Gamma Function $\Gamma(n)$

Definitions:

$$\boxed{\Gamma(n) = \int_0^\infty x^{n-1} e^{-x}\, dx}$$ الصيغة العامة - (1)

$$\Gamma(n) = 2\int_0^\infty x^{2n-1} e^{-x^2}\, dx \quad \ldots \quad (2)$$

$$\Gamma(n) = \int_0^1 \left[\ln\left(\tfrac{1}{x}\right)\right]^{n-1} dx \quad \ldots \quad (3)$$

$$\Gamma(n+1) = n! \quad \ldots \quad (4)$$

$$\Gamma(n+1) = n\,\Gamma(n) \quad \ldots \quad (5)$$

How to evaluate $\Gamma(n)$

$\Gamma(n)$

$n=0$	n: عدد صحيح		الاعداد الكسرية
	n موجب	n سالب	
$\Gamma(0)=\infty$	$\Gamma(n)=(n-1)!$	$\Gamma(-1)=-\infty$	$\Gamma(\frac{1}{2})=\sqrt{\pi}$
	$\Gamma(3)=(3-1)! = 2!$	$\Gamma(-2)=\infty$	$\Gamma(n)=(n-1)\,\Gamma(n-1)$
	$\Gamma(2)=1$	$\Gamma(-3)=-\infty$	نطرح 1 : كسر موجب
	$\Gamma(1)=1$	$\vdots$	$\Gamma(\frac{3}{2})=(\frac{3}{2}-1)\,\Gamma(\frac{3}{2}-1) = \frac{1}{2}\,\Gamma(\frac{1}{2})$
			$\Gamma(n+1)=n\,\Gamma(n)$
			نضيف 1 : سالب
			$\therefore \Gamma(n)=\frac{\Gamma(n+1)}{n}$
			$\Gamma(-\frac{1}{2})=\frac{\Gamma(-\frac{1}{2}+1)}{-\frac{1}{2}} = \frac{\Gamma(\frac{1}{2})}{-\frac{1}{2}} = -2\,\Gamma(\frac{1}{2}) = -2\sqrt{\pi}$

$$\boxed{\Gamma(n) = \int_0^\infty x^{n-1}\, e^{-x}\, dx}$$

الصيغة العامة $\Gamma(n)$
يمكن اشتقاق العلاقات
2 ، 3 ، 4 ، 5 من
هذه الصيغة لـ Γn

الشروط الواجبة للحل بدالة كاما

١- التكامل محدد من 0 الى ∞
٢- تواجد الدالة الاسية e^{-x}
٣- اس الدالة e هو $-x$ وليس ($-x^2$ ، $-2x$ ، $x+1$)
٤- تواجد x مرفوعة للاس الرمزي (x^{n-1}) مثل x^2 ، x^3 ، ... $x^{1/2}$

القيمة العددية لدالة كاما .

عندما يكون n عدد صحيح $\Gamma(n) = (n-1)!$

" " n كسري موجب $\Gamma(n) = (n-1)\Gamma(n)$..

" " " " سالب $\Gamma(n) = \dfrac{\Gamma(n+1)}{n}$

حالات دالة كاما

الحالة	الفرض المناسب	حدود التكامل
$e^{f(x)}$	Let $f(x) = -t$	$0 \to \infty$
$\ln(x)$	" $\ln x = -t$	$0 \to 1$
$\ln(\frac{1}{x})$	" $\ln x = +t$	$0 \to 1$
$a^{f(x)}$	$f(x)\ln a = t$	$0 \to \infty$

قيم الـ ln

$$\ln(0) = -\infty$$
$$\ln(1) = 0$$
$$\ln(e) = 1$$
$$\ln(\infty) = \infty$$

Evaluate the integral

$$I = \int_0^a y^4 \sqrt{a^2-y^2}\, dy$$

Solution: $\sqrt{a^2-y^2}$ نبسط المقدار

$$I = \int_0^a y^4\, a\left(1-\frac{y^2}{a^2}\right)^{\frac{1}{2}} dy$$

$$= a\int_0^a y^4 \left(1-\frac{y^2}{a^2}\right) dy \quad \text{—} \ (1)$$

$$\because \beta(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx \quad \text{—} \ (2)$$

