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Analytical Mechanics

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4. change of coordinates system

11- Change of Coordinates system :-

$$\vec{A} = \hat{i}A_x + \hat{j}A_y + \hat{k}A_z \quad \text{......(1)}$$

Anew Vector ($\hat{i}'\hat{j}'\hat{k}'$) having a different orientation from that $\hat{i}\,\hat{j}\,\hat{k}$

$$\vec{A} = \hat{i}'A_x' + \hat{j}'A_y' + \hat{k}'A_z' \quad \text{......(2)}$$

$$\left.\begin{cases} A_x = \vec{A}\cdot\hat{i}' = (\hat{i}\cdot\hat{i}')A_x + (\hat{j}\cdot\hat{i}')A_y + (k\cdot\hat{i}')A_z \\ A_y = \vec{A}\cdot\hat{j}' = (\hat{i}\cdot\hat{j}')A_x + (\hat{j}\cdot\hat{j}')A_y + (\hat{k}\cdot\hat{j}')A_z \\ A_z = \vec{A}\cdot\hat{k}' = (i\cdot\hat{k}')A_x + (\hat{j}\cdot\hat{k}')A_y + (k\cdot\hat{k}')A_z \end{cases}\right\} \text{......(3)}$$

Similarity unprimed components are expressed as

$$\left.\begin{cases} A_x = \vec{A}\cdot\hat{i} = (\hat{i}'\cdot\hat{i})A_x' + (\hat{j}'\cdot\hat{i})A_y' + (\hat{k}'\cdot\hat{i})A_z' \\ A_y = \vec{A}\cdot\hat{j} = (\hat{i}'\cdot j)A_x' + (\hat{j}'\cdot\hat{j})A_y' + (\hat{k}'\cdot\hat{j})A_z' \\ A_z = \vec{A}\cdot\hat{k} = (\hat{i}'\cdot\hat{k})A_x' + (\hat{j}'\cdot\hat{k})A_y' + (\hat{k}'\cdot\hat{k})A_z' \end{cases}\right\} \text{......(4)}$$

All of the coefficients of tranformation in eq. (3 ~~and~~ also appear eq (4

$$\hat{i}\cdot\hat{i}' = \hat{i}'\cdot\hat{i} \;;\; \hat{j}'\cdot\hat{i} = \hat{i}\cdot\hat{j}' \;;\; k\cdot\hat{i}' = \hat{i}'\cdot\hat{k}$$

The equation of transformation are convenientlly expressed in matrix notation.

Thus equation (3) is written

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \hat{i}\cdot\hat{i}' & \hat{j}\cdot\hat{i}' & \hat{k}\cdot\hat{i}' \\ \hat{i}\cdot\hat{j}' & \hat{j}\cdot\hat{j}' & \hat{k}\cdot\hat{j}' \\ \hat{i}\cdot\hat{k}' & \hat{j}\cdot\hat{k} & \hat{k}\cdot\hat{k}' \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} \quad \text{------} (5)$$

Example :- Express the vector $\vec{A} = 3\hat{i} + 2\hat{j} + \hat{k}$ interms of the triad $\hat{i}'\hat{j}'\hat{k}'$, Where the $x'y'$ axis are rotated 45° around Z- axis, the Z & Z' axis coinciding

