- System of linear algebra equations:-

One of the most important use of matrices occurs in solution of system algebra equ.s.

- non homogenous equ.s:

$$\left.\begin{array}{l} a_{11}X_1 + a_{12}X_2 + \cdots + a_{1n}X_n = h_1 \\ a_{21}X_1 + a_{22}X_2 + \cdots + a_{2n}X_n = h_2 \\ \vdots \\ a_{n1}X_1 + a_{n2}X_2 + \cdots + a_{nn}X_n = h_n \end{array}\right\}$$ non-homogenous equ.

Where $a_{11}\ a_{12} \cdots$ are the coefficients and $h_1, h_2 \cdots$ are const..

The set of equ. can written in a matrix form

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & \cdots & \cdots & a_{nn} \end{pmatrix} \ ; \ X = \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{pmatrix} \ ; \ h = \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \end{pmatrix}$$

or

$$AX = H$$

$$\underbrace{A^{-1}A}_{I}X = A^{-1}H$$

$$\therefore \boxed{X = A^{-1}H}$$

1 Solve by matrix inverse ($X = A^{-1}H$)

Example: Solve the equs.

$$x + y + z = 6$$
$$x + 2y + 3z = 14$$
$$x + 4y + 9z = 36$$

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{pmatrix} \; ; \; X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \; ; \; H = \begin{pmatrix} 6 \\ 14 \\ 36 \end{pmatrix}$$

$$\therefore X = A^{-1}H$$

$$A^{-1} = \frac{\operatorname{adj}(A)}{|A|}$$

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix} = 1\begin{vmatrix} 2 & 3 \\ 4 & 9 \end{vmatrix} - 1\begin{vmatrix} 1 & 3 \\ 1 & 9 \end{vmatrix} + 1\begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix} = 2$$

$$\operatorname{adj}(A) = (\text{Cofactor } A)^T$$

$$= \begin{pmatrix} 6 & -5 & 1 \\ -6 & 8 & -2 \\ 2 & -3 & 1 \end{pmatrix}$$

$$\therefore A^{-1} = \begin{pmatrix} 3 & \frac{-5}{2} & \frac{1}{2} \\ -3 & 4 & -1 \\ 1 & \frac{-3}{2} & \frac{1}{2} \end{pmatrix}$$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 & -\frac{5}{2} & \frac{1}{2} \\ -3 & 4 & -1 \\ 1 & -\frac{3}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 6 \\ 14 \\ 36 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

2 Solve by Cramer's rule.

$$X = A^{-1} H$$

$$= \frac{adj(A) \cdot H}{|A|} = \frac{H \cdot adj(A)}{|A|}$$

$$x_1 = \frac{1}{|A|} \left[h_1 A_{11} + h_2 A_{21} + \cdots + h_n A_{n1} \right]$$

$$x_2 = \frac{1}{|A|} \left[h_1 A_{12} + h_2 A_{22} + \cdots h_n A_{n2} \right]$$

$$\vdots$$

$$x_n = \frac{1}{|A|} \left[h_1 A_{1n} + h_2 A_{2n} + \cdots h_n A_{nn} \right]$$

$$X_1 = \frac{\begin{vmatrix} h_1 & a_{12} & \cdots & a_{1n} \\ h_2 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ h_n & a_{2n} & & a_{nn} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}}, \text{ and so on}$$

$$X_1 = \frac{D_1}{|A|} = \frac{D_1}{D}$$

$$X_2 = \frac{D_2}{|A|} = \frac{D_2}{D}$$

$$X_3 = \frac{D_3}{|A|} = \frac{D_3}{D}$$

$$\vdots$$

Solve the following equ.s.

$$X + y + Z = 6$$

$$X + 2y + 3Z = 14$$

$$X + 4y + 9Z = 36$$

Using Cramer's rule

$$D = |A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix}$$

$$D_1 = \begin{vmatrix} 6 & 1 & 1 \\ 14 & 2 & 3 \\ 36 & 4 & 9 \end{vmatrix}$$

$$D_2 = \begin{vmatrix} 1 & 6 & 1 \\ 1 & 14 & 3 \\ 1 & 36 & 9 \end{vmatrix}$$

$$D_3 = \begin{vmatrix} 1 & 1 & 6 \\ 1 & 2 & 14 \\ 1 & 4 & 36 \end{vmatrix}$$

$$\therefore \quad x_1 = \frac{D_1}{D} = 1$$

$$x_2 = \frac{D_2}{D} = 2$$

$$x_3 = \frac{D_3}{D} = 3$$

Conclusion (condition):

for $X = A^{-1} H$ & $A^{-1} = \frac{\operatorname{adj} A}{|A|}$

i.e. $|A| \neq 0 \longrightarrow A^{-1}$ exist.

and there is only one solution.

Quiz: Show that $AX = H$ has only one solution if $|A| \neq 0$

$\because AX = H$

first sol. (X_1)

$AX_1 = H$ ---- ①

Second sol. (X_2)

$AX_2 = H$ ---- ②

$\therefore$ from ① & ②

$\therefore AX_1 = AX_2$

$\therefore X_1 = X_2$

$\therefore$ there is only one sol.

The conditions for non-homogenous equ.s:-

① If $|A| = 0$ & $D_1, D_2, D_3 \neq 0$

$\downarrow$

$D = 0$

$\therefore$ no finite solution of the set of the equ., and the equ.s are then said to be inconsistent.

Example: Test the consistency of the set of equ.s.

$$3x + 2y = 2$$
$$3x + 2y = 6$$

$$D = \begin{vmatrix} 3 & 2 \\ 3 & 2 \end{vmatrix} = 6 - 6 = 0$$

$$D_1 = \begin{vmatrix} 2 & 2 \\ 6 & 2 \end{vmatrix} = -8$$

$$D_2 = \begin{vmatrix} 3 & 2 \\ 3 & 6 \end{vmatrix} = 12$$

$$\therefore \quad x_1 = \frac{D_1}{D} = \frac{-8}{0} = \infty$$

$$x_2 = \frac{D_2}{D} = \frac{12}{0} = \infty$$

$\therefore$ No finite sol.

$\therefore$ the condition of equ.s with 3 variable is

- $D \neq 0$
  - Consistent with one sol.
- $D = 0$
  - $D_1 = D_2 = D_3 = 0$
    - Consistent with infinitly many sol.
  - D_1 or D_2 or $D_3 \neq 0$
    - inconsistent