يجب تبسيط المقدار $\left(1-\frac{y^2}{a^2}\right)$ ليكون مشابه

لـ $(1-x)$

$$\therefore \text{Let } x = \frac{y^2}{a^2}$$

$$y^2 = a^2 x \rightarrow y = a\, x^{\frac{1}{2}} \rightarrow dy = \frac{a}{2} x^{-\frac{1}{2}} dx$$

Limit: $\because x = \frac{y^2}{a^2}$

y	x
0	0
a	1

$$\therefore I = a\int_0^1 (a x^{\frac{1}{2}})^4 (1-x)^{\frac{1}{2}} \cdot \left(\frac{a}{2} x^{-\frac{1}{2}} dx\right)$$

$$= \frac{a^6}{2}\int_0^1 x^2 (1-x)^{\frac{1}{2}} x^{-\frac{1}{2}} dx$$

$$= \frac{a^6}{2}\int_0^1 x^{2-\frac{1}{2}} (1-x)^{\frac{1}{2}} dx$$

$$= \frac{a^6}{2}\int_0^1 x^{\frac{3}{2}} (1-x)^{\frac{1}{2}} dx \quad \ldots (3)$$

مقارنة المعادلة (3) بالمعادلة (2)

$\therefore\quad m-1=\frac{3}{2} \longrightarrow m=1+\frac{3}{2}=\frac{5}{2}$

$n-1=\frac{1}{2} \longrightarrow n=\frac{3}{2}$

$$\because \quad (m,n)=\frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$$

$$=\frac{\Gamma(\frac{5}{2})\,\Gamma(\frac{3}{2})}{\Gamma(\frac{5}{2}+\frac{3}{2})}=\frac{\Gamma(\frac{5}{2})\,\Gamma(\frac{3}{2})}{\Gamma(4)}$$

$$\Gamma(\frac{5}{3})=\Gamma(\frac{3}{2}+1)=\frac{3}{2}\Gamma(\frac{3}{2})=\frac{3}{2}\Gamma(1+\frac{1}{2})$$

$$=\frac{3}{2}\cdot\frac{1}{2}\Gamma(\frac{1}{2})$$

$$\Gamma(\frac{3}{2})=\frac{1}{2}\Gamma(\frac{1}{2}) \quad ; \quad \Gamma(2)=1$$

$$\therefore\quad I=\frac{a^6}{2}\,\frac{\Gamma(\frac{5}{2})\,\Gamma(\frac{3}{2})}{\Gamma(4)}=\frac{a^6}{2}\,\frac{(\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi})(\frac{1}{2}\sqrt{\pi})}{3!}$$

$$=\frac{a^6}{32}\pi$$

H W Evaluate the integral

$$I=\int_0^\infty 4\,x^4\,e^{-x^4}\,dx$$

Answer $I=\frac{1}{4}\Gamma(\frac{1}{4})$

prove that

$$\int_0^1 x^m (\ln x)^n \, dx = \frac{(-1)^n \, n!}{(m+1)^{n+1}} \quad \cdots (1)$$

