College of Science
Department of Physics
Fourth Class
Lecture 9

Quantum Mechanics

2023-2024

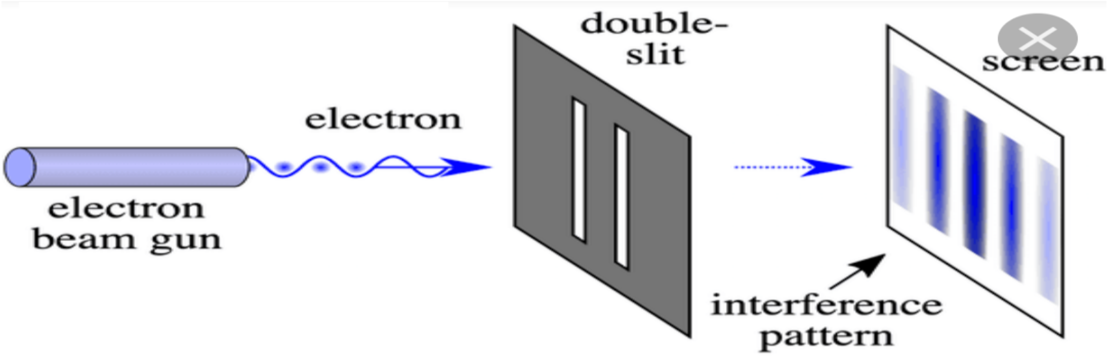
Lecture 9: The uncertainty relations

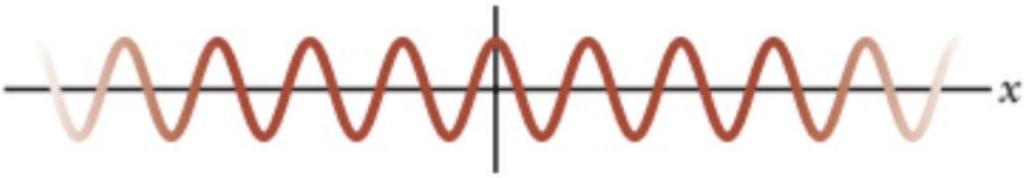
Preparation

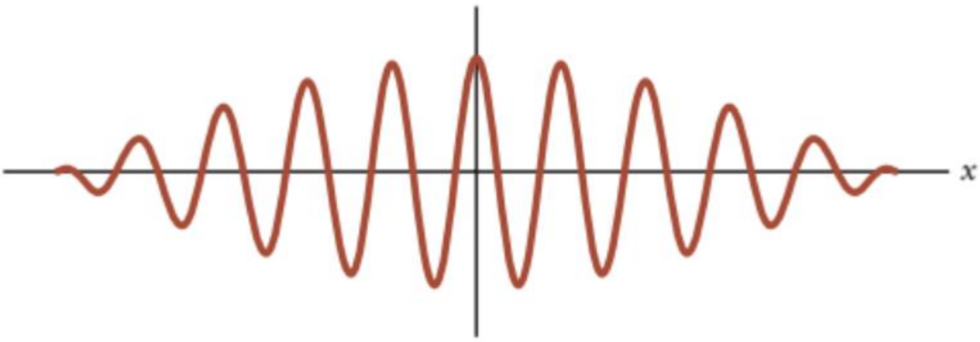
Assist. prof. Alaa Abdul Hakeim

4.2 The uncertainty relations

The width of a wave packet $\Psi_o(x)$ and the width of its amplitude $\emptyset(k)$ are not independent, they are correlated by a reciprocal relationship. The reciprocal relationship between the widths in the x and k spaces has a direct connection to Heisenberg's uncertainty relation ($\Delta x \Delta P_x \geq \frac{\hbar}{2}$) .



