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Analytical Mechanics

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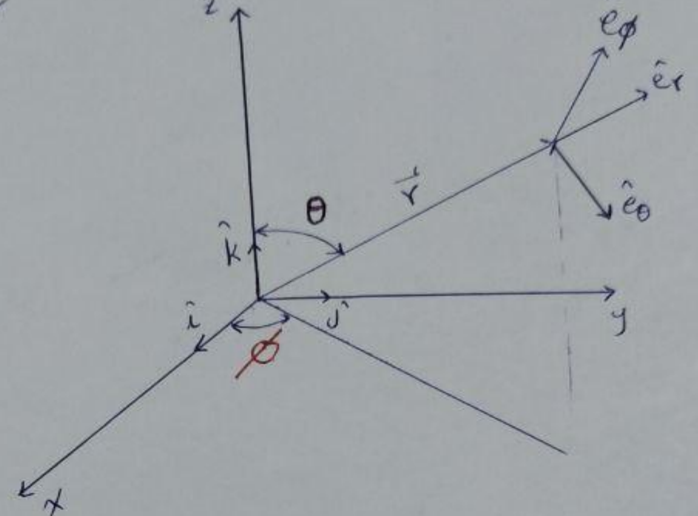
Lec.7: Spherical coordinates

spherical coordinates :-

position vector in spherical coordinates gives as

$$\vec{r} = r\hat{e}_r$$



The velocity is $\vec{v} = \dfrac{d\vec{r}}{dt} = \dot{r}\hat{e}_r + r\dfrac{d\hat{e}_r}{dt}$ ------- (1)

How to express $\dfrac{d\hat{e}_r}{dt}$ in-terms of the unit vectors in the rotated traid $(\hat{e}_r, \hat{e}_\theta, \hat{e}_\phi)$

$$\left.\begin{aligned} \hat{e}_r &= \hat{i}\sin\theta\cos\phi + \hat{j}\sin\theta\sin\phi + \hat{k}\cos\theta \\ \hat{e}_\theta &= \hat{i}\cos\theta\cos\phi + \hat{j}\cos\theta\sin\phi - \hat{k}\sin\theta \\ \hat{e}_\phi &= -\hat{i}\sin\phi + \hat{j}\cos\phi \end{aligned}\right\} \quad \text{------- (2)}$$

$$\therefore \frac{d\hat{e}_r}{dt} = \hat{i}(\dot{\theta}\cos\theta\cos\phi - \dot{\phi}\sin\theta\sin\phi) + \hat{j}(\dot{\theta}\cos\theta\sin\phi + \dot{\phi}\sin\theta\cos\phi) - \hat{k}\dot{\theta}\sin\theta \quad \text{------} (3)$$

By using the expressions for $\hat{e}_\phi$ and $\hat{e}_\theta$ in eq. (3), we get

$$\frac{d\hat{e}_r}{dt} = \dot{\phi}\,\hat{e}_\phi\sin\theta + \dot{\theta}\,\hat{e}_\theta \quad \text{------} (4)$$

$$\frac{d\hat{e}_\theta}{dt} = -\dot{\theta}\,\hat{e}_r + \dot{\phi}\,\hat{e}_\phi\cos\theta \quad \text{------} (5)$$

$$\frac{d\hat{e}_\phi}{dt} = -\dot{\phi}\,\hat{e}_r\sin\theta - \dot{\phi}\,\hat{e}_\theta\cos\theta \quad \text{------} (6)$$

sub. eq. (4) in eq. (1), we obtain

$$\vec{v} = \hat{e}_r\dot{r} + \hat{e}_\phi r\dot{\phi}\sin\theta + \hat{e}_\theta r\dot{\theta} \quad \text{------} (7)$$

To find the acceleration

$$\vec{a} = \frac{d\vec{v}}{dt} = \hat{e}_r\ddot{r} + \dot{r}\frac{d\hat{e}_r}{dt} + \hat{e}_\phi\frac{d}{dt}(r\dot{\phi}\sin\theta) + r\dot{\phi}\sin\theta\frac{d\hat{e}_\phi}{dt} + \hat{e}_\theta\frac{d}{dt}(r\dot{\theta}) + r\dot{\theta}\frac{d\hat{e}_\theta}{dt}$$

Sub. eq. (4), eq (5) & and (6), we obtain

$$\vec{a} = (\ddot{r} - r\dot{\phi}^2\sin^2\theta - r\dot{\theta}^2)\hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2\sin\theta\cos\theta) + (r\ddot{\phi}\sin\theta + 2\dot{r}\dot{\phi}\sin\theta + 2r\dot{\theta}\dot{\phi}\cos\theta)\hat{e}_\phi$$