Boolean Algebra and Logic Simplification

Laws and Rules of Boolean Algebra

Boolean algebra is the mathematics of digital logic.

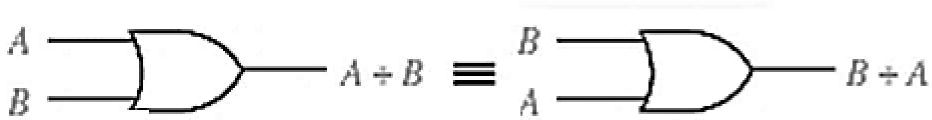
1-) Laws of Boolean Algebra

1- <u>Commutative Laws:</u> قانون التبادل

The commutative law of addition for two variables is written as:

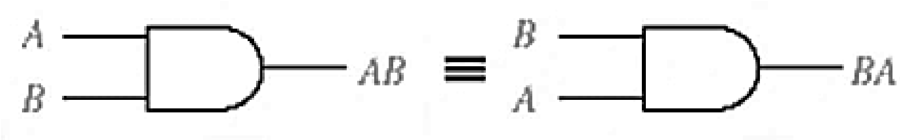
$$\boldsymbol{A + B = B + A}$$

This law states that the order in which the variables are OR gate makes no difference.



Application of Commutative Law

Also, $AB = BA$



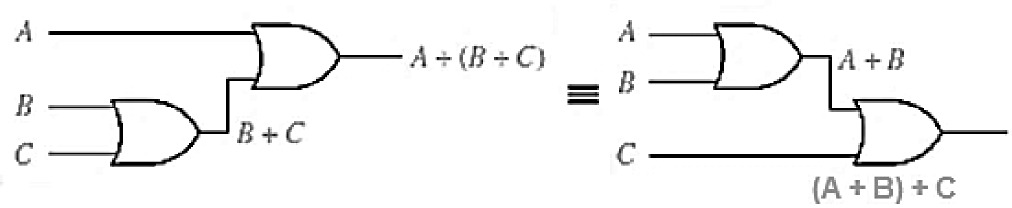
Application of Commutative Law

2- <u>Associative Laws:</u>

The associative law of addition is written as follows for three variables:

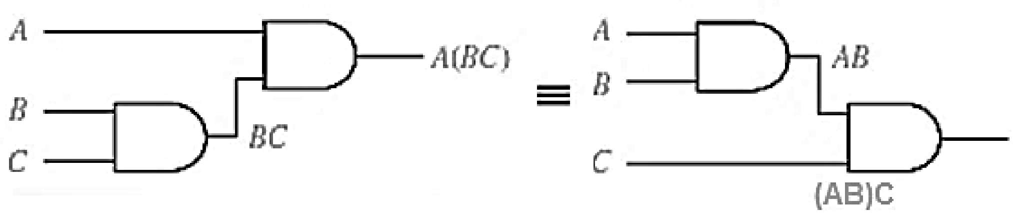
$$\boldsymbol{A + (B + C) = (A + B) + C}$$

This law states that when OR more than two variables, the result is the same regardless of the grouping of the variables.



Application of Associative Low

Also, AND gate $A(BC) = (AB)C$

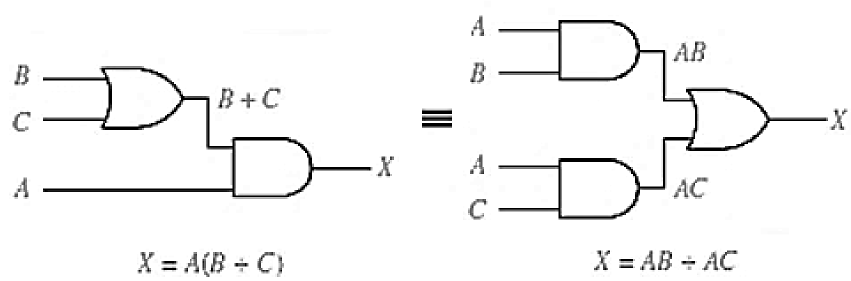


Application of Associative Low

<u>3- Distributive Law</u>

The distributive law is written for three variables as follows:

$$A(B + C) = AB + AC$$



Application of Distributive Low

<u>2-) Rules of Boolean Algebra</u>

lists 12 basic rules that are useful in manipulating and simplifying Boolean expressions. Rules 1 through 9 will be viewed in terms of their application to logic gates:

Basic rules of Boolean algebra.

1. $A + 0 = A$	**7.** $A \cdot A = A$
2. $A + 1 = 1$	**8.** $A \cdot \overline{A} = 0$
3. $A \cdot 0 = 0$	**9.** $\overline{\overline{A}} = A$
4. $A \cdot 1 = A$	**10.** $A + AB = A$
5. $A + A = A$	**11.** $A + \overline{A}B = A + B$
6. $A + \overline{A} = 1$	**12.** $(A + B)(A + C) = A + BC$

A, *B*, or *C* can represent a single variable or a combination of variables.

