Chapter Two

Limit and Continuity

Distance between points and Distance between point and the line

Distance between two points A=(x_1, y_1) and B=(x_2, y_2) is defined by

$$d(A, B) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Example.

The distance between two points (-1,2) and (3,4) is

$$d = \sqrt{(3 - (-1))^2 + (4 - 2)^2} = \sqrt{20}$$

Distance between point and the line defined by

$$d = \frac{|Ax + By + c|}{\sqrt{A^2 + B^2}}$$

Example.

The distance between $P = (3,4)$ and the line $2x + 3y + 1$ is

$$d = \frac{|2(3) + 3(4) + 1|}{\sqrt{2^2 + 3^2}} = \frac{19}{\sqrt{13}}$$

Slope.

The Slope of non-vertical line is

$$m = \frac{rise}{run} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

The slope of a line is the tangent of the lines angle of inclination. If m denotes the slope and Ø is the angle, then: $m = tan\emptyset$.

Example.

Find the slope of the line that crosses the x-axis with angle $\emptyset = \frac{\pi}{4}$.

Solution.

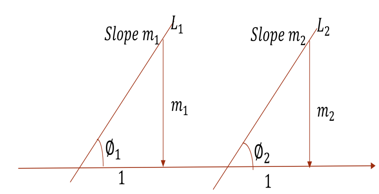
$m = tan\emptyset = tan\ 45^{\circ} = 1$

Lines that are parallel or perpendicular

Parallel lines have equal angles of inclination. Hence, if they are not vertical, parallel lines have the same slope.

Conversely is true if neither of two perpendicular lines L_1 and L_2 is vertical, their slope m_1 *and* m_2 are related by the equation

$$m_1 . m_2 = -1.$$



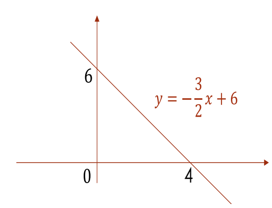
If $m_1 = m_2$, then $\emptyset_1 = \emptyset_2$ and the lines are parallel

Point – Slope equation

The equation $y - y_1 = m(x - x_1)$ is the point- slope equations of the line that passes through the point (x_1, y_1) with slope m.

Examples. Write the equation for the line that passes the point

i- (2,3) with the slope $-\frac{3}{2}$.



Solution:

$$y - y_1 = m(x - x_1)$$

$$y - 3 = -\frac{3}{2}(x - 2)$$

$$y = -\frac{3}{2}x + 6$$

Slope –Intercept Equations

If we take the point $(x_1, y_1) = (0, b)$ in the point slope equation for the line, we find that

$$y - b = m(x - 0)$$

Hence $y = mx + b$

Hence, the equation $y = mx + b$ is the slope – intercept equation of the line with slope m *and* y –intercept b .

Example.

