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Analytical Mechanics

2023-2024

5. Tangential and normal components of acceleration

Tangential and normal components of acceleration :-

The velocity vector of a moving particle can be written as

$$\vec{V} = v\,\hat{\tau} \quad \text{......} \quad ①$$

where $v \rightarrow$ the particle speed

$\hat{\tau} \longrightarrow$ a unit vector that gives the direction of the particle's motion

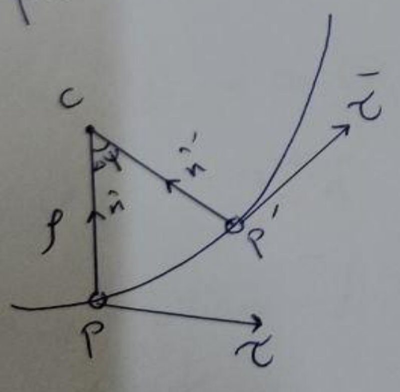
$$\vec{a} = \frac{d\vec{V}}{dt} = \frac{d}{dt}(v\,\tau) = \dot{v}\,\hat{\tau} + v\,\frac{d\tau}{dt} \quad \text{......} \quad ②$$

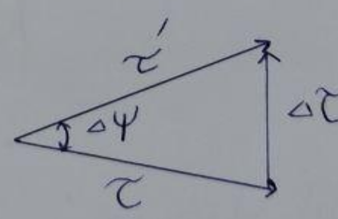
To evaluate $\frac{d\tau}{dt}$

Let the particle is initially on its path of motion at point P. In a time interval Δt the particle moves to another point P' a certain distance Δs along the path.

$$P \xrightarrow[\Delta s]{\text{moves at } \Delta t} P'$$

$$\tau \longrightarrow \tau'$$





unit tangent vector

The other component of magnitude

$a_n = \frac{v^2}{\rho}$ is called centripetal acceleration

The magnitude $|\vec{a}| = \left|\frac{d\vec{V}}{dt}\right| = \left(\dot{v}^2 + \frac{v^4}{\rho^2}\right)^{1/2}$ ---- (3)

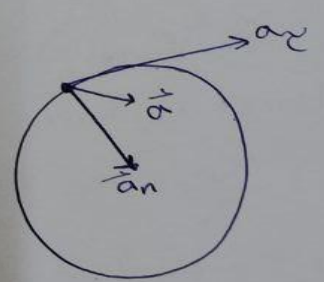
For example if a particle moves on a circle with constant speed v, the acceleration vector magnitude $\frac{v^2}{R_o}$

$$|\vec{a}| = \sqrt{\dot{v}^2 + \frac{v^4}{R^2}}$$

$$|\vec{a}| = \sqrt{0 + \frac{v^4}{R^2}} = \frac{v^2}{R}$$

If the speed is not constant (irregular motion)

$$v \longrightarrow \dot{v} \neq 0$$



For small $\Delta \Psi$

$$\Delta \tau \xrightarrow[\text{approach}]{} \Delta \Psi$$

$$\lim_{\Delta\Psi \to 0} \frac{\Delta \hat{\tau}}{\Delta \Psi} = \frac{d\hat{\tau}}{d\Psi} = 1$$

Therefore, call it the unit normal vector ($\hat{n}$)

$$\frac{d\hat{\tau}}{d\Psi} = \hat{n}$$

To find $\frac{d\hat{\tau}}{d\Psi}$, we use the chain rule as follows

$$\frac{d\tau}{dt} = \frac{d\tau}{d\Psi} \cdot \frac{d\Psi}{dt} = \hat{n}\,\frac{d\Psi}{dt} = \hat{n}\,\frac{d\Psi}{ds} \cdot \frac{ds}{dt} = \hat{n}\,\frac{v}{\rho}$$

$$\frac{d\tau}{dt} = \hat{n}\,\frac{v}{\rho} \quad \text{------} \quad (2)$$

sub eq. (2) in eq. (1), we get

$$\vec{a} = \dot{v}\,\tau + \frac{v^2}{\rho}\,\hat{n}$$

The acceleration of a moving particle has a component of magnitude

$a_\tau = \dot{v} = \ddot{S}$ in the direction of motion.

This the tangential acceleration

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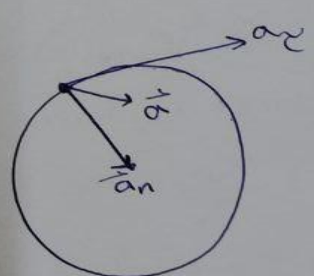


Ex / If a particle moves with constant speed, but its direction is changed contineously. Prove that $\vec{a} \perp \vec{v}$

a) Using tangential and normal component of speed and acceleration rel.

b) Using the cartesian coordinates.

Solution:-

$$\vec{v} = v\hat{\tau}$$

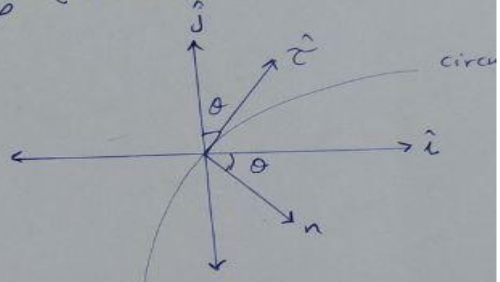
$$\vec{a} = \dot{v}\hat{\tau} + v\frac{d\hat{\tau}}{dt} \quad ; \quad \dot{v} \to 0 \quad \text{constant speed}$$

$$\vec{a} = v\frac{d\hat{\tau}}{dt}$$

$$\vec{a} = v\hat{n}\frac{d\psi}{ds}\cdot\frac{ds}{dt} = v\hat{n}\cdot\frac{v}{\rho} = \frac{v^2}{\rho}\hat{n}$$

$$\hat{n} \perp \hat{\tau}$$

$$\vec{a}\cdot\vec{v} = \frac{v^2}{\rho}\hat{n}\cdot v\hat{\tau} = \frac{v^3}{\rho}(\hat{n}\cdot\hat{\tau}) = 0$$



$$\hat{\tau} = \hat{i}\sin\theta + \hat{j}\cos\theta$$

$$\hat{n} = \hat{i}\cos\theta - \hat{j}\sin\theta$$

$$\vec{v} = v\hat{\tau} = v(\hat{i}\sin\theta + \hat{j}\cos\theta)$$

$$\vec{a} = \dot{v}\hat{\tau} + \frac{v^2}{\rho}\hat{n}$$

if the speed is constant

$$\vec{a} = \frac{v^2}{\rho}\hat{n} = \frac{v^2}{\rho}(\hat{i}\cos\theta - \hat{j}\sin\theta)$$

$$\vec{v}\cdot\vec{a} = v(\hat{i}\sin\theta + \hat{j}\cos\theta)\cdot\frac{v^2}{\rho}(\hat{i}\cos\theta - \hat{j}\sin\theta)$$

$$= \frac{v^3}{\rho}(\sin\theta\cos\theta - \cos\theta\sin\theta) = 0$$

$$\vec{v}\cdot\vec{a} = 0$$

$$\vec{v} \perp \vec{a}$$