College of Science
Department of Physics
Fourth Class
Lecture 20

Quantum Mechanics

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Lecture 20: Spin angular momentum

Preparation

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9.4 Ladder operator

$$\hat{\boldsymbol{l}}_+ = \hat{\boldsymbol{l}}_{\boldsymbol{x}} + \boldsymbol{i}\hat{\boldsymbol{l}}_{\boldsymbol{y}} \qquad and \qquad \hat{\boldsymbol{l}}_- = \hat{\boldsymbol{l}}_{\boldsymbol{x}} - \boldsymbol{i}\hat{\boldsymbol{l}}_{\boldsymbol{y}}$$

$$\hat{\boldsymbol{l}}_+ \ is\ raising\ operator$$

$$\hat{\boldsymbol{l}}_- \ is\ lowering\ operator$$

Ladder operators are using to raise or lower the eigen value of angular momentum.

Commutation relations of $\hat{\boldsymbol{l}}_{\boldsymbol{z}}$ with $\hat{\boldsymbol{l}}_+$ and $\hat{\boldsymbol{l}}_-$

$$[l_z, l_+] = [l_z, l_x + il_y] = [l_z, l_x] + i[l_z, l_y] = i\hbar\,\hat{\boldsymbol{l}}_{\boldsymbol{y}} + i(-i\hbar\hat{\boldsymbol{l}}_{\boldsymbol{x}})$$

$$= i\hbar\hat{\boldsymbol{l}}_{\boldsymbol{y}} + \hbar\hat{\boldsymbol{l}}_{\boldsymbol{x}} = \hbar(\hat{\boldsymbol{l}}_{\boldsymbol{x}} + \boldsymbol{i}\hat{\boldsymbol{l}}_{\boldsymbol{y}}) = \hbar\hat{\boldsymbol{l}}_+$$

And

$$[l_z, l_-] = [l_z, l_x - il_y] = [l_z, l_x] - i[l_z, l_y] = i\hbar\,\hat{\boldsymbol{l}}_{\boldsymbol{y}} - i(-i\hbar\hat{\boldsymbol{l}}_{\boldsymbol{x}})$$

$$= i\hbar\hat{\boldsymbol{l}}_{\boldsymbol{y}} - \hbar\hat{\boldsymbol{l}}_{\boldsymbol{x}} = -\hbar(\hat{\boldsymbol{l}}_{\boldsymbol{x}} - \boldsymbol{i}\hat{\boldsymbol{l}}_{\boldsymbol{y}}) = -\hbar\hat{\boldsymbol{l}}_-$$

Generally, $[l_z, l_\pm] = \pm\hbar\hat{\boldsymbol{l}}_\pm$

Commutation relations of $\hat{\boldsymbol{l}}_+$ and $\hat{\boldsymbol{l}}_-$ mutually

$$[l_+, l_-] = l_+l_- - l_-l_+ = (\hat{\boldsymbol{l}}_{\boldsymbol{x}} + \boldsymbol{i}\hat{\boldsymbol{l}}_{\boldsymbol{y}})(\hat{\boldsymbol{l}}_{\boldsymbol{x}} - \boldsymbol{i}\hat{\boldsymbol{l}}_{\boldsymbol{y}}) - (\hat{\boldsymbol{l}}_{\boldsymbol{x}} - \boldsymbol{i}\hat{\boldsymbol{l}}_{\boldsymbol{y}})(\hat{\boldsymbol{l}}_{\boldsymbol{x}} + \boldsymbol{i}\hat{\boldsymbol{l}}_{\boldsymbol{y}})$$

$$= l_x(\hat{\boldsymbol{l}}_{\boldsymbol{x}} - \boldsymbol{i}\hat{\boldsymbol{l}}_{\boldsymbol{y}}) + il_y(\hat{\boldsymbol{l}}_{\boldsymbol{x}} - \boldsymbol{i}\hat{\boldsymbol{l}}_{\boldsymbol{y}}) - \boldsymbol{l}_{\boldsymbol{x}}(\hat{\boldsymbol{l}}_{\boldsymbol{x}} + \boldsymbol{i}\hat{\boldsymbol{l}}_{\boldsymbol{y}}) + \boldsymbol{i}\boldsymbol{l}_{\boldsymbol{y}}(\hat{\boldsymbol{l}}_{\boldsymbol{x}} + \boldsymbol{i}\hat{\boldsymbol{l}}_{\boldsymbol{y}})$$

$$= 2i(l_yl_x - l_xl_y) = 2i[l_y, l_x] = 2i(-i\hbar l_z) = 2\hbar l_z$$

Similarly, $[l_-, l_+] = -2\hbar l_z$

9.5 Spin angular momentum

The spin operator $\hat{S}_x, \hat{S}_y and\ \hat{S}_z$ associated with the components of spin angular momentum satisfy the commutation relation

$$[S_x, S_y] = S_xS_y - S_yS_x = i\,\hbar S_z$$

$$[S_y, S_z] = S_yS_z - S_zS_y = i\,\hbar S_x \qquad [S_z, S_x] = S_zS_x - S_xS_z = i\,\hbar S_y$$

