Maxwell-Boltzmann Statistics

In order that a consistent picture may be presented for the different types of statistics, the concept of energy states has been introduced even in the case of classical statistics. This approach can be justified either by considering the classical case as representing the limit where the separation of the energy levels goes to zero or by noting that the classical statistics are, strictly, only a limiting case of one of the types of quantum statistics. In any case, the results obtained for the classical assemblies by this method will be seen to have the same form as those derived on the assumption of continuous energy levels. The statistical distribution which is now derived by determining the most probable state of a given classical assembly of non-interacting systems forms the basis of the classical or Maxwell-Boltzmann statistics.

2.1. Distribution over energies

As mentioned in the introduction it is possible to describe the state of an assembly at a given instant of time by specifying the position and momentum of each system in the assembly. However, where the systems are non-interacting, it is more useful for the purpose of the statistical analysis to specify the distribution of the systems over the various energies which are available. The detailed distribution may be given by specifying the exact energy of each of the N *systems of* the assembly as, for example,

system 1 with energy ε_1
system 2 with energy ε_2
system i with energy ε_i
system N with energy ε_N

The energies of the systems will then be related to the total energy by the condition.$\sum_i \varepsilon_i = E$

Alternatively, a less detailed distribution· may be given by specifying the number of systems which have energies in a range ε to $\varepsilon + d\varepsilon$. This latter type of distribution is obviously more suitable for the purpose of statistical calculations and will still give all the information required about the state of the assembly.

Consider that the energies of the systems can be divided into *'sheets'* so that the sheet *s* will include all the energy states in the range ε_s to $\varepsilon_s + d\varepsilon_s$ and the effective energy of a system in that sheet is ε_s. The number of energy states available to the systems in the sheet *s*, g_s , is called the ***weight*** of the sheet.

The distribution of the systems over the various energies is then given by specifying the *occupation number* n_s for the number of systems with energy ε_s in the sheet *s*. If the energies of the systems are spread over a total of *r* energy sheets then the distribution may be written in terms of the occupation numbers as follows:

Sheet number	1	2	3	...	s	...	r
Sheet energy	ϵ_1	ϵ_2	ϵ_3	...	ϵ_s	...	ϵ_r
Weight of sheet	g_1	g_2	g_3	...	g_s	...	g_r
Occupation number	n_1	n_2	n_3	...	n_s	...	n_r

where the total of the occupation numbers,

$$\sum_{s=1}^{r} n_s$$

is equal to the total number of systems *N.* The energy of the systems in the sheet *s* is $n_s\varepsilon_s$ and the total energy of the assembly is.

$$\sum_{s=1}^{r} n_s\epsilon_s$$

This schematic distribution will represent one of the possible ***configurations*** of the assembly and each configuration of the assembly will correspond to a number of different arrangements· of the systems among the energy sheets. Thus, in a given configuration, it is evidently possible to exchange two systems between two sheets and obtain a different arrangement of the systems while maintaining the same overall configuration. Similarly, a new arrangement may be produced by transferring a. system from a given energy state within a sheet to another state within the same sheet although this transfer again does not produce a new configuration.

Some of the different types of arrangement which correspond to the same configuration are illustrated in Fig. 2. Here four systems, labelled *a, b,* c and *d,* are shown distributed over two energy sheets of weight *g* = 3 and *g* = 4 respectively.

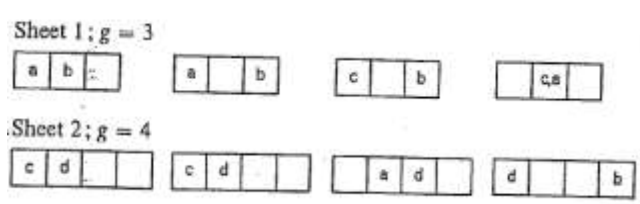

Fig. 2. Four systems a, b, c, and d distributed over two sheets with two systems in sheet 1 and two systems in sheet 2. The sheets have weights 3 and 4 respectively. Calculation will show that, in the classical case where the systems are completely distinguishable,

there are 864 possible arrangements corresponding to the given configuration only 4 of which are shown here.

It is necessary to note, however, that those new arrangements which are produced by interchanging two systems are only counted when the systems are classically distinguishable. When the systems are taken to be identical, as is the case in quantum statistics, it will be seen that the interchange of two such systems does not produce a new arrangement.

At this point it is necessary to introduce one of the fundamental assumptions of statistical physics , namely that

the probability that an assembly is in a particular, allowed arrangement is the same for all such arrangements.

(This assumption may be seen to be equivalent to a statement involving the 6N-dimensional phase space. Thus, if the state of the assembly is represented by a point in Γ_{6N} space the probability that this point is within a given volume of Γ_{6N} space is the same for all equal volumes.) The condition implied by the term ***allowed arrangements*** arises from the conditions which may be imposed on the assembly, e.g. fixed volume and fixed energy. The justification of this assumption, however reasonable it may appear, will clearly lie in the results which are obtained from its application.

While all arrangements of the systems are assumed to be equally probable, all configurations are not. Thus a configuration in which all the *N* systems of an assembly are in the same energy state can be produced in only one way. On the other hand, a configuration in which the *N* systems are only specified as being distributed among the g states of a particular sheet will have g^N different arrangements since each system may be positioned in the sheet in different ways. A comparison of these two configurations will show that, on the basis of equal probabilities for each arrangement, the latter configuration is g^N times more probable than the former.