1-) $A + 0 = A$

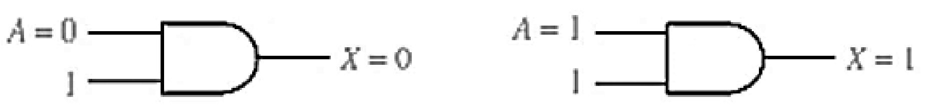
$$X = A + 0 = A$$

2-) $A + 1 = 1$

$$X = A + 1 = 1$$

3-) $A \cdot 0 = 0$

4-) $A \cdot 1 = A$



5-) $A + A = A$

6-) $A + \overline{A} = 1$

7-) $\boldsymbol{A \,.\, A = A}$

8-) $\boldsymbol{A \,.\, \overline{A} = 0}$

9-) $\boldsymbol{\overline{\overline{A}} = A}$

10-) $\boldsymbol{A + AB = A}$

This rule can be proved by applying the distributive law, rule 2, and rule 4 as follows:

$$\begin{aligned} A + AB &= A \cdot 1 + AB = A(1 + B) && \text{Factoring (distributive law)} \\ &= A \cdot 1 && \text{Rule 2: } (1 + B) = 1 \\ &= A && \text{Rule 4: } A \cdot 1 = A \end{aligned}$$

11-) $\boldsymbol{A + \overline{A}B = A + B}$

This rule can be proved as follows:

$$\begin{aligned} A + \overline{A}B &= (A + AB) + \overline{A}B && \text{Rule 10: } A = A + AB \\ &= (AA + AB) + \overline{A}B && \text{Rule 7: } A = AA \\ &= AA + AB + A\overline{A} + \overline{A}B && \text{Rule 8: adding } A\overline{A} = 0 \\ &= (A + \overline{A})(A + B) && \text{Factoring} \\ &= 1 \cdot (A + B) && \text{Rule 6: } A + \overline{A} = 1 \\ &= A + B && \text{Rule 4: drop the 1} \end{aligned}$$

12-) $(A+B)(A+C)=A+BC$

This rule can be proved as follows:

$(A+B)(A+C)$	$= AA + AC + AB + BC$	Distributive law
	$= A + AC + AB + BC$	Rule 7: $AA = A$
	$= A(1 + C) + AB + BC$	Factoring (distributive law)
	$= A \cdot 1 + AB + BC$	Rule 2: $1 + C = 1$
	$= A(1 + B) + BC$	Factoring (distributive law)
	$= A \cdot 1 + BC$	Rule 2: $1 + B = 1$
	$= A + BC$	Rule 4: $A \cdot 1 = A$

3-) De Morgan's Theorems

De Morgan's first theorem is stated as follows:

● **The complement of a product of variables is equal to the sum of the complements of the variables.**

Means that " $\overline{AB} = \overline{A} + \overline{B}$

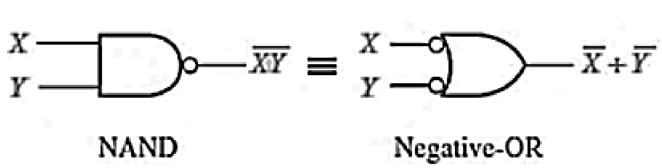
Stated another way,

The complement of two or more AND variables is equivalent to the OR of the complements of the individual variables.

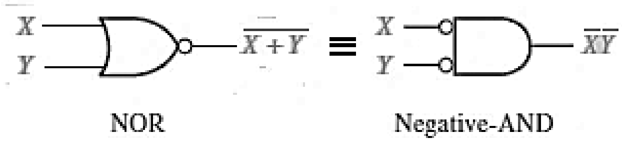
De Morgan's second theorem is stated as follows:

●**The complement of a sum of variables is equal to the product of the complements of the variables.**

Means that , $\overline{A+B} = \overline{A}\,\overline{B}$



Inputs		Output	
X	Y	$\overline{XY}$	$\overline{X}+\overline{Y}$
0	0	1	1
0	1	1	1
1	0	1	1
1	1	0	0



Inputs		Output	
X	Y	$\overline{X+Y}$	$\overline{X}\,\overline{Y}$
0	0	1	1
0	1	0	0
1	0	0	0
1	1	0	0

Example:

Apply De Morgan's theorems to the expressions**:**

$\overline{ABC}$, $\overline{A+B+C}$

Solution: $\overline{A} + \overline{B} + \overline{C}$ $\overline{A}\,\overline{B}\,\overline{C}$

Problem:

Apply DeMorgan's theorem to the expression $\overline{\overline{X} + \overline{Y} + \overline{Z}}$.

Apply DeMorgan's theorem to the expression $\overline{\overline{W}\,\overline{X}\,\overline{Y}\,\overline{Z}}$.

