University of Mosul

College of Science

Department of Physics

Second stage

Lecture 6

Sound and wave Motion

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Lecture 6: Lissajous Figures

Preparation

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Lissajous Figures:

When an object undergoes two simple harmonic motions that are perpendicular to each other, the resulting motion of the object follows a curved path. This curved shape is called a Lissajous figure. The shape depends on the amplitudes and frequencies of each of the simple harmonic motions and the phase difference between them. For more details, if we have a simple pendulum suspended from a point and undergoing a simple harmonic motion along the vertical axis only, and if we allow the pendulum to swing at the same time with a small amplitude along the horizontal axis, then the pendulum will undergo two perpendicular motions simultaneously. As a result, the pendulum will move in two dimensions along a path determined by the combination of these two motions. If the ratio of the frequencies of the movements is a rational number and the phase difference between them is a specific angle, then the shape of the path will be a closed curve, and this can be explained analytically and graphically with specific examples.

1. Combining two simple harmonic motions in perpendicular directions.

Suppose there is a particle that is simultaneously affected by two simple harmonic motions. One affects the length of the x-axis, and the other affects the y-axis. First, we will consider the case where the frequencies are equal. Assuming that the instantaneous displacement of the particle along the x-axis is.

$x = asin(\omega t + \theta)$…………………………..1

The instantaneous displacement of the same particle along the y-axis is

$y = bsin \quad t$ ………………………………..2

Here we notice that the two harmonic motions x and y differ in amplitude and initial phase of the motion. To obtain the general equation of the motion path, we eliminate the time between equations (1) and (2). From equation (1), we obtain:

$\frac{x}{a} = sin\omega t cos\theta + cos\omega t sin\theta$ ………………..3

From equation (2), we get:

$\frac{y}{b} = sin\omega t$ ……………………………………..4

And by taking advantage of the matched ($\cos^2\theta+\sin^2\theta=1$) Equation (4) becomes:

$\cos wt = \sqrt{1-\frac{y^2}{b^2}}$ ……………………………..5

We substitute equations (4) and (5) into equation (3), and it turns out that:

$\frac{x}{a} = \frac{y}{b}\cos\theta + \sqrt{1-\frac{y^2}{b^2}}\sin\theta$ ……………………….6

We arrange this rate to become

$\frac{x}{a} - \frac{y}{b} = \cos\theta + \sqrt{1-\frac{y^2}{b^2}}\sin\theta$…………………7

Square both sides of the equation (7)

$[\frac{x}{a} - \frac{y}{b}\cos\theta\ ]^2 = (\sin^2\theta\ 1-(\frac{y^2}{b^2}))$

$(\frac{x}{a})^2 - 2(\frac{xy}{ab})\cos\theta + (\frac{y^2}{b^2})\cos\ ^2\theta = (\ \sin^2\theta - (\frac{y^2}{b^2})\ \sin^2\theta)$

We reduce this equation and arrange it to become as follows:

$$\frac{x^2}{y^2} + \frac{y^2}{b^2} + \frac{2xy}{ab}\cos\theta = \sin\ ^2\theta \text{ …………………..8}$$

Equation (8) represents the general equation of the ellipse, which represents the path taken by a particle when exposed to the influence of two simple harmonic motions of the same frequency, different amplitude, and phase difference. Different Lissajous modes can be derived from the above equation (8).

1. When θ =0, 2 π, then equation 8 becomes.

$\{(\frac{x}{a}) - (\frac{y}{b})\}^2 = 0$

$y = (\frac{b}{a})x$ …………………………...9   prove that?

The mentioned equation indicates that the path followed by the particle is a straight line inclined at a positive slope equal to (a/b) as shown in the figure (a and z). Therefore, it can be inferred that both x and y always have the same algebraic sign. Specifically, both are either positive or both are negative. This case corresponds to linearly polarized oscillation in optics.

