(3-2) Bernoulli's Theorem

Bernoulli's theorem is a fundamental theory of hydrodynamics that describes a fluid in motion. It is really the application of the law of conservation of energy to fluid flow. Let us consider the fluid flowing in the pipe of figure 3-2.

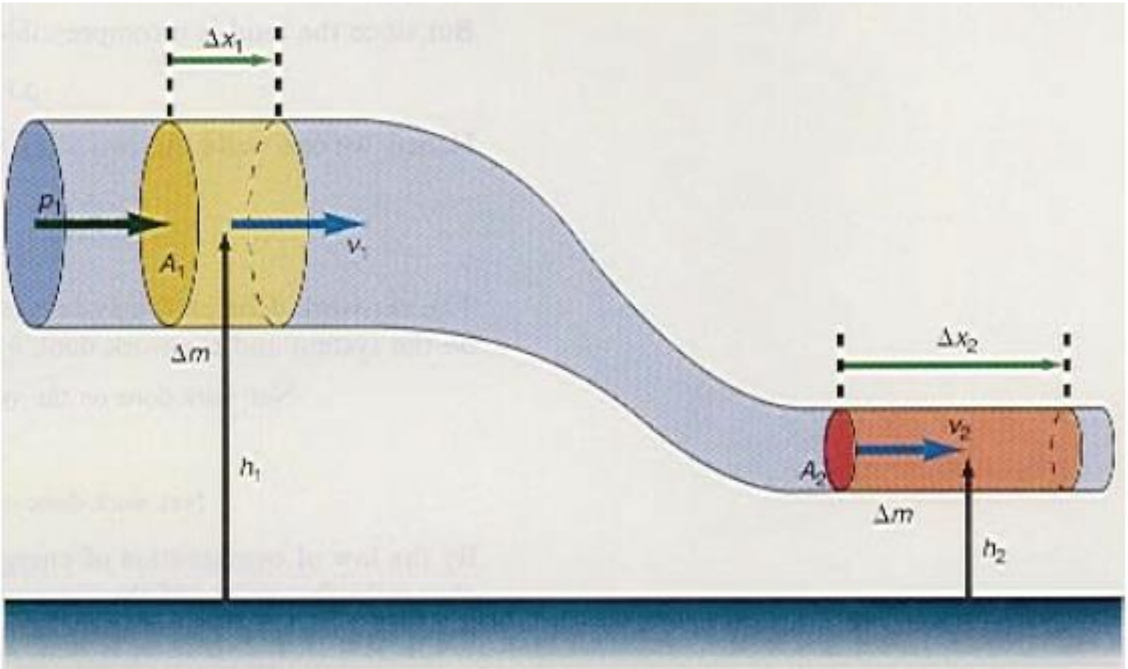

Figure (3-2)

A_1 = is the cross-section area of left hand side.
A_2 = is the cross-section area of the right hand side.
The pipe is filled with a nonviscous, incompressible fluid. A uniform pressure $\mathbf{p_1}$ is applied, such as from a piston, to a small element of mass of the fluid $\mathbf{\Delta m}$ and causes this mass to move through a distance $\mathbf{\Delta x_1}$ of the pipe. Because the fluid is incompressible, the fluid moves throughout the rest of the pipe. The same small mass $\mathbf{\Delta m}$, at the right-hand side of the pipe, moves through a distance $\mathbf{\Delta x_2}$. The work done on the system by moving the small mass through the distance $\mathbf{\Delta x_1}$ is given by the definition of work as

$$W_1 = F_1 \Delta x_1$$

Using equation 2-2, we can express the force F_1 moving the mass to the right in terms of the pressure exerted on the fluid as

$$F_1 = P_1 A_1$$

Hence,

$$W_1 = P_1 A_1 \Delta x_1$$

But $A_1 \Delta x_1 = \Delta V_1$

The volume of the fluid moved through the pipe. Thus, we can write the work done on the system as

$$W_1 = P_1 \Delta V_1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3-12)$$

As this fluid moves through the system, the fluid itself does work by exerting a force $\boldsymbol{F_2}$ on the mass **Δm** on the right side, moving it through the distance $\boldsymbol{\Delta x_2}$. Hence, the work done by the fluid system is

$$W_2 = F_2 \Delta x_2$$

But we can express the force F_2 in terms of the pressure P_2 on the right side by

$$F_2 = P_2 A_2$$

Therefore, the work done by the system is

$$W_2 = P_2 A_2 \Delta x_2$$

But, $A_2 \Delta x_2 = \Delta V_2$

the volume moved through the right side of the pipe. Thus, the work done by the system becomes

$$W_2 = P_2 \Delta V_2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3-13)$$