$\overline{\overline{A + B\overline{C}} + D(\overline{E + \overline{F}})}$

Examples

Apply DeMorgan's theorems to each of the following expressions:

(a) $\overline{(A + B + C)D}$
(b) $\overline{ABC + DEF}$
(c) $\overline{A\overline{B} + \overline{C}D + EF}$

Solution

(a) Let $A + B + C = X$ and $D = Y$. The expression $\overline{(A + B + C)D}$ is of the form $\overline{XY} = \overline{X} + \overline{Y}$ and can be rewritten as

$$\overline{(A + B + C)D} = \overline{A + B + C} + \overline{D}$$

Next, apply DeMorgan's theorem to the term $\overline{A + B + C}$.

$$\overline{A + B + C} + \overline{D} = \overline{A}\overline{B}\overline{C} + \overline{D}$$

(b) Let $ABC = X$ and $DEF = Y$. The expression $\overline{ABC + DEF}$ is of the form $\overline{X + Y} = \overline{X}\overline{Y}$ and can be rewritten as

$$\overline{ABC + DEF} = (\overline{ABC})(\overline{DEF})$$

Next, apply DeMorgan's theorem to each of the terms $\overline{ABC}$ and $\overline{DEF}$.

$$(\overline{ABC})(\overline{DEF}) = (\overline{A} + \overline{B} + \overline{C})(\overline{D} + \overline{E} + \overline{F})$$

(c) Let $A\overline{B} = X$, $\overline{C}D = Y$, and $EF = Z$. The expression $\overline{A\overline{B} + \overline{C}D + EF}$ is of the form $\overline{X + Y + Z} = \overline{X}\overline{Y}\overline{Z}$ and can be rewritten as

$$\overline{A\overline{B} + \overline{C}D + EF} = (\overline{A\overline{B}})(\overline{\overline{C}D})(\overline{EF})$$

Next, apply DeMorgan's theorem to each of the terms $\overline{A\overline{B}}$, $\overline{\overline{C}D}$, and $\overline{EF}$.

$$(\overline{A\overline{B}})(\overline{\overline{C}D})(\overline{EF}) = (\overline{A} + B)(C + \overline{D})(\overline{E} + \overline{F})$$

<u>Logic Simplification Using Boolean Algebra</u>

Using Boolean algebra techniques, simplify this expression:

Simplify the following Boolean expression:

$$[A\overline{B}(C + BD) + \overline{A}\overline{B}]C$$

Solution

Step 1: Apply the distributive law to the terms within the brackets.

$$(A\overline{B}C + A\overline{B}BD + \overline{A}\,\overline{B})C$$

Step 2: Apply rule 8 ($\overline{B}B = 0$) to the second term within the parentheses.

$$(A\overline{B}C + A \cdot 0 \cdot D + \overline{A}\,\overline{B})C$$

Step 3: Apply rule 3 ($A \cdot 0 \cdot D = 0$) to the second term within the parentheses.

$$(A\overline{B}C + 0 + \overline{A}\,\overline{B})C$$

Step 4: Apply rule 1 (drop the 0) within the parentheses.

$$(A\overline{B}C + \overline{A}\,\overline{B})C$$

Step 5: Apply the distributive law.

$$A\overline{B}CC + \overline{A}\,\overline{B}C$$

Step 6: Apply rule 7 ($CC = C$) to the first term.

$$A\overline{B}C + \overline{A}\,\overline{B}C$$

Step 7: Factor out $\overline{B}C$.

$$\overline{B}C(A + \overline{A})$$

Step 8: Apply rule 6 ($A + \overline{A} = 1$).

$$\overline{B}C \cdot 1$$

Step 9: Apply rule 4 (drop the 1).

$$\overline{B}C$$

Example 2

Simplify the following Boolean expression:

$$\overline{A}BC + A\overline{B}\,\overline{C} + \overline{A}\,\overline{B}\,\overline{C} + A\overline{B}C + ABC$$

Solution

Step 1: Factor BC out of the first and last terms.

$$BC(\overline{A} + A) + A\overline{B}\,\overline{C} + \overline{A}\,\overline{B}\,\overline{C} + A\overline{B}C$$

Step 2: Apply rule 6 ($\overline{A} + A = 1$) to the term in parentheses, and factor $A\overline{B}$ from the second and last terms.

$$BC \cdot 1 + A\overline{B}(\overline{C} + C) + \overline{A}\,\overline{B}\,\overline{C}$$

Step 3: Apply rule 4 (drop the 1) to the first term and rule 6 ($\overline{C} + C = 1$) to the term in parentheses.

$$BC + A\overline{B} \cdot 1 + \overline{A}\,\overline{B}\,\overline{C}$$

Step 4: Apply rule 4 (drop the 1) to the second term.

$$BC + A\overline{B} + \overline{A}\,\overline{B}\,\overline{C}$$

Step 5: Factor $\overline{B}$ from the second and third terms.

$$BC + \overline{B}(A + \overline{A}\,\overline{C})$$

Step 6: Apply rule 11 ($A + \overline{A}\,\overline{C} = A + \overline{C}$) to the term in parentheses.