Solution: Let $\ln x = -t \quad \boxed{x = e^{-t}}$

$$dx = -e^{-t} dt$$

Limits . $\boxed{x = 0} \rightarrow \ln(x) = -t \rightarrow \ln(0) = -t$

$$\ln(0) = -t \rightarrow \infty = -t \rightarrow \boxed{t = \infty}$$

$$\boxed{x = 1} \rightarrow \ln(1) = -t \rightarrow \boxed{t = 0}$$

نعوض في المعادلة (1)

$$I = \int_\infty^0 (e^{-t})^m \, (-t)^n \, (-e^{-t}) \, dt$$

$$= \cancel{-}\int_0^\infty \left[e^{-mt} \right] \left((-1)^n t^n \right) (\cancel{-}e^{-t}) \, dt$$

$$= \int_0^\infty (-1)^n \, e^{-(m+1)t} \, t^n \, dt$$

$$I = (-1)^n \int_0^\infty e^{-(m+1)t} \, t^n \, dt \quad \cdots (2)$$

بالمقارنة مع دالة كاما

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-t} \, dt$$

لا يمكن المقارنة لوجود $(m+1)$

Let $-(m+1)t = -y \rightarrow (m+1)t = y \rightarrow dt = \frac{dy}{(m+1)}$

Limits $t = 0 \rightarrow y = 0$ & $t = \infty \rightarrow y = \infty$

نعوض في المعادلة (2)

$$I = (-1)^n \int_0^\infty e^{-y} \left(\frac{y}{m+1} \right)^n \cdot \left(\frac{dy}{m+1} \right)$$

$$I = (-1)^n \int_0^\infty e^{-y} \left[\frac{y^n}{(m+1)^n}\right] \frac{dy}{(m+1)^1}$$

$$= \frac{(-1)^n}{(m+1)^{n+1}} \underbrace{\int_0^\infty e^{-y} y^n \, dy}_{\Gamma(n+1) = n!}$$

$$\therefore I = \left[(-1)^n / (m+1)^{n+1}\right] n! \, .$$

$$I = \int_0^\infty \frac{x^a}{a^x} \, dx \qquad a > 1$$

solution: $I = \int_0^\infty x^a a^{-x} \, dx$ $\qquad \int_0^\infty x^3 \left(\frac{1}{2}\right)^x dx$

Let $\boxed{x \ln a = t} \longrightarrow dx = \frac{dt}{\ln a}$

at $x = 0 \longrightarrow t = 0 \qquad x = \infty \longrightarrow t = \infty$

$$I = \int_0^\infty \left(\frac{t}{\ln a}\right)^a e^{-t} \frac{dt}{\ln a}$$

$$= \frac{1}{(\ln a)^{a+1}} \int_0^\infty t^a e^{-t} \, dt$$

$$= \frac{\Gamma(a+1)}{(\ln a)^{a+1}}$$

Evaluate the following integrals:

$$I = \int_0^1 \frac{dx}{\sqrt{\ln\left(\frac{1}{x}\right)}} \equiv \int_0^1 \frac{dx}{\sqrt{-\ln x}} \quad \dots (1)$$

Solution: Let $\ln\left(\frac{1}{x}\right) = t$

$$\ln 1 - \ln x = t \longrightarrow -\ln x = t \qquad \ln(1) = 0$$

$$+\ln x = -t \longrightarrow e^{\ln x} = e^{-t} \rightarrow x = e^{-t}$$

$$dx = -e^{-t}\, dt$$

Limits: at $\boxed{x = 0} \longrightarrow \because \ln\left(\frac{1}{x}\right) = t \rightarrow \ln\left(\frac{1}{0}\right) = t$

$$\ln(\infty) = t \longrightarrow \boxed{t = \infty}$$

Note: $\ln \infty = \infty \qquad \ln(1) = 0$

$\ln 0 = -\infty \qquad \ln e = 1$

at $\boxed{x = 1} \longrightarrow \ln\left(\frac{1}{x}\right) = t \rightarrow \ln\left(\frac{1}{1}\right) = t \rightarrow \boxed{t = 0}$

نعوض بالمعادلة (1)

$$\therefore I = \int_{\infty}^{0} \frac{1}{\sqrt{t}} \left(-e^{-t}\, dt\right)$$

$$= -\int_0^{\infty} t^{-\frac{1}{2}} \left(-e^{-t}\, dt\right) = \int_0^{\infty} t^{-\frac{1}{2}}\, e^{-t}\, dt$$