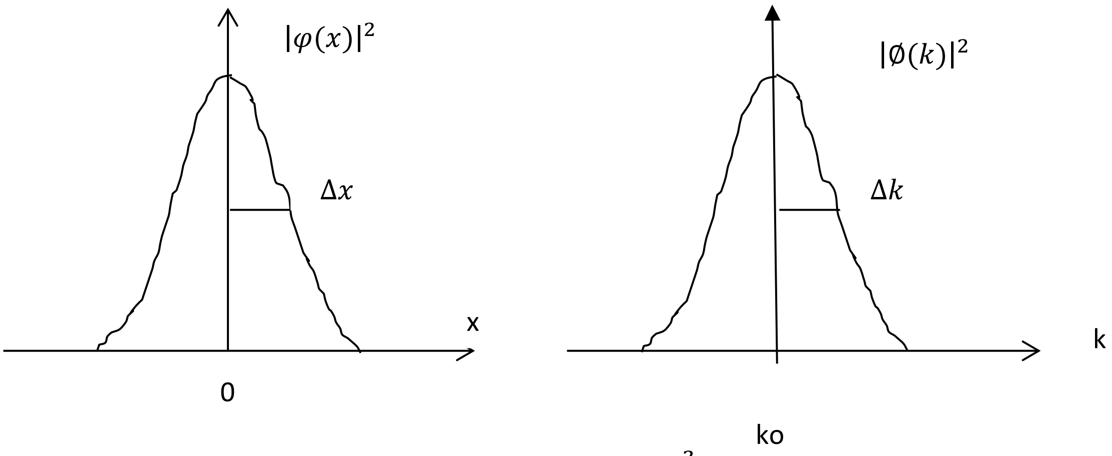




For simplicity, let us illustrate the main ideas on the Gaussian wave packet.

$$\Psi_o(x) = \left(\frac{2}{\pi a^2}\right)^{\frac{1}{4}} e^{-\frac{x^2}{a^2}} e^{ik_o x} \quad \text{---------------8}$$

$$\emptyset(k) = \left(\frac{a}{2\pi}\right)^{\frac{1}{4}} e^{-\frac{a^2}{4}(k-k_o)^2} \quad \text{----------------9}$$



Two localized wave packet $\Psi_o(x) = (\frac{2}{\pi a^2})^{\frac{1}{4}}\, e^{-\frac{x^2}{a^2}} e^{ik_o x}$ and $\emptyset(k) = (\frac{a}{2\pi})^{\frac{1}{4}}\, e^{-\frac{a^2}{4}(k-k_o)^2}$ peak at x=0 and $k=k_o$ respectively, and vanish far away.

In fig. above, $|\Psi_o(x)|^2$ and $|\emptyset(k)|^2$ are centered at x=0 and $k=k_o$

Respectively.

It is convenient to define.

Δx → half width corresponding to half max. $|\Psi_o(x)|^2$

Δk → half width corresponding to half max. $|\emptyset(k)|^2$

In this way when x varies from 0 → $\pm\Delta x$

k varies from k_o→ $\pm\Delta k$

the functions $|\Psi_o(x)|^2$ and $|\emptyset(k)|^2$ drop to $e^{-\frac{1}{2}}$

$$\frac{|\Psi(0, \pm\Delta x)|^2}{|\Psi(0)|^2} = e^{-\frac{1}{2}} \quad , \qquad \frac{|\emptyset(k_o, \pm\Delta k)|^2}{|\emptyset(k_o)|^2} = e^{-\frac{1}{2}} - - - - 10$$

These combined with eq8&9, lead to

$$e^{-2\frac{\Delta x^2}{a^2}} = e^{-\frac{1}{2}} \quad and \quad e^{-\frac{a^2}{2}\Delta k^2} = e^{-\frac{1}{2}}$$

$$-2\frac{\Delta x^2}{a^2} = -\frac{1}{2} \qquad so \qquad \Delta x = \frac{a}{2}$$

Also $-\frac{a^2}{2}\Delta k^2 = -\frac{1}{2} \qquad so \quad \Delta k = \frac{1}{a}$

Hence $\Delta x \Delta k = \frac{1}{2} \qquad\qquad \Delta k = \frac{\Delta P}{\hbar}$ we have

$$\boxed{\Delta x \Delta P = \frac{\hbar}{2}} ------11$$

A comparison of eq.11 with Heisenberg's relation

$\boxed{\Delta x \Delta P_x \geq \frac{\hbar}{2}}$ $\boxed{\Delta y \Delta P_y \geq \frac{\hbar}{2}}$ $\boxed{\Delta z \Delta P_z \geq \frac{\hbar}{2}}$ -----------12