$$BC + \overline{B}(A + \overline{C})$$

Step 7: Use the distributive and commutative laws to get the following expression:

$$BC + A\overline{B} + \overline{B}\,\overline{C}$$

Simplify the

$AB+A(B+C)+B(B+C)$

- (distributive law)
  - $AB+AB+AC+BB+BC$
- (rule 7; BB=B)
  - $AB+AB+AC+B+BC$
- (rule 5; AB+AB=AB)
  - $AB+AC+B+BC$
- (rule 10; B+BC=B)
  - $AB+AC+B$
- (rule 10; AB+B=B)
  - $B+AC$

Solution

Step 1: Ap

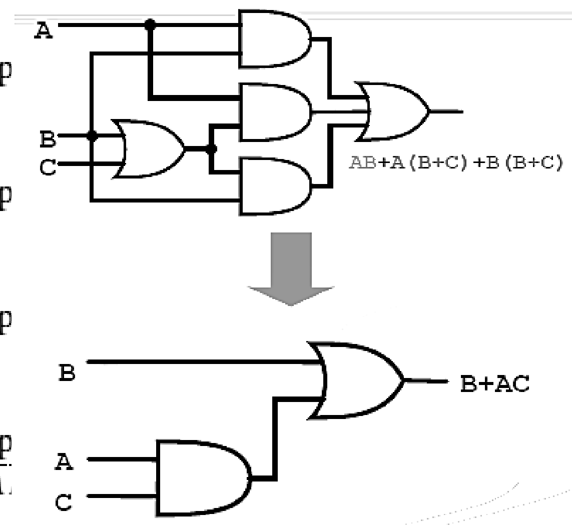
Step 2: Ap

Step 3: Ap

Step 4: Ap
[A

Step 5: Ap

Step 6: Ap



- (rule 10; AB+B=B)
  - $B+AC$

Simplification Using Boolean Algebra

Standard Forms of Boolean Expressions

All Boolean expressions, regardless of their form, can be converted into either of two standard forms:

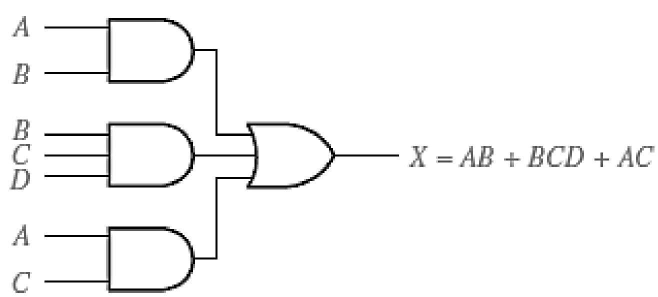
The sum-of-products (SOP) form

The product-of-sums (POS) form

Standardization makes the evaluation,
simplification, and implementation of Boolean expressions much more systematic and easier.

$$AB + ABC$$
$$ABC + CDE + \overline{B}C\overline{D}$$
$$\overline{A}B + \overline{A}B\overline{C} + AC$$

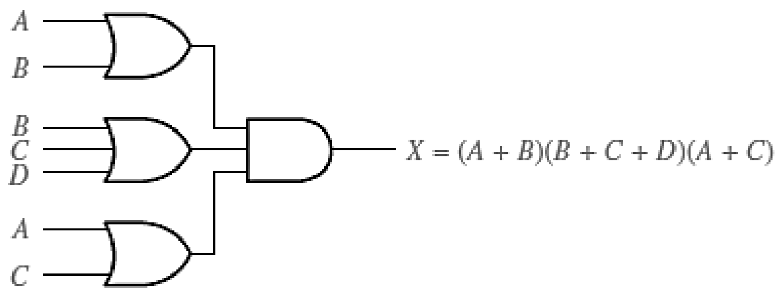
SOP expression can be implemented by AND-OR logic in which the outputs of a number (equal to the number of product terms in the expression) of AND gates connect to the inputs of an OR gate equals the SOP expression.
An SOP expression can be implemented with one OR gate and
two or more AND gates.



lementation of the SOP expression $AB + BCD + A$

Implementation of a POS Expression

Implementing a POS expression simply requires ANDing the outputs of two or more OR gates. A sum term is produced by an OR operation, and the product of two or more sum terms is produced by an AND operation. Therefore, a POS expression can be implemented by logic in which the outputs of a number (equal to the number of sum terms in the expression) of OR gates connect to the inputs of an AND gate the expression (A + B)(B + C + D)(A + C). The output X of the AND gate equals the POS expression.

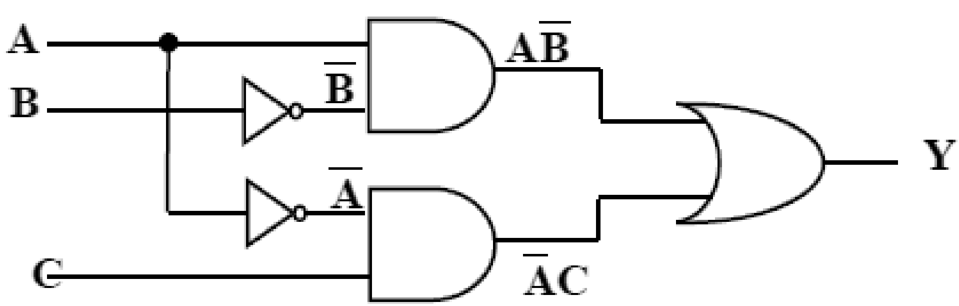


Implementation of the POS expression $(A + B)(B + C + D)(A + C)$.