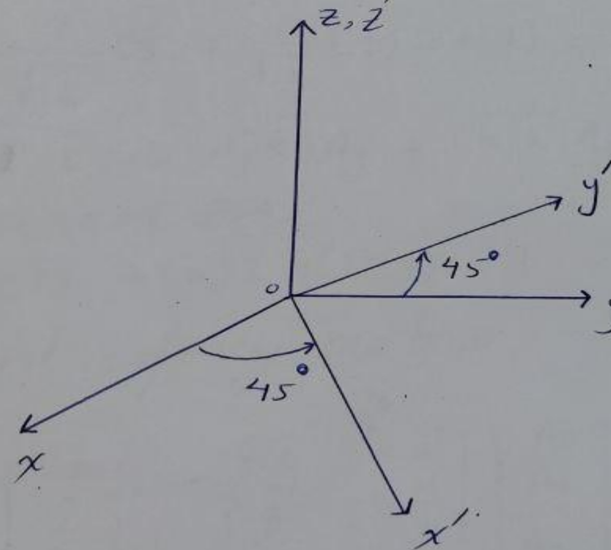

Fig : The primed axes $O'x'y'z'$ are rotated 45° around Z - axis

Referring to Figure, we have for the coefficients of transformation

$$\hat{i}\cdot\hat{i}' = \overset{\cos 45}{\frac{1}{\sqrt{2}}} \qquad \hat{j}\cdot\hat{i}' = \overset{\cos(-45)}{\frac{1}{\sqrt{2}}} \qquad \hat{k}\cdot\hat{i}' = \overset{\cos 90}{0}$$

$$\hat{i}\cdot\hat{j}' = -\frac{1}{\sqrt{2}}\ {\scriptstyle\cos 135} \qquad \hat{j}\cdot\hat{j}' = \overset{\cos 45}{\frac{1}{\sqrt{2}}} \qquad \hat{k}\cdot\hat{j}' = \overset{\cos 90}{0}$$

$$\hat{i}\cdot\hat{k}' = \underset{\cos 90}{0} \qquad \hat{j}\cdot\hat{k}' = \underset{\cos 90}{0} \qquad \hat{k}\cdot\hat{k}' = \underset{\cos 0}{1}$$

These give

$$A_x' = \vec{A}\cdot\hat{i}' = (\hat{i}\cdot\hat{i}')A_x + (\hat{j}\cdot\hat{i}')A_y + (\hat{k}\cdot\hat{i}')A_z$$

$$A_x' = \frac{1}{\sqrt{2}}(3) + \frac{1}{\sqrt{2}}(2) + 0(1) = \frac{5}{\sqrt{2}}$$

$$A_y' = \vec{A}\cdot\hat{j}' = (\hat{i}\cdot\hat{j}')A_x + (\hat{j}\cdot\hat{j}')A_y + (\hat{k}\cdot\hat{j}')A_z$$

$$A_y' = \vec{A}\cdot\hat{j}' = -\frac{1}{\sqrt{2}}(3) + \frac{1}{\sqrt{2}}(2) + 0(1) = -\frac{1}{\sqrt{2}}$$

$$A_z' = \vec{A}\cdot\hat{k}' = (\hat{i}\cdot\hat{k}')A_x + (\hat{j}\cdot\hat{k}')A_y + (\hat{k}\cdot\hat{k}')A_z$$

$$\cancel{= 0(3) + 3(0}$$

$$= 0(3) + (0)2 + (1)(1) = 1$$

another method is to use matrix

$$\begin{bmatrix} A_x' \\ A_y' \\ A_z' \end{bmatrix} = \begin{bmatrix} \hat{i}\cdot\hat{i}' & \hat{j}\cdot\hat{i}' & \hat{k}\cdot\hat{i}' \\ \hat{i}\cdot\hat{j}' & \hat{j}\cdot\hat{j}' & \hat{k}\cdot\hat{j}' \\ \hat{k}\cdot\hat{k}' & \hat{j}\cdot\hat{k}' & \hat{k}\cdot\hat{k}' \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

Example :- Find the transformation matrix, the first being about the z-axis (angle ϕ), and the second about the new y' axis (angle θ) is given by the matrix product.

solution:-

$$\begin{bmatrix} \cos\phi & 0 & -\sin\phi \\ 0 & 1 & 0 \\ \sin\phi & 0 & \cos\phi \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \cos\phi\cos\theta & \cos\phi\sin\theta & -\sin\phi \\ -\sin\theta & \cos\theta & 0 \\ \sin\phi\cos\theta & \sin\phi\sin\theta & \cos\phi \end{bmatrix}$$

$$\begin{bmatrix} A_x' \\ A_y' \\ A_z' \end{bmatrix} = \begin{bmatrix} \cos\phi\cos\theta & \cos\phi\sin\theta & -\sin\phi \\ -\sin\theta & \cos\theta & 0 \\ \sin\phi\cos\theta & \sin\phi\sin\theta & \cos\phi \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$