1-The slope –intercept equation of line with slope 2 and $y -$ intercept 5 is

$$y = 2x + 5$$

2-Find the slope and y –intercept of the line $8x + 5y = 20$

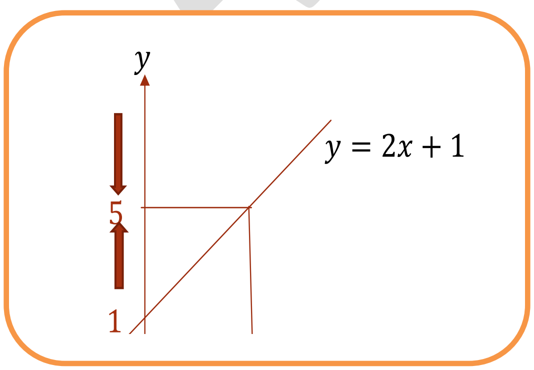
Solution: H.W

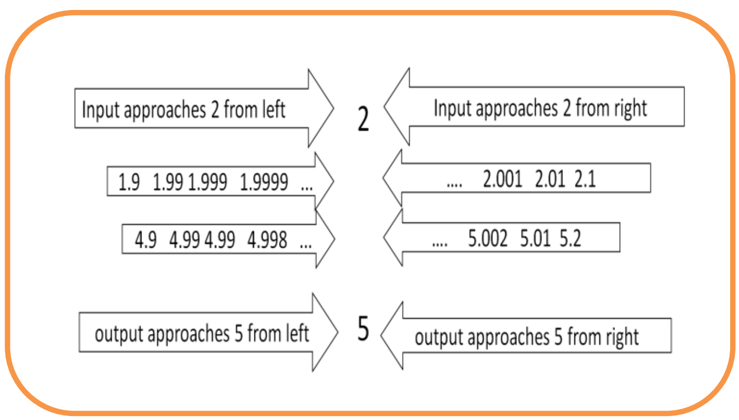
Limit and Law of Limit

Definition: if the values of a function f of x approach the value L as x approaches c we say f has limit L as x approaches to c and we write

$$\lim_{x \to c} f(x) = L$$

Example. As table and Fig. suggest $\lim_{x \to 2}(2x + 1) = 5$





The Limit Law. If L, M, c, and k are real numbers and $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$,

then:

1. Sum Rule: $\lim_{x\to c}(f(x) + g(x)) = L + M$
2. Difference Rule: $\lim_{x\to c}(f(x) - g(x)) = L - M$
3. Constant Multiple Rule: $\lim_{x\to c}(K.f(x)) = K.M$, *any number K*
4. Product Rule: $\lim_{x\to c}\big(f(x).g(x)\big) = L.M$
5. Quotient Rule: $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{L}{M}, \ M \neq 0$
6. Power Rule: $\lim_{x\to c}\big(f(x)\big)^{\frac{r}{n}} = (L)^{\frac{r}{n}}$, $n \neq 0$

Remark.

1. If $f(x) = k$, (f is constant function), then $\lim_{x\to c} f(x) = \lim_{x\to c} k = k$.
2. If $f(x) = x$ (f is identity function), then $\lim_{x\to c} f(x) = \lim_{x\to c} x = c$.
3. If $f(x) = ax^2 + bx + k$ (f is quadratic function), then
4. $\lim_{x\to c} f(x) = \lim_{x\to c} ax^2 + \lim_{x\to c} bx + \lim_{x\to c} k = ac^2 + bc + k$

Examples. Find

1). $\lim_{x\to 3} x^2(2 - x)$

2). $\lim_{x\to 2} \frac{x^2+4x+4}{x+2}$

Solution 1). $\lim_{x\to 3} x^2(2 - x) = 3^2.(2 - 3) = -9$

2) $\lim_{x\to 2} \frac{x^2+2x+4}{x+2} = \frac{4+8+4}{4} = 4$

or $\lim\limits_{x\to 2} \frac{x^2+4x+4}{x+2} = \lim\limits_{x\to 2} \frac{(x+2)^2}{x+2} = \lim\limits_{x\to 2} x + 2 = 4.$

Right-hand and Left-hand limit

Right hand limit:

$\lim\limits_{x\to c^+} f(x) = L,$ means that $f(x) \to L$ as $x \to c$ from right.

- Left hand limit:

$\lim\limits_{x\to c^-} f(x) = L,$ means that $f(x) \to L$ as $x \to c$ from left.

- Limit exist:

$\lim\limits_{x\to c} f(x)$ exist and equal to L , if and only if $\lim\limits_{x\to c^+} f(x) = \lim\limits_{x\to c^-} f(x) = L.$

Example 1. If $f(x) = \begin{cases} x^2 - 2 \,, & x \geq 2 \\ x \,, & x < 2 \end{cases}$ then

$\lim\limits_{x\to 2^+} f(x) = \lim\limits_{x\to 2^+} (x^2 - 2) = 2,$ and $\lim\limits_{x\to 2^-} f(x) = 2,$

$i.e$ Limit exists. Since $\lim\limits_{x\to 2^+} f(x) = \lim\limits_{x\to 2^-} f(x) = 2.$