If we consider an electron (spin=1/2) each operator $\hat{S}_x, \hat{S}_y and\ \hat{S}_z$ must have just two eigen values (1/2$\hbar$, -1/2$\hbar$)

$$S_x = \frac{1}{2}\,\hbar\,\hat{\sigma}_x \qquad S_y = \frac{1}{2}\,\hbar\,\hat{\sigma}_y \qquad S_z = \frac{1}{2}\,\hbar\,\hat{\sigma}_z$$

Where $\hat{\sigma}_x, \hat{\sigma}_y and\ \hat{\sigma}_z$are new operators with

$$\hat{\sigma}_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \hat{\sigma}_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \qquad \hat{\sigma}_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

The matrices associated with $\hat{\sigma}_x, \hat{\sigma}_y and\ \hat{\sigma}_z$are Pauli matrices.

$$\therefore S_x = \frac{1}{2}\,\hbar \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \ , \ S_y = \frac{1}{2}\,\hbar \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \ , \ S_z = \frac{1}{2}\,\hbar \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$S_x = \frac{1}{2}(S_+ + S_-) \qquad S_y = \frac{1}{2i}(S_+ - S_-)$$

9.6 Eigen value of $\hat{L}^2, \hat{L}_z$

If we represent a stat function Ψ(n,l,m) where l and m are orbital and magnetic quantum number

$$L^2\Psi\,(n,l,m) = l(l+1)\,\Psi\,(n,l,m)$$

$$l_z\Psi\,(n,l,m) = m\,\Psi\,(n,l,m)$$

The eigen values of $\hat{l}_+$ and $\hat{l}_-$

$$l_+\Psi\,(n,l,m) = \sqrt{l(l+1) - m(m+1)}\ \ (n,l,m+1)$$

$$l_-\Psi\,(n,l,m) = \sqrt{l(l+1) - m(m-1)}\ \ (n,l,m-1)$$

Example (1):

Evaluate $\left[\sigma_x, \sigma_y\right]$

$$\left[\sigma_x, \sigma_y\right] = \sigma_x\sigma_y - \sigma_y\sigma_x$$

$$\hat{\sigma}_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \hat{\sigma}_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \qquad \hat{\sigma}_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\left[\sigma_x, \sigma_y\right] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} - \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\left[\sigma_x, \sigma_y\right] = \begin{bmatrix} i & 0 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} = 2\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} = 2i\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = 2i\hat{\sigma}_z$$

In Dirac notation

$$|\varphi\rangle \; ket \; state \; function \; \varphi$$

$$\langle\varphi| \; bra \; is \; complex \; conjugate \; \varphi^*$$

So the eigen value of orbital angular momentum

$$l_+|l,m\rangle = \sqrt{l(l+1) - m(m+1)} \;\; |l,m+1\rangle$$

$$l_-|l,m\rangle = \sqrt{l(l+1) - m(m-1)} \;\; |l,m-1\rangle$$

$$l_z|l,m\rangle = m|l,m\rangle$$

$$l^2|l,m\rangle = l(l+1)|l,m\rangle$$

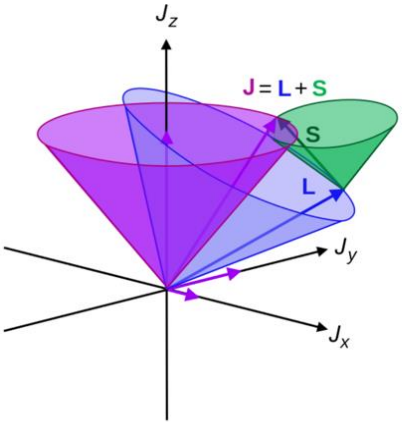
Also, the eigen value of spin operators

$$S_z|\pm m_s\rangle = \pm m_s|\pm m_s\rangle$$

$S_+\left|m_s\right\rangle = \sqrt{s(S+1) - m_s(m_s+1)}|m_s+1\rangle$

$S_-\left|m_s\right\rangle = \sqrt{s(S+1) - m_s(m_s-1)}|m_s-1\rangle$

9.7 Spin – Orbit interpretation



Spin -orbit coupling J=L+S

$J^2 = (\hat{L} + \hat{S})^2 = L^2 + S^2 + 2L.S$

$L.S = l_z s_z + l_x s_x + l_y s_y = \frac{1}{2}(l_+ s_- + l_- s_+)$

Example (2):