Reveals that the Gaussian wave packet yields an equality, not inequality relation. In fact, eq.11 is the lowest limit of Heisenberg's inequality. As a result, the Gaussian wave packet is called the minimum uncertainty wave packet. All other wave packets yield higher values for the product of x and P uncertainties: $\Delta x \Delta P > \frac{\hbar}{2}$. In conclusion the value of the uncertainty's product varies with the choice of Ψ, but the lowest bound, $\frac{\hbar}{2}$, is provided by a Gaussian wave function also

$$\Delta E \Delta t \geq \frac{\hbar}{2} \quad , \quad \Delta L \Delta \theta \geq \frac{\hbar}{2}$$

Problem

Calculate the uncertainty in position of (a) a neutron moving at 5x 10^6m/sec and (b) a 50 Kg person moving at 2 m/sec.

Solution:

(a) $\Delta x \Delta P_x \geq \frac{ħ}{2}$

$$\Delta x \geq \frac{ħ}{2\Delta P} \approx \frac{ħ}{2mv} = \frac{1.05 \times 10^{-34}}{2 \times 1.65 \times 10^{-27} \times 5 \times 10^{6}}$$

$=6.4 \times 10^{-15}$ m

This distance is comparable to the size of a nucleus.

(b) $\Delta x \geq \frac{ħ}{2\Delta P} \approx \frac{ħ}{2mv} = \frac{1.05\times10^{-34}}{2\times50\times2} = 0.5 \ \times 10^{-36}\, m$

This magnitude is beyond human detection (neglected)

4.3 Application of uncertainty principle:

The principle of uncertainty explains a large number of facts which could not be explained by classical ideas. Some of its applications are discussed here.

1- The non- existence of the electron in the nucleus:

Radius of nucleus is of the order 10^{-12} cm, so that if any electron is confined with nucleus, the uncertainty in its position must not be greater than 10^{-12} cm.

According to uncertainty principle

$$\Delta q \Delta p \sim ħ$$

Where

Δq is uncertainty in position

Δp is uncertainty in momentum

$$ħ = \frac{h}{2\pi} \qquad , \qquad \Delta P = \frac{ħ}{\Delta q}$$

1 joule = 10^7erg ħ= 1.055x 10^{-34} j.sec = 1.055 x 10^{-27} erg.sec

Δq = 2r = 2 x 10^{-12} cm diameter of nucleus

$$\Delta P = \frac{ħ}{\Delta q} = \frac{1.055 \times 10^{-27}}{2 \times 10^{-12}} \approx 5.275 \ \times 10^{-16} gm.cm/sec$$

If this is the uncertainty in momentum of the electron, the momentum of the electron must be at least comparable with its magnitude, i.e

P≈ 5.275 x 10^{-16} gm.cm/sec

The K.E of the electron of mass m is given by

$$T = \frac{P^2}{2m} = \frac{(5.275 \times 10^{-16})^2}{2 \times 9.1 \times 10^{-28}}\ erg$$

$$= \frac{(5.275 \times 10^{-16})^2}{2 \times 9.1 \times 10^{-28} \times 1.6 \times 10^{-12}}$$ =0.95 x 10^8 ev = 95 Mev

This means that if the electron exists inside the nucleus, their kinetic energy must be of the order of 95 Mev. But experimental observation shows that no electron in the atom possess energy greater than 4 Mev. Clearly the conclusion is that electrons do not exist in the nucleus.

2- The radius of the Bohr's first orbit

If Δq & Δp are the uncertainty in the position and momentum of the electron in the first orbit, then we have

$$\Delta q \Delta p \sim \hbar \qquad or\ \Delta p \sim \frac{\hbar}{\Delta q}$$

The uncertainty in the K.E. of the electron may be written as

$$\Delta T = \frac{1}{2} m (\Delta v)^2 = \frac{1}{2} \frac{(m\Delta v)^2}{m} = \frac{1}{2} \frac{(\Delta p)^2}{m} \sim \frac{1}{2m} (\frac{\hbar}{\Delta q})^2 \sim \frac{\hbar^2}{2m(\Delta q)^2}$$

