H.W: Show that

$$AA^{\dagger} = A^{\dagger}A = I$$

- The inverse of a matrix:

The inverse of a matrix is defined as

$$A^{-1} = \frac{\text{adj}(A)}{\det(A)} \quad \text{------} \quad (*)$$

Where adj(A) is the adjoint of A, and det(A) is the determinant of A.

Note: Any matrix has an inverse if and only if A is non-singular or $\det(A) \neq 0$.

المحدد

- Determinant:

The determinant of A, denoted by det(A), is the sum of all product or is the following sum which is summed over a permutation (+1 for even permutation of 1 2 3 --- n, -1 for odd permutation and zero if any index is repeated)

تبادل، تبديل

$$D = \sum_{i,j,k} \varepsilon_{ijk}\, a_i\, b_j\, c_k$$

where $\varepsilon_{ijk} = \begin{cases} 1 & \text{for even permutation} \\ -1 & \text{" odd "} \\ 0 & \text{if any index is repeated} \end{cases}$

$\varepsilon_{112} = 0 \quad \varepsilon_{101} = 0$

Example: Find ε_{123}, ε_{321}, ε_{231}

$\varepsilon_{123} = +1$

$$\varepsilon_{321} \rightarrow \underset{1}{\varepsilon_{312}} \rightarrow \underset{2}{\varepsilon_{132}} \rightarrow \underset{3}{\varepsilon_{123}} = -1$$

$$\varepsilon_{231} \rightarrow \underset{1}{\varepsilon_{213}} \rightarrow \underset{2}{\varepsilon_{123}} = +1$$

H.W.

1. prove that

$A^{-1}A = AA^{-1} = I$

$(AB)(AB)^{-1} = I$

2. Find the determinant of

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

is A singular or not

∴ The general form of the determinant

$$\boxed{D = (-1)^{i+j} \quad a_{ij}\, D_{ij}}$$

<u>Example:</u> Find the determinant of

$$A = \begin{pmatrix} 1 & 3 & 4 \\ 2 & 1 & 0 \\ 3 & 1 & 1 \end{pmatrix} \equiv \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \; D_i$$

$$\text{Det}(A) = (-1)^{1+1} \cdot 1 \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} + (-1)^{1+2} \cdot 3 \begin{vmatrix} 2 & 0 \\ 3 & 1 \end{vmatrix} + (-1)^{1+3} \cdot 4 \begin{vmatrix} 2 \\ 3 \end{vmatrix}$$

$$= 1 \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} - 3 \begin{vmatrix} 2 & 0 \\ 3 & 1 \end{vmatrix} + 4 \begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix}$$

$$= (1-0) - 3(2-0) + 4(2-3)$$

$$= 1 - 6 - 4 = -9$$

Since $\text{Det}(A) = -9 \neq 0$, then according to equ. (*) our conclusion is that A has an inverse denoted by A^{-1}. Otherwise ($\text{Det}(A) = 0$), then A is a singular matrix.

Properties of determinant :-

1) The determinant of A and its transpose A^T are equal

i.e. $|A| = |A^T|$ or $\det(A) = \det(A^T)$ اذا كانت المصفوفة ابعادها 2X2 فقط.

<u>example:</u> If $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

$$\therefore \det(A) = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$$

$$= 4 - 6 = -2$$

and

$$A^T = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$$

and $\det(A^T) = \begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix}$

$$= 4 - 6 = -2$$

2) If A has a row (column) of Zero then $|A| = 0$.

example:

$$A=\begin{pmatrix}1&0&1\\0&0&0\\0&1&0\end{pmatrix} \leftarrow \text{2nd row has Zero elements}$$

$$\det(A)=\begin{vmatrix}1&0&1\\0&0&0\\0&1&0\end{vmatrix}$$

$$=1\begin{vmatrix}0&0\\1&0\end{vmatrix}-0\begin{vmatrix}0&0\\0&0\end{vmatrix}+1\begin{vmatrix}0&0\\0&1\end{vmatrix}$$

$$=1\times 0-0+0=0$$

3) If A has two identical (متماثل، متطابق) rows (columns) then $|A|=0$.

4) If A is triangular, i.e. A has Zero above or below the diagonal, "then $|A|$ = product of diagonal elements"

Example:

$$A=\begin{pmatrix}1&0&0\\1&2&0\\2&4&3\end{pmatrix}$$

prove:

5) $|AB| = |A||B|$

6) If the determinant of a given matrix A, is not Zero, so A is invertable (non-singular) i.e. A has an inverse, otherwise (if $|A| = 0$) then A is singular.

Now from equ. ⊛

$$A^{-1} = \frac{adj(A)}{|A|}$$

- Adjoint of A:

$$Adj(A) = (\text{Cofactor } A)^{T}$$

the transpose of the matrix of cofactor of the elements a_{ij} of A is denoted by adj A is called the classical adjoint of A.

Cofactor of A:

Let $A_{3\times3} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

then the cofactor of A is

$$\text{Cofactor}(A) = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$$

where $A_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$

$A_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$

and so on

Example: Find the cofactor of the following matrix.

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

$$\text{Cofactor}(A) = \begin{pmatrix} +\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & 3 \\ 3 & 1 \end{vmatrix} & +\begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} \\ -\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} & +\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 3 & 2 \end{vmatrix} \\ +\begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} & +\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} -5 & 7 & 1 \\ 3 & -5 & 1 \\ 1 & 1 & -1 \end{pmatrix}$$

H.W. Find the inverse of A.

