University of Mosul
College of Science
Department of Physics
Fourth Class
Lecture 2

Quantum Mechanics

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Lecture 2: Operators

Preparation

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Unit 2

Operators

2.1 Why do we need operators

Since the physical act of observation may affect the system being observed, any mathematical description of observation must represent both the system and the act of observation. In other words it must include a mathematical operation in turn acts on the mathematical quantity which described the system.

2.2 What is an operator?

An operator "Â" is a mathematical entity which act on a function Ψ(x), give another function say Φ(x)

$$\Phi(x) = \hat{A}\,\Psi(x)$$

The operator for the sake of this argument is always a function of X and $\frac{\partial}{\partial x}$ .We use $\frac{\partial}{\partial x}$ and not $\frac{d}{dx}$ because latter we shall work in 3-dimension and use also $\frac{\partial}{\partial y}$ and $\frac{\partial}{\partial z}$.

Every specific mathematical operation like adding "6" or multiply by C, or extracting the third root………etc. can be represented by a characteristic symbol which is the called an operator.

Operators are

6+C, $\sqrt[3]{x}$, $\frac{d}{dx}$,$\int_a^b dt$, A$\frac{d^2}{dx^2} + \frac{d}{dx} + c$ and so forth

If for example Â= $\frac{\partial}{\partial x}$ x and Ψ(x)= sin x then

$$\Phi(x) = \hat{A}\,\Psi(x)$$

$$= \frac{\partial}{\partial x} x \sin x$$

$$= \sin x + x \cos x \text{ --------------1}$$

The very important point about operator equations, that if say

$$\hat{A} = \hat{B} + \hat{C}$$

Then any Ψ(x) we must have

$$\hat{A}\,\Psi(x)=\hat{B}\,\Psi(x)+\hat{C}\,\Psi(x)$$

Generally

$$\frac{\partial}{\partial x}x\,\Psi(x)=\Psi(x)+x\frac{\partial}{\partial x}\Psi(x)$$

Thus, we have the operator eq.

$$\frac{\partial}{\partial x}x=1+x\frac{\partial}{\partial x}\text{ ---------------2}$$

Note: parial $\frac{\partial}{\partial x}\Psi(x,y,z)$, exact $\frac{d}{dx}\Psi(x)$

Linear Operator:

In Q.M. we are interested almost exclusively with linear operators. An operator Â is said to be linear if it satisfy the following condition

$$\hat{A}(u+\upsilon)=\hat{A}u+\hat{A}\,\upsilon$$

Where u and υ arbitrary operand

$$C\hat{A}=\hat{A}C \quad , \quad C=\text{constant}$$

Problem 2.1

Verify the following operator eqs.

$$\left(\frac{\partial}{\partial x}-x\right)\left(\frac{\partial}{\partial x}+x\right)=\frac{\partial^2}{\partial x^2}-x^2+1$$

Solution

$$\left(\frac{\partial}{\partial x}-x\right)\left(\frac{\partial}{\partial x}+x\right)\Psi(x)=\left(\frac{\partial^2}{\partial x^2}+\frac{\partial}{\partial x}x-x\frac{\partial}{\partial x}-x^2\right)\Psi(x)$$

$$=\frac{\partial^2\Psi}{\partial x^2}+\frac{\partial}{\partial x}x\Psi(x)-x\frac{\partial\Psi}{\partial x}-x^2\Psi(x)$$

$$=\frac{\partial^2\Psi}{\partial x^2}+\Psi(x)-x^2\Psi(x)$$

$$\left(\frac{\partial}{\partial x}-x\right)\left(\frac{\partial}{\partial x}+x\right)\Psi(x)=\frac{\partial^2}{\partial x^2}+1-x^2\,\Psi(x)$$

$$\left(\frac{\partial}{\partial x}-x\right)\left(\frac{\partial}{\partial x}+x\right)=\frac{\partial^2}{\partial x^2}+1-x^2$$

Problem 2.2

Establish the operator equation

$$\frac{\partial}{\partial x}\, x^n = n\, x^{n-1} + x^n \frac{\partial}{\partial x}$$

Solution:

$\frac{\partial}{\partial x}\, x^n \Psi(\mathrm{x}) = n\, x^{n-1} \Psi(\mathrm{x}) + x^n \frac{\partial}{\partial x} \Psi(\mathrm{x})$

$$\frac{\partial}{\partial x}\, x^n = n\, x^{n-1} + x^n \frac{\partial}{\partial x}$$

2.3 Eigen function and eigen values

If the effect of an operator (Â) on a function Ψ(x)is merely to multiply Ψ(x) by a constant "a" i.e.

