College of Science
Department of Physics
Fourth Class
Lecture 8

Quantum Mechanics

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Lecture 8: Wave packets and the uncertainty principle

Preparation

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Unit 4

Wave packets and the uncertainty principle

4.1 Wave packet

The first attempt to explain the dual "wave – particle "nature of matter was made by schr*ö*edinger by treating each of the corpus and particularly electrons as a wave packet.

A wave packet comprises a group of waves, each with slightly different velocity and wave length, with phases and amplitudes so chosen that they interfere constructively over only a small region of space where the particle can be located ,outside of which they produce destructive interference so that the amplitude reduce to zero rapidly .The amplitude of one dimensional wave –packet in general, resembles the curve of fig 4.1

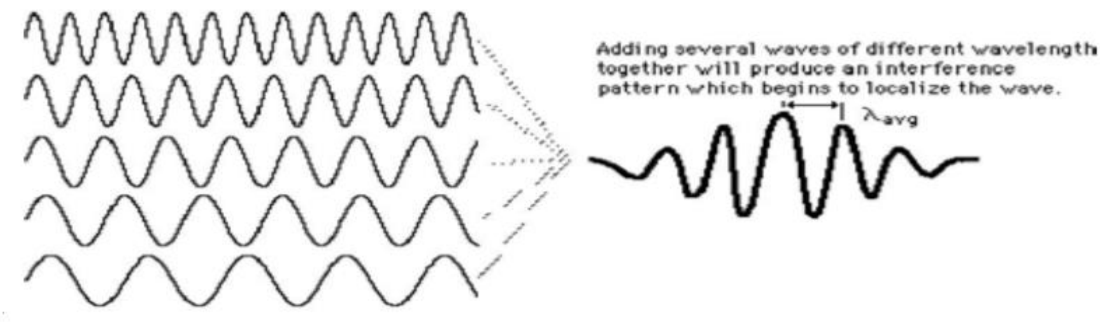


Fig 4.1

Such a packet moves with its own velocity group called group velocity. The individual waves forming the packet possess an average velocity called the phase velocity υ.

This packet is intended to describe a "classical" particle confined to a one-dimensional region, for instance, a particle moving along the x-axis. We can construct the packet Ψ(x,t) by superposing plane waves (propagating along the x-axis)of different frequencies (or wave lengths):

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \emptyset(k) e^{i(kx-\omega t)} dk -------1$$

$\emptyset(k)$ → is the amplitude of the wave packet.

Choosing this time to be t=0 and let Ψ(x,0) → $\Psi_o(x)$ eq.1 becomes

$\Psi_o(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \emptyset(k) e^{i(kx)} dk - - - - - 2$ wave function in x- space

Where $\emptyset(k)$ is the Fourier transform of $\Psi_o(x)$

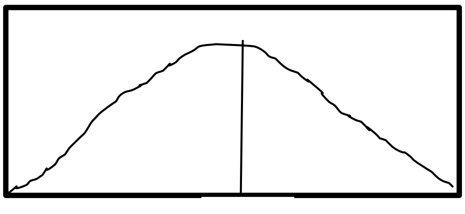
$\emptyset(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Psi_o(x) e^{-i(kx)} dx - - - - - 3$ wave function in momentum space or k- space.

The relations 2&3 show that $\emptyset(k)$ determine $\Psi_o(x)$ and vice versa. The packet eq.2, whose form is determined by the x-dependence of $\Psi_o(x)$, dose indeed have the required property of localization:

$|\Psi_o(x)|$ peaks at x=0

Vanishes for away from x=0.

On one hand $x \to 0$, $e^{ikx} \to 1$



Hence the waves of different frequency interfere constructively (i.e. The various k-integrations in eq.2 add constructively.

On the other hand, far away from x=0 $(i.e, |x| > 0)$ the phase e^{ikx} goes through many periods leading to violent oscillation there by yielding destructive interference (i.e. The various k-integrations in eq.2 add up to zero).

This implies, in the language of Born's probabilistic interpretation, that the particle has a greater probability of being found near x=0 and a low chance of being far away from x=0.

The same comments apply to the amplitude $\emptyset(k)$ as well:

$\emptyset(k)$ peaks at k=0 and vanishes far away

In summary the particle is represented not by a single DE Broglie wave of well-defined frequency and wavelength, but by a wave packet that is obtained by adding many waves of different frequencies.

The physical interpretation of the wave packet is obvious:

$\Psi_o(x) \to$ is the wave function or probability, amplitude for finding the particle at position x

$|\Psi_o(x)|^2 \rightarrow$ gives the probability density for finding the particle at x

P(x)dx= $|\Psi_o(x)|^2 dx \rightarrow$ gives the prob. of finding the particle between x & x+dx

What about the physical interpretation of $\emptyset(k)$?

From eq.2 & eq.3 it follows that

$$\int_{-\infty}^{\infty} |\Psi_o(x)|^2 dx = \int_{-\infty}^{\infty} |\emptyset(k)\ |^2 dk - - - - - 4$$

$\emptyset(k)$ interpreted as prob. amplitude for measuring a wave vector "k" for a particle in the state $\emptyset(k)$.

$|\emptyset(k)\ |^2 \rightarrow$ represents the prob. density for measuring k as the particle wave vector.

P(k)dk = $|\emptyset(k)\ |^2 dk$ the prob. of finding the particle wave vector between k and k+dk

We can extract information about the particle's motion by simply expressing its corresponding matter wave in terms of the particles energy, E, and momentum, P, using

$$k = \frac{P}{\hbar} , \rightarrow dk = \frac{dP}{\hbar} \ , \ E = \hbar\omega$$

And redefining

$\tilde{\emptyset}(P) = \emptyset\ (\hbar k)$ momentum amplitude of the packet.

We can rewrite eq.1 to eq.3 as follows

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \tilde{\emptyset}(P) e^{i(Px-Et)/\hbar} dP - - - - - - - - 5$$

$$\Psi_o(x) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \tilde{\emptyset}(P) e^{i(Px)/\hbar} dP - - - - - - - - 6$$

$$\tilde{\emptyset}(P) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \Psi_o(x) e^{-i(Px)/\hbar} dx - - - - - - - - 7$$

Where E(P) is the total energy of the particle described by the wave packet Ψ(x,t) and $\widetilde{\emptyset}(P)$ is the momentum amplitude of the packet.

Problem:

a- Find $\varphi(x,0)$ for a Gaussian wave packet $\emptyset(k) = A \exp\left[-a^2 \frac{(k-k_0)^2}{4}\right]$, where A is a normalization factor to be found. Calculate the probability of finding the particles in the region $-\frac{a}{2} \leq x \leq \frac{a}{2}$.

b- Find $\emptyset(k)$ for a square wave packet $\varphi_o(x) = \begin{cases} Ae^{-k_o x} & x \leq a \\ 0 & x > a \end{cases}$, find the factor A so that $\varphi_o(x)$ is normalized.