H.W. Test the consistency of the

(1) set of equ.s.

$$3x + 3y + 2z = 1$$
$$x + 2y + 0z = 4$$
$$0x + 10y + 3z = -2$$

Solve by Cramer's

(2)
$$8x - 4y + z = 8$$
$$4x + 2y - 2z = 0$$
$$2x + 7y - 4z = 0$$

(3)
$$x + y + z = 1$$
$$x - y + 2z = 5$$
$$2x - 2y + 3z = 1$$

Another important information about the solution of system equ.s is the :-

- Eigenvalues and Eigenvectors:

Eigenvectors make understanding linear transformation easy. There are axes (direction) along which a linear transformation acts simply by 'stretching/compressing and

If $A = (a_{ij})$ is $n \times n$ matrix then

$$AX = \lambda X$$

A: is $n \times n$ (n by n) matrix.

X: is non-Zero $n \times 1$ vector.

λ: is a scalar (which may be either real or complex).

Any value of λ for which this equ. has a sol. is Known as an eigenvalues of the matrix A. It is some times also called the characteristic value, and the vector $\vec{X}$ which correspond to this value is called eigenvector.

$$\therefore \vec{A} \cdot \vec{X} - \lambda \vec{X} = 0$$

$$(\vec{A} - \lambda \cdot \vec{I})X = 0$$

$\because$ X is non-Zero vector, this equ. which only have a sol. if

$$\boxed{|\vec{A} - \lambda \cdot I| = 0 \quad \text{characteristic equ.}}$$

note: recall that the homogenous system ($Ax_i - \lambda_i x_i = 0$) has non-zero sol. if and only if the determinant of the matrix of coefficient is zero.

This equ. ($|A - \lambda . I| = 0$) is called the characteristic equ. of A and is an n^{th} order polynomial in λ with n roots, these roots are called the eigenvalues of A.

∴ 1- Eigenvalue equ. (characteristic equ.)

$$\boxed{|A - I\lambda| = 0}$$

Example: Find the eigenvalues of the matrix

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

The characteristic equ. is

$$|A - \lambda I| = 0$$

$$\left| \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0$$

$$\left| \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = 0$$

$$\left| \begin{bmatrix} -\lambda & 1 \\ -2 & (-3-\lambda) \end{bmatrix} \right| = 0$$

$$-\lambda(-3-\lambda) - (1 * -2) = 0$$

$$3\lambda + \lambda^2 + 2 = 0$$

$$\lambda^2 + 3\lambda + 2 = 0$$

$$(\lambda+1)(\lambda+2) = 0$$

$$\therefore \left.\begin{aligned} \lambda_1 &= -1 \\ \lambda_2 &= -2 \end{aligned}\right\} \text{the eigenvalues of the matrix } A$$

2- The eigenvectors:

To find the eigenvector associated with the eigenvalue, we have

$$AX - \lambda I X = 0$$

$$\boxed{(A-\lambda I)X = 0}$$ Eigenvectors equs.

From the prievous example

$\lambda_1 = -1 \quad , \lambda_2 = -2$

then the eigenvectors associated with the first eigenvalue $\lambda_1 = -1$

$$\because (A - \lambda_i I)X = 0$$

$$\underset{2\times2}{\left[\begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} - \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\right]} \underset{2\times1}{\begin{bmatrix} X_{1,1} \\ X_{2,1} \end{bmatrix}} = 0$$

$$\therefore \begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix} \begin{pmatrix} X_{1,1} \\ X_{2,1} \end{pmatrix} = 0$$

and for simplicity we drop the subscript 1 in $X_{1,1}$ & $X_{2,1}$

$$\left.\begin{aligned} X_1 + X_2 &= 0 \\ -2X_1 - 2X_2 &= 0 \end{aligned}\right\} \quad \text{i.e. } X_1 = -X_2$$

let $X_1 = K$, where K is an arbitrary const.