Example 2. Find $\lim\limits_{x\to 0} \frac{|x|}{x}$

Solution: since $|x| = \begin{cases} x, & x \geq 0 \\ x, & x < 0 \end{cases}$

then $\lim\limits_{x\to 0^+} \left(\frac{|x|}{x}\right) = \lim\limits_{x\to 0^+} \left(\frac{x}{x}\right) = 1,$ and

$\lim\limits_{x\to 0^-} \left(\frac{|x|}{x}\right) = \lim\limits_{x\to 0^-} \left(\frac{-x}{x}\right) = -1,$

$i.e \lim\limits_{x\to 0} \left(\frac{|x|}{x}\right)$ does not exist.

Because $\lim\limits_{x\to 0^+} f(x) = 1 \neq -1 = \lim\limits_{x\to 0^-} f(x)$.

Example 3. Find $\lim\limits_{x\to 0} [x]$. **Solution**: H.W

Note. To find Limits, use substitution

a) If you get real number, then the limit exists.

b) If you get ∞, $-\infty$, then the limit does not exist.

c) If you get $\frac{0}{0}$, $\frac{\infty}{\infty}$, 1^{∞}, ∞^{0}, $\infty-\infty$, you have some operations before decided the limit exist or not (these operations: factors analysis, multiply with conjugate, simplification).

Example 4. Find $\lim_{x\to 2}\left(\frac{x^2-4}{x-2}\right)$.

Solution. $\lim_{x\to 2}\left(\frac{x^2-4}{x-2}\right)=\frac{0}{0}$

$$\therefore \lim_{x\to 2}\left(\frac{x^2-4}{x-2}\right)=\lim_{x\to 2}\left(\frac{(x+2)(x-2)}{x-2}\right)=\lim_{x\to 2}(x+2)=4$$

Example 5 Find $\lim_{x\to 1}\left(\frac{x-1}{\sqrt{x}-1}\right)$.

Solution. $\lim_{x\to 1}\left(\frac{x-1}{\sqrt{x}-1}\right)=\frac{0}{0}$

$$\therefore \lim_{x\to 1}\left(\frac{x-1}{\sqrt{x}-1}\right)=\lim_{x\to 1}\left(\frac{(\sqrt{x}+1)(\sqrt{x}-1)}{\sqrt{x}-1}\right)=\lim_{x\to 2}\left(\sqrt{x}+1\right)=2$$

Example 6. Find $\lim_{x\to 2}\left(\frac{1}{x^2-4}\right)$

Solution. $\lim_{x\to 2}\left(\frac{1}{x^2-4}\right)=\frac{1}{0}=\infty$

Example 7. Find $\lim_{x\to 0}\left(\frac{\sqrt{x+9}-3}{x}\right)$.

Solution:$\lim_{x\to 0}\left(\frac{\sqrt{x+9}-3}{x}\right)=\frac{0}{0}$

$$\therefore \lim_{x\to 0}\left(\frac{\sqrt{x+9}-3}{x}\right) = \lim_{x\to 0}\left(\frac{(\sqrt{x+9}-9)(\sqrt{x+9}+3)}{x\sqrt{x+9}+3}\right)$$

$$= \lim_{x\to 0}\left(\frac{x+9-9}{x\sqrt{x+9}+3}\right) = \lim_{x\to 0}\left(\frac{1}{\sqrt{x+9}+3}\right) = \frac{1}{6}\ .$$

Example 8. find $\mathop{lim}\limits_{x\to 3^-}(x-[x])$

Solution. $\mathop{lim}\limits_{x\to 3^-}(x-[x]) = \mathop{lim}\limits_{x\to 3^-}(x) - \mathop{lim}\limits_{x\to 3^-}([x]) = 3-2 = 1$

Example 9 Find $\lim\limits_{x\to 3}([x-1])$.