Â Ψ(x) = a Ψ(x) --------3 a= constant

Then we call Ψ(x) an eigen function of operator Â and a an eigen value of Â

Equation 3 is true only for certain special functions Ψ(x), and not for all Ψ(x). We cannot therefore deduce from eq. (3) an operator equation Â=a

The eigen function of an operator are thus those special functions which remain unaltered under the operation of operator, apart from multiplication by a constant " eigen value"

Problem 2.3

Which of the following function are eigen function of

(a) $\frac{d}{dx}$ (b) $\frac{d^2}{dx^2}$

$$e^{ax}, e^{ax^2}, x, x^2, ax+b, sinx$$

Solution:

1- $\frac{d}{dx} e^{ax} = a\, e^{ax}$ eigen function, a- eigen value

$\frac{d^2}{dx^2} e^{ax} = a^2\, e^{ax}$ eigen function, a^2- eigen value

2- $\frac{d}{dx}e^{ax^2} = 2a\, xe^{ax^2}$ not eigen function

$\frac{d^2}{dx^2}e^{ax^2} = \frac{d}{dx}(2axe^{ax^2})$

$\frac{d^2}{dx^2}e^{ax^2}$ $=2a\, e^{ax^2} + (2ax)(2ax)e^{ax^2}$

$\frac{d^2}{dx^2}e^{ax^2}$ $=2a\, e^{ax^2}(1 + 2a\, x^2)$ not eigen function

3- $\frac{d}{dx}x = 1$ not eigen function

$\frac{d^2}{dx^2}x = \frac{d}{dx}1 = 0$ not eigen function

4- $\frac{d}{dx}x^2 = 2x$ not eigen function

$\frac{d^2}{dx^2}x^2 = \frac{d}{dx}(2x) = 2$ not eigen function

5- $\frac{d}{dx}(ax + b) = a$ not eigen function

$\frac{d^2}{dx^2}(ax + b) = \frac{d}{dx}a = 0$ not eigen function

6- $\frac{d}{dx}\sin x = cosx$ not eigen function

$\frac{d^2}{dx^2}\ sinx = \frac{d}{dx}(cosx) = -sinx$ eigen function & (-1) eigen value

Problem 2.4

Show that Ψ(x) =A $e^{\alpha x}$is an eigen function for the operator

$$\hat{F} = \frac{d^2}{dx^2} + \frac{2}{x}\frac{d}{dx} + \frac{2\alpha}{x}$$

And find it is eigen value

Solution:

$$\hat{F}\Psi(x) = \left(\frac{d^2}{dx^2} + \frac{2}{x}\frac{d}{dx} + \frac{2\alpha}{x}\right)Ae^{-\alpha x}$$

$$= A\alpha^2 e^{-\alpha x} - \frac{2\alpha}{x}\ Ae^{-\alpha x} + \frac{2\alpha}{x}Ae^{-\alpha x}$$

$$= \alpha^2\, A\, e^{-\alpha x}$$

The function is an eigen function with eigen value α^2

Problem 2.5

Find the constant "C" which make the function f(x)=A $e^{-\beta x}$ an eigen function for the operator

$\hat{F} = \frac{d^2}{dx^2} + \frac{2}{x}\frac{d}{dx} + \frac{c}{x}$ calculate the eigen value

Solution:

If f(x) is an eigen function of $\widehat{F}$ then $\hat{F}\ f(x) = const\ f(x)$

$$\hat{F}f(x) = \left(\frac{d^2}{dx^2} + \frac{2}{x}\frac{d}{dx} + \frac{c}{x}\right)Ae^{-\beta x}$$

$$= \beta^2 A\ e^{-\beta x} - \frac{2}{x}\beta Ae^{-\beta x} + \frac{C}{x}Ae^{-\beta x}$$

f(x) is an eigen function when

$\frac{-2}{x}\beta + \frac{C}{x} = 0$ and C=2 β

The eigen value = β^2

When $\hat{F}f(x) = \beta^2 A\ e^{-\beta x} = \beta^2\ f(x)$