4-) The Boolean Expression for a Logic Circuit

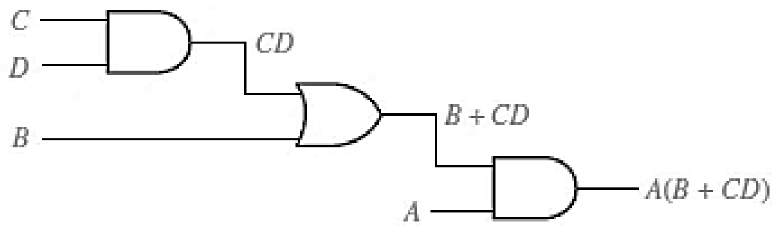
To derive the Boolean expression for a given combinational logic circuit, begin at the left-most
inputs and work toward the final output, writing the expression for each gate. For the example circuit in Figure

Example- 1 $Y = A\overline{B} + \overline{A}C$



Example- 2

1. The expression for the left-most **AND** gate with inputs C and D is CD.
2. The output of the left-most **AND** gate is one of the inputs to the OR gate and B is the other input. Therefore, the expression for the **OR** gate is B + CD.
3. The output of the OR gate is one of the inputs to the right-most AND gate and A is the other input. Therefore, the expression for this **AND** gate is A(B + CD), which is the final output expression for the entire circuit.

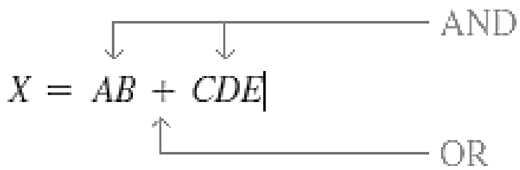


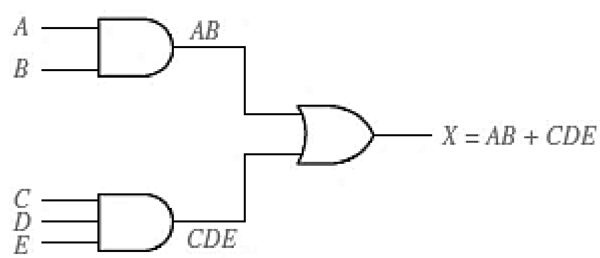
5-) Implementation of a Logic Circuit Using a Boolean Expression

●Let's examine the following Boolean expression:

X = AB + CDE

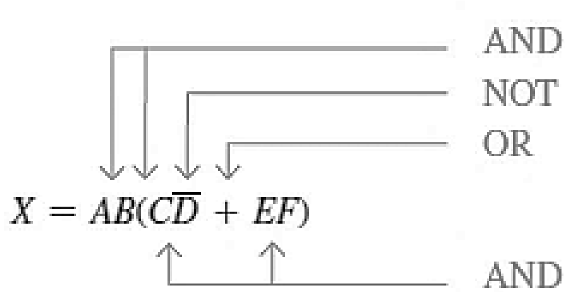
A brief inspection shows that this expression is composed of two terms, AB and CDE, with a domain of five variables. The first term is formed by AND A with B, and the second term is formed by AND C, D, and E. The two terms are then OR to form the output X. These operations are indicated in the structure of the expression as follows:



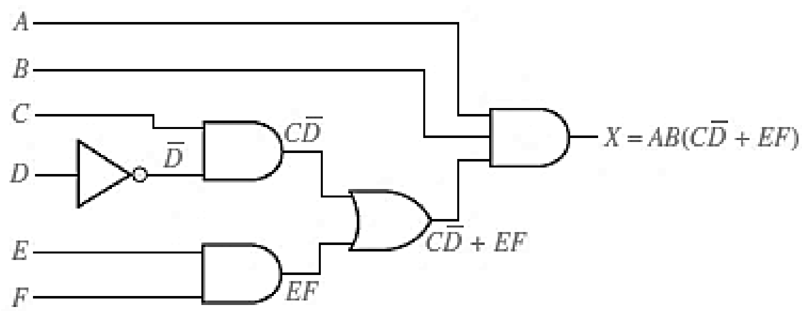


Logic circuit for $X = AB + CDE$.