Solution: $\therefore \lim\limits_{x\to 3^+}([x-1]) = \lim\limits_{x\to 3^+} 2 = 2$

And $\lim\limits_{x\to 3^-}([x-1]) = \lim\limits_{x\to 3^-} 1 = 1$

$\therefore \lim\limits_{x\to 3}([x-1])$ does not exist.

Example 10. Evaluate $\lim\limits_{x\to 0}\frac{\sqrt{x^2+100}-10}{x^2}$.

Solution.

$$\frac{\sqrt{x^2+100}-10}{x^2} = \frac{\sqrt{x^2+100}-10}{x^2}\cdot\frac{\sqrt{x^2+100}+10}{\sqrt{x^2+100}+10}$$

$$= \frac{x^2+100-100}{x^2(\sqrt{x^2+100}+10)} = \frac{x^2}{x^2(\sqrt{x^2+100}+10)}$$

$$= \frac{1}{(\sqrt{x^2+100}+10)}$$

here fore, $\lim\limits_{x\to 0}\frac{\sqrt{x^2+100}-10}{x^2} = \lim\limits_{x\to 0}\frac{1}{(\sqrt{x^2+100}+10)} = \frac{1}{20} = 0.05$

Exercise. Find

1- $\lim\limits_{x\to 0}\left(\frac{tanx-sinx}{sin^3x}\right)$.

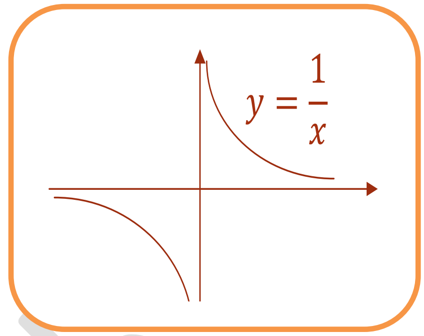
2- $\lim\limits_{x\to 5}\left(\frac{x^2-25}{3x-15}\right)$.

3-Find $\lim\limits_{x\to 0}\left(\frac{1-cosx}{sinx}\right)$.

4- if $f(x) = \begin{cases} 3-x, & x<2 \\ \frac{x}{2}+1, & x>2 \end{cases}$, Is $\lim\limits_{x\to 2} f(x)$ exist?

Limit involving infinity

- Limit as $x \to \infty$ or $x \to -\infty$,



for example

$\lim_{x\to\infty}\left(\frac{1}{x}\right) = 0,$ $\lim_{x\to-\infty}\left(\frac{1}{x}\right) = 0$

Examples.

- $\lim_{x\to\infty}\left(5+\frac{1}{x}\right) = \lim_{x\to\infty}\left(5+\frac{1}{x}\right) = \lim_{x\to\infty} 5 + \lim_{x\to\infty}\left(\frac{1}{x}\right) = 5+0 = 5$
- $\lim_{x\to-\infty}\left(\frac{4}{x^2}\right) = \lim_{x\to-\infty}\left(\frac{4}{x^2}\right) = \lim_{x\to-\infty} 4 \,.\, \lim_{x\to-\infty}\left(\frac{1}{x}\right).\, \lim_{x\to-\infty}\left(\frac{1}{x}\right) = (4)(0)(0) = 0$

Limits of Rational Functions as x → ±∞

1. $\lim_{x\to\pm\infty}\left(\frac{f(x)}{g(x)}\right) = 0$ if $\deg(f) < \deg(g)$
2. $\lim_{x\to\pm\infty}\left(\frac{f(x)}{g(x)}\right) = k$ (k *is* finite number) if $\deg(f) = \deg(g)$
3. $\lim_{x\to\pm\infty}\left(\frac{f(x)}{g(x)}\right) = \infty$ if $\deg(f) > \deg(g)$.

