### *1.3. The average properties of an assembly*

Consider an assembly of *N* systems which has a total energy *E* and is contained in a ·volume *V.* As the state of the assembly is defined by the values of the *6N* coordinates, the manner in which the state of the assembly is changing with time will be described by the motion of the point representing these *6N* coordinates in $\Gamma_{6N}$ . Although the illustration of such a motion can only be given in two dimensions, an attempt is made in Fig. 1 to indicate the changing state of an assembly. In this figure p(N) is taken to represent the momentum coordinates and x(N) to represent the position coordinates.

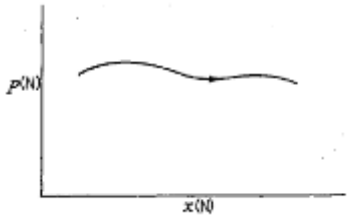


***Fig. 1. Motion of a point in the $\Gamma_{6N}$ space, the arrow indicating the increase Of time***

If the properties of an assembly are known as functions of the position x(N), p(N) in $\Gamma_{6N}$ space (i.e. of the *6N* coordinates of the systems) then the average properties of the assembly will be found by averaging the known functions over all allowed positions x(N), p(N). Reference is here made to allowed positions because, even in the absence of other restrictions, all the spatial coordinates x(N) will necessarily correspond to systems inside the volume *V* of the assembly and the momentum coordinates must satisfy the equation 1.4 for the total energy of the assembly.

Consider some property *X* of an assembly which can be written as a function of the *6N* coordinates, i.e. as *X(x(N),* p(N)). If the probability that the point representing the assembly lies in the element of volume $d\Gamma_{6N}$ at (x(N), p(N)) is P(x(N), p(N)) $d\Gamma_{6N}$ then the average of this property, *X,* will be given by the normal statistical value.

$$\overline{X} = \int_{\Gamma_{6N}} X(\mathbf{x}(N), \mathbf{p}(N)) P(\mathbf{x}(N), \mathbf{p}(N))\, d\Gamma_{6N} \qquad 1.5a$$

or, if the total probability has not been normalized to unity over the whole of the space $\Gamma_{6N}$

$$\overline{X} = \frac{\int_{\Gamma_{6N}} X(\mathbf{x}(N), \mathbf{p}(N)) P(\mathbf{x}(N), \mathbf{p}(N))\, d\Gamma_{6N}}{\int_{\Gamma_{6N}} P(\mathbf{x}(N), \mathbf{p}(N))\, d\Gamma_{6N}} \qquad 1.5b$$

These averages may also be taken as a summation over all states of the assembly. If the property has a value $X_i$ of when the assembly is in the state *i*

and the probability that the assembly is in the state *i* is *Pi*, then the average value of *X* will be.

$$\bar{X} = \frac{\sum_i p_i X_i}{\sum_i p_i} \qquad 1.6a$$

or, if the probabilities are normalized so that $\Sigma p_i = 1$,

$$\bar{X} = \sum_i p_i X_i \qquad 1.6b$$

where the summation is over all the possible states.

### *1.4. Classical and quantum assemblies*

From the point of view of statistical physics the results obtained for the properties of an assembly will depend on whether the component systems are considered to obey classical or quantum mechanics. The differences in the results will arise from the fundamental differences in the 'assumptions which are made regarding the behavior of the different types of systems.

If the systems in an assembly obey classical mechanics then a limitation will be imposed on the energies of the systems only if there is a definite total energy for the assembly. Also each of these classical systems will be completely distinguishable from every other system in the assembly even if the systems all belong to the same species of particle.

If, on the other hand, the systems in the assembly obey quantum mechanics, there will only be certain discrete energy levels which are available to the systems rather than the continuum of energies which are available to the classical systems. Thus, for example, in the case of the simple harmonic oscillator the only values of energy which the oscillator can take are given by $(n+\frac{1}{2})h\nu$ where *n* is an integer, *h* is Planck's constant and $\nu$ is the oscillator frequency. Also, unless they are considered to be localized in space as in the case of atoms at particular sites in a crystal lattice, two identical quantum mechanical systems must be taken as being completely indistinguishable.

There are two types of quantum mechanical system. If a system has an angular momentum which is half-integral in units of $h/2\pi$ or, which is equivalent, has an antisymmetric wave function, then it will obey the Pauli exclusion principle. Such a system (e.g. an electron or proton) is known as a *fermion* and will be restricted in its occupation of energy states in that no single state can be occupied by more than one such system. On the other hand, a system which has an integral value of angular momentum, and hence a symmetric wave function, will not obey the Pauli exclusion principle. This type of

system (e.g. a photon or an alpha-particle) is · known as a *boson* and there is no restriction on the number of such systems which may occupy a given energy state.

Because of the difference in the fundamental properties of the two types of quantum mechanical system, the fermion and the boson, it follows that the form of quantum statistics which applies to a given quantum mechanical assembly will -depend on the particular nature of the component systems.
While the above discussion indicates that there will be three types of statistics, one classical and two quantum mechanical, in practice the classical case will only exist as an approximation to one of the two quantum cases since all systems will strictly obey quantum mechanics. However, there are many cases where classical statistics will give a good description of the assembly under consideration, especially where the systems are localized in space, and it is therefore convenient to commence the study of statistical physics by considering the properties of a classical assembly.