$$\therefore X_1 = \begin{pmatrix} K_1 \\ -K_1 \end{pmatrix}$$

Going through the same procedure for the second eigenvalue $\lambda_2 = -2$, then

$$X_2 = \begin{pmatrix} K_2 \\ -2K_2 \end{pmatrix}$$ H.W.

$$\therefore X = \begin{pmatrix} K_1 & K_2 \\ -K_1 & -2K_2 \end{pmatrix}$$

H.W: Find the eigenvalues and the eigenvectors of

$$B = \begin{bmatrix} 0 & 16 \\ -1 & -4 \end{bmatrix}$$

$$X_1 = \begin{bmatrix} k_1 \\ -k_1 \end{bmatrix}$$

$$X_2 = \begin{bmatrix} k_2 \\ -2k_2 \end{bmatrix}$$

To find the numerical values of the eigenvectors, X_1 and X_2 we normalize the length of the eigenvectors to unity, we require

$$(X_1)^2 + (X_2)^2 + \cdots + (X_n)^2 = 1$$ (for the eigenvectors corresponding to eigenvalue

or more generally for any eigenvalue either for real or imaginary

$$\boxed{X_1 X_1^* + X_2 X_2^* + \cdots + X_n X_n^* = 1}$$

Therefore the normalization of X_1' & X_2' where

$$X_1 = \begin{bmatrix} k_1 \\ -k_1 \end{bmatrix} \equiv \begin{bmatrix} X_1 \\ -X_1 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \longrightarrow X_2 = -X_1$$

$$\therefore X_1^* X_1 + X_2^* X_2 = 1$$

$$(X_1)^2 + (X_2)^2 = 1$$

$$(X_1)^2 + (-X_1)^2 = 1$$

$$2X_1^2 = 1 \Rightarrow X_1 = \frac{1}{\sqrt{2}}$$

$$X_2 = \frac{-1}{\ }$$

$$X_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix}$$ The normalized eigenvectors corresp to $\lambda_1 = -$

Going through the same procedure we can find the numerical values of the eigenvectors corresponding to eigenvalue $\lambda = -2$.

i.e. to find

$$X_2 = \begin{pmatrix} k_2 \\ -2k_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ -2x_1 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longrightarrow x_1 = \frac{-1}{2} x_2$$

the normalized case

$$x_1^2 (x_1^2)^* + x_2^2 (x_2^2)^* = 1 \quad \text{(by dropping the superscript 2)}$$

or $\quad (x_1)^2 + (x_2)^2 = 1$

$$\left(\frac{-1}{2} x_2\right)^2 + x_2^2 = 1$$

$$\frac{x_2^2}{4} + x_2^2 = 1 \longrightarrow x_2 = \frac{2}{\sqrt{5}} \;\;\&\;\; x_1 = \frac{-1}{\sqrt{5}}$$

$$\therefore X_2 = \begin{pmatrix} -\frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix}$$ the normalized eigenvectors corresponding to $\lambda_2 = -2$

Therefore the full normalized eigenvectors (of A) corresponding to eigenvalues $\lambda_1 = -1, \lambda_2 = -2$

$$\therefore \quad X = (X_1 \;\; X_2)$$

$$X = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{5}} \\ \frac{-1}{\sqrt{2}} & \frac{2}{\sqrt{5}} \end{pmatrix}$$

ملخص
Summary: Given the matrix $A_{n\times n}$, find the eigenvalues and the associated normalized eigenvectors.

$$A_{n\times n} \xrightarrow{\text{find}} ① \ \lambda_n (\lambda_1, \lambda_2, \dots)$$

② find the eigenvectors corresponding to λ_1

$$X_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

③ normalize X to find its numerical values

i.e. $x_1 x_1^* + x_2 x_2^* = 1$

then $X_1 = (\quad)_{\lambda_1}$

④ Similarly find

$$X_2 = (\quad)_{\lambda_2}$$

⑤ $X = (X_1 \ X_2)$