Examples

1- $\lim_{x\to\infty}\left(\frac{-x}{7x+4}\right) = \lim_{x\to\infty}\left(\frac{-x}{7x+4}\right) = \lim_{x\to\infty}\left(\frac{-\frac{x}{x}}{\frac{7x}{x}+\frac{4}{x}}\right)$

$$= \lim_{x\to\infty}\left(\frac{-1}{7+\frac{4x}{x}}\right) = \frac{-1}{7+0} = \frac{-1}{7}$$

2- $\lim_{x\to\infty}\left(\frac{5x+2}{2x^3-1}\right) = \lim_{x\to\infty}\left(\frac{5x+2}{2x^3-1}\right) = \lim_{x\to\infty}\left(\frac{\frac{5x}{x^3}+\frac{2}{x^3}}{\frac{2x^3}{x^3}-\frac{1}{x^3}}\right) = \lim_{x\to\infty}\left(\frac{\frac{5}{x^2}+\frac{2}{x^3}}{2-\frac{1}{x^3}}\right) = \frac{0+0}{2-0} = 0.$

3-. $\lim_{x\to\infty}\left(\frac{x^2+1}{x-1}\right) = \lim_{x\to\infty}\left(\frac{x^2+1}{x-1}\right) = \lim_{x\to\infty}\left(\frac{\frac{x^2}{x^2}+\frac{1}{x^2}}{\frac{x}{x^2}-\frac{1}{x^2}}\right) = \lim_{x\to\infty}\left(\frac{1+\frac{1}{x^2}}{\frac{1}{x}-\frac{1}{x^2}}\right) = \frac{1+0}{0-0} = \infty.$

Some Important Limits:

1. $\lim\limits_{x\to 0} \sin x = 0$
2. $\lim\limits_{x\to 0} \cos x = 0$
3. $\lim\limits_{x\to 0} \frac{\sin x}{x} = 1 = \lim\limits_{x\to 0} \frac{x}{\sin x}$
4. $\lim\limits_{x\to 0} \frac{\tan x}{x} = 1 = \lim\limits_{x\to 0} \frac{x}{\tan x}$
5. $\lim\limits_{x\to 0} \frac{\log(1+x)}{x} = 1$
6. $\lim\limits_{x\to 0} e^{x} = 1$
7. $\lim\limits_{x\to 0} \frac{e^{x}-1}{x} = 1$
8. $\lim\limits_{x\to 0} \frac{e^{x}-1}{x} = 1$
9. $\lim\limits_{x\to 0} \frac{a^{x}-1}{x} = log_e a$
10. $\lim\limits_{x\to \infty} \left(1+\frac{1}{x}\right)^{x} = e$
11. $\lim\limits_{x\to \infty} \left(1+\frac{1}{x}\right)^{x} = e = \lim\limits_{x\to -\infty} \left(1+\frac{1}{x}\right)^{x}$

In Exercises 5–22, find the limit.

5. $\lim\limits_{x\to 2} x^4$ **6.** $\lim\limits_{x\to -2} x^3$

7. $\lim\limits_{x\to 0} (2x-1)$ **8.** $\lim\limits_{x\to -3} (3x+2)$

9. $\lim\limits_{x\to -3} (x^2+3x)$ **10.** $\lim\limits_{x\to 1} (-x^2+1)$

11. $\lim\limits_{x\to -3} (2x^2+4x+1)$ **12.** $\lim\limits_{x\to 1} (3x^3-2x^2+4)$

13. $\lim\limits_{x\to 2} \frac{1}{x}$ **14.** $\lim\limits_{x\to -3} \frac{2}{x+2}$

15. $\lim\limits_{x\to 1} \frac{x-3}{x^2+4}$ **16.** $\lim\limits_{x\to 3} \frac{2x-3}{x+5}$

17. $\lim\limits_{x\to 7} \frac{5x}{\sqrt{x+2}}$ **18.** $\lim\limits_{x\to 3} \frac{\sqrt{x+1}}{x-4}$

19. $\lim\limits_{x\to 3} \sqrt{x+1}$ **20.** $\lim\limits_{x\to 4} \sqrt[3]{x+4}$

21. $\lim\limits_{x\to -4} (x+3)^2$ **22.** $\lim\limits_{x\to 0} (2x-1)^3$

In Exercises 23–26, find the limits.