2- when it is $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$ The equation 8 becomes:

$\frac{x^2}{y^2} + \frac{y^2}{b^2} = 1$ …………………………..10

The equation represents an complete ellipse with its principal axes aligned along the x and y axes, respectively. The path of a particle moving along this trajectory can be determined according to θ=π/2 as follows: When the particle starts moving at time t = 0, x starts decreasing from its maximum positive value, while y immediately begins to ascend from zero. As a result, the incomplete trajectory occurs in the opposite direction to the clockwise direction. This is evident in the subsequent graph (c). However, if the angle θ=3π/2, the direction of the particle's movement will be in the clockwise direction, as shown in figure (z). It is noted that equation (11) resolves into a circular shape when a=b.

3- When $\theta = \pi$, equation (8) becomes

$\{(\frac{x}{a}) + (\frac{y}{b})\}^2 = 0$

$$y = \left(-\frac{b}{a}\right)x \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 11$$

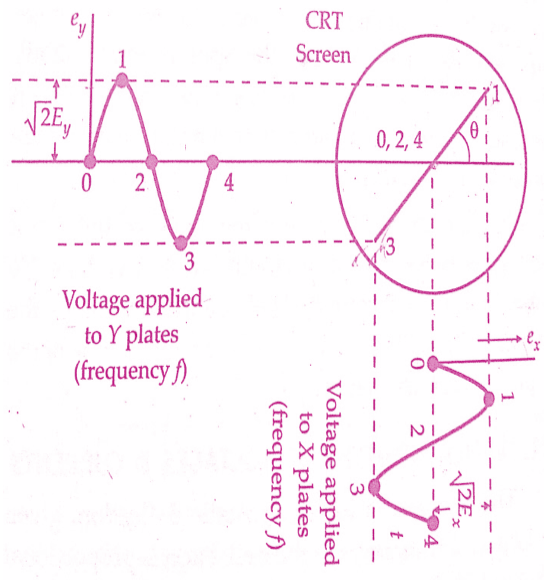
This solution represents the case in the previous figure (a), but the slope of the straight line is a negative value equal to (-b/a).

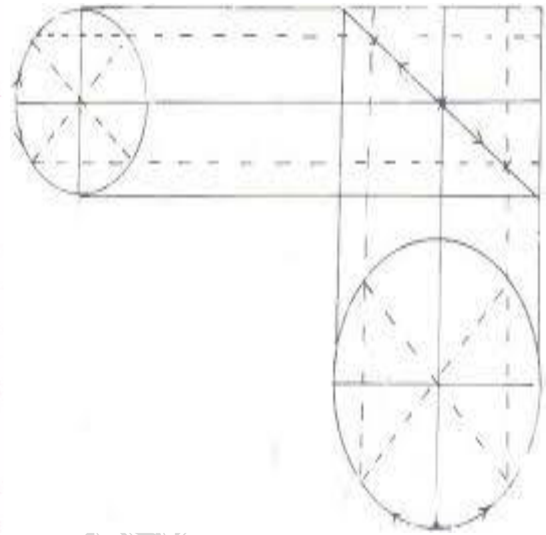
4- When $\theta = \pi/4, 7\pi/4$, equation (8) takes the form of an inclined ellipse.