نقارن مع دالة كاما

$$\Gamma(n) = \int_0^{\infty} x^{n-1} e^{-x}\, dt$$

$$\therefore n - 1 = -\frac{1}{2} \longrightarrow n = \frac{1}{2}$$

$$\therefore I = \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

H.W. $I = \displaystyle\int_0^{1/3} \frac{dx}{\sqrt{\ln\left(\frac{1}{3x}\right)}}$

$$I = \int_0^1 [x \ln(x)]^2 \, dx \quad —(1)$$

Solution:

① Let $\boxed{\ln(x) = -t} \rightarrow x = e^{-t}$

② نفاضل $\quad dx = -e^{-t} dt$

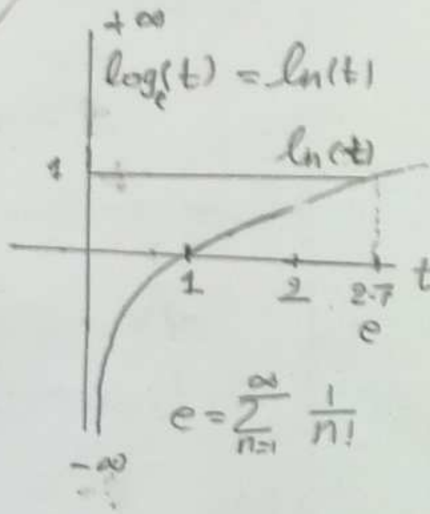
③ New limits

at $\boxed{x=0} \rightarrow \ln(0) = -\infty = -t$

$\therefore$ at $x=0 \rightarrow \boxed{t=\infty}$

at $\boxed{x=1} \rightarrow \ln(1) = 0$

$\rightarrow \boxed{t=0}$



$e = \sum_{n=1}^{\infty} \frac{1}{n!}$

نعوض في المعادلة (1)

$$\therefore I = \int_\infty^0 [e^{-t}(-t)]^2 (-e^{-t} dt)$$

$$= \int_\infty^0 - e^{-2t} t^2 e^{-t} dt$$

$$I = \int_0^\infty e^{-3t} t^2 \, dt = \int_0^\infty t^2 e^{-3t} dt$$

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} dx$$ نقارن مع دالة جاما

نفرض مرة اخرى

Let $-3t = -y \rightarrow 3t = y \rightarrow 3\,dt = dy$

$dt = \frac{dy}{3}$

Limits : $t = 0 \rightarrow y = 0 \quad ; \quad t = \infty \rightarrow y = \infty$

$$\therefore I = \int_0^\infty \left(\frac{y}{3}\right)^2 e^{-y} \frac{dy}{3} = \frac{1}{27} \int_0^\infty y^2 e^{-y} dy$$

نقارن مع دالة $\Gamma(n)$

$\because n-1 = 2 \rightarrow n = 3 \quad \therefore I = \frac{1}{27} \Gamma(3)$

$= \frac{1}{27} 2! = \frac{2}{27}$

Beta Function $\beta(m,n)$

1. $\beta(m,n) = \int_0^1 x^{n-1} (1-x)^{m-1} dx$

2. $\beta(m,n) = \frac{1}{a^{m+n-1}} \int_0^a x^{n-1} (a-x)^{m-1} dx$

3. $\beta(m,n) = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} dx$

4. $\beta(m,n) = 2 \int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$

5. $\beta(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$

حل التكاملات بدالة بيتا (محتويات المسألة)

دوال مثلثية Cosine، Sine
الحل بالمعادلة (4)

الحد العلوي = ∞
الحل بالمعادلة (3)

ارقام عادية

الرقم العلوي ≠ 1 $a =$
(2)

الرقم العلوي = 1
(1)

Example: Evaluate $I = \int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta$ (4)

$$I = \int_0^{\pi/2} \sin^{\frac{1}{2}}\theta \, . \cos^{-\frac{1}{2}}\theta \, d\theta$$