For $|l, m_l\rangle$ state evaluate $l_+|1,0\rangle$,$|1,1\rangle$,$|1,-1\rangle$

$$l_+|l, m_l\rangle = \sqrt{l(l+1) - m_l(m_l+1)}\ \ |l, m_l + 1\rangle$$

$$l_+\big|1,0\rangle = \sqrt{2}\ \ |1,1\rangle$$

$$l_+\big|1,1\rangle = \sqrt{2-2}\ \ |1,2\rangle = 0\ impossible\ state$$

<u>H.W.</u>

1- For $|S, m_S\rangle$ state evaluate $S_+\left|\frac{1}{2}, -\frac{1}{2}\right\rangle, S_-\left|\frac{1}{2}, \frac{1}{2}\right\rangle$

2- Evaluate $l_z|1,0\rangle$

9.8 Zeeman Effect

As we mentioned in Chapter 3, the splitting of spectral lines when an atom is placed in an external magnetic field was looked for by Faraday, predicted on the basis of classical theory by Lorentz, and first observed by Zeeman,[17] for whom the effect is now named.

In quantum mechanics, a shift in the frequency and wavelength of a spectral line implies a shift in the energy level of one or both of the states involved in the transition. The Zeeman effect that occurs for spectral lines resulting from a transition between *singlet* states is traditionally called the *normal* effect, while that which occurs when the total spin of either the initial or final states, or both, is nonzero is called the *anomalous* effect.[18] There is no fundamental difference between the two, however, so we will generally not distinguish between them, save for one exception: the large value of the electron's magnetic moment complicates the explanation of the anomalous effect a bit, so we will discuss the Zeeman effect in transitions between singlet states first.

Normal Zeeman Effect

For singlet states, the spin is zero and the total angular momentum **J** is equal to the orbital angular momentum **L**. When placed in an external magnetic field, the energy of the atom changes because of the energy of its magnetic moment in the field, which is given by

$$\Delta E = -\boldsymbol{\mu} \cdot \mathbf{B} = -\mu_z B \qquad \textbf{7-68}$$

where the z direction is defined by the direction of **B** (compare with Equation 7-54). Using Equation 7-45 for μ_z, we have $\mu_z = -m_\ell \mu_B = -m_\ell(e\hbar/2m_e)$, and

$$\Delta E = +m_\ell \frac{e\hbar}{2m_e} B = m_\ell \mu_B B \qquad \textbf{7-69}$$

Since there are $2\ell + 1$ values of m_ℓ, each energy level splits into $2\ell + 1$ levels. Figure 7-28 shows the splitting of the levels for the case of a transition between a state with $\ell = 2$ and one with $\ell = 1$. The selection rule $\Delta m_\ell = \pm 1$ restricts the number of possible lines to the nine shown.

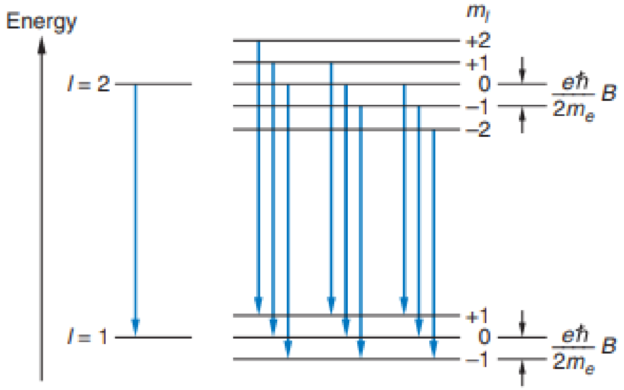


FIGURE 7-28 Energy-level splitting in the normal Zeeman effect for singlet levels $\ell = 2$ and $\ell = 1$. Each level is split into $2\ell + 1$ terms. The nine transitions consistent with the selection rule $\Delta m = 0, \pm 1$, give only three different energies because the energy difference between adjacent terms is $e\hbar B/2m_e$ independent of ℓ.

Because of the uniform splitting of the levels, there are only three different transition energies: $E_0 + e\hbar B/2m_e$, E_0, *and* $E_0 - e\hbar B/2m_e$, *corresponding to the transitions with* $\Delta m_\ell = +1$, $\Delta m_\ell = 0$, *and* $\Delta m_\ell = -1$. We can see that there will only be these energies for any initial and final values of ℓ. The change in the frequency of the emitted spectral line is the energy change divided by h. The frequency changes are therefore $\pm eB/2m_e$ or 0.