The uncertainty in the potential energy of the same electron is

$$\Delta V = -\frac{Ze^2}{\Delta q}$$

So that the uncertainty in the total energy is $\Delta E = \Delta T + \Delta V$

$$\Delta E = \frac{\hbar^2}{2m(\Delta q)^2} - \frac{Ze^2}{\Delta q}$$

The uncertainty in the energy will be minimum if

$$\frac{d(\Delta E)}{d(\Delta q)} = 0\ and\ \frac{d^2(\Delta E)}{d(\Delta q)^2}\ is + ve$$

Thus $\frac{d(\Delta E)}{d(\Delta q)} = -\frac{\hbar^2}{m(\Delta q)^3} + \frac{Ze^2}{(\Delta q)^2}$ ----1 , for minimum E put $\frac{d(\Delta E)}{d(\Delta q)} = 0$

$0 = -\frac{\hbar^2}{m(\Delta q)^3} - \frac{Ze^2}{(\Delta q)^2}$ or $\frac{\hbar^2}{m(\Delta q)^3} = \frac{Ze^2}{(\Delta q)^2}$ or $\Delta q = \frac{\hbar^2}{mZe^2}$

Differentiating eq. 1 again, we get

$$\frac{d^2(\Delta E)}{d(\Delta q)^2} = \frac{3\hbar^2}{m(\Delta q)^4} - 2\frac{Ze^2}{(\Delta q)^3}$$

$$= \frac{3\hbar^2}{m(\Delta q)^3 \Delta q} - 2\frac{Ze^2}{(\Delta q)^3} = \frac{3\hbar^2}{m(\Delta q)^3 \frac{\hbar^2}{mZe^2}} - 2\frac{Ze^2}{(\Delta q)^3}$$

$$\frac{d^2(\Delta E)}{d(\Delta q)^2} = \frac{3Ze^2}{(\Delta q)^3} - \frac{2Ze^2}{(\Delta q)^3} \cong \frac{Ze^2}{(\Delta q)^3} = \textit{is positive}$$

i.e. the relation $\Delta q \sim \frac{\hbar^2}{mZe^2}$ represent the condition of minimum in the first orbit. Therefore, the radius of the first orbit is given by

r = $\Delta q \sim \frac{\hbar^2}{mZe^2} = \frac{h^2}{4\pi^2 mZe^2}$ which is just the radius of Bohr's first orbit.

Problem 1:

An electron is confined to a box of length 10^{-8} m , calculate the minimum uncertainty in its velocity.

Solution:

$m_e = 9.1 \times 10^{-31}$ Kg , ħ= 1.054×10^{-34} j.sec

$$\Delta q \Delta p \sim ħ$$

There fore $(\Delta q)_{max}(\Delta P)_{min} \sim ħ$

Given $(\Delta q)_{max} = 10^{-8}\, m$

$$(\Delta P)_{min} = \frac{ħ}{(\Delta q)_{max}} = \frac{1.05 \times 10^{-34}}{18^{-8}} = 1.05 \times 10^{-26} Kg.m/sec$$

But

$$(\Delta P)_{min} = m\,(\Delta v)_{min}$$

$\therefore\ m\,(\Delta v)_{min} = 1.05\ \times 10^{-26}$

$$(\Delta v)_{min} = \frac{1.05 \times 10^{-26}}{m} = \frac{1.05 \times 10^{-26}}{9.1\text{x}\,10^{-31}}$$

$= 1.15 \times 10^4$ m/sec

Problem 2:

Find the uncertainty in the momentum of a particle when its position is determined within 0.01cm. Find also the uncertainty in the velocity of an electron and an α- particle respectively when they are located within $5x10^{-8}$ cm.