● this expression shows that the terms AB and (CD + EF) are AND.
The term CD + EF is formed by first AND C and D and AND and F, and then OR these two terms. This structure is indicated in relation to the expression as follows:



the logic operations must be done in the proper order.
The logic gates required to implement X = AB(CD + EF) are as follows:
1. One inverter to form D
2. Two 2-input AND gates to form CD and EF
3. One 2-input OR gate to form CD + EF
4. One 3-input AND gate to form X

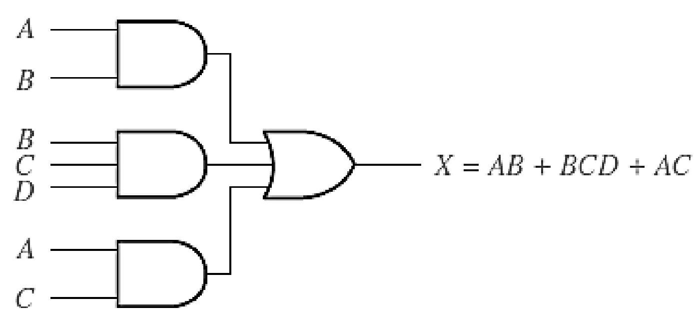


From a Boolean Expression to a Logic Circuit

Find how many gates you need from these expressions ??

$X = \overline{ABC} + B(EF + \overline{G})$ 1
$X = A[BC(A + B + C + D)]$ 2
$X = B(C\overline{D}E + \overline{E}FG)(\overline{AB} + C)$ 3

Binary Representation of a Standard Product Term



Implementation of the SOP expression $AB + BCD + AC$.

A standard product term is equal to 1 for only one combination of variable values. For example, the product term ABCD is equal to 1 when A = 1, B = 0, C = 1, D = 0, as shown below, and is 0 for all other combinations of values for the variables.

$$A\overline{B}C\overline{D} = 1 \cdot \overline{0} \cdot 1 \cdot \overline{0} = 1 \cdot 1 \cdot 1 \cdot 1 = 1$$

In this case, the product term has a binary value of 1010 (decimal ten).
Remember, a product term is implemented with an AND gate whose output is 1 only if each of its inputs is 1. Inverters are used to produce the complements of the variables as required.

An SOP expression is equal to 1 only if one or more of the product terms in the expression is equal to 1.

Determine the binary values for which the following standard SOP expression is equal to 1:

$$ABCD + A\overline{B}\,\overline{C}D + \overline{A}\,\overline{B}\,\overline{C}\,\overline{D}$$

The term $ABCD$ is equal to 1 when $A = 1$, $B = 1$, $C = 1$, and $D = 1$.

$$ABCD = 1 \cdot 1 \cdot 1 \cdot 1 = 1$$

The term $A\overline{B}\,\overline{C}D$ is equal to 1 when $A = 1$, $B = 0$, $C = 0$, and $D = 1$.

$$A\overline{B}\,\overline{C}D = 1 \cdot \overline{0} \cdot \overline{0} \cdot 1 = 1 \cdot 1 \cdot 1 \cdot 1 = 1$$

The term $\overline{A}\,\overline{B}\,\overline{C}\,\overline{D}$ is equal to 1 when $A = 0$, $B = 0$, $C = 0$, and $D = 0$.

$$\overline{A}\,\overline{B}\,\overline{C}\,\overline{D} = \overline{0} \cdot \overline{0} \cdot \overline{0} \cdot \overline{0} = 1 \cdot 1 \cdot 1 \cdot 1 = 1$$

Q/ What is the standard product term for a 1 in cell 0010?

Find the values of the variables that make each product term 1

(a) ABC (b) $\overline{A}\,\overline{B}C$

Conversion of a General Expression to SOP Form

Examples:

(a) $AB + B(CD + EF) = AB + BCD + BEF$

(b) $(A + B)(B + C + D) = AB + AC + AD + BB + BC + BD$

(c) $\overline{\overline{(A + B)} + C} = (\overline{\overline{A + B}})\overline{C} = (A + B)\overline{C} = A\overline{C} + B\overline{C}$

<u>The Standard SOP Form</u>

A standard SOP expression is one in which all the variables in the domain appear in each product term in the expression. Standard SOP expressions are important in constructing truth tables, Convert each SOP expression to standard SOP form.

(a) $(C + D)(A + \overline{D})$ (b) $A(A\overline{D} + C)$ (c) $(A + C)(CD + AC)$

a) $CA + C\overline{D} + DA + D\overline{D}$ b) $AA\overline{D} + AC$ c) $ACD + AAC + CCD + CAC$

Boolean Expressions and Truth Tables

Develop a truth table for the standard SOP expression $\overline{A}\,\overline{B}C + A\overline{B}\,\overline{C} + ABC$.

Inputs			Output	
A	*B*	*C*	y	Product Term
0	0	0	0	
0	0	1	0	
0	1	0	1	$\overline{A}B\overline{C}$
0	1	1	0	
1	0	0	0	
1	0	1	1	$A\overline{B}C$
1	1	0	0	
1	1	1	0	

Inputs			Output	
A	*B*	*C*	y	Product Term
0	0	0		
0	0	1		$\overline{A}\,\overline{B}C$
0	1	0		
0	1	1		
1	0	0		$A\overline{B}\,\overline{C}$
1	0	1		
1	1	0		
1	1	1		ABC

Inputs			Output	
A	*B*	*C*	y	Product Term
0	0	0	0	
0	0	1	1	$\overline{A}\,\overline{B}C$
0	1	0	0	
0	1	1	0	
1	0	0	1	$A\overline{B}\,\overline{C}$
1	0	1	0	
1	1	0	0	
1	1	1	1	ABC

Create a truth table for the standard SOP expression $\overline{A}B\overline{C} + A\overline{B}C$.

