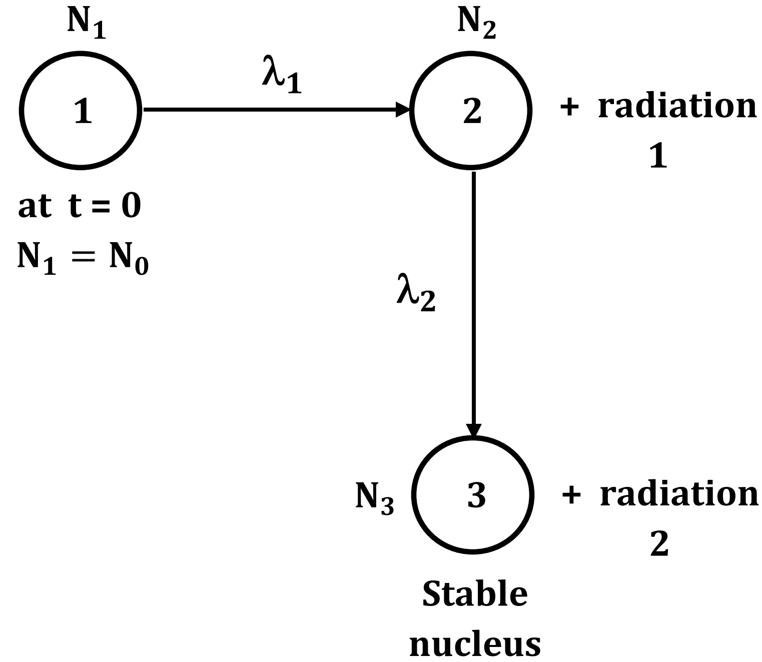
<u>**Production of Radioisotope by a Decaying Parent**</u>

<u>**(Radioactive – Series Decay)**</u>

Suppose a parent nucleus (1) decays and produced radioactive daughter (2) and radiation (1), and that the daughter (2) producing a stable nucleus (3) and radiation (2). Note that every decaying nucleus produces <u>one</u> daughter nucleus.



$$\frac{dN_1}{dt} = -\lambda_1 N_1 \quad \text{-------------} \quad ①$$

$$\frac{dN_2}{dt} = \lambda_1 N_1 - \lambda_2 N_2 \quad \text{-------------} \quad ②$$

$$\frac{dN_3}{dt} = \lambda_2 N_2 \quad \text{-------------} \quad ③$$

<u>**at time $t = 0$**</u>

$$N_1 = N_0$$

$$N_2 = N_3 = 0$$

<u>**at time $t = \infty$**</u>

$$N_1 = N_2 = 0$$

$$N_3 = N_0$$

if N_0 is the original number of parent nucleus (1) at time ($t = 0$) then :

$$N_1 = N_0\, e^{-\lambda_1 t} \quad \text{-------------} \quad ④$$

From eq. ② and ④ we have :

$$\frac{dN_2}{dt} = +\lambda_2 N_2 = N_0\, e^{-\lambda_1 t} \quad \text{-------------} \quad ⑤$$

if daughter ② is stable nucleus (i.e. $\lambda_2 = 0$) then $N_2 = N_0(1 - e^{-\lambda_1 t})$ at any time (t) :

$$N_1 + N_2 = N_0$$

The total number of nuclei constant

This is a differential equation consist of a general solution homogeneous solution equation. The solution for N_2 as a function of time will be :

$$N_2 = N_0\left(C_1\, e^{-\lambda_1 t} + C_2\, e^{-\lambda_2 t}\right) \quad \text{---------} \quad ⑥$$

In order to evaluate the coefficients $\mathbf{C_1}$ and $\mathbf{C_2}$ we substitute $\mathbf{N_2}$ and $\frac{\mathbf{dN_2}}{\mathbf{dt}}$ from **eq.** ⑥ in **eq.** ⑤.