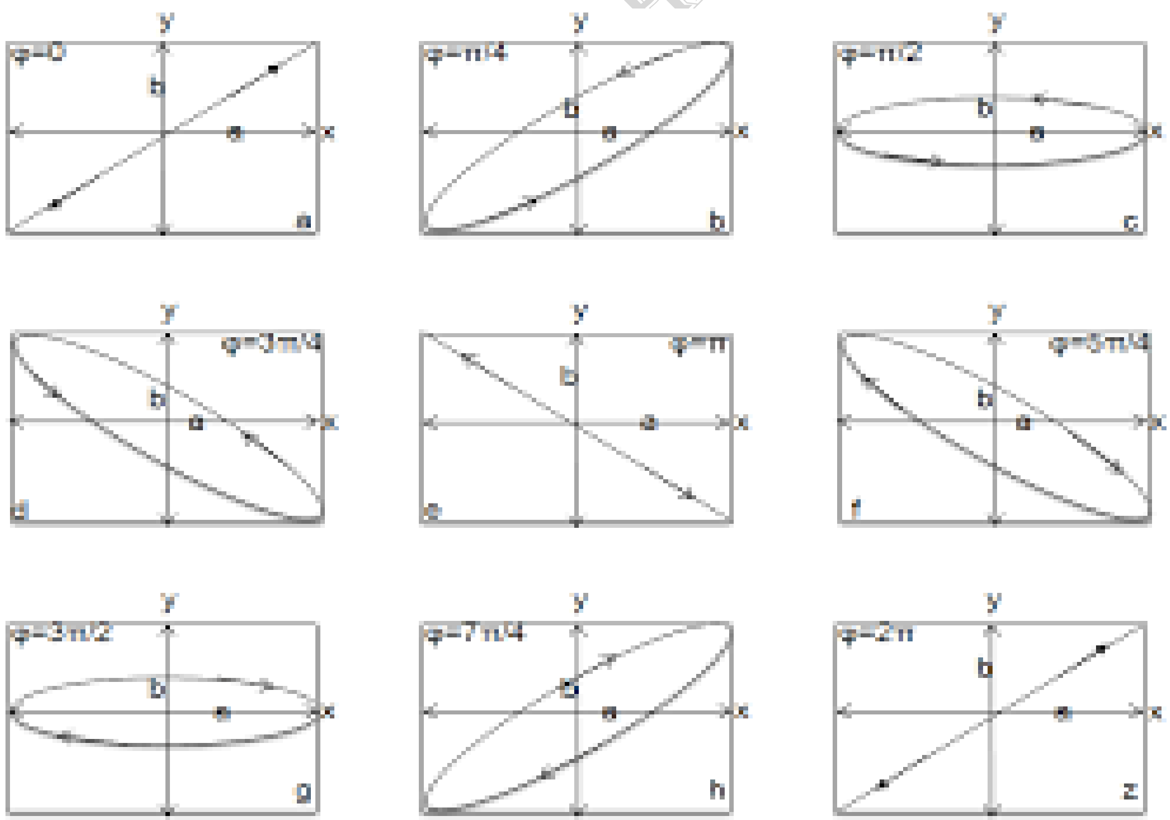
When $\theta = \pi/(4)$, the direction of the particle's movement is opposite to the direction of clockwise movement, as shown in the following figure (b). In fact, this figure represents the case where φ is between zero and $\pi/2$, and thus this figure is an intermediate between cases (a) and (c). When $\theta=7\pi/4$, the direction of movement of the particle is the same as the direction of movement of the clock and time hands, as shown in the following figure (h).

5-When $3\pi/4, 5\pi/4$, equation (8) also takes the form of an inclined ellipse and the resultant motion path of the particle is as shown in the following figure (f and d).

[1] This series of changes in the Lissajous figure repeats in the same way at each periodic time. Hence, we conclude that the [2] resultant motion of combining any two perpendicular harmonic motions with identical frequencies takes the form of an ellipse in all cases, [3] considering the circle and the straight line as two special cases of the ellipse.







The particle's genuine movement is either in the clockwise or counterclockwise direction, contingent upon which of the two motions precedes the other.

Example: Two waves move in two perpendicular directions, represented by the equations x = asin(ωt-60) and y = asin(ωt). Find their resultant and direction.

1. Synthesis of two simple orthogonal harmonic motions with a frequency ratio of 2 to 1.

Suppose we have a particle that undergoes two orthogonal harmonic motions represented by the two equation :

$x = a\sin(2\omega t + \theta)$ …………………………1

$y = b\sin\omega t$ ……………………………………2

Now we try to connect the two equations by eliminating time t. From equation (2), we get:

$(\frac{y}{b}) = \sin\omega t$ ………………………………3

It is the equation $(\sin^2\omega t + \cos^2\omega t = 1)$ we find that:

$$\cos wt = \sqrt{1 - \sin^2\omega t}$$

$$\cos wt = \sqrt{1 - \frac{y^2}{b^2}} \text{……………………….4}$$

From equation (1), we get:

$$\frac{x}{a} = \sin(2\omega t + \theta) = \sin 2\omega t \cos\theta + \cos 2\omega t \sin\theta \text{……………5}$$

But $(\sin 2\omega t = 2\sin\omega t\cos\omega t)$,($\cos 2\omega t = 1\text{-}2\sin^2\omega t$) Substituting into equation (5), we find that