Develop a truth table for each of the following standard SOP expressions:

(a) $A\overline{B}C\overline{D} + AB\overline{C}\,\overline{D} + \overline{A}\,\overline{B}CD + \overline{A}\,\overline{B}\,\overline{C}\,\overline{D}$

Develop a truth table for each of the SOP expressions:

$\overline{A}B + AB\overline{C} + \overline{A}\,\overline{C} + A\overline{B}C$

<u>For each truth table derive a standard SOP expression.</u>

(a)

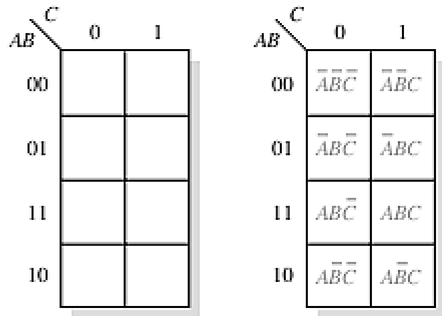
A B C	X
0 0 0	0
0 0 1	1
0 1 0	0
0 1 1	0
1 0 0	1
1 0 1	1
1 1 0	0
1 1 1	1

(b)

A B C	X
0 0 0	0
0 0 1	0
0 1 0	0
0 1 1	0
1 0 0	0
1 0 1	1
1 1 0	1
1 1 1	1

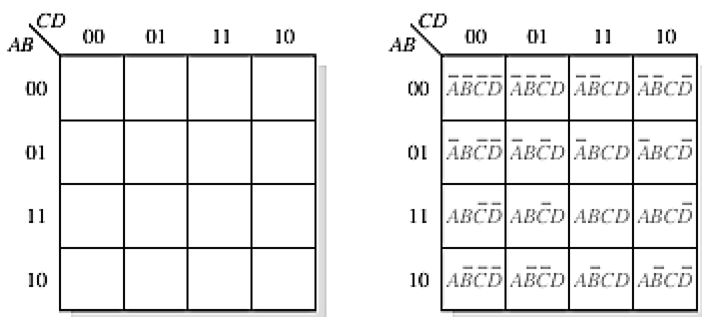
The Karnaugh Map

The 2-Variable Karnaugh Map
The 3-Variable Karnaugh Map
The 4-Variable Karnaugh Map
The 5-Variable Karnaugh Map



(a) (b)

A 3-variable Karnaugh map showing Boolean product terms for each cell



(a) (b)

A 4-variable Karnaugh map

Example

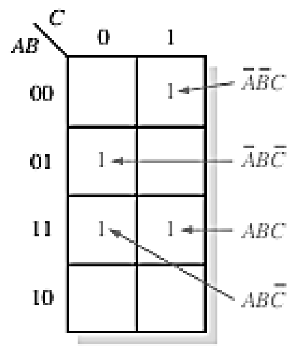
Map the following standard SOP expression on a Karnaugh map:

$$\overline{A}\,\overline{B}C + \overline{A}B\overline{C} + AB\overline{C} + ABC$$

Evaluate the expression as shown below. Place a 1 on the 3-variable Karnaugh map in Figure 4–29 for each standard product term in the expression.

$$\overline{A}\,\overline{B}C + \overline{A}B\overline{C} + AB\overline{C} + ABC$$

$$001 \quad 010 \quad 110 \quad 111$$



Example 2

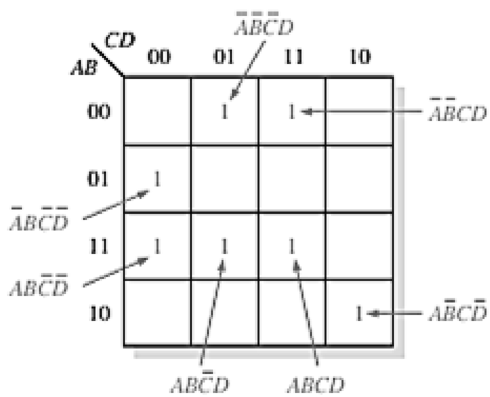
Map the following standard SOP expression on a Karnaugh map:

$$\overline{A}\,\overline{B}CD + \overline{A}B\overline{C}\,\overline{D} + AB\overline{C}D + ABCD + AB\overline{C}\,\overline{D} + \overline{A}\,\overline{B}\,\overline{C}D + A\overline{B}C\overline{D}$$