H.W. given

$$\begin{pmatrix} 5 & 4 \\ 1 & 1 \end{pmatrix} X = \begin{pmatrix} 1 & -2 \\ 1 & 3 \end{pmatrix}$$

find X

The inverse of a matrix A

$$A^{-1} = \frac{adj(A)}{|A|} = \frac{adj(A)}{det(A)}$$

$$adj(A) = (Cofactor)^T$$

$$Cofactor(A) = (-1)^{i+j}\ Minor$$

Minor: The minor of an element is defined as a determinant obtained by deleting the row and the column containing the element.

Thus for a matrix (A)

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

the minors

$$A'_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

$$A'_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

$$A'_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$\vdots$$

Then the cofactor of A_{11}

$$A_{11} = (-1)^{1+1} A'_{11}$$

$$A_{12} = (-1)^{1+2} A'_{12}$$

$$A_{13} = (-1)^{1+3} A'_{13}$$

$$\vdots$$

$$\therefore \text{Cofactor of } A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$$

$$\text{adj} = (\text{cofactor})^T = \begin{pmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{pmatrix}$$

Example: Find the inverse of

$$A = \begin{pmatrix} 7 & 4 & 6 \\ 9 & 5 & 8 \\ 3 & 1 & 2 \end{pmatrix}$$

$$A^{-1} = \frac{adj(A)}{|A|}$$

$$|A| = (-1)^{i+j}\, a_{ij}\, D_{ij}$$

$$= (-1)^{1+1} \cdot 7 \cdot \begin{vmatrix} 5 & 8 \\ 1 & 2 \end{vmatrix} + (-1)^{1+2} \cdot 4 \begin{vmatrix} 9 & 8 \\ 3 & 2 \end{vmatrix} + (-1)^{1+3} \cdot 6 \begin{vmatrix} 9 & 5 \\ 3 & 1 \end{vmatrix}$$

$$= +7 \begin{vmatrix} 5 & 8 \\ 1 & 2 \end{vmatrix} - 4 \begin{vmatrix} 9 & 8 \\ 3 & 2 \end{vmatrix} + 6 \begin{vmatrix} 9 & 5 \\ 3 & 1 \end{vmatrix}$$

$$|A| = 14 + 24 - 36$$

$$|A| = 2$$

The conclusion is that has an inverse or (non-singular)

$$adj(A) = \text{Cofactor } A = (-1)^{i+j} \text{ Minor}$$

$$\text{Cofactor}(A) = \begin{bmatrix} (-1)^{1+1}\begin{vmatrix} 5 & 8 \\ 1 & 2 \end{vmatrix} & (-1)^{1+2}\begin{vmatrix} 9 & 8 \\ 3 & 2 \end{vmatrix} & (-1)^{1+3}\begin{vmatrix} 9 & 5 \\ 3 & 1 \end{vmatrix} \\ (-1)^{2+1}\begin{vmatrix} 4 & 6 \\ 1 & 2 \end{vmatrix} & (-1)^{2+2}\begin{vmatrix} 7 & 6 \\ 3 & 2 \end{vmatrix} & (-1)^{2+3}\begin{vmatrix} 7 & 4 \\ 3 & 1 \end{vmatrix} \\ (-1)^{3+1}\begin{vmatrix} 4 & 6 \\ 5 & 8 \end{vmatrix} & (-1)^{3+2}\begin{vmatrix} 7 & 6 \\ 9 & 8 \end{vmatrix} & (-1)^{3+3}\begin{vmatrix} 7 & 4 \\ 9 & 5 \end{vmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 6 & -6 \\ -2 & -4 & -5 \\ 2 & -2 & -1 \end{bmatrix}$$

$$Adj(A) = (\text{cofactor } A)^T$$

$$= \begin{pmatrix} 2 & -2 & 2 \\ 6 & -4 & -2 \\ -6 & 5 & -1 \end{pmatrix}$$

$$\therefore A^{-1} = \frac{\begin{pmatrix} 2 & -2 & 2 \\ 6 & -4 & -2 \\ -6 & 5 & -1 \end{pmatrix}}{2}$$

$$A^{-1} = \begin{pmatrix} 1 & -1 & 1 \\ 3 & -2 & -1 \\ -3 & 5/2 & -1/2 \end{pmatrix}$$

H.W: prove that

$$AA^{-1} = I$$

Properties of determinant:

7) If two rows (or two columns) of a determinant are interchanged, the sign of the value of the determinant changed.

Example:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 3 & 1 & 2 \end{pmatrix}$$

H.W.

إيجاد المحدد لها
ثم تبديل الصف الاول بدل
الصف الثاني ثم ايجاد المحدد
مرة اخرى.

- Elementary transformations:

Any one of the following operations on the matrix is called Elementary transformations.

1- Interchanging any two rows (or column).
This transformation in indicated by $R_{i,j}$ if the ith and jth rows are interchanged.

2- Multiplication of the elements of any row (or column) by a non-zero scalar quantity to get $K R_j$

3- Addition of const. multiplication of the elements of any row to the corresponding elements of any other row $(R_i + K R_j)$.

Therefore two matrices are said to be equivalent if one is obtained from the other by elementary transformations.

The symbol $\sim$ is used for equivalance.

اختزل

<u>Example</u>:- Reduce the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ 3 & 1 & 2 \end{bmatrix} \quad \text{to} \quad \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{bmatrix}$$

Using elementary transformation.

The transformations:

$$[\quad] \to R_2 \longrightarrow R_2 - 2R_1 \longrightarrow [\quad]$$

$$R_3 = R_3 - 3R_1 \qquad [\quad]$$

$$R_3 \longrightarrow R_3 + 5R_2$$