– Power of a Matrix

Diagonalization is an important concept and is useful in many ways. For example, the elements of the K^{th} power (k a positive integer) of a given square matrix A are usually difficult to obtain except for small values of k.

$\because$ $D = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$, then $D^k = \begin{pmatrix} \alpha^k & 0 \\ 0 & \beta^k \end{pmatrix}$

suppose A is an arbitrary matrix of order 2, which is similar to the diagonal matrix D, then. $D = X^{-1} A X$: X: non-singular matrix i.e. $|X| \neq 0$

$$\therefore D^2 = (X^{-1} A X)(X^{-1} A X)$$
$$= X^{-1} A \underbrace{X X^{-1}}_{I} A X = X^{-1} A^2 X$$

similarly $D^3 = X^{-1} A^3 X$, and in general

$$D^k = X^{-1} A^k X$$

To find A^k from the above equation, a pre-multiplication by X and post-multiplication by X^{-1}, so that $X D^k X^{-1} = X X^{-1} A^k X X^{-1}$

$$\therefore \boxed{A^k = X D^k X^{-1}}$$

Hence, since D^k may be easily calculated, A^k may be obtained.

Similar matrix

The two matrices A and D are said to be similar if there exist a non-singular matrix, such that $D = X^{-1} A X$

or in general for any two similar matrices A and B with a non-sigular matrix X; then

$$D = X^{-1} A X .$$

proof: Let $R1 = AR$, multiply by X

$$X R1 = X A R$$

$$= X A (X^{-1} X) R$$

$$X R1 = (X A X^{-1}) X R$$

$$X R_1 = D X R$$

$$\therefore D = X A X^{-1}$$

Exercise: Show that similar matrices have the same eigenvalues.

solution: $D = X^{-1} A X$

$$\therefore D - \lambda I = X^{-1} A X - \lambda I$$

$$= X^{-1} A X - \lambda X^{-1} X$$

$$= X^{-1} (A - \lambda I) X$$

and hence

$$| D - \lambda I | = | X^{-1} (A - \lambda I) X |$$

$$= | X^{-1} | | A - \lambda I | | X | = | X^{-1} | | X | | A - \lambda I |$$

$$= | A - \lambda I |$$

Q/ Find the diagonal matrix of the matrix

$$A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$$

$$|A - \lambda I| = 0 \qquad \begin{vmatrix} 5-\lambda & 4 \\ 1 & -2\lambda \end{vmatrix} = 0 \qquad \underbrace{\lambda_1 = 1,\ \lambda_2 = 6}_{\text{Eigen-values}}$$

The Characteristic equation

Eigenvectors : $\lambda_1 = 1$

$$\left[\begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right]\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{bmatrix} 4 & 4 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \left.\begin{aligned} 4X1 + 4X2 &= 0 \\ X1 + X2 &= 0 \end{aligned}\right\} X_1 = -X_2$$

The normalized eigenvectors

$$X_1^2 + X_2^2 = 1 \longrightarrow X_1^2 + (-X_1)^2 = 1 \rightarrow X_1 = \frac{1}{\sqrt{2}},\ X_2 = \frac{1}{\sqrt{2}}$$

$$X_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$$

Eigenvector : $\lambda = 6$:

$$\left[\begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix} - \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}\right]\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{bmatrix} -1 & 4 \\ 1 & -4 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \left.\begin{aligned} -X_1 + 4X_2 &= 0 \\ X_1 - 4X_2 &= 0 \end{aligned}\right\} X_2 = \frac{1}{4}X_1$$

The normalized eigenvectors $\quad X_1^2 + X_2^2 = 1$

$$X_1^2 + \left(\frac{1}{4}X_1\right)^2 = 1 \longrightarrow X_1 = \frac{4}{\sqrt{17}},\ X_2 = \frac{1}{\sqrt{17}}$$

$$\therefore X_2 = \begin{pmatrix} \frac{4}{\sqrt{17}} \\ \frac{1}{\sqrt{17}} \end{pmatrix} \longrightarrow X = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{4}{\sqrt{17}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{17}} \end{pmatrix}$$

H.W. Find X^{-1} & $D = X^{-1} D X$

- The Cayley - Hamilton Theorem.

Theorem: Every square matrix satisfies its own Characteristic equation $|A-\lambda I|=0$

That is only if

$$\begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix} = 0$$ Characteristic equation

Then the expansion of $|A-I\lambda|$ give rise to an n^{th} degree polynomial in λ say $f(\lambda)$, Called the Characteristic polynomial and the roots $\lambda_1, \lambda_2, \cdots, \lambda_n$ of the Charateristic equation $f(\lambda)=0$ are Called the eigenvalues (Characteristic roots, latent roots or proper values) of the matrix A. To each root λ_i $(i=1,2,\cdots,n)$ there is a non-trivial Solution X_i called the eigenvectors (Characteristic vector, latent vector). For any other value of $\lambda \neq \lambda_i$ the only solution of $(A-\lambda I)X=0$ is the trivial one $X=0$

$\therefore$ The left hand side of $(A-\lambda I)X=0$ my be written as

$$f(\lambda)=|A-\lambda I|=(-1)^n\left[\lambda^n-\alpha_1\lambda^{n-1}+\alpha_2\lambda^{n-2}-\cdots+(-1)^n a_n\right]$$

Where $\alpha_1, \alpha_2, \ldots, \alpha_n$ defined in terms of the elements a_{ik}, and hence

$$f(A) = (-1)^n (A^n - \alpha_1 A^{n-1} + \alpha_2 A^{n-2} - \cdots + (-1)^n \alpha_n I)$$