$\frac{x}{a}$ =2sin ωt cos ωt cos θ+(1-2 sin ^{2}ωt) sin θ …………6

Substituting equations (3) and (4) into equation (6), we get

$$\frac{x}{a}=2\left(\frac{y}{b}\right)\left(1-\left(\frac{y^2}{b^2}\right)\right)^{\frac{1}{2}}\cos\theta+\left(1-2\left(\frac{y^2}{b^2}\right)\right)\sin\theta \ldots\ldots\ldots 7$$

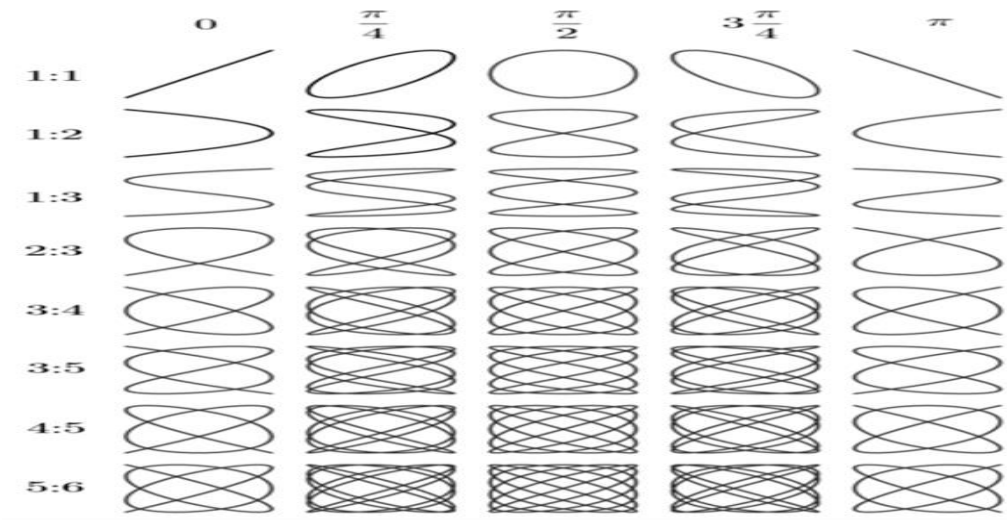
We arrange Equation (7) so that it becomes:

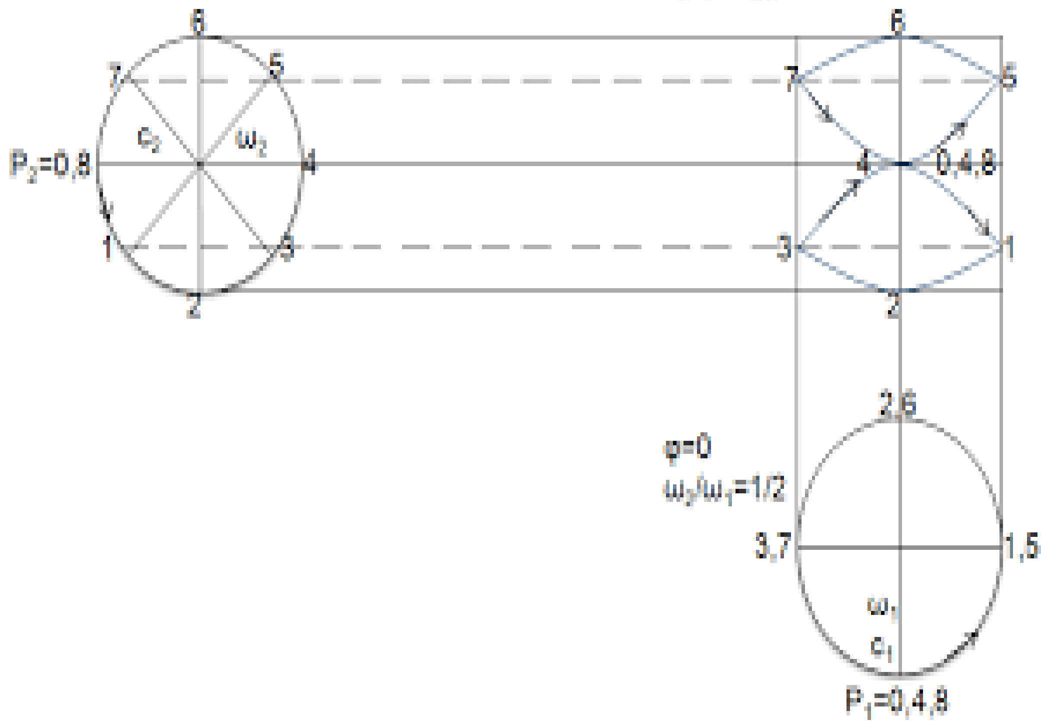
$$\left\{\frac{x}{a}-\left(1-2\left(\frac{y^2}{b^2}\right)\right)\sin\theta\right\}=2\left(\frac{y}{b}\right)\sqrt{1-\left(\frac{y^2}{b^2}\right)}\cos\theta \ldots\ldots\ldots\ldots 8$$