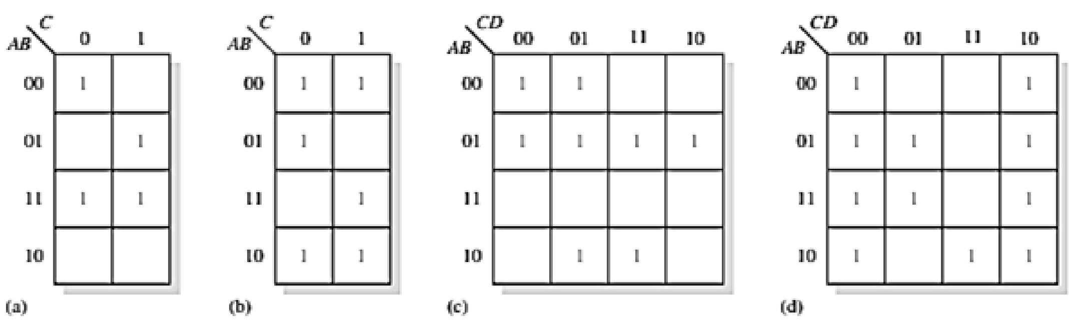
Evaluate the expression as shown below. Place a 1 on the 4-variable Karnaugh map in Figure 4–30 for each standard product term in the expression.

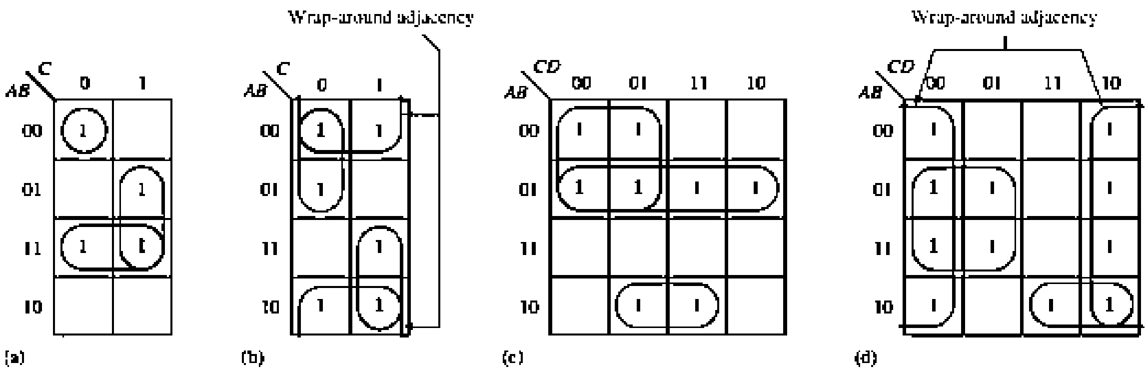
$$\overline{A}\,\overline{B}CD + \overline{A}B\overline{C}\,\overline{D} + AB\overline{C}D + ABCD + AB\overline{C}\,\overline{D} + \overline{A}\,\overline{B}\,\overline{C}D + A\overline{B}C\overline{D}$$

$$0011 \quad 0100 \quad 1101 \quad 1111 \quad 1100 \quad 0001 \quad 1010$$

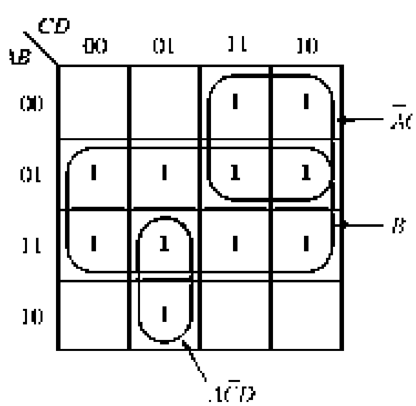


Group the 1s in each of the Karnaugh maps in Figure





Example: determine the product terms for the Karnaugh map in Figure and write the resulting minimum SOP expression



$$B + \overline{A}C + A\overline{C}D$$

Problems:

For each truth table in Table, derive a standard SOP and a standard expression

(a)

ABC	X
000	0
001	1
010	0
011	0
100	1
101	1
110	0
111	1

(b)

ABC	X
000	0
001	0
010	0
011	0
100	0
101	1
110	1
111	1

(c)

ABCD	X
0000	1
0001	1
0010	0
0011	1
0100	0
0101	1
0110	1
0111	0
1000	0
1001	1
1010	0
1011	0
1100	1
1101	0
1110	0
1111	0

(d)

ABCD	X
0000	0
0001	0
0010	1
0011	0
0100	1
0101	1
0110	0
0111	1
1000	0
1001	0
1010	0
1011	1
1100	1
1101	0
1110	0
1111	1

Use a Karnaugh map to find the minimum SOP form for each expression:

(a) $\overline{A}\,\overline{B}\,\overline{C} + \overline{A}\,\overline{B}C + A\overline{B}C$ **(b)** $AC(\overline{B} + C)$

(c) $\overline{A}(BC + B\overline{C}) + A(BC + B\overline{C})$ **(d)** $\overline{A}\,\overline{B}\,\overline{C} + A\overline{B}\,\overline{C} + \overline{A}B\overline{C} + AB\overline{C}$