23. $f(x) = 5-x,\ g(x) = x^3$
(a) $\lim\limits_{x\to 1} f(x)$ (b) $\lim\limits_{x\to 4} g(x)$ (c) $\lim\limits_{x\to 1} g(f(x))$

24. $f(x) = x+7,\ g(x) = x^2$
(a) $\lim\limits_{x\to -3} f(x)$ (b) $\lim\limits_{x\to 4} g(x)$ (c) $\lim\limits_{x\to -3} g(f(x))$

25. $f(x) = 4-x^2,\ g(x) = \sqrt{x+1}$
(a) $\lim\limits_{x\to 1} f(x)$ (b) $\lim\limits_{x\to 3} g(x)$ (c) $\lim\limits_{x\to 1} g(f(x))$

26. $f(x) = 2x^2-3x+1,\ g(x) = \sqrt[3]{x+6}$
(a) $\lim\limits_{x\to 4} f(x)$ (b) $\lim\limits_{x\to 21} g(x)$ (c) $\lim\limits_{x\to 4} g(f(x))$

Continuity

➨ Definition.

The function $y = f(x)$ is continuous at $x = c$ if and only if all three of the following statements are true

1. $f(c)$ exist (c lies in domain of f)
2. $\lim\limits_{x\to c} f(x)\ exist$ (f has a limit as $x \to c$)
3. $\lim\limits_{x\to c} f(x) = f(c)$ (the limit equals the function value).

➨ Definition.

The function $f(x)$ is continuous if it is continuous at each point of its Domain. Otherwise f is discontinuous (not continuous).

Examples.

1. Show that $f(x) = x^2 + x + 1$ is continuous at $x = -1$.

Solution.

1. $f(-1) = (-1)^2 + (-1) + 1 = 1$ exist

2. $\lim_{x\to -1}(x^2 + x + 1) = 1$ exist

3. $\lim_{x\to -1} f(x) = f(-1) = 1$

$\therefore f(x)$ is continuous at $x = -1$

2. Let $f(x) = \begin{cases} 2 & if\ x = 1 \\ \frac{x^2-1}{x-1} & if\ x \neq 1 \end{cases}$, Is f(x) continuous at $x = 1$.

Solution.

1. $f(1) = 2$ exist

2. $\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1} \frac{(x+1)(x-1)}{x-1} = \lim_{x\to 1}(x+1) = 2$ exist

3. $\lim_{x\to 1} f(x) = f(1) = 2$ $\therefore f(x)$ is continuous at $x = 1$.

3. Let $f(x) = \begin{cases} 2x & if\ x \leq 3 \\ \frac{x^2-9}{x-3} & if\ x > 3 \end{cases}$

- Is $f(x)$ continuous at $x = 3$.
- Is $f(x)$ continuous at $x = 4$.
- Is $f(x)$ continuous at $x = 2$.

Solution. H.W

4- Let $f(x) = \begin{cases} \frac{x^2-9}{x-3} & if\ x < 3 \\ kx - 3 & if\ x \geq 3 \end{cases}$, find the value of k that makes f(x) is continuous at $x = 3$.

Solution.

Since $f(x)$ is continuous at x=3 then $\lim_{x\to 3} f(x) = f(3)$

$$\lim_{x\to 3}\frac{x^2-9}{x-3} = 3k-3 \Longrightarrow \lim_{x\to 3}\frac{(x+3)(x-3)}{x-3} = 3k-3 \Longrightarrow \lim_{x\to 3} x+3 = 3k-3$$

$$6 = 3k-3 \Longrightarrow 3k = 9 \Longrightarrow k = 3.$$

Exercise.

1- Let $f(x) = \begin{cases} 2x+1 & if\ x > 2 \\ 3x-1 & if\ x < 2 \end{cases}$, Is $f(x)$ continuous.

2- if $f(x) = \begin{cases} 2x & if\ x > 3 \\ x^2 & if\ x \le 3 \end{cases}$, Is f(x) continuous or not.

