College of Science
Department of Physics
Fourth Class
Lecture 10

Quantum Mechanics

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Lecture 10: Problems

Preparation

Assist. prof. Alaa Abdul Hakeim

Problem 2

Calculate the expectation value $\langle E \rangle$ of the total energy of a particle whose associated wave function is of the general form

$$\Psi(x,t) = \sum_{n=1}^{\infty} a_n e^{\frac{-iE_n t}{\hbar}} \varphi_n(x)$$

Discuss the physical meanings of your result

Solution:

$$\langle E \rangle = \int_{-\infty}^{\infty} \Psi^*(x,t)\ \hat{H}\ \Psi(x,t) dx$$

From sch. eq. $\hat{H}\Psi(x,t) = i\hbar \frac{\partial}{\partial t} \Psi(x,t)$

$$\langle E \rangle = \int_{-\infty}^{\infty} \Psi^*(x,t)\ i\hbar \frac{\partial}{\partial t}\ \Psi(x,t) dx$$

$$\int_{-\infty}^{\infty} \sum_{l=1}^{\infty} a_l^*\ e^{\frac{iE_l t}{\hbar}} \varphi_l^*(x) i\ \hbar \frac{\partial}{\partial t} \sum_{n=1}^{\infty} a_n\ e^{\frac{-iE_n t}{\hbar}} \varphi_n(x) dx$$

$$\int_{-\infty}^{\infty} \sum_{l=1}^{\infty} a_l^*\ e^{\frac{-iE_l t}{\hbar}} \varphi_l^*(x) i\ \hbar \sum_{n=1}^{\infty} a_n \frac{-iE_n}{\hbar} e^{\frac{-iE_n t}{\hbar}} \varphi_n(x) dx$$

$$\langle E \rangle = \sum_{n=1}^{\infty} \sum_{l=1}^{\infty} E_n a_l^* a_n e^{\frac{i(E_l - E_n)t}{\hbar}} \int_{-\infty}^{\infty} \varphi_l^*(x) \varphi_n(x) dx$$

$$\int_{-\infty}^{\infty} \varphi_l^*(x) \varphi_n(x) dx = \begin{cases} 1 & n = l \\ 0 & n \neq l \end{cases}$$

The summation over (l) contributes only the single term (l=n), and we have

$\langle E \rangle = \sum_{n=1}^{\infty} E_n a_n^* a_n e^{\frac{i(E_n - E_n)t}{\hbar}}$ $= \sum_{n=1}^{\infty} E_n a_n a_n^*$

Since the total energy of the particle is quantized and can be equal to one of the eigen values E_n , any single measurement of this quantity can only yield one of these values. However, as the wave function

$$\Psi(x,t) = \sum_{n=1}^{\infty} a_n \Psi_n(x,t)$$

Is a sum of a number of different wave function $\Psi_n(x,t)$, each corresponding to a different eigen value, any one of these eigen values could be found in the measurement. Considering this fact, it is clear that we should interpret the equation

$$\langle E \rangle = \sum_{n=1}^{\infty} E_n a_n^* a_n$$

As that the quantity $a_n^* a_n$ is equal to the probability that a single measurement of the total energy will yield the value E.

Problem 3:

Verify that

$$\Psi(x,y,z,t) = exp\left[-i\,\frac{E(t-to)}{\hbar}\right] \emptyset(x,y,z)$$

Is an eigen state of the Hamiltonian for all time and that it is correctly always normalized if it is correctly normalized at one time.

Solution:

For $\Psi(x,y,z,t)$ to be an eigen state for all time we want

$\hat{H}\Psi$=const Ψ

Where const. independent of time

$\hat{H}\ exp\left[-i\,\frac{E(t-to)}{\hbar}\right] \emptyset(x,y,z)$

$= exp\left[-i\,\frac{E(t-to)}{\hbar}\right] \hat{H}\ \emptyset(x,y,z)$

$= exp\left[-i\,\frac{E(t-to)}{\hbar}\right]\ E\ \emptyset(x,y,z)$

=E $exp\left[-i\,\frac{E(t-to)}{\hbar}\right] \emptyset(x,y,z) = E\ \Psi(x,y,z,t)$

And E is a stationary eigen value, and therefore independent of time.