The differentiation of **eq.** ⑥ give :

$$\frac{\mathbf{dN_2}}{\mathbf{dt}} = \mathbf{N_0}\,(-\,\lambda_1 \mathbf{C_1}\, \mathbf{e}^{-\lambda_1 t} \;-\; \lambda_2 \mathbf{C_2}\, \mathbf{e}^{-\lambda_2 t})$$

Substituting into **eq.** ⑤, we obtain :

$$\cancel{\mathbf{N_0}}\left(-\,\lambda_1 \mathbf{C_1}\, \mathbf{e}^{-\lambda_1 t} - \lambda_2 \mathbf{C_2}\, \mathbf{e}^{-\lambda_2 t}\right) + \lambda_2 \cancel{\mathbf{N_0}} \left(\mathbf{C_1}\, \mathbf{e}^{-\lambda_1 t} + \mathbf{C_2}\, \mathbf{e}^{-\lambda_2 t}\right) = \lambda_1 \cancel{\mathbf{N_0}}\, \mathbf{e}^{-\lambda_1 t}$$

$$\therefore -\,\lambda_1 \mathbf{C_1}\, \mathbf{e}^{-\lambda_1 t} - \lambda_2 \mathbf{C_2}\, \mathbf{e}^{-\lambda_2 t} + \lambda_2 \mathbf{C_1}\, \mathbf{e}^{-\lambda_1 t} + \lambda_2 \mathbf{C_2}\, \mathbf{e}^{-\lambda_2 t} = \lambda_1\, \mathbf{e}^{-\lambda_1 t}$$

$$\mathbf{e}^{-\lambda_1 t}\,(-\,\lambda_1 \mathbf{C_1} + \lambda_2 \mathbf{C_1} - \lambda_1) = \mathbf{0} \qquad \text{---------} \quad ⑦$$

Since $\mathbf{e}^{-\lambda_1 t} \neq \mathbf{0}$, then $(-\,\lambda_1 \mathbf{C_1} + \lambda_2 \mathbf{C_1} - \lambda_1) = \mathbf{0}$

$$\therefore\ \mathbf{C_1} = \frac{\lambda_1}{\lambda_2 - \lambda_1}$$

The coefficient $\mathbf{C_2}$ depends on the value $\mathbf{N_2}$, at time $\mathbf{t = 0}$, $\mathbf{N_2 = 0}$. From **eq.** ⑥

$\mathbf{0 = N_0(C_1 + C_2)}$, and since $\mathbf{N_0 \neq 0}$, then

$\mathbf{C_1 + C_2 = 0}$ or $\mathbf{C_2} = -\,\mathbf{C_1} = -\,\frac{\lambda_1}{\lambda_2 - \lambda_1}$

then the total number of daughter nuclei ($\mathbf{N_2}$) will be :

$$\mathbf{N_2} = \mathbf{N_0}\,\frac{\lambda_1}{\lambda_2 - \lambda_1}\,(\mathbf{e}^{-\lambda_1 t} - \mathbf{e}^{-\lambda_2 t}) \qquad \text{---------} \quad ⑧$$

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The activity of radioactive daughter $(\mathbf{2})$ is $\mathbf{A_2} = \lambda_2 \mathbf{N_2}$

$$\therefore\ \mathbf{A_2} = \lambda_2 \mathbf{N_2} = \lambda_2\, \mathbf{N_0}\,\frac{\lambda_1}{\lambda_2 - \lambda_1}(\mathbf{e}^{-\lambda_1 t} - \mathbf{e}^{-\lambda_2 t}) \qquad \text{---------} \quad ⑨$$

$$\mathbf{A_2} = (\mathbf{N_0}\,\lambda_1)\,\frac{\lambda_2}{\lambda_2 - \lambda_1}(\mathbf{e}^{-\lambda_1 t} - \mathbf{e}^{-\lambda_2 t})$$

And since the activity of the parent $(\mathbf{1})$ at time $(\mathbf{t})$ is

$$\mathbf{A_1} = (\mathbf{N_1}\,\lambda_1) = (\mathbf{N_0}\,\lambda_1)\,\mathbf{e}^{-\lambda_1 t}$$

$$\therefore\ (\mathbf{N_0}\,\lambda_1) = (\mathbf{N_1}\,\lambda_1)\ \mathbf{e}^{+\lambda_1 t}$$

∴ **The daughter activity**

$$A_2 = A_1 \frac{\lambda_2}{\lambda_2 - \lambda_1}\left[1 - e^{-(\lambda_2-\lambda_1)t}\right] \quad \text{---------} \quad (10)$$

