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Analytical Mechanics

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Lec.13: Motion of charged particles in electric and magnetic fields

In the presense of a static magnetic field $\vec{B}$ (magnetic induction)

The force acting on a moving charged particle is

$$\vec{F} = q(\vec{v} \times \vec{B})$$

The differential eq. of motion of a particle moving in a purely magnetic field is

$$m\frac{d^2\vec{r}}{d^2t} = q(\vec{v} \times \vec{B})$$

The above eq. states that the acceleration of the particle is always at right angle to the direction of motion.

This means that the tangential component of the acceleration ($\dot{v}$) is zero, and so the particle moves with constant speed.

This is ture even if $\vec{B}$ is a varying function of the position $\vec{r}$ as long as it does not vary with time.

Ex/ Let us examine the motion of a charged particle in a uniform constant field; that is, we shall write

sol.

$$\vec{B} = \hat{k}B$$

The differential eq. of motion now reads

$$m\frac{d^2\vec{r}}{dt^2} = q(\vec{v} \times \hat{k}B)$$

$$= qB\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dot{x} & \dot{y} & \dot{z} \\ 0 & 0 & 1 \end{vmatrix}$$

$$m(\hat{i}\ddot{x} + \hat{j}\ddot{y} + \hat{k}\ddot{z}) = qB(\hat{i}\dot{y} - \hat{j}\dot{x})$$

Equating components, we have

$$m\ddot{x} = qB\dot{y} \quad \text{------ (a}$$

$$m\ddot{y} = -qB\dot{x} \quad \text{------ (b}$$

$$\ddot{z} = 0 \quad \text{------ (c}$$

These eq.s are not of seperated type. But solution can be done by integration

$$\left.\begin{aligned} m\dot{x} &= qBy + c_1 \quad \text{—— (a} \\ m\dot{y} &= -qBx + c_2 \quad \text{—— (b} \\ \dot{z} &= \text{constant} = \dot{z}_0 \quad \text{—— (c} \end{aligned}\right\} \text{—— (1}$$

Or

كابتل

$\dot{x} = \omega y + C_1$ ——— (a

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$\dot{y} = -\omega x + C_2$ ——— (b ⎫ ——— (2

$\dot{z} = \dot{z}_0$ ——— (c

where $\omega = \frac{qB}{m}$

The C's are constants of integration, and

$C_1 = \frac{c_1}{m}$, $C_2 = \frac{c_2}{m}$

sub. (2b eq.) in (1a eq.)

$m\ddot{x} = qB(-\omega x + C_2)$

$\ddot{x} = -\omega^2 x + \omega^2 a$ ——— (3

where $a = \frac{C_2}{\omega}$

~~where The solution~~

$\ddot{x} + \omega^2 x = \omega^2 a$

The solution is

$x = a + A\cos(\omega t + \theta_0)$ ——— (4

where A & θ_0 are constants of integration

$\dot{x} = -A\omega\sin(\omega t + \theta_0)$ ——— (5

sub. eq. (5 in eq. (2a), we get

$$y = b - A\sin(\omega t + \theta_0) \quad ---- (6)$$

where $b = -\frac{C_1}{\omega}$

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prove that $x = a + A\cos(\omega t + \theta_0)$ is solution of the equation $\ddot{x} + \omega^2 x = \omega^2 a$

To find the form of the path of motion, we eliminate "t" between eq. 4 and eq. 6, we obtain

$$(x-a)^2 + (y-b)^2 = A^2$$

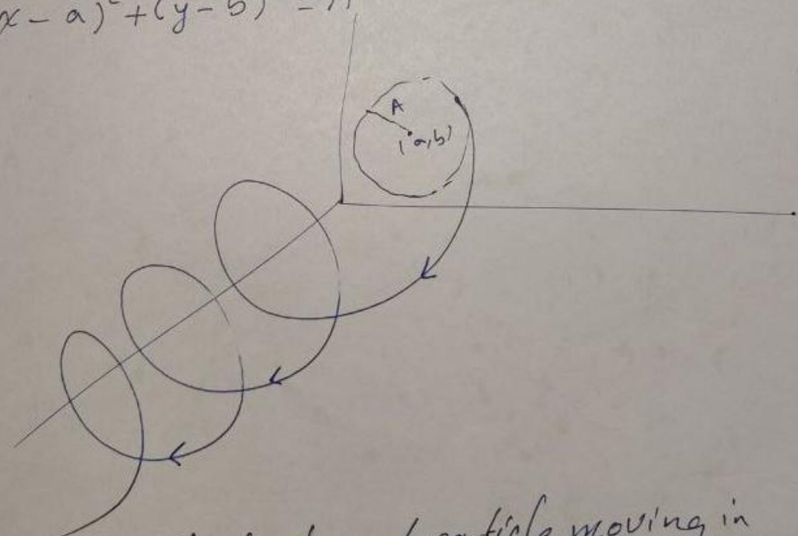

Fig. Helical path of a charged particle moving in a magnetic field.

Consider the simple case

- Let z-axis in the direction of the field, then

$$E_x = E_y = 0 \quad \& \quad E = E_z$$

The differential equations of motion of a particle of charge q moving in this field are then

$$\ddot{x} = 0 \quad , \quad \ddot{y} = 0 \quad \ddot{z} = \frac{qE}{m} = \text{constant}$$

- These are exactly the same as those for a projectile in the uniform gravitational field. The path is therefore ~~parap~~ parabola

In the electromagnetic theory

$$\vec{\nabla} \times \vec{E} = 0$$

$\vec{E} \longrightarrow$ is due to static charges

The motion in such a field is conservative

& the potential energy of a particle of charge q in the field is $q\Phi$

$$\text{Total energy} = \frac{1}{2} m v^2 + q\Phi \Longrightarrow \text{constant}$$

- Motion of charged particles in electric and magnetic fields :-

A charged particle in the field of other electric charge, it experiences a force

$$F = q\vec{E} \quad \text{——— (1)}$$

The eq. of motion of the particle is

$$m\frac{d^2\vec{r}}{dt^2} = q\vec{E} \quad \text{——— (2)}$$

or in components form

$$m\ddot{x} = qE_x$$
$$m\ddot{y} = qE_y$$
$$m\ddot{z} = qE_z$$

The field components are, in general, functions of the position coordinates x, y, z.

In the case of time-varying fields that is, if the charges producing $\vec{E}$ are moving the components of course, also involve E