Now If X_i is an eigenvectors of A Corresponding to the i^{th} eigenvalues λ_i then

$$A X_i = \lambda_i X_i \quad , \text{ and}$$

$$f(A) X_i = (-1)^n \left[A^n - \alpha_1 A^{n-1} + \alpha_2 A^{n-2} - \cdots + (-1)^n \alpha_n I\right] X_i$$

i.e. for eigenvalues

$$\boxed{\lambda^n + \alpha_1 \lambda^{n-1} + \alpha_2 \lambda^{n-2} + \cdots + \alpha^n = 0}$$

and Cayley - Hamilton theorem

$$\boxed{A^n + \alpha_1 A^{n-1} + \alpha_2 A^{n-2} + \cdots + \alpha_n I = 0}$$

The following example illustrate the theorem.

Example: Given $A = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}$

Show that "A" must satisfy the C-H. Theorem:

Solution: $|A - I\lambda| = 0$ the characteristic equation

$$\left| \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right| = 0 \qquad \begin{vmatrix} 4-\lambda & 1 \\ 2 & 3-\lambda \end{vmatrix} = 0$$

$\therefore \lambda^2 - 7\lambda + 10 = 0$

Hence, by the Cayley - Hamilton theorem, A must satisfy the relation

$$A^2 - 7A + 10I = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$A^2 = \begin{bmatrix} 18 & 7 \\ 14 & 11 \end{bmatrix},$$

$$7A = \begin{pmatrix} 28 & 7 \\ 14 & 21 \end{pmatrix}$$

$$\begin{bmatrix} 18 & 7 \\ 14 & 11 \end{bmatrix} - \begin{bmatrix} 28 & 7 \\ 14 & 21 \end{bmatrix} + \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} =$$

$$= \begin{bmatrix} 18-28+10 & 7-7+0 \\ 14-14+0 & 11-21+10 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Further more , since A is non-singular ($\det A \neq 0$) and $A^2 - 7A + 10 = I$, find A^{-1}

$$\therefore \; A^{-1}A^2 - 7AA^{-1} + 10A^{-1} = 0$$

$$A - 7I + 10A^{-1} = 0$$

$$\therefore A^{-1} = \frac{1}{10}\left[\begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix} - \begin{pmatrix} 7 & 0 \\ 0 & 7 \end{pmatrix}\right] = \frac{1}{10}\begin{vmatrix} -3 & 1 \\ 2 & -4 \end{vmatrix} = \begin{vmatrix} \frac{-3}{10} & \frac{1}{10} \\ \frac{2}{10} & \frac{-4}{10} \end{vmatrix}$$

This is a very useful way of evaluating the inverse of a matrix, and may readily extended to higher negative power. : Find A^{-2}

$$A^{-2} = A^{-1}A^{-1} = \; ?$$

Furthermore, Since A is non-singular and

$$A^2 - 7A + 10I = 0$$

This equation may be written as (multiply by A^{-1}

$$A^{-1}A^2 - 7A^{-1}A + 10A^{-1} = 0$$

$$A - 7I + 10A^{-1} = 0$$

$$\therefore \quad A^{-1} = \frac{1}{10}(A - 7I)$$

$$A^{-1} = \frac{1}{10}\left[\begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix} - \begin{pmatrix} 7 & 0 \\ 0 & 7 \end{pmatrix}\right] = \frac{1}{10}\begin{bmatrix} -3 & 1 \\ 2 & -4 \end{bmatrix} = \begin{bmatrix} -\frac{3}{10} & \frac{1}{10} \\ \frac{2}{10} & \frac{-4}{10} \end{bmatrix}$$

This is a very useful way of evaluating the inverse of a matrix, and may be readily extended to higher negative powers; to find A^{-2}

$$A^{-2} = A^{-1}A^{-1} = \begin{bmatrix} -\frac{3}{10} & \frac{1}{10} \\ \frac{2}{10} & \frac{-4}{10} \end{bmatrix}\begin{bmatrix} -\frac{3}{10} & \frac{1}{10} \\ \frac{2}{10} & \frac{-4}{10} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{9}{100} + \frac{2}{100} & -\frac{3}{100} - \frac{4}{100} \\ -\frac{6}{100} + \left(-\frac{8}{100}\right) & \frac{2}{100} + \frac{16}{100} \end{bmatrix} = \begin{bmatrix} \frac{11}{100} & -\frac{7}{100} \\ \frac{-14}{100} & \frac{18}{100} \end{bmatrix}.$$

- Power of matrices

- Using the eigenvalues of a given matrix $A_{2\times 2}$ to find the powers of (2x2) matrices

$$A^n = \frac{\lambda_1^n \lambda_2 - \lambda_1 \lambda_2^n}{\lambda_2 - \lambda_1} \overbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}^{I} + \frac{\lambda_2^n - \lambda_1^n}{\lambda_2 - \lambda_1} A$$

for $\lambda_1 \neq \lambda_2$

and

$$A^n = (1-n)\lambda_1^n I + n\lambda_1^{n-1} A \qquad \text{for} \quad \lambda_1 = \lambda_2$$

Note: ① A square matrix A which satisfies the relation

$$A^2 = A$$

is called idempotent.

proof: If $AB = A$ & $BA = B$

for then $ABA = (AB)A = AA = A^2$ —①

and $ABA = A(BA) = AB = A$ —②

$\therefore$ from 1 & 2 then $A = A^2$

② A square matrix which satisfies the relation

$$A^k = 0$$

where k is a positive integer, is said to nilpotent of order k.