The total number of nucleus **3** (i.e. N_3) can be obtained from **eq.** (8) and (3) if $N_3 = 0$ at time $t = 0$:

$$N_3 = N_0 \frac{\lambda_1\lambda_2}{\lambda_2 - \lambda_1}\left(\frac{1 - e^{-\lambda_1 t}}{\lambda_1} - \frac{1 - e^{-\lambda_2 t}}{\lambda_2}\right)$$

* if parent (1) is **short – lived** compared to the daughter (2), i.e. $T_1 \ll T_2$ or ($\lambda_1 \gg \lambda_2$), then after a **long – time** ($t \gg \frac{1}{\lambda_1}$) ; $e^{-\lambda_1 t} \ll e^{-\lambda_2 t}$, and from **equation** (8) we have:

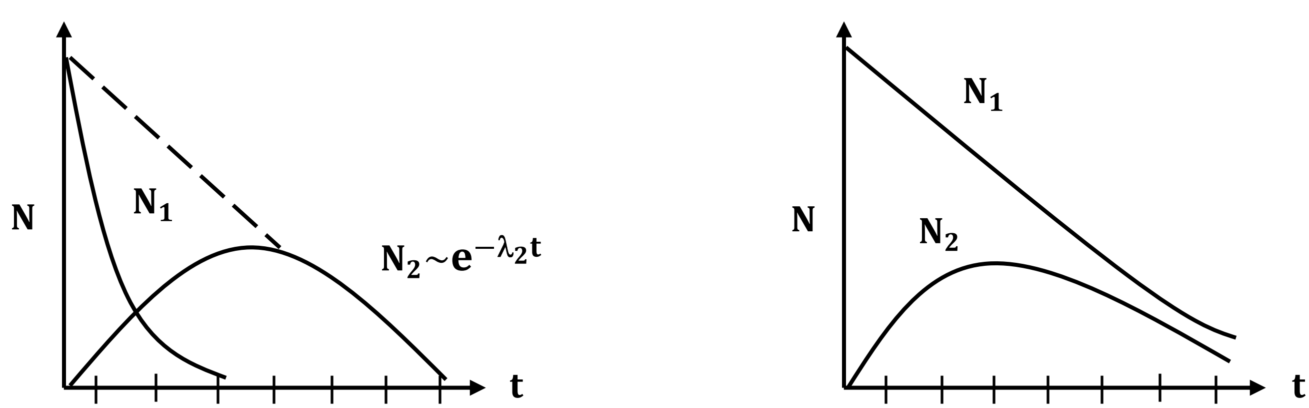
$$N_2 = N_0 \frac{\lambda_1}{\lambda_1 - \lambda_2}\left(e^{-\lambda_2 t}\right)$$

This means that the decay of daughter (2) is determined by its own **half – life** T_2 .

* if the parent (1) is **long – lived** compared to its daughter (2) i.e. $T_1 \gg T_2$ or ($\lambda_1 \ll \lambda_2$), then after a **long – time** ($t \gg \frac{1}{\lambda_1}$) , $e^{-\lambda_2 t} \approx 0$ and N_2 from **eq.** (8) will be:

$$N_2 = N_0 \frac{\lambda_1}{\lambda_2 - \lambda_1}\left(e^{-\lambda_1 t}\right)$$

This means that the decay of daughter (2) is determined by the **half – life** of parent (1).



Short – lived Parent
$T_1 < T_2$; ($\lambda_1 > \lambda_2$)

Long – lived Parent
$T_1 > T_2$; ($\lambda_1 < \lambda_2$)

Fig. Decay of Radioactive Daughter

Transient Equilibrium : if the **parent** is **longer – lived** than the daughter ($T_1 > T_2$) but the **half – life** of parent is **not** very long , then $e^{-\lambda_2 t} \ll e^{-\lambda_1 t}$, therefore **;**

$$N_2 = N_0 \frac{\lambda_1}{\lambda_2 - \lambda_1} e^{-\lambda_1 t} = \left(N_0\, e^{-\lambda_1 t}\right) \frac{\lambda_1}{\lambda_2 - \lambda_1} = N_1 \frac{\lambda_1}{\lambda_2 - \lambda_1}$$

