Eigenvalues and Eigenvectors (Eigen-function)

Another important information about the solution of linear of system equation are the eigenvalues and the eigenvector.

Consider a matrix A, for an example one representation a physical transformation, When this matrix is used to transforms a given vector $\vec{x}$, the result is

$$y = AX$$

Now an interesting question is : Are there any vector which does not change its direction under this transformation. Howeve allow the vector magnitude vary by scalar λ, such a question is of the form

$$A X_i = \lambda_i X_i$$

nxn matrix non-zero matrix

eigenvectors

eigenvalues

So such special $\vec{x}_i$ are called eigenvectors and the change in magnitude depends on the eigenvalue λ, if $A = (a_{ij})_{n \times n}$, then

$$\vec{A} \cdot \vec{x} = \lambda \vec{I} \cdot \vec{x}$$

$\vec{A}$: is $n \times n$ (n by n) matrix

$\vec{x}_i$: is a non-zero $n \times 1$ vector

λ_i : is a scalar (either real or complex)

Any value of λ for which this equation has a solution is known as an eigenvalue of the matrix A. It is sometimes also called the characteristic value, and the vector $\vec{x}$ is which corresponds to this value is called eigenvector. The eigenvalue problem can be written as.

$$\vec{A} \cdot \vec{x}_i - \lambda_i \vec{I} \cdot \vec{x}_i = 0$$

$$(\vec{A} - \lambda_i \vec{I}) \cdot \vec{x}_i = 0$$, & for simplicity we drop the vector arrow.

If $\vec{x}$ is non-zero, this equation which only have a solution if

$$\boxed{|\vec{A} - \lambda \vec{I}| = 0}$$ The characteristic equation

Example: Find the eigenvalues $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$

Solution: The characteristic equation is

$$|A - \lambda I| = 0$$

$$\left| \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} - \lambda \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \right| = 0$$

$$\begin{vmatrix} -\lambda & 1 \\ -2 & -3-\lambda \end{vmatrix} = \lambda^2 - 3\lambda + 2\lambda = 0$$

$$\lambda^2 - 3\lambda + 2\lambda = 0 \rightarrow (\lambda + 1)(\lambda + 2) = 0$$

$\therefore$ The two eigenvalues are $\lambda_1 = -1$, $\lambda_2 = -2$

- The eigenvectors

In order to find the eigenvectors corresponding to each eigenvalue, then

$$\boxed{A X_i = \lambda_i X_i \qquad \text{Eigenvector equations}}$$

Example: From the previous example

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$ has an eigenvalues $\lambda_1 = -1, \lambda_2 = -$

Find their associated non-zero eigenvectors

solution: The eigenvectors X_1 associated with the eigenvalue $\lambda = -1$

$$(A - \lambda_i I) X_i = 0$$

$$(A - \lambda_1 I) X_1 = 0$$

$$\left(\begin{bmatrix} 1 & 0 \\ -2 & -3 \end{bmatrix} - \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \right) \begin{bmatrix} X_1^1 \\ X_2^1 \end{bmatrix} = 0$$

Where $X_1^{(1)}$ and $X_2^{(2)}$ are the eigenvectors of the eigenvalue λ_1, and for simplicity we drop the superscripts (1) & (2) $X_1^{(1)}$ & $X_2^{(2)} \longrightarrow$

X_1 & X_2

$$\begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = 0$$

$X_1 + X_2 = 0 \longrightarrow X_1 = -X_2$, and

$-2X_1 - 2X_2 = 0 \longrightarrow X_1 = -X_2$

Let $X1 = K1$; $X2 = -k_1$

$$\therefore X_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1^{(1)} \\ -x_1 \end{bmatrix} = \begin{bmatrix} K1 \\ -K1 \end{bmatrix}$$

Then the eigenvectors for $\lambda = -2$

$$A X_2 = \lambda_2 X_2$$

$$(A - \lambda_2) X_2 = 0$$

$$\left(\begin{bmatrix} 1 & 0 \\ -2 & -3 \end{bmatrix} - (-2) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{pmatrix} x_1^{(2)} \\ x_2^{(2)} \end{pmatrix} = 0$$

$$\begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\left. \begin{array}{l} 2x_1 + x_2 = 0 \\ -2x_1 - x_2 = 0 \end{array} \right] \longrightarrow 2x_1 = -x_2$$

let $x1 = K_2$; $x_2 = -2K_2$

$$\therefore X_2 = \begin{bmatrix} x_1 \\ -2x_1 \end{bmatrix} = \begin{bmatrix} K_2 \\ -2K_2 \end{bmatrix}$$

$$\therefore X = \begin{bmatrix} K_1 & K_2 \\ -K_1 & -2K_2 \end{bmatrix}$$

Then $K1$ & K_2 Can be found using normalization.

H.W. : Find the eigenvalues and the associtated eigenvector for the matrix

$$B = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$$

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Answer : $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 3$ and the associated eigenvectors $\begin{bmatrix} K_1 & K_2 & K_1 \\ -K_1 & -K_1 & -K_1 \\ 0 & -K_2 & -K_2 \end{bmatrix}$

To find the numerical values of the eigenvectors X_1 and X_2, we normalize the length of the eigenvectors to unity

$$(X_1^1)^2 + (X_2^1)^2 + \cdots + (X_n^1)^2 = 1$$

More generally for real and Complex eigenvectors

$$(X_1)(X_1)^* + X_2(X_2)^* + \cdots + X_n(X_n)^* = 1$$

The numerical eigenvector for

$$X_1 = \begin{bmatrix} X_1^1 \\ X_2^1 \end{bmatrix} = \begin{bmatrix} k_1 \\ -k_1 \end{bmatrix}$$

