properties of determinant:

7) If two rows (or two columns) of a determinant are interchanged, the sign of the value of the determinant changed.

Example:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 3 & 1 & 2 \end{pmatrix}$$

- Elementary transformations:

Any one of the following operations on the matrix is called Elementary transformations.

1- Interchanging any two rows (or column).
This transformation in indicated by $R_{i,j}$ if the ith and jth rows are interchanged.

2- Multiplication of the elements of any row (or column) by a non-zero scalar quantity to get $K R_j$

3- Addition of const. multiplication of the elements of any row to the corresponding elements of any other row $(R_i + K R_j)$.

Therefore two matrices are said to be equivalent if one is obtained from the other by elementary transformations.

The symbol $\sim$ is used for equivalance.

اختزل

<u>Example:</u> Reduce the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ 3 & 1 & 2 \end{bmatrix} \text{ to } \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{bmatrix}$$

Using elementary transformation.

The transformations:

$$\begin{bmatrix} & \end{bmatrix} \sim R_2 \longrightarrow R_2 - 2R_1 \longrightarrow \begin{bmatrix} & \end{bmatrix}$$

$$R_3 = R_3 - 3R_1 \quad \begin{bmatrix} & \end{bmatrix}$$

$$R_3 \longrightarrow R_3 + 5R_2$$

Sol.

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ 3 & 1 & 2 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 3 & 1 & 2 \end{bmatrix} \xrightarrow{R_3 \to R_3 - 3R_1}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & -5 & -7 \end{bmatrix} \xrightarrow{R_3 \to R_3 + 5R_2} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{bmatrix}$$

H.W.

① Reduce the matrix

$$\begin{bmatrix} 1 & 3 & 3 \\ 2 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix} \text{ to a unit matrix } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

② Correct the following transformation

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \xrightarrow{R_1 \to -R_2,\ R_2 \to -R_1} \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}$$

Example:

Transform $\begin{bmatrix} 1 & 3 & 3 \\ 2 & 4 & 10 \\ 3 & 8 & 4 \end{bmatrix}$ into a unit matrix.

$$A = \begin{bmatrix} 1 & 3 & 3 \\ 2 & 4 & 10 \\ 3 & 8 & 4 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 3 & 3 \\ 0 & -2 & 4 \\ 3 & 8 & 4 \end{bmatrix} \xrightarrow{R_3 \to R_3 - 3R_1}$$

$$\begin{bmatrix} 1 & 3 & 3 \\ 0 & -2 & 4 \\ 0 & -1 & -5 \end{bmatrix} \xrightarrow{R_2 \to -\frac{1}{2} R_2} \begin{bmatrix} 1 & 3 & 3 \\ 0 & 1 & -2 \\ 0 & -1 & -5 \end{bmatrix} \xrightarrow{R_1 \to R_1 - 3R_2}$$

$$\begin{bmatrix} 1 & 0 & 9 \\ 0 & 1 & -2 \\ 0 & -1 & -5 \end{bmatrix} \xrightarrow{R_3 \to R_3 + R_2} \begin{bmatrix} 1 & 0 & 9 \\ 0 & 1 & -2 \\ 0 & 0 & -7 \end{bmatrix} \xrightarrow{R_3 \to -\frac{1}{7} R_3}$$

$$\begin{bmatrix} 1 & 0 & 9 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \to R_1 - 9R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 \to R_2 + 2R_3}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Elementary matrices :-

A matrix obtained from a unit matrix by a single elementary transformation called elementary matrix.

$$I = \begin{pmatrix} 1 & 0 & 0 & \cdots & \\ 0 & 1 & \cdots & & \\ 0 & 0 & 1 & \cdots & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}_{n \times n}$$

and for 3x3 matrix

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \equiv \begin{pmatrix} \delta_{11} & 0 & 0 \\ 0 & \delta_{22} & 0 \\ 0 & 0 & \delta_{33} \end{pmatrix}$$

for example the matrix

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ is obtained from the unit matrix

under the transformation

$$R_2 + 3R_1 \longrightarrow R_2$$

H.W.

Write down the steps of transformation of the following matrix

$$B = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Using the elementary matrix.

