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Analytical Mechanics

2023-2024

Lec.12: Vertical motion in a resistance medium: Terminal velocity

2.10 Vertical Motion in a Resistance Medium :- Terminal Velocity

If the resistance is $\propto v$ (Linear case), we can express this force as $(-cv)$ regardless of the sign of v, because the resistance is always opposite to the direction of motion.

* The constant of proportional (c) depends on

1- The size of the object
2- Shape of the object
3- Viscosity of the fluid.

* Let us take the x-axis to be positive up word.

The differential eq. of motion is

$$\vec{F} = m\frac{d\vec{v}}{dt}$$

$$-mg - cv = m\frac{d\vec{v}}{dt}$$

$$t = \int_{v_0}^{v} \frac{m\,dv}{-mg - cv} \Rightarrow t = -\frac{m}{c}\ln\frac{mg + cv}{mg + cv_0}$$

$$-\frac{ct}{m} = \ln\frac{mg + cv}{mg + cv_0} \longrightarrow e^{-\frac{c}{m}t} = \frac{mg + cv}{mg + cv_0}$$

$$v = -\frac{mg}{c} + \left(\frac{mg}{c} + v_0\right)e^{-\frac{c}{m}t} \quad \text{......} (1)$$

velocity

Terminal velocity: - The highest velocity attained by an object falling through a fluid.

Eq. ① expresses v as a function of (t), So a second integration will give x as a function of t:

$$\frac{dx}{dt} = -\frac{mg}{c} + \left(\frac{mg}{c} + v_0\right) e^{-\frac{c}{m}t}$$

$$\int_{x_0}^{x} dx = \int_0^t \left\{-\frac{mg}{c} + \left(\frac{mg}{c} + v_0\right) e^{-\frac{ct}{m}}\right\} dt$$

$$x = -\frac{mg}{c} t + \left(\frac{mg}{c} + v_0\right)\left(-\frac{m}{c}\right) e^{-\frac{ct}{m}} \Big]_0^t$$

$$x = -\frac{mg\,t}{c} + \left(\frac{mg}{c} + v_0\right)\left(-\frac{m}{c}\right)\left[e^{-\frac{ct}{m}} - 1\right]$$

$$x = -\frac{mg\,t}{c} + \left(\frac{m^2 g}{c^2} + \frac{m v_0}{c}\right)\left(1 - e^{-\frac{c}{m}t}\right) \quad \text{--- ②}$$

$\therefore v_t = \frac{mg}{c}$ (terminal velocity)

$\therefore \tau = \frac{m}{c}$ (characteristic time)

$$v = -v_t + (v_t + v_0)\, e^{-\frac{t}{\tau}}$$

Thus, an object dropped from rest $(v_0 = 0)$

$$v = -v_t + v_t\, e^{-\frac{t}{\tau}}$$

when $(t = \tau)$

$$v = -v_t + v_t\, e^{-1}$$

ocity

If the viscous resistance $\propto v^2$ (quadratic case)

* positive direction up word

So
$$-mg \pm cv^2 = m\frac{dv}{dt}$$

$-cv^2$ (resistance term) for v positive $\Longrightarrow$ up word motion

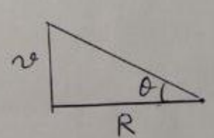
$+cv^2$ (resistance term) for v negative $\longrightarrow$ downword motion

The double sign is necessary for any resistive force that involves an even power of v

$$\int_{t_0}^{t} dt = \int_{v_0}^{v} \frac{m\,dv}{-mg - cv^2}$$

$$t - t_0 = -\int_{v_0}^{v} \frac{m\,dv}{mg + cv^2} \longrightarrow t - t_0 = -\frac{m}{c}\int_{v_0}^{v} \frac{dv}{\frac{m}{c}g + v^2}$$

$$\tan\theta = \frac{v}{R}$$



$v = R\tan\theta$ ——— ①

$dv = R\sec^2\theta\, d\theta$ ——— ②

$R^2 + v^2 = R^2 + R^2\tan^2\theta$ ——— ③

$R^2 + v^2 = R^2\sec^2\theta$ اكمل

$$t - t_0 = -\frac{m}{c}\int_{v_0}^{v} \frac{dv}{R^2+v^2}$$

$$t - t_0 = \frac{-m}{c}\int \frac{R \cdot \sec^2\theta \, d\theta}{R^2 \sec^2\theta} = -\frac{m}{c}\int_{\tan^{-1}\frac{v_0}{R}}^{\tan^{-1}\frac{v}{R}} \frac{R\sec^2\theta \, d\theta}{R^2\sec^2\theta}$$

$$t - t_0 = -\frac{m}{Rc}\int_{\tan^{-1}\frac{v_0}{R}}^{\tan^{-1}\frac{v}{R}} d\theta \longrightarrow t - t_0 = -\frac{m}{c}\sqrt{\frac{c}{mg}}\tan^{-1}\frac{v}{v_t}$$

$$= -\tau \tan^{-1}\frac{v}{v_t}$$

$$v_t = \sqrt{\frac{mg}{c}}$$

$$\tau = \sqrt{\frac{m}{cg}}$$

$t = -\tau \tan^{-1}\frac{v}{v_t} + t_0$ upwand

$t = -\tau \tanh^{-1}\frac{v}{v_t} + t_0'$ down word