But since the fluid is incompressible,

$$\Delta V_1 = \Delta V_2 = \Delta V$$

Hence, we can write the two work terms, equations 3-12 and 3-13, as

$$W_1 = P_1 \Delta V$$

$$W_2 = P_2 \Delta V$$

The net work done on the system is equal to the difference between the work done **on** the system and the work done **by** the system. Hence,

$$\text{Net work done on the system} = W_{on} - W_{by}$$
$$= W_1 - W_2$$
$$= P_1 \Delta V - P_2 \Delta V$$

$$\text{Net work done on the system} = (P_1 - P_2)\,\Delta V \ \ldots\ldots\ldots\ldots (3\text{-}14)$$

By the law of conservation of energy, the net work done on the system produces a change in the energy of the system. The fluid at position 1 is at a height h_1 above the reference level and therefore possesses a potential energy given by

$$PE_1 = \Delta mgh_1 \quad \dots\dots\dots\dots\dots\dots\dots (3-15)$$

Because this same fluid is in motion at a velocity v_1, it possesses a kinetic energy given by

$$kE_1 = \frac{1}{2}\Delta mv_1^2 \quad \dots\dots\dots\dots\dots\dots\dots (3-16)$$

Similarly at position 2, the fluid possesses the potential energy

$$PE_2 = \Delta mgh_2 \quad \dots\dots\dots\dots\dots\dots\dots (3-17)$$

And the kinetic energy

$$kE_2 = \frac{1}{2}\Delta mv_2^2 \quad \dots\dots\dots\dots\dots\dots\dots (3-18)$$

Therefore, we can now write the law of conservation of energy as

Net work done on the system = Change in energy of the system

Net work done on the system = $(E_{tot})_2 - (E_{tot})_1$

Net work done on the system = $(PE_2 + KE_2) - (PE_1 + KE_1)$ ....(3-19)

Substituting equations 3-14 through 3-18 into equation 3-19 we get

$$(P_1 - P_2)\Delta V = \left[\Delta mgh_2 + \frac{1}{2}\Delta mv_2^2\right] - \left[\Delta mgh_1 + \frac{1}{2}\Delta mv_1^2\right] \dots\dots\dots..(3-20)$$

But the total mass of fluid moved Δm is given by

$$\Delta m = \rho\Delta V \quad \dots\dots\dots\dots\dots(3\text{-}21)$$

Substituting equation 3-21 back into equation 3-20, gives

$$(P_1 - P_2)\Delta V = \rho\Delta Vgh_2 + \frac{1}{2}\rho\Delta Vv_2^2 - \rho\Delta Vgh_1 - \frac{1}{2}\rho\Delta Vv_1^2$$

Dividing each term by ΔV gives

$$P_1 - P_2 = \rho gh_2 + \frac{1}{2}\rho v_2^2 - \rho gh_1 - \frac{1}{2}\rho v_1^2$$

If we place all the terms associated with the fluid at position 1 on the left-hand side of the equation and all the terms associated with the fluid at position 2 on the right-hand side, we obtain

$$P_1 + \rho gh_1 + \frac{1}{2}\rho v_1^2 = P_2 + \rho gh_2 + \frac{1}{2}\rho v_2^2$$

This Equation is the mathematical statement of **Bernoulli's theorem.**

Bernoulli's theorem: It says that the sum of the pressure, the potential energy per unit volume, and the kinetic energy per unit volume at any one location of the fluid is equal to the sum of the pressure, the potential energy per unit volume, and the kinetic energy per unit volume at any other location in the fluid, for a nonviscous, incompressible fluid in streamlined flow.

Example 4: In figure 3-2, the pressure $P_1 = 2.94 \times 10^3$ **N/m²**, whereas the velocity of the water is $v_1 = 0.322$ **m/s**. The diameter of the pipe at location 1 is **10 cm** and it is **5 m** above the ground. If the diameter of the pipe at location 2 is **4 cm**, and the pipe is **2 m** above the ground, find the velocity of the water v_2 at position 2, and the pressure P_2 of the water at position 2.

Example 5: A fluid with density of **1.65×10^3 kg/m³** flow in two part of horizontal tube connected from their side, the cross-sectional area of the first tube is **10 cm²** and flow of velocity is **275 cm/sec** with pressure of **1.2×10^5 Pa**. If the cross-sectional area of the other tube is **2.5 cm²**

a) What is the velocity of flow of the small tube?

b) What is the pressure of the small tube?

Example 6: water flows horizontally through a garden hose of radius **1 cm** at speed of **1.4 m/sec**. The water shoots horizontally out of a nozzle of radius **0.25 cm**. What is the change pressure of the water inside the nose?

Example 7: tank of water open to the atmosphere height of **0.5 m** above an open hall what is the velocity of the flow of water from the hall if it is opened?