reduced number of elements $= \frac{n^2-n}{2} + n$

$$= \frac{1}{2} n(n+1)$$

for example: If $n=3$

$\therefore$ no. of elements $= \frac{1}{2} 3(3+1) = 6$

b- Skew-symmetric matrix:

1. It is also $n \times n$ matrix.

<u>example:</u>

$$A = \begin{pmatrix} 0 & 2 & 1 \\ -2 & 0 & 3 \\ -1 & -3 & 0 \end{pmatrix}$$

i.e. $a_{ij} = -a_{ji}$

and the leading diagonal is of zero elements.

~~In the previ~~ therefore:

no. of elements $= \frac{1}{2} n(n-1)$

<u>Example:</u> If $n=3$ in the prevous example

no. of elements $= \frac{1}{2} n(n-1)$

$$= \frac{1}{2} 3(3-1)$$

$$= \frac{3}{2} \times 2 = 3$$

- Any $n \times n$ matrix A can be expressed interms of symmetric matrix A and its skew-symmetric matrix.

Let $M = A^T$ for symmetric

and $S = -A^T$ for skew-symmetric

$$\therefore A = M + S \quad \text{-----} \; ①$$

$$A^T = (M+S)^T = M^T + S^T = M - S \quad \text{---} \; ②$$

adding and subtracting equ.s. ① and ②, we get

$$M = \frac{1}{2}(A + A^T) \quad \text{symmetric matrix}$$

$$S = \frac{1}{2}(A - A^T) \quad \text{skew-symmetric matrix}$$

- Hermiatian matrix:-

The condition for the Hermitian matrix is

$$A = (A^*)^T = A^\dagger \leftarrow \begin{matrix} A \text{ dagger} \\ A \text{ adjoint} \end{matrix}$$

<u>Example:</u> Is the following matrix Hermitian, and symmetric

$$A = \begin{pmatrix} 1 & 1+i & i \\ 1-i & 2 & 4 \\ -i & 4 & 3 \end{pmatrix} \quad a_{ij} \neq a_{ji}$$

It is a non-symmetric matrix

$$A^* = \begin{pmatrix} 1 & 1-i & -i \\ 1+i & 2 & 4 \\ +i & 4 & 3 \end{pmatrix}$$

take the transpose of A^*

$$(A^*)^T = \begin{pmatrix} 1 & 1+i & +i \\ 1-i & 2 & 4 \\ -i & 4 & 3 \end{pmatrix}$$

$$= A$$

$\therefore$ A is a Hermitian matrix.

Note: If $a_{ij}^{*} = -a_{ji}$,
then A is called skew-Hermitian matrix.

H.W.

Example: Is A Hermitian, skew-Hermitian.

$$A = \begin{pmatrix} i & 1 & 1-i \\ -1 & 0 & i \\ -1-i & i & 2i \end{pmatrix}$$

مسار

- Trace of a square matrix:

Let $A = [a_{ij}]_{n,n}$,
then the trace of the square matrix A is defined as:

$\sum_{i=1}^{n} a_{ii}$ and denoted by $tr(A)$.

i.e. $\sum_{i=1}^{n} a_{ii} = a_{11} + a_{22} + a_{33} + \cdots + a_{nn}$

i.e. the sum of the leading diagonal.

<u>Example:</u>

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

$$\therefore \ tr(A) = 1 + 5 + 9 = 15$$

properties of tr:

1) $tr(kA) = K \, tr(A)$ K: scalar.
2) $tr(A+B) = tr(A) + tr(B)$
3) $tr(AB) = tr(BA)$ ← H.W.

متعامد

- Orthognal matrix:

A square matrix A (with real elements) is said to be orthognal, if

$$A^T = A^{-1}$$

or a real matrix A is orthognal if and only if

$$A^T A = A A^T = I$$

Prove that : $A^{-1} = A^{T}$

Proof:

Let $x' = Ax$ (transformation) تحويل ، تحول

this transformation depends on the behavior of the operator A (translation, rotation, inversion, etc.) نقل ، انتقال

multiple both sides from right by A^{-1}, then

$\therefore \; x = A^{-1} x'$

$x = A^{-1} A x$

$\therefore \; AA^{-1} = I$ ---- ①

Since $A^{T}_{ij} = A_{ji}$ ----- ②

from orthogonality condition

$\underbrace{a_{ij}\, a_{ik}} = \delta_{jk}$

for $j = k = 1$

$\therefore \; \underbrace{a_{i1}\, a_{i1}} = \delta_{11}$

or $a_{i1}\, a_{i1} = I$

Using the property in equ. ② , then

$$a_{ii} \; a_{ii}^{T} = I$$
$$\downarrow \quad \downarrow$$
$$A \; A^{T} = I \quad ---- (3)$$

By comparing equ.s (1) and (3), we get

$$AA^{T} = AA^{-1} = I$$

or $A^{T} = A^{-1}$ or $A^{-1} = A^{T}$

$\therefore$ A is an orthogonal matrix.

<u>Example:</u> If A, B are orthogonal matrices each of order n then AB and BA are orthogonal matrices.

<u>Sol.</u>

1st. A is orthogonal matrix, then

$$AA^{T} = I$$

and also

$$BB^{T} = I$$

we want to prove that AB is also orthogonal matrix.

To prove that AB is orthogonal, then we must have

$$(AB)(AB)^T = I$$

or $(AB)^T (AB) = I$

L.H.S.

$$(AB)^T(AB) = (B^T A^T)(AB)$$

$$= B^T \underbrace{A^T A}_{I} B$$

$$= B^T I B = B^T B = I$$

$\therefore (AB)^T (AB) = I$

$\therefore (AB)$ is orthogonal

وحدة، وحدية

- Unitary matrix:

A square matrix A (with complex elements) is said to be unitary if

$$A^{*T} = A^{-1}$$

or $\boxed{A^{\dagger} = A^{-1}}$

$$\boxed{A^{\dagger} A = A A^{\dagger} = I}$$

the condition of unitary matrix