College of Science
Department of Physics
Fourth Class
Lecture 5

Quantum Mechanics

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Lecture 5: Problems

Preparation

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3.4 Postulate 4

When a given system is in a state Ψ, the expected mean ($\bar{a}$) of a sequence of measurements on the observable whose corresponding operator $\hat{A}$ is given by

$$\bar{a} = \frac{\int \Psi^* \hat{A}\, \Psi dv}{\int \Psi^* \Psi dv}$$

There will be more detailed discussion about this in later unit.

3.5 Problems

Problem 3.1

Show that $[\hat{x}, \hat{p}_x] = i\hbar$

Solution: $\boldsymbol{p_x} = -i\hbar \frac{\partial}{\partial x}$

$[\hat{x}, \hat{p}_x]\Psi = -i\hbar(\hat{x}\frac{\partial}{\partial x} - \frac{\partial}{\partial x}\hat{x})\Psi$

$= -i\hbar(\hat{x}\frac{\partial \Psi}{\partial x} - \frac{\partial}{\partial x}(x\Psi))$

$= -i\hbar(\hat{x}\frac{\partial \Psi}{\partial x} - \Psi - x\frac{\partial \Psi}{\partial x}) = -i\hbar(-\Psi) = i\hbar\Psi$

$[\hat{x}, \hat{p}_x] = +i\hbar$

Problem 3.2

Show that if $\hat{l}_x, \hat{l}_y, \hat{l}_z$ denote the x,y,z components of angular momentum

$$[\hat{l}_x, \hat{l}_y] = i\hbar\hat{l}_z$$

Solution: $\vec{L} = \hat{r} \times \hat{p}$ $\qquad \hat{r} = \hat{\imath}\hat{x} + \hat{\jmath}\hat{y} + \hat{k}\hat{z}$

$$\hat{p} = \hat{\imath}\left(-i\hbar\frac{\partial}{\partial x}\right) + \hat{\jmath}\left(-i\hbar\frac{\partial}{\partial y}\right) + \hat{k}(-i\hbar\frac{\partial}{\partial z})$$

$$\vec{L} = -i\hbar\begin{vmatrix} i & j & k \\ x & y & z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{vmatrix}$$

$\hat{l}_x = -i\hbar(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y})$

$$\hat{l}_y = -i\hbar(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z})$$

$$\hat{l}_z = -i\hbar(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x})$$

Thus $\quad [\hat{l}_x, \hat{l}_y] = \hat{l}_x\hat{l}_y - \hat{l}_y\hat{l}_x$

$$= \left[(-i\hbar)\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)(-i\hbar)\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)\right] - \left[(-i\hbar)\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)(-i\hbar)\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)\right]$$

$$= -\hbar^2\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right) + \hbar^2\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)$$

$$= +\hbar^2(-y\frac{\partial}{\partial z}z\frac{\partial}{\partial x} + y\frac{\partial}{\partial z}x\frac{\partial}{\partial z} + z\frac{\partial}{\partial y}z\frac{\partial}{\partial x} - z\frac{\partial}{\partial y}x\frac{\partial}{\partial z} + z\frac{\partial}{\partial x}y\frac{\partial}{\partial z} - z\frac{\partial}{\partial x}z\frac{\partial}{\partial y} - x\frac{\partial}{\partial z}y\frac{\partial}{\partial z} + x\frac{\partial}{\partial z}z\frac{\partial}{\partial y}$$

$$[\hat{l}_x, \hat{l}_y] = \hbar^2(-y\frac{\partial}{\partial x} - yz\frac{\partial^2}{\partial z\partial x} + 0 + yx\frac{\partial^2}{\partial z^2} + 0 + z^2\frac{\partial^2}{\partial y\partial x} - 0 - zx\frac{\partial^2}{\partial y\partial z} + 0 + zy\frac{\partial^2}{\partial x\partial z} - 0 - z^2\frac{\partial^2}{\partial x\partial y} - 0 - xy\frac{\partial^2}{\partial z^2} + x\frac{\partial}{\partial y} + xz\frac{\partial^2}{\partial z\partial y})$$

$$= \hbar^2(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x})$$

But $\hat{l}_z = -i\hbar(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x})$