نقارن مع المعادلة (4): نلاحظ I هو $\frac{1}{2}$ المعادلة (4)

$2m-1 = \frac{1}{2} \rightarrow m = \frac{3}{4}$ $\quad 2n-1 = -\frac{1}{2}$ $\quad n = \frac{1}{4}$

$$\therefore I = \frac{1}{2} \beta\left(\frac{3}{4}, \frac{1}{4}\right) = \frac{1}{2} \frac{\Gamma(\frac{3}{4})\,\Gamma(\frac{1}{4})}{\Gamma(1)}$$

يمكن ايجاد قيم $\Gamma(\frac{3}{4})$ و $\Gamma(\frac{1}{4})$ من قانون حاصل الضرب

$$\Gamma(p)\,\Gamma(1-p) = \frac{\pi}{\sin(p\pi)}$$

بحيث يكون مجموع p و $1-p$ يساوي واحد

اي $\frac{1}{4} + \frac{3}{4} = 1$ وان p هو اصغر عدد

$$\therefore \quad \Gamma(\tfrac{1}{4})\,\Gamma(1-\tfrac{1}{4}) = \frac{\pi}{\sin(\frac{1}{4}\pi)} = \frac{\pi}{\sin(\frac{\pi}{4})} = \frac{\pi}{\frac{1}{\sqrt{2}}}$$

$$= \sqrt{2}\,\pi$$

$$\because \quad I = \frac{1}{2}\,\beta(\tfrac{3}{4},\tfrac{1}{4})$$

$$= \frac{1}{2}\sqrt{2}\,\pi = \frac{\pi}{\sqrt{2}}$$

1/ $I = \int_0^1 \frac{dx}{\sqrt{-\ln x}}$

Solution: $y = -\ln x \longrightarrow x = e^{-y} \rightarrow dx = -e^{-y} dy$

$x = 0 \longrightarrow y = \infty$
$x = \infty \longrightarrow y = 0$

$$I = -\int_\infty^0 \frac{e^{-y}}{\sqrt{y}}\, dy = \int_0^\infty y^{-\frac{1}{2}} e^{-y} dy = \Gamma(\tfrac{1}{2}) = \sqrt{\pi}$$

---

2 $I = \int_0^2 \frac{x^2}{\sqrt{2-x}}\, dx$

Solution:

$$I = \int_0^2 \frac{x^2}{\sqrt{2}(1-\frac{x}{2})^{1/2}} dx = \frac{1}{\sqrt{2}} \int_0^2 x^2 (1-\tfrac{x}{2})^{-\frac{1}{2}} dx$$

Let $y = \frac{x}{2} \rightarrow x = 2y$; $x^2 = 2y^2 \rightarrow dx = 2dy$

Limits : $x = 2y$ $\quad x = 0 \longrightarrow y = 0$
$\quad x = 2 \longrightarrow y = 1$

$$\therefore I = \frac{1}{\sqrt{2}} \int_0^1 (4y^2)(1-y)^{-\frac{1}{2}}\, 2\, dy$$

$$I = 4\sqrt{2} \int_0^1 y^2 (1-y)^{-\frac{1}{2}} dy \quad \cdots (1)$$

$$\because \beta(m,n) = \int_0^1 y^{m-1} (1-y)^{n-1} dy \quad \cdots (2)$$

Comparing equations (1) and (2), then

$m - 1 = 2 \longrightarrow m = \frac{3}{2}$

$n - 1 = -\frac{1}{2} \longrightarrow n = \frac{1}{2}$

$$\because \beta(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$$

$$\therefore \beta(3, \tfrac{1}{2}) = \frac{\Gamma(3)\ \Gamma(\frac{1}{2})}{\Gamma(3+\frac{1}{2})} = \frac{2!\ \sqrt{\pi}}{\Gamma(1+\frac{5}{2})}$$