To normalize the wave function, we write

$\int N^2\Psi^* \Psi d\vartheta = 1$ when N is eigen value

$$exp\left[i\frac{E(t-to)}{\hbar}\right]exp\left[-i\frac{E(t-to)}{\hbar}\right] = 1$$

$$\therefore\ N^2\int \emptyset^* \emptyset\ d\vartheta = 1$$

But this integral is time independent.

Once the solution is normalized it is always normalized.

Problem 4

The Hamiltonian for a one-dimensional harmonic oscillator is

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{1}{2}m\omega^2 x^2$$

Verify that two wave function are solutions of the sch. eq. are

$\Psi_1 = \exp\left[\frac{-(m\omega x^2+i\hbar\omega t)}{2\hbar}\right]$ *and* $\Psi_2 = x\exp\left[\frac{-(m\omega x^2+3i\hbar\omega t)}{2\hbar}\right]$

If a and b are real constants, verify that Ψ_3 is also a solution, where

$\Psi_3 = a\,\Psi_1 + b\,\Psi_2$

Without evaluating the integrals explain how you would normalize Ψ_3.

Solution:

$\hat{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{1}{2}m\omega^2 x^2$ -------- 1

$$\Psi_1 = \exp\left[\frac{-(m\omega x^2 + i\hbar\omega t)}{2\hbar}\right]$$

$$\Psi_1 = \exp\left[\frac{-(m\omega x^2)}{2\hbar}\right]\exp\left[\frac{-i\hbar\omega t}{2\hbar}\right]$$

$$\frac{\partial^2\Psi_1}{\partial x^2} = \frac{\partial^2}{\partial x^2}\exp\left[\frac{-(m\omega x^2)}{2\hbar}\right]\exp\left[\frac{-i\omega t}{2}\right] \quad ----2$$

But

$$\frac{\partial^2}{\partial x^2}\exp\left[\frac{-(m\omega x^2)}{2\hbar}\right]=\frac{\partial}{\partial x}\left[-\frac{m\omega}{2\hbar}2x\, e^{\frac{-m\omega x^2}{2\hbar}}\right]$$

$$=-\frac{m\omega}{\hbar}\frac{\partial}{\partial x}\left[x\, exp\left[\frac{-(m\omega x^2)}{2\hbar}\right]\right]$$

$$=-\frac{m\omega}{\hbar}\, e^{\frac{-m\omega x^2}{2\hbar}}-\frac{m^2\omega^2}{2\hbar^2}x(-2x)e^{\frac{-m\omega x^2}{2\hbar}}-----3$$

Substitute eq.3 into eq2

$$\frac{\partial^2\Psi_1}{\partial x^2}=\left(\frac{-m\omega}{\hbar}+\frac{m^2\omega^2}{\hbar^2}x^2\right)e^{\frac{-m\omega x^2}{2\hbar}}e^{\frac{-i\omega t}{2}}-----4$$

Sub. eq.4 into eq.1

$$\hat{H}\,\Psi_1=-\frac{\hbar^2}{2m}\left(\frac{-m\omega}{\hbar}+\frac{m^2\omega^2}{\hbar^2}x^2\right)e^{\frac{-m\omega x^2}{2\hbar}}e^{\frac{-i\omega t}{2}}+\frac{1}{2}m\omega^2x^2\,\Psi_1$$

$$\hat{H}\,\Psi_1=-\frac{\hbar^2}{2m}\left(\frac{-m\omega}{\hbar}+\frac{m^2\omega^2}{\hbar^2}x^2\right)\Psi_1+\frac{1}{2}m\omega^2x^2\,\Psi_1$$

$$=\left(\frac{1}{2}\hbar\omega-\frac{1}{2}m\omega^2x^2+\frac{1}{2}m\omega^2x^2\right)\Psi_1$$