3- Show that $f(x) = \frac{\sqrt{x}}{x^2+1}$ is continuous at $x = 1$

4- Show that $f(x) = |x-2|$ is continuous at $x = 2$

5- Let $f(x) = \begin{cases} x+2 & if\ x \ge 2 \\ 3x-2 & if\ x < 2 \end{cases}$, is continuous.

6- Let $f(x) = \begin{cases} \frac{sin2x}{x} & if\ x \ne 0 \\ 2 & if\ x = 0 \end{cases}$, is continuous.

7- Let $f(x) = ax + 5$ is continuous at $x = 1$, and $f\ (1)\ = 4$ find a.

8- Prove that $f(x) = tanx$ is continuous at $x \in \left[0, \frac{\pi}{2}\right]$.

9- Prove that $f(x) = cosx$ is continuous at $x \in \boldsymbol{R}$

10- Prove that $f(x) = e^x$ is continuous at $x \in \boldsymbol{R}$

11- Let $f(x) = \begin{cases} kx^2 & \text{if } x \le 2 \\ 3 & \text{if } x > 2 \end{cases}$, is continuous at x=2, find k.

- The polynomials. $p(x) = a_\circ + a_1x + a_2x^2 + \cdots + a_nx^n$ are continuous at every point of its domain of definition.
- The rational function is continuous at every point where it's denominator is nonzero.

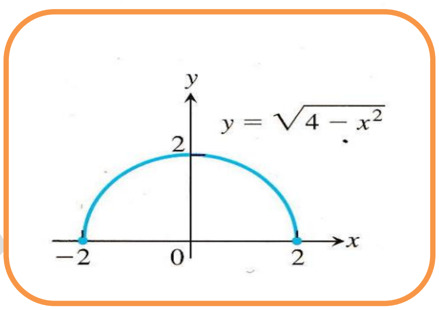
1. $f(x) = \frac{2x}{x-2}$ is continuous at $R\backslash\{2\}$
2. $f(x) = \frac{1}{x}$ is continuous at $R\backslash\{0\}$

Example. Show that $f(x) = x + |x|$ is continuous at $x = 2$.

Solution. Clearly that $f_1(x) = x$ is continuous at $x = 2$ (since $f_1(x)$ is polynomial) and $f_2(x) = |x| = \begin{cases} x, & x \ge 0 \\ -x, & x < 0 \end{cases}$, is continuous at $x = 2$

$\therefore f(x) = f_1(x) + f_2(x) = x + |x|$ is continuous at $x = 2$.

EXAMPLE The function $f(x) = \sqrt{4 - x^2}$ is continuous over its domain $[-2, 2]$. It is right-continuous at $x = -2$, and left-continuous at $x = 2$.



➨ Signum Function.

The function f(x)=sign(x) is defined by

$$f(x) = \begin{cases} -1 & x < 0 \\ 0 & x = 0 \\ 1 & x > 1 \end{cases}$$

Is discontinuous at $x = 0$ because $f(0) = 0$ exist but

$$\lim_{x\to 1^+} f(x) = 1 \neq -1 = \lim_{x\to 1^-} f(x)$$

➨ Great Integer Function.

The $f(x) = [x]$ is continuous at $x = \frac{1}{2}$ but discontinuous at x=1.

$$\lim_{x\to 1^+} [x] = [1.1] = 1 \neq 0 = [0.9] = \lim_{x\to 1^-} [x]$$

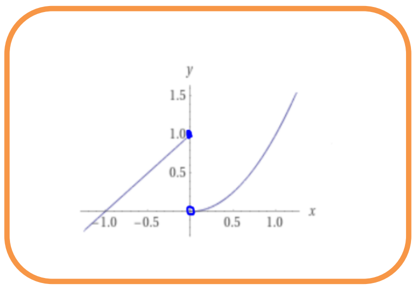
$$\lim_{x\to \frac{1}{2}^+} [x] = [0.6] = 0 = [0.4] = \lim_{x\to \frac{1}{2}^-} [x]$$

Types of Discontinuous Functions

➨ 1- Jump Discontinuity.