$\frac{N_2}{N_1} = \frac{\lambda_1}{\lambda_2 - \lambda_1}$. Multiplying by $\left(\frac{\lambda_2}{\lambda_1}\right)$ we obtain $\frac{N_2 \lambda_2}{N_1 \lambda_1} = \frac{\lambda_2}{\lambda_2 - \lambda_1}$

$$\therefore \frac{\text{Activity of 2}}{\text{Activity of 1}} = \frac{\lambda_2}{\lambda_2 - \lambda_1} \quad \text{or} \quad \frac{A_2}{A_1} = \frac{\lambda_2}{\lambda_2 - \lambda_1}$$

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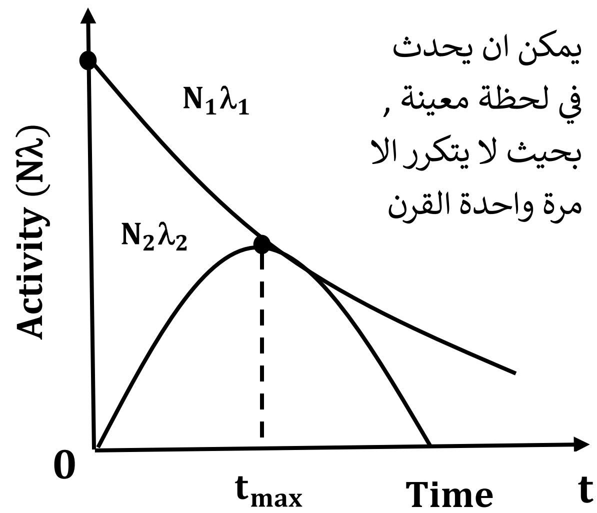
Secular (Ideal) Equilibrium :

When $\mathbf{T_1 \gg T_2}$, i.e. when $\boldsymbol{\lambda_1 \ll \lambda_2}$, the activity of the residual parent ($\boldsymbol{\lambda_1 N_1}$) and of the accumulated daughter ($\boldsymbol{\lambda_2 N_2}$) are equal since from transient equilibrium we have **:**



$$\frac{\textbf{Activity of 2}}{\textbf{Activity of 1}} = \frac{A_2}{A_1} = \frac{\lambda_2 N_2}{\lambda_1 N_1} = \frac{\lambda_2}{\lambda_2 - \lambda_1}$$

and since $\boldsymbol{\lambda_1 \ll \lambda_2}$, therefore $\boldsymbol{\lambda_2 - \lambda_1 \approx \lambda_2}$

$$\therefore \frac{\textbf{Activity of 2}}{\textbf{Activity of 1}} = \mathbf{1}$$ **; i.e. Activity of daughter 2 = Activity of Parent 1**

Note that **eq.** ② shows immediately that $\mathbf{dN_2/dt = 0}$

$$\frac{dN_2}{dt} = \lambda_1 N_1 - \lambda_2 N_2 = 0 \qquad \therefore \lambda_2 N_2 = \lambda_1 N_1$$

In this case the equilibrium is called **"Secular"** or **"Ideal"** (in which type 2 nuclei is decaying at the same rate in which is formed) and exist only at time when the activity of daughter ② is maximum , which is $\mathbf{t_{max}}$ **.**

<u>Time of Maximum Activity of Daughter Product</u>

From **eq.** ⑧ the amount of $\mathbf{N_2}$ is **<u>zero</u>** both at $\mathbf{t = 0}$ and $\mathbf{t = \infty}$, when all the atoms of both parent ① and daughter ② have decayed. At some intermediate time $\mathbf{t_{max}}$, the amount of daughter ② and hence its activity $(\lambda_2 N_2)$, pass through a maximum value . $\mathbf{dN_2/dt = 0}$ for time $(\mathbf{t_{max}})$. From differential of **eq.** ⑧ we find $(\mathbf{t_{max}})$ for maximum activity of $(\mathbf{N_2})$.

$$\mathbf{N_2 = N_0 \frac{\lambda_1}{\lambda_2 - \lambda_1} \left(e^{-\lambda_1 t} - e^{-\lambda_2 t}\right)}$$

$$\therefore \ \mathbf{\frac{dN_2}{dt} = N_0 \frac{\lambda_1}{\lambda_2 - \lambda_1} \left(-\lambda_1 e^{-\lambda_1 t} + \lambda_2 e^{-\lambda_2 t}\right) = 0}$$

$$\therefore \ \mathbf{-\lambda_1 e^{-\lambda_1 t_{max}} + \lambda_2 e^{-\lambda_2 t_{max}} = 0 \quad ; \quad \frac{\lambda_2}{\lambda_1} = e^{(\lambda_2 - \lambda_1)\, t_{max}}}$$