$$\hat{H}\,\Psi_1=\frac{1}{2}\hbar\omega\,\Psi_1--------5$$

On the other hand

$$i\hbar\frac{\partial}{\partial t}\Psi_1=i\hbar\exp\frac{-(m\omega x^2)}{2\hbar}\;\frac{\partial}{\partial t}e^{\frac{-i\omega t}{2}}$$

$$=i\hbar\exp\frac{-(m\omega x^2)}{2\hbar}\;\left(\frac{-i\omega}{2}\right)\exp\left(\frac{-i\omega t}{2}\right)$$

$$=\frac{1}{2}\hbar\omega\Psi_1------6$$

From eq.5 and eq.6

$\hat{H}\,\Psi_1 = i\,\hbar\frac{\partial\Psi_1}{\partial t}$ $\qquad\therefore\ \Psi_1\ is\ a\ solution$

Now for Ψ_2

$$\Psi_2 = x\, exp\left[\frac{-(m\omega x^2 + 3i\hbar\omega t)}{2\hbar}\right]$$

$$=x\, exp\left[\frac{-m\omega x^2}{2\hbar}\right] exp\left[\frac{-3i\omega t}{2}\right]$$

But $\frac{\partial^2}{\partial x^2}\, x\, exp\left[\frac{-m\omega}{2\hbar}x^2\right]$

$$= \frac{\partial}{\partial x}\left\{1 \times exp\left(\frac{-m\omega x^2}{2\hbar}\right) + x\left(\frac{-m\omega}{2\hbar}\right)2x\, exp\left(\frac{-m\omega x^2}{2\hbar}\right)\right\}$$

$$= \frac{\partial}{\partial x}\left\{exp\left(\frac{-m\omega x^2}{2\hbar}\right) - \left(\frac{m\omega}{\hbar}\right)x^2\, exp\left(\frac{-m\omega x^2}{2\hbar}\right)\right\}$$

$$= \frac{-m\omega}{\hbar}x\, exp\left(\frac{-m\omega x^2}{2\hbar}\right) - \frac{m\omega}{\hbar}2x\, exp\left(\frac{-m\omega x^2}{2\hbar}\right) - x^2\left(\frac{-m^2\omega^2}{2\hbar^2}\right)2x\, exp\left(\frac{-m\omega x^2}{2\hbar}\right)$$

$$\frac{\partial^2}{\partial x^2}\, x\, exp\left[\frac{-m\omega}{2\hbar}x^2\right] = \left(\frac{-m\omega}{\hbar}x - 2\frac{m\omega}{\hbar}x + \frac{m^2\omega^2}{\hbar^2}x^3\right) exp\left(\frac{-m\omega x^2}{2\hbar}\right)$$

$$\frac{\partial^2}{\partial x^2}\, x\, exp\left[\frac{-m\omega}{2\hbar}x^2\right] = \left(\frac{-3m\omega}{\hbar}x + \frac{m^2\omega^2}{\hbar^2}x^3\right)e^{\frac{-m\omega x^2}{2\hbar}}$$

$$=\left(\frac{-3m\omega}{\hbar} + \frac{m^2\omega^2}{\hbar^2}x^2\right)xe^{\frac{-m\omega x^2}{2\hbar}} - - - 7$$

From eq.1 and eq.7 for Ψ_2

$$\hat{H}\,\Psi_2 = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi_2}{\partial x^2} + \frac{1}{2}m\omega^2x^2\Psi_2$$

$$\hat{H}\,\Psi_2 = -\frac{\hbar^2}{2m}\exp\left(\frac{-i3\omega t}{2}\right)\left(\frac{-3m\omega}{\hbar} + \frac{m^2\omega^2}{\hbar^2}x^2\right)xe^{\frac{-m\omega x^2}{2\hbar}} + \frac{1}{2}m\omega^2x^2\exp\left(\frac{-i3\omega t}{2}\right)x\, e^{\frac{-m\omega x^2}{2\hbar}}$$

$$=\exp\left(\frac{-i3\omega t}{2}\right)\left(\frac{3}{2}\hbar\omega - \frac{m\omega^2}{2}x^2 + \frac{m\omega^2}{2}x^2\right)x\, e^{\frac{-m\omega x^2}{2\hbar}}$$

$$= \frac{3}{2}\hbar\omega\, .\, xe^{\frac{-m\omega x^2}{2\hbar}}.\exp\left(\frac{-i3\omega t}{2}\right)$$