A function f has a jump discontinuity at x=c if $\lim_{x\to c^+} f(x) \neq \lim_{x\to c^-} f(x)$

Example. If $f(x) = \begin{cases} x + 1 & \text{if } x \leq 0 \\ x^2 & \text{if } x > 0 \end{cases}$,



Is $f(x)$ continuous at $x = 0$.

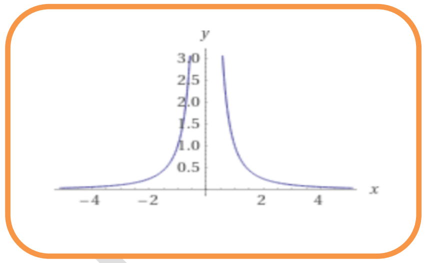
Solution.

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x^2 = 0 \neq 1 = \lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (x + 1)$$

f is discontinuous at $x = 0$ because the left not equal right hand limit.

➨ 2- Infinite Discontinuity

A function f has an infinite discontinuity at $x = c$ if it is unbounded in every interval about c.

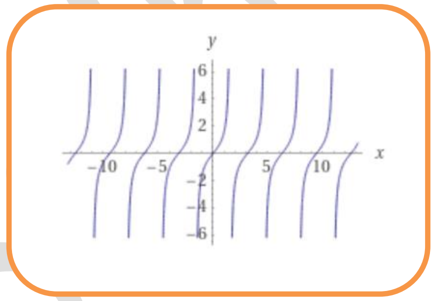


Example. If $f(x) = \frac{1}{x^2}$ Is f continuous.

Solution.

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{1}{x^2} = \frac{1}{0} = \infty$$

f has infinite discontinuous at $x = 0$.



Example. If $f(x) = tanx$ Is f continuous.

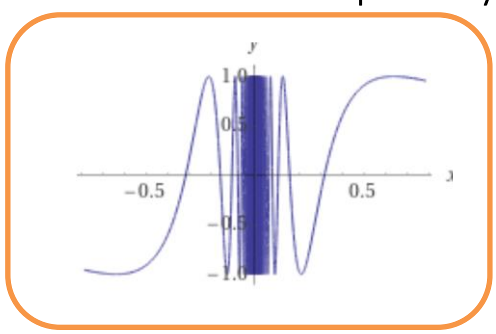
Solution.

$$\lim_{x\to \frac{\pi}{2}} f(x) = \lim_{x\to \frac{\pi}{2}} tanx = \infty$$

f has infinite discontinuous at $x = \pm\frac{(2n+1)\pi}{2}$.

➨ **3-Oscillatory Discontinuity.**

A function f has an Oscillatory discontinuity at x=c that has more complicate type



Example. If $\mathrm{f(x)} = \sin\frac{1}{x}$ Is f continuous.

Solution.

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} sin\frac{1}{x} = sin\frac{1}{0} = \infty$$

Limit not exists

f has Oscillatory discontinuous at $x = 0$.

4-Removable Discontinuity

A function f has a removable discontinuity at $x = c$ if $\lim_{x \to 0} f(x)$ exists and equal L. *and* f either dose not exists or $f(c) \neq L$.

Example 1. If $f(x) = \begin{cases} x^2 & x < 1 \\ 2 - x & x > 1 \end{cases}$ Is f continuous.

Solution.

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} 2 - x = 1 = \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x^2$$

Limit exists and equal 1. but $f\ (1)$ not exists

f has Removable discontinuous at $x = 1$.

Example 2. If $f(x) = \begin{cases} \frac{1 - cosx}{x} & x \neq 0 \\ 3 & x = 0 \end{cases}$ Is f continuous.

Solution. $\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{1 - cosx}{x} = 0$

Limit exists and equal 0. but $f\ (0)\ = 3$ not equal limit

f has Removable discontinuous at $x = 0$.