- The inverse of a matrix from elementary matrices (Gauss - Jordan method) :-

If A is reduced to I by elementary transformation, then

$PA = I$

$\therefore P = A^{-1}$ = elementary matrix

The rule:

1. Write the matrix A

i.e. $A = IA = AI$

2. Perform elementary transformation on A of the left hand side and on I on the right hand side.

3. So that A is reduced to I and I of the right hand side, is reduced to p, getting,

$$I = pA$$

<u>Example:</u> Find the inverse of

$$A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$$

Using Gauss-Jordan method

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} = B$$

B is the inverse of A

i.e. $AB = I$

$$A=\begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\because \underset{A^{-1}}{P}A = I$$

$$\because A = IA$$

$$\begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$$\begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} \xrightarrow[\substack{\cancel{3 \quad -3 \quad 4} \\ 1 \quad 1 \quad \frac{4}{3}}]{R_1 \rightarrow \cancel{R_1} + \frac{1}{3}R_1} \begin{bmatrix} 1 & -1 & 4/3 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_2 \rightarrow R_2 - 2R_1} \begin{bmatrix} 1 & -1 & \frac{4}{3} \\ 0 & -1 & \frac{4}{3} \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ -\frac{2}{3} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$$\begin{array}{ccc} 2 & -3 & 4 \\ -2 & \pm 2 & \mp 8/3 \\ \hline 0 & -1 & \frac{4}{3} \end{array}$$

$$\xrightarrow{R_2 \rightarrow -R_2} \begin{bmatrix} 1 & -1 & \frac{4}{3} \\ 0 & +1 & -\frac{4}{3} \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ \frac{2}{3} & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$$\begin{bmatrix} 1 & -1 & \frac{4}{3} \\ 0 & 1 & -\frac{4}{3} \\ 0 & -1 & 1 \end{bmatrix} \begin{matrix} R_1 \xrightarrow{} R_1+R_2 \\ \\ R_3 \rightarrow R_3+R_2 \end{matrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{4}{3} \\ 0 & 0 & -\frac{1}{3} \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ \frac{2}{3} & -1 & 0 \\ \frac{2}{3} & -1 & 1 \end{bmatrix}$$

$$R_2 \rightarrow -3R_3 \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{4}{3} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ \frac{2}{3} & -1 & 0 \\ -2 & 3 & -3 \end{bmatrix} A$$

$$R_2 \longrightarrow R_2 + \frac{4}{3}R_3 \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix} A$$

$$\therefore A^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$$

H.W. Find the inverse (Use $(A = IA)$) for the following matrices.

$$\begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix} \xrightarrow{\text{answer}} \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 1 \\ 4 & 1 & 0 \\ 8 & 1 & 1 \end{bmatrix} \xrightarrow{\text{answer}} \begin{bmatrix} 1 & 2 & -1 \\ -4 & -7 & 4 \\ -4 & -9 & 5 \end{bmatrix}$$

- The rank of a matrix :-

The rank of a matrix is the number of nonzero rows in A_R (A_R : the reduced matrix) whose rows are equivalent to A

Example:

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix} \text{ has 3-ranks}$$

$$\begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix} \text{ has 3-ranks}$$

- Solution of simutaneous equations using elementary transformations :-

We can solve a set of equ.s by writting them in a matrix form and the matrix of the coefficients of $x, y, z, \ldots$ is reduced using E.T. .

Example: solve the following equ.s.

$$x - y + 2z = 3$$
$$x + 2y + 3z = 5$$
$$3x - 4y - 5z = -13$$

47

1- Write the equ.s in the form of a matrix.

$$\begin{bmatrix} 1 & -1 & 2 \\ 1 & 2 & 3 \\ 3 & -4 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ -13 \end{bmatrix}$$

$\downarrow$

$R_2 \to R_2 - R_1$
$R_3 \to R_3 - 3R_1$ $\longrightarrow$

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & 1 \\ 0 & -1 & -11 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -22 \end{bmatrix}$$

$R_3 \longrightarrow R_3 - \frac{1}{3} R_2 \longrightarrow$

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & -\frac{32}{3} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -\frac{64}{3} \end{bmatrix}$$