$= \frac{3}{2}\hbar\omega \;\; x \exp \left[\frac{-(m\omega x^2 + i3\omega t)}{2\hbar}\right]$

$\hat{H}\, \Psi_2 = \frac{3}{2}\hbar\omega \;\; \Psi_2 \qquad ------8$

On the other hand

$i\,\hbar\frac{\partial \Psi_2}{\partial t} = i\hbar x \,.\, e^{\frac{-m\omega x^2}{2\hbar}} \;\frac{\partial}{\partial t} e^{\frac{-3i\omega t}{2}}$

$i\,\hbar\frac{\partial \Psi_2}{\partial t} = i\hbar x \,.\, e^{\frac{-m\omega x^2}{2\hbar}} \left(\frac{-3i\omega}{2}\right) e^{\frac{-3i\omega t}{2}}$

$i\,\hbar\frac{\partial \Psi_2}{\partial t} = \frac{3}{2}\hbar\omega \;\; \Psi_2 \qquad --------9$

from eq.8 &9

$\hat{H}\, \Psi_2 = i\,\hbar\frac{\partial \Psi_2}{\partial t} \qquad\qquad \therefore \; \Psi_2 \; is\ a\ solution$

$\Psi_3 = a\, \Psi_1 + b\, \Psi_2$

$\hat{H}\Psi_3 = \hat{H}\, a\, \Psi_1 + \hat{H}\, b\, \Psi_2$

From eq.5 and eq.8

$\hat{H}\, \Psi_3 = a\;\frac{1}{2}\hbar\omega\Psi_1 + b\frac{3}{2}\hbar\omega \;\; \Psi_2 \;\; ------10$

On the other hand, from eq. 6&9

$i\,\hbar\frac{\partial \Psi_3}{\partial t} = i\,\hbar\frac{\partial}{\partial t} a\Psi_1 + i\,\hbar\frac{\partial}{\partial t} b\Psi_2$

$i\,\hbar\frac{\partial \Psi_3}{\partial t} = \frac{1}{2}\hbar\omega\; a\Psi_1 + \frac{3}{2}\hbar\omega\; b\; \Psi_2 \;\; ----11$

From eq.10 &11

Ψ_3 is also a solution of the schr$\ddot{o}$edinger eq.

To normalize Ψ_3 we must have

$$\int \Psi_3^* \Psi_3 \, d\vartheta = 1$$

$$\int (a\Psi_1^* + b\Psi_2^*)(a\Psi_1 + b\Psi_2)\, d\vartheta = 1$$

A and b are real as given in the problem i.e.

$$\int [a^2\Psi_1^*\Psi_1 + b^2\Psi_2^*\Psi_2 + ab(\Psi_1^*\Psi_2 + \Psi_2^*\Psi_1)]d\vartheta = 1 - - - 12$$

If Ψ had been orthonormal, then we would have

$\int \Psi_1^*\Psi_1 d\vartheta = 1$ $\int \Psi_2^*\Psi_2 d\vartheta = 1$ normalization

$\int \Psi_1^*\Psi_2 d\vartheta = \int \Psi_2^*\Psi_1 d\vartheta = 0$ orthogonal

$\int (a^2\Psi_1^*\Psi_1 + b^2\Psi_2^*\Psi_2)\, d\vartheta = 1$

$$a^2 \int \Psi_1^*\Psi_1 d\vartheta + b^2 \int \Psi_2^*\Psi_2 d\vartheta = 1$$

Thus for normalization of Ψ_3 we would have to satisfy

$a^2 + b^2 = 1$