$\therefore \quad X_1 X_1^* + X_2 X_2^* = 1$ and for real X

$X_1^2 + X_2^2 = 1 \qquad \therefore \quad X_2 = -X_1$

$X_1^2 + (-X_1)^2 = 1 \longrightarrow 2X_1^2 = 1 \qquad X = \pm \frac{1}{\sqrt{2}}$

or $X_1 = \frac{1}{\sqrt{2}}$ for positive value & $X_2 = -\frac{1}{\sqrt{2}}$,

$$\therefore \quad X_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix},$$

similarly $\quad X_2 = \begin{bmatrix} \frac{-1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}$

Therefore the full normalized eigenvectors corresponding to $\lambda_1 = -1$, $\lambda_2 = -2$ is,

$$X = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{5}} \\ -\frac{1}{\sqrt{2}} & \frac{2}{\sqrt{5}} \end{bmatrix} \quad (5)$$

Diagonalization of Matrices

For a given (n by n) matrix "A" there exist an invertable matrix X (Whose columns are are formed from the eigenvectors of "A"), Such that $D = X^{-1} A X$ is diagonal matrix D, then the two matrices A and the diagonal matrix D are said to Similar, through the equation

$$D = X^{-1} A X \quad \text{(similarity transformation)}$$

which means that the matrix A has been transformed into diagonal D, by similarity transformation.

<u>H.W.</u> prove that $D = X^{-1} A X$.

Note: The square matrix X which diagonalizes A, is found by grouping the eigen vector of A into a square matrix and the resulting diagonal matrix has eigenvalues of A as its diagonal elements, and the transformation of a matrix A to $X^{-1} A X$ is known as <u>similarity transformation</u>.

<u>H.W.</u> : If $X = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{5}} \\ -\frac{1}{\sqrt{2}} & \frac{2}{\sqrt{5}} \end{bmatrix}$ then,

$$D = X^{-1} A X = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$$

Example: Find the eigenvalues and associated normalized eigenvectors, then find the diagonalized matrix D, of the matrix
$$\begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$$

Solution:

1. The eigenvalues : $|A - I\lambda| = 0$ Characteristic equation.

$$\left|\begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right| \quad \begin{vmatrix} 4-\lambda & 1 \\ 2 & 3-\lambda \end{vmatrix} = (\lambda - 2)(\lambda - 5) = 0$$

Hence $\boxed{\lambda_1 = 2 \quad ; \quad \lambda_2 = 5}$

2. The eigenvectors belong to λ_1 & λ_2

$$(A - \lambda_i I) X_i = 0$$

a for $\lambda_1 = 2$

$$\left[\begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix} - 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right] \begin{bmatrix} X_1^{(1)} \\ X_2^{(2)} \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \longrightarrow \left.\begin{matrix} 2X_1 + X_1 = 0 \\ 2X_1 + X_1 = 0 \end{matrix}\right\} \text{twice}$$

$$\therefore \ X_1 = -\tfrac{1}{2} X_2$$

The normalization of the real eigenvectors to unity is $X_1^2 + X_2^2 = 1 \longrightarrow (-\tfrac{1}{2}X_2)^2 + X_2^2 = 1$

$$\frac{X_2^2}{4} + X_2 = 1 \longrightarrow X_2 = \frac{2}{\sqrt{5}} \quad ; \quad X_1 = -\frac{1}{\sqrt{5}}$$

$$\therefore X_1^{(1)} = \begin{pmatrix} -\frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix}$$

b/ $\lambda_2 = 5$

$$\left[\begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix} - 5\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right]\begin{bmatrix} X_1^{(2)} \\ X_2^{(2)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \longrightarrow \left.\begin{matrix} -X_1 + X_2 = 0 \\ 2X_1 - 2X_2 = 0 \end{matrix}\right\} \text{then } X1 = X_2$$

The normalized of the eigenvectors to unity is

$$X_1^2 + X_2^2 = 1 \longrightarrow X_1^2 + X_1^2 = 1 \rightarrow X_1 = X_2 = \frac{1}{\sqrt{2}}$$

$$\therefore X_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

Then the full matrix X, whose Column are formed from the normalized eigenvectors of A is

$$X = \begin{pmatrix} -\frac{1}{\sqrt{5}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

3. The diagonalized matrix $D = X^{-1} A X$

$$X^{-1} = \frac{adj\,(X)}{|X|}$$

$$|X| = -\frac{1}{\sqrt{5}}\frac{1}{\sqrt{2}} - \frac{2}{\sqrt{5}\sqrt{2}} = -\frac{3}{\sqrt{2}\sqrt{5}}$$

$$\operatorname{adj}(X) = [\text{Cofactor } X]^T$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-2}{\sqrt{5}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{5}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{-2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \end{bmatrix}$$

$$\therefore X^{-1} = -\frac{2\sqrt{2}\sqrt{5}}{3}\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{-2}{\sqrt{2}} & \frac{1}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} -\frac{\sqrt{5}}{3} & \frac{\sqrt{5}}{3} \\ -\frac{2\sqrt{2}}{3} & \frac{\sqrt{2}}{3} \end{bmatrix}$$

A striaght forwarad calculation now gives.

$$X^{-1}AX = \begin{bmatrix} -\frac{\sqrt{5}}{3} & \frac{\sqrt{5}}{3} \\ -\frac{2\sqrt{2}}{3} & \frac{\sqrt{2}}{3} \end{bmatrix} \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{5}} & \frac{1}{\sqrt{2}} \\ \frac{2}{-\sqrt{5}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} = D$$

The elements of the leading diagonal of the matrix D, represent the eigenvalues $\lambda_1 = 2$, $\lambda_2 = 5$ of the matrix A