Therefore ; $\mathbf{t_{max} = \frac{\ln(\lambda_2/\lambda_1)}{\lambda_2 - \lambda_1}}$ or $\mathbf{t_{max} = \tau_2 \left(\frac{T_1}{T_1 - T_2}\right) \ln\left(\frac{T_1}{T_2}\right)}$

This is important result shows that $\mathbf{t_{max}}$ is positive and real for either $\mathbf{T_1 > T_2}$ or $\mathbf{T_1 < T_2}$.

Production (Yield) of Radioactive Nuclei by Nuclear Bombardment

Let's assume that we place a target of stable nuclei into a reactor or an accelerator such as a cyclotron. The nuclei of the target will capture (**absorb**) a neutron or a charged particle to produce a radioactive species.

Suppose that **Q** atoms (**Nuclei**) are produced per second at constant rate

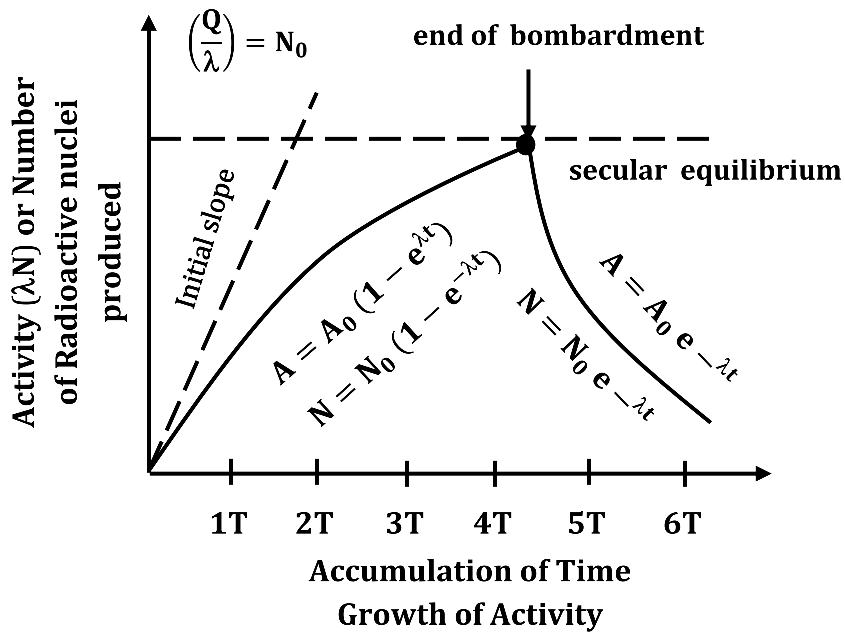
$$\mathbf{Q = N_0\, \sigma\, \Phi = constant}$$

Where

$\mathbf{N_0} \Rightarrow$ **target nuclei**

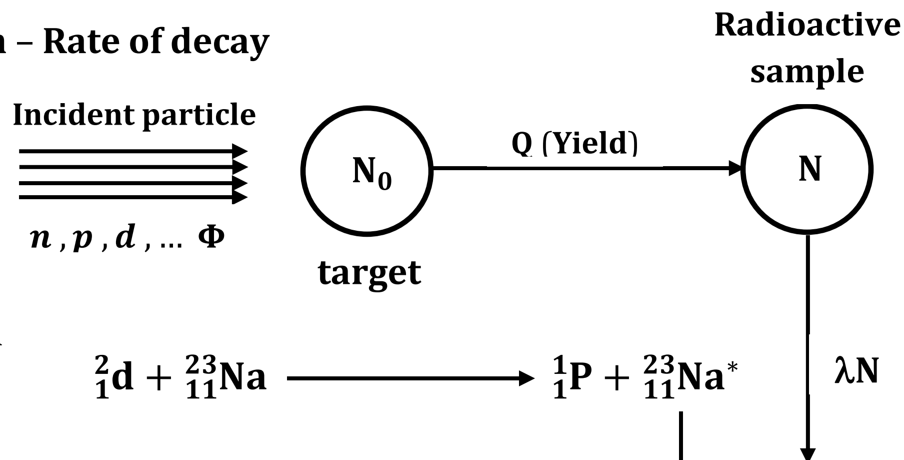
$\boldsymbol{\sigma} \Rightarrow$ **cross – section , i.e. probability production of radioactive nuclei (capture probability).**