$\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) = \frac{\hat{l}_z}{-i\hbar}$ $\qquad$ $[\hat{l}_x, \hat{l}_y] = \hbar^2\frac{\hat{l}_z}{-i\hbar} = \frac{\hbar}{-i}\hat{l}_z$

$$[\hat{l}_x, \hat{l}_y] = i\hbar\hat{l}_z$$

Note : similarly we can obtain

$$[\hat{l}_x, \hat{l}_y] = i\hbar\hat{l}_z$$

$$[\hat{l}_y, \hat{l}_z] = i\hbar\hat{l}_x$$

$$[\hat{l}_z, \hat{l}_x] = i\hbar\hat{l}_y$$

Problem 3.3

A particle is moving in a potential V(x)=$\frac{1}{2}m\omega^2x^2$

a- State the energy operator.
b- Verify that Ψ(x) is a stationary energy eigen function where $\Psi(x) = Nexp(-\frac{m\omega}{2ħ}x^2)$
c- What is the energy eigen values corresponding to Ψ(x).

Solution:

a- Energy operator is $\hat{H} = -\frac{ħ^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{1}{2}m\omega^2x^2$
b- If Ψ(x) is a stationary energy eigen function

$\hat{H}$ Ψ(x) = const. Ψ(x)

$$\left[-\frac{ħ^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{1}{2}m\omega^2x^2\right]\Psi(x) = -\frac{ħ^2}{2m}\frac{\partial^2}{\partial x^2}\Psi(x) + \frac{1}{2}m\omega^2x^2\Psi(x)$$

But

$$\frac{\partial^2}{\partial x^2}\ \Psi(x) = \frac{\partial^2}{\partial x^2}\ Nexp\left(-\frac{m\omega}{2ħ}x^2\right)$$

$$=\text{N}\left[\frac{\partial}{\partial x}\left(\frac{-m\omega}{2ħ}\right)2x \exp\left(\frac{-m\omega}{2ħ}x^2\right)\right]$$

$$\frac{\partial^2}{\partial x^2}\ \Psi(x) = \text{N}\left[\left(\frac{-m\omega}{ħ}\right)\exp\left(\frac{-m\omega}{2ħ}x^2\right) + \frac{m\omega}{ħ}x\frac{m\omega}{2ħ}2x\exp\left(\frac{-m\omega}{2ħ}x^2\right)\right]$$

$$=-N\left[1-\frac{m\omega}{ħ}x^2\right]\frac{m\omega}{ħ}\ exp\left(\frac{-m\omega}{2ħ}x^2\right)$$

$$\hat{H}\ \Psi(x) = -\frac{ħ^2}{2m}\frac{\partial^2}{\partial x^2}\Psi(x) + V(x)\Psi(x)$$

$$\hat{H}\ \Psi(x)= N\left(1-\frac{m\omega}{ħ}x^2\right)\frac{\omega}{2}ħex\,p\left(\frac{-m\omega}{2ħ}x^2\right) + \frac{1}{2}Nm\omega^2x^2exp\left(\frac{-m\omega}{2ħ}x^2\right)$$

$$\hat{H}\ \Psi(x)= \left(1-\frac{m\omega}{ħ}x^2 + \frac{m\omega}{ħ}x^2\right)\frac{\omega ħ}{2}Nexp\left(\frac{-m\omega}{2ħ}x^2\right)$$

i.e. $\hat{H}$ Ψ(x)=$\frac{\omega ħ}{2}\Psi$(x)

c- Energy eigen value = $\frac{\omega ħ}{2}$

problem 3.4

A particle m in one dimensional box $0 \le x \le a$ is in the ground state $\varphi(x) = \sqrt{\frac{2}{a}}\ \sin\left(\frac{\pi x}{a}\right)$. Find the expectation value of position $\langle x \rangle$.

$$\langle x \rangle = \int_{-\infty}^{+\infty} \varphi^* \, \varphi \; dx = \int_0^a \sqrt{\frac{2}{a}} \sin\left(\frac{\pi x}{a}\right) x \sqrt{\frac{2}{a}} \sin\left(\frac{\pi x}{a}\right) dx$$

$$= \frac{2}{a} \int_0^a \mathrm{x}\ (\sin\left(\frac{\pi x}{a}\right))^2 \, dx = \frac{2}{a} \int_0^a x \left(\frac{1 - \cos\left(\frac{2\pi x}{a}\right)}{2}\right) dx$$

$$= \frac{1}{a} \int_0^a x \;\; dx - \frac{1}{a} \int_0^a x \cos\left(\frac{2\pi x}{a}\right) \, dx$$

$$\int_0^a x \;\; dx = \frac{a^2}{2}$$

$$= \frac{1}{a} \int_0^a x \;\; dx = \frac{1}{a} \left[\frac{x^2}{2}\right]_0^a = \frac{a}{2}$$