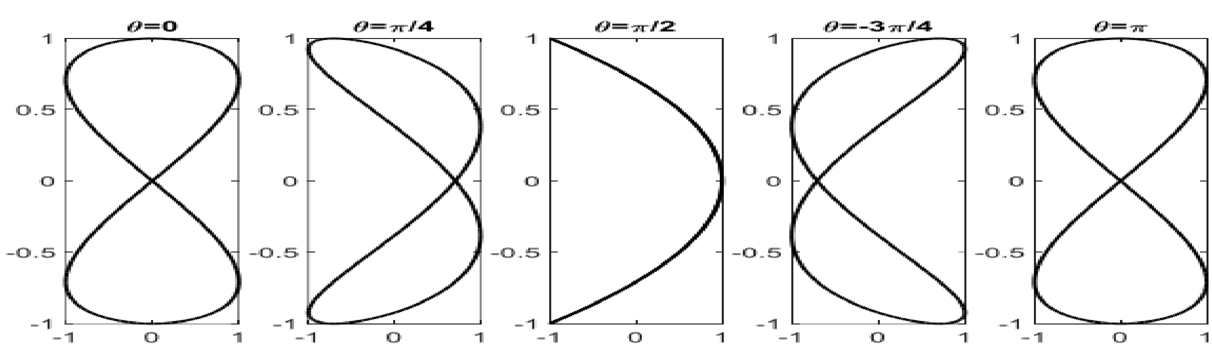
We square both sides of equation (8) and simplify the terms, so the resulting equation becomes as follows:

$$\left(\frac{x}{a}-\sin\theta\right)^2+\left(4\frac{y^2}{b^2}\right)\left\{\frac{y^2}{b^2}+\frac{x}{a}\sin\theta-1\right\}=0 \ldots\ldots\ldots\ldots\ldots 9$$

This is the general equation for the curve that contains two closed loops, and it determines the shape of the path that the particle takes For different values of θ and the following figure (1) shows the shapes of the curves that we get when you take the values $(\theta=\pi, \frac{3\pi}{4}, \frac{\pi}{2}, \frac{\pi}{4}, 0)$







Q: What are the practical benefits of Lissajous?

Lissajous figures offer a way to compare two frequencies or the periodic times of two harmonic motions. They can be used to find the value of an unknown frequency when a known frequency is available. They are also useful for detecting phase variations resulting from the combination of two orthogonal vibrations. Various practical methods, including mechanical and optical approaches, can be used to obtain Lissajous figures. However, the most important method is the electronic approach using an oscilloscope cathode.

Q1- Prove that the following equation satisfies the given equation? H.W

$$\frac{x}{a} - \left(1 - 2\left(\frac{y^2}{b^2}\right)\right)\sin\theta = 2\left(\frac{y}{b}\right)\sqrt{1 - \left(\frac{y^2}{b^2}\right)}\cos\theta$$

Check the following equation

$$\left(\left(\frac{x}{a}\right) - \sin\theta\right)^2 + \left(\frac{2y}{b}\right)^2\left[\left(\frac{x}{a}\right)\sin\theta + \left(\frac{y}{b}\right)^2 - 1\right] = 0$$

Sixth lecture