$\boldsymbol{\Phi} \Rightarrow$ **flux of the incident beam (neutron / cm^2. sec).**



And let **N** be the number of radioactive atoms present at time **t** . The **net rate** of increase of radioactive nuclei

Net Rate = Rate of production – Rate of decay



$$\frac{dN}{dt} = Q - \lambda N$$

$$\frac{dN}{dt} + \lambda N = Q$$

Multipying by $e^{+\lambda t}$, we obtain

$$e^{+\lambda t}\,\frac{dN}{dt} + \lambda N e^{+\lambda t} = Q e^{+\lambda t}$$

$$\frac{d}{dt}\left(N e^{\lambda t}\right) = Q e^{+\lambda t}$$

By integration , we have

$$N e^{\lambda t} = \frac{Q}{\lambda}\, e^{\lambda t} + C \quad \text{or}$$

$${}^{2}_{1}d + {}^{23}_{11}Na \longrightarrow {}^{1}_{1}P + {}^{23}_{11}Na^{*}$$

$${}^{23}_{11}Na(d,p){}^{24}Na^{*} \xrightarrow{\beta^-} {}^{24}_{12}Mg$$

$$\downarrow \beta^-$$

$${}^{24}_{12}Mg$$

$\mathbf{N} = \frac{\mathbf{Q}}{\lambda} + \mathbf{C}\,\mathbf{e}^{-\lambda \mathbf{t}}$. the integration constant **C** can be determined from the condition that : when **t = 0** , **N = 0** , which gives : $\mathbf{C} = -\frac{\mathbf{Q}}{\lambda}$

$$\therefore\ \mathbf{N} = \frac{\mathbf{Q}}{\lambda} - \frac{\mathbf{Q}}{\lambda}\,\mathbf{e}^{-\lambda \mathbf{t}} \quad \text{or} \quad \mathbf{N} = \frac{\mathbf{Q}}{\lambda}(\mathbf{1} - \mathbf{e}^{-\lambda \mathbf{t}}) \quad ; \quad \mathbf{N_0} = \frac{\mathbf{Q}}{\lambda} = \mathbf{Q}\tau$$

$$\therefore\ \mathbf{N} = \mathbf{N_0}\,(\mathbf{1} - \mathbf{e}^{-\lambda \mathbf{t}})$$

N approaches approximately a limiting value of $\frac{\mathbf{Q}}{\lambda}$**,** it reaches **50%** of this value after **one half – life** (**1T**) , **75%** after (**2T**) and ≈ **88%** after (**3T**) **three half – lives**. Since $\mathbf{N} = \frac{\mathbf{Q}}{\lambda}(\mathbf{1} - \mathbf{e}^{-\lambda \mathbf{t}})$, then $\mathbf{N}\lambda = \mathbf{Q}(\mathbf{1} - \mathbf{e}^{-\lambda \mathbf{t}})$ or the activity **A** will be **:** $\mathbf{A} = \mathbf{Q}(\mathbf{1} - \mathbf{e}^{-\lambda \mathbf{t}})$, which is the strength of the radioactive source immediately after the end of its production from the curve of growth of activity , it is not **practical** to bombard the target for longer than **2** to **3 half – lives** since **75%** or **88%** of the maximum number or (of the maximum activity) of radioactive nuclei produced.

Note that for irradiation times long compared with the **half – life** ($\mathbf{t} \gg \mathbf{T_{1/2}}$) **,** the exponential ($\mathbf{e}^{-\lambda \mathbf{t}}$) approaches **Zero** and the activity is approximately constant **:** ($\mathbf{A(t)} = \mathbf{Q}$ **at** $\mathbf{t} \gg \mathbf{T_{1/2}}$).

In this case new activity is being formed at the same rate at which the odder activity decays. This is an example of secular equilibrium.