Solution:

$$\Delta q \Delta p \sim \hbar$$

$$\Delta P \sim \frac{\hbar}{\Delta q} = \frac{1.05 \times 10^{-27}}{0.01} = 1.05 \times 10^{-25}\ gm.cm/sec$$

If Δv is the uncertainty in the velocity of a particle of mass m, we have

$\Delta P = m\Delta v \qquad \therefore \Delta q \Delta P = \Delta q. m\ \Delta v \sim \hbar \qquad \therefore \Delta v = \frac{\hbar}{m\Delta q}$

Uncertainty in the velocity of electron:

m_e= 9.1 x 10^{-28} gm $\qquad \Delta q = 5\ \times 10^{-8}\ cm$

$$\therefore \Delta v = \frac{\hbar}{m\Delta q} = \frac{1.05 \times 10^{-27}}{9.1 \times 10^{-28} \times 5 \times 10^{-8}} = 2.3 \times 10^{7} cm/sec$$

Uncertainty in the velocity of α- particle

m_α = 4 x mass of proton= 4 x 1.67 x 10^{-24} gm = 6.68 x 10^{-24} gm

$$\therefore \Delta v = \frac{\hbar}{m\Delta q} = \frac{1.05 \times 10^{-27}}{6.68 \times 10^{-24} \times 5 \times 10^{-8}} = 3.14\ \times 10^{3} cm/sec$$

Problem 3:

A particle of mass m in a one-dimensional box is found to be in the ground state:

$$\psi(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{\pi x}{a}\right)$$

Find $\Delta x \Delta p$ for this state.

Using $p = i\hbar d/dx$ we have:

$$p\psi(x) = i\hbar \frac{d}{dx}\left[\sqrt{\frac{2}{a}} \sin\left(\frac{\pi x}{a}\right)\right] = \frac{-\hbar\pi}{a}\sqrt{\frac{2}{a}} \cos\left(\frac{\pi x}{a}\right)$$

and:

$$p^2\psi(x) = i\hbar \frac{d}{dx}\left(\frac{-\hbar\pi}{a}\sqrt{\frac{2}{a}} \cos\left(\frac{\pi x}{a}\right)\right) = \frac{-\hbar\pi}{a}\sqrt{\frac{2}{a}} \sin\left(\frac{\pi x}{a}\right)$$

$$\Rightarrow \langle p\rangle = \int \psi^*(x) p \psi(x) dx$$

We found in the example above that $\langle p\rangle = 0$ for this state.

$$\begin{aligned}
\langle p^2\rangle &= \int \psi^*(x) p^2 \psi(x) dx \\
&= \int_0^a \sqrt{\frac{2}{a}} \sin\left(\frac{\pi x}{a}\right) \frac{\hbar^2\pi^2}{a^2}\sqrt{\frac{2}{a}} \sin\left(\frac{\pi x}{a}\right) dx \\
&= \frac{\hbar^2\pi^2}{a^2}\int_0^a \left(\frac{2}{a}\right)\left[\sin\left(\frac{\pi x}{a}\right)\right]^2 dx \\
&= \frac{2\hbar^2\pi^2}{a^3}\int_0^a \frac{1-\cos\left(\frac{2\pi x}{a}\right)}{2} dx \\
&= \frac{\hbar^2\pi^2}{a^3} x\Big|_0^a = \frac{\hbar^2\pi^2}{a^2}
\end{aligned}$$

$$\Delta p = \sqrt{\langle p^2\rangle - \langle p\rangle^2} = \sqrt{\frac{\hbar^2\pi^2}{a^2}} = \frac{\hbar\pi}{a}$$

$$\langle x\rangle = \int \psi^*(x) x \psi(x) dx = \int_0^a \sqrt{\frac{2}{a}} \sin\left(\frac{\pi x}{a}\right) x \sqrt{\frac{2}{a}} \sin\left(\frac{\pi x}{a}\right) dx$$

$$= \frac{2}{a}\int_0^a x\left(\sin\left(\frac{\pi x}{a}\right)\right)^2 dx$$

$$= \frac{2}{a}\frac{a^2}{4} = \frac{a}{2}$$

$$\langle x^2\rangle = \int \psi^*(x) x^2 \psi(x) dx = \int_0^a \sqrt{\frac{2}{a}} x^2 \left(\sin\left(\frac{\pi x}{a}\right)\right)^2 dx = \frac{a^2}{6}\left(2 - \frac{3}{\pi^2}\right)$$

$$\Delta x = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = \sqrt{\frac{a^2(\pi^2-6)}{12\pi^2}} = \frac{a}{\pi}\sqrt{\frac{(\pi^2-6)}{12}}$$

Notice that $\sqrt{(\pi^2-6)/12} = 0.57$, and so $\Delta x \Delta p > \hbar/2$