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Analytical Mechanics

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Lec.4: Forced harmonic motion resonance

2.15 Forced Harmonic Motion Resonance

study a damped H.O that driven by an external harmonic force, i.e a force that varies sinusoidally with time.

Let the applied force

$$F_{ext} = F_0 \cos(\omega t + \theta)$$

ω - angular freq.

F_0 - amplitude

or it is more convient to use exponential form

$$F_{ext} = F_0 e^{i(\omega t + \theta)}$$

The total force $m\ddot{x}$ is

$$m\ddot{x} = \underbrace{-kx}_{\text{elastic restoring force}} \underbrace{- c\dot{x}}_{\text{viscous damping force}} + F_{ext}$$

$$m\ddot{x} + c\dot{x} + kx = F_{ext} = F_0 e^{i(\omega t + \theta)} \quad \text{.......} (1)$$

The solution of eq. (1) is given by the sum of two part

i) the solution of the homogenous eq.

$$m\ddot{x} + c\dot{x} + kx = 0$$

Which we have already solved in previous section and it represents an oscillation which eventually decays to zero.

ii) A solution that depends on the nature of applied force.

∴ this force is constant in amplitud and varies sinusoidally with time, therefore we shall try a solution of the form

$$x = A e^{i(\omega t + \theta')}$$

If this guess is correct, we must have

$$m \frac{d^2}{dt^2}\left[A e^{i(\omega t + \theta')}\right] + c\frac{d}{dt}\left[A e^{i(\omega t + \theta')}\right] + kA e^{i(\omega t + \theta')} = F_0 e^{i(\omega t + \theta)}$$

For all values of t.

this reduced to

$$-m\omega^2 A + i\omega c A + kA = F_0 e^{i(\theta - \theta')}$$

$$= F_0\left[\cos(\theta - \theta') + i\sin(\theta - \theta')\right]$$

Equating the real and imaginary parts

$$A\,(k - m\omega^2) = F_0 \cos\phi \quad \text{------} (2)$$

$$c\omega A = F_0 \sin\phi \quad \text{------} (3)$$

where phase difference $\theta - \theta' = \phi$

Dividing eq (3) by eq (2), we obtain

$$\tan\phi = \frac{c\omega}{k - m\omega^2}$$

By squaring both sides of eq (2) and (3) and adding we find

$$A^2(k - m\omega^2)^2 + c^2\omega^2 A^2 = F_0^2$$

solving for A

$$A = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + c^2\omega^2}} \quad \text{------} (4)$$

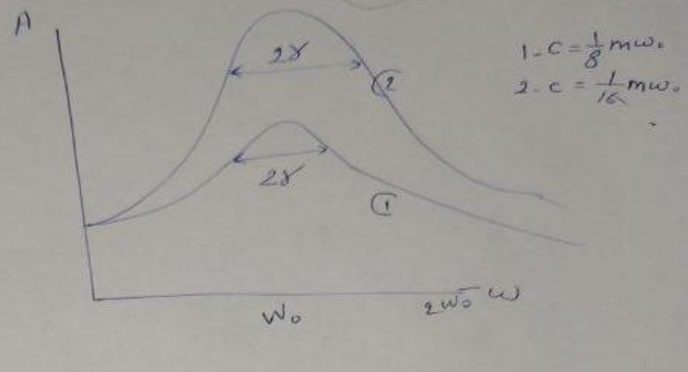
$$\omega_0 = \sqrt{\frac{k}{m}} \;\; \& \;\; \gamma = \frac{c}{2m} \text{, then}$$

$$\tan\phi = \frac{2\gamma\omega}{\omega_0^2 - \omega^2} \quad \text{------} (5)$$

and

$$A = \frac{\frac{F_0}{m}}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2}} \quad \text{------} (6)$$

Eq. 6) is of fundamental importance, because it relating the amplitude A to the driving freq. of applied driving force



To find w_r, calculate $\frac{dA}{d\omega}$ & set it equal zero.

From eq. 6)

$$A = \frac{\frac{F_0}{m}}{\sqrt{(w_0^2 - w^2)^2 + 4\gamma^2 w^2}}$$

$$\frac{dA}{dW} \overset{W \to W_r}{=} 0 = -\frac{1}{2}\frac{F_0}{m}\left[(w_0^2 - w_r^2)^2 + 4\gamma^2 w_r^2\right]^{-3/2}\left[2(w_0^2 - w_r^2)(-2w_r) + 8\gamma^2 w_r\right]$$

$$= \frac{-\frac{F_0}{m}\left[-4W_r(w_0^2 - W_r^2) + 8\gamma^2 W_r\right]}{2\left[(W_0^2 - W_r^2)^2 + 4\gamma^2 W_r^2\right]^{3/2}} = 0$$

$$\frac{F_0}{m}(4W_r)\left[W_0^2 - W_r^2 - 2\gamma^2\right] = 0$$

$$W_0^2 - W_r^2 - 2\gamma^2 = 0 \longrightarrow W_r^2 = W_0^2 - 2\gamma^2$$

$$W_r = \sqrt{W_0^2 - 2\gamma^2} \longrightarrow W_r = W_0\left(1 - \frac{2\gamma^2}{W_0^2}\right)^{1/2} \quad (7)$$

2- The steady-state amplitude at the maximum frequency, A_{max} is obtained from eq. (6) and (7)

$$A_{max} = \frac{\frac{F_o}{m}}{2\gamma\sqrt{w_o^2-\gamma^2}} = \frac{F_o}{c\sqrt{w_o^2-\gamma^2}}$$

solution

$$A = \frac{\frac{F_o}{m}}{\sqrt{(W_o^2 - W_r^2)^2 + 4\gamma^2 W_r^2}} \text{ ---- (6)}$$

$$w_r = w_o\left(1 - \frac{2\gamma^2}{W_o^2}\right)^{1/2} \text{ ---- (7)}$$

$$A_{max} = \frac{\frac{F_o}{m}}{\left[\left(w_o^2 - w_o^2\left(1-\frac{2\gamma^2}{w_o^2}\right)\right)^2 + 4\gamma^2 w_o^2\left(1-\frac{2\gamma^2}{w_o^2}\right)\right]^{1/2}}$$

$$A_{max} = \frac{\frac{F_o}{m}}{\left[(\cancel{w_o^2} - \cancel{w_o^2} + 2\gamma^2)^2 + 4\gamma^2 w_o^2 - 8\gamma^4\right]^{1/2}}$$

$$A_{max} = \frac{\frac{F_o}{m}}{\left[4\gamma^4 + 4\gamma^2 W_o^2 - 8\gamma^4\right]^{1/2}}$$

$$A_{max} = \frac{\frac{F_o}{m}}{\left[4\gamma^2 W_o^2 - 4\gamma^4\right]^{1/2}} = \frac{\frac{F_o}{m}}{2\gamma\sqrt{W_o^2-\gamma^2}} = \frac{F_o}{c\sqrt{W_o^2-\gamma^2}}$$