m+n)-1} e^{-r^2}\, dr}_{\Gamma(m+n)}\Big]$$

$$\therefore\ \Gamma(m)\Gamma(n) = \beta(m,n)\,\Gamma(m+n)$$

$$\therefore\ \beta(m,n) = \frac{\Gamma(n)\Gamma(m)}{\Gamma(m+n)}$$

<u>Exersices:</u> Find the value of the following integrals

1. $\int_{\pi/2}^{0} \sin^2\theta\, d\theta$

2. $\int_0^{\pi/2} \tan^2\theta\, d\theta$

Find the following integrals.

$$I = \int_{\pi/2}^{0} \sqrt{\sin^3\theta}\; d\theta$$

$$I = \int_{\pi/2}^{0} \sin^{3/2}\theta \; d\theta$$

$$I = -\int_{0}^{\pi/2} \sin^{3/2}\theta \underbrace{\cos^0\theta}_{=1} \, d\theta \text{ ------- } (1)$$

since $\beta(m,n) = 2\int_{0}^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$ ---- (2)

Compare equ.s (1) and (2), then

$$2m-1 = \frac{3}{2} \longrightarrow 2m = \frac{5}{2} \qquad \therefore m = \frac{5}{4}$$

$$2n-1 = 0 \longrightarrow 2n = 1 \qquad \therefore n = \frac{1}{2}$$

then

$$I = -\frac{\beta(\frac{5}{4}, \frac{1}{2})}{2}$$

$$\because \beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

$$\therefore I = -\frac{1}{2} \frac{\Gamma(\frac{5}{4})\Gamma(\frac{1}{2})}{\Gamma(\frac{5}{4}+\frac{1}{2})} = -\frac{1}{2}\sqrt{\pi} \frac{\Gamma(\frac{5}{4})}{\Gamma(\frac{7}{4})}$$

Find $\Gamma(\frac{5}{4})$ & $\Gamma(\frac{7}{4})$

$$I=\int_0^1 (1-Z^2)^{3/2}\,dZ$$

$$\because\ \beta(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}\,dx \quad \text{------}\ (1)$$

Let $x=Z^2 \longrightarrow dx=2Z\,dZ$

$Z=x^{1/2} \longrightarrow dZ=\frac{1}{2}x^{-1/2}dx$

Limit: $\because\ Z=x^{1/2}$ at $Z=0 \quad x=0$

at $Z=1$

$1=x^{1/2}$

$x=1^{-1/2} \Longrightarrow x=1$

$$\therefore\ I=\int_0^1 (1-x)^{3/2}\left(\frac{1}{2}x^{-1/2}dx\right)$$

$$=\frac{1}{2}\int_0^1 x^{-1/2}(1-x)^{3/2}\,dx \quad \text{-----}\ (2)$$

$\therefore\ n-1=\frac{3}{2} \Longrightarrow n=\frac{5}{2}$

$m-1=-\frac{1}{2} \longrightarrow m=\frac{1}{2}$

$$\therefore\ I=\frac{1}{2}\beta\left(\frac{5}{2},\frac{1}{2}\right)$$

$$=\frac{1}{2}\,\frac{\Gamma(\frac{5}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{5}{2}+\frac{1}{2})}$$

$$I = \int_0^1 x^4 (1-x)^3 \, dx \quad \text{-------} \; ①$$

$$\because \beta(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} \, dx \quad \text{-------} \; ②$$

Compare equs ① & ②

$$\therefore \; m-1 = 4 \longrightarrow m = 5$$

$$n-1 = 3 \longrightarrow n = 4$$

$$\because \beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

$$\therefore I = \frac{\Gamma(5)\Gamma(4)}{\Gamma(5+4)}$$

$$= \frac{4!\,3!}{8!} = \frac{4!\,\cancel{3!}}{8\times7\times6\times5\times4\times\cancel{3!}}$$

$$= \frac{\cancel{4}\times\cancel{3}\times2\times1}{8\times7\times\cancel{6}\times5\times\cancel{4}}$$

$$= \frac{1}{280}$$

$$I = \int_0^\infty 3^{-4x^2} \, dx$$

$$\because a^{\alpha x} = e^{\ln a^{\alpha x}} \qquad a \text{; is a const.}$$

$$\therefore I = \int_0^\infty e^{\ln(3^{-4x^2})} \, dx$$

$$= \int_0^\infty e^{-4x^2 \ln(3)} \, dx$$

Let $y = 4x^2 \ln(3)$

$$\therefore \quad x^2 = \frac{y}{4\ln(3)}$$

$$x = \frac{y^{1/2}}{\sqrt{4\ln 3}}$$

$$dx = \frac{\frac{1}{2}\, y^{-1/2}}{\sqrt{4\ln 3}}\, dy$$

$$\therefore \quad I = \frac{1}{4\sqrt{\ln 3}} \int_0^\infty \underbrace{y^{-1/2}\, e^{-y}\, dy}_{\Gamma(\frac{1}{2}) = \sqrt{\pi}}$$

$$\therefore \quad I = \frac{\sqrt{\pi}}{4\sqrt{\ln(3)}}$$