$$\Gamma(1+\tfrac{5}{2}) = \tfrac{5}{2}\Gamma(\tfrac{5}{2}) = \tfrac{5}{2}\ \tfrac{3}{2}\ \Gamma(\tfrac{3}{2})$$

$$= \tfrac{5}{2}\ \tfrac{3}{2}\cdot\tfrac{1}{2}\ \Gamma(\tfrac{1}{2})$$

$$\therefore \beta(3, \tfrac{1}{2}) = \frac{2\ \cancel{\sqrt{\pi}}}{\frac{5}{2}\ \frac{3}{2}\ \frac{1}{2}\ \cancel{\sqrt{\pi}}} = \frac{16}{15}$$

$$\therefore I = 4\sqrt{2}\ \frac{16}{15} = \frac{32\sqrt{2}}{15}$$

---

3. $$I = \int_0^a y^4 \sqrt{a^2 - y^2}\ dy$$

Solution

$$I = \int_0^a y^4\ a(1 - y^2/a^2)^{1/2}\ dy$$

Let $X = \frac{y^2}{a^2} \longrightarrow y = a X^{1/2} \longrightarrow dy = \frac{a}{2} X^{-1/2} dx$

Limits, $y = 0 \longrightarrow X = 0$

$y = a \longrightarrow a = a X^{1/2} \longrightarrow X = 1$

$$\therefore I = \int_0^1 (a^4 x^2) .\ a(1-x)^{1/2} . \frac{a}{2} X^{-\frac{1}{2}} dx$$

$$I = \frac{a^6}{2} \int_0^1 X^{-\frac{1}{2}} (1 - x)^{1/2}\ dx$$

$$\because \beta(m,n) = \int_0^1 y^{m-1} (1-y)^{n-1} dy$$

$\therefore \quad m - 1 = -\frac{1}{2} \quad m = \frac{1}{2}\ ; \quad n - 1 = \frac{1}{2} \quad n = \frac{3}{2}$

$$\therefore I = \frac{a^6}{2}\ \beta(\tfrac{1}{2}, \tfrac{3}{2}) \quad \ldots$$

$$I = \int_0^a y^{-1} (a^2 - y^2)^{1/2} \, dy$$

Solution:

$$I = a \int_0^a y^{+1} \left(1 - \frac{y^2}{a^2}\right)^{1/2} dy$$

Let $x = \frac{y^2}{a^2} \rightarrow y^2 = a^2 x$

$\rightarrow y = a x^{1/2} \rightarrow dy = \frac{1}{2} a x^{-1/2} dx$

Limits: $x = \frac{y^2}{a^2}$ $\quad y = 0 \rightarrow x = 0$

$y = a \rightarrow x = 1$

$$\therefore I = a \int_0^1 (a \cancel{x^{1/2}})^{+1} (1 - x)^{1/2} \cdot \frac{1}{2} a \cancel{x^{-1/2}} \, dx$$

$$= \frac{a^3}{2} \int_0^1 x^0 (1 - x)^{1/2} \, dx$$

$$\because \beta(m,n) = \int_0^1 x^{\boxed{m-1}} (1 - x)^{\boxed{n-1}} \, dx$$

$\therefore m - 1 = 0 \rightarrow m = 1$

$n - 1 = \frac{1}{2} \rightarrow n = \frac{3}{2}$

$$\therefore \beta(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$$

$$\beta\left(1, \frac{3}{2}\right) = \frac{\Gamma(1)\,\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(1 + \frac{3}{2}\right)} = \frac{1 \cdot \Gamma\left(\frac{3}{2}\right)}{\Gamma(5/2)}$$

$$= \frac{\cancel{\Gamma\left(\frac{3}{2}\right)}}{\frac{3}{2}\,\cancel{\Gamma\left(\frac{3}{2}\right)}} = \frac{2}{3}$$

$$\therefore I = \frac{a^3}{2} \cdot \frac{